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THE GREEN RINGS OF MINIMAL HOPF QUIVERS

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Abstract Let $\Bbbk$ be a field and let Q be a minimal Hopf quiver, i.e. a cyclic quiver or the infinite linear quiver, and let $\mathrm{rep}^{\mathrm{ln}}(Q)$ denote the category of locally nilpotent finite-dimensional $\Bbbk$-representations of Q. The category $\mathrm{rep}^{\mathrm{ln}}(Q)$ has natural tensor structures induced from graded Hopf structures on the path coalgebra $\Bbbk Q$. Tensor categories of the form $\mathrm{rep}^{\mathrm{ln}}(Q)$ are an interesting class of tame hereditary pointed tensor categories that are not finite. The aim of this paper is to compute the Clebsch–Gordan formulae and Green rings of such tensor categories.

Keywords: Green ring; tensor category; quiver representation

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1. Introduction

Throughout the paper we work over an algebraically closed field $\Bbbk$ of characteristic 0. Vector spaces, (co)algebras, (co)modules, Hopf algebras, categories, morphisms and unadorned $\otimes$ are over $\Bbbk$. By a tensor category, we mean a locally finite abelian rigid monoidal category in which the neutral object is simple; see [**4**] for unexplained notions of tensor categories. Recall that the Green ring of a tensor category $\mathcal{C}$, denoted by $\mathcal{GR}(\mathcal{C})$, is the free abelian group generated by the isomorphism classes $[X]$ of objects in $\mathcal{C}$, with multiplication given by tensor product $[X] \cdot [Y] = [X \otimes Y]$ modulo all split short exact sequences. It is well known that Green rings are a convenient way of organizing information about direct sums and tensor products of tensor categories.

Given a tensor category $\mathcal{C}$, it is certainly interesting to determine its Green ring $\mathcal{GR}(\mathcal{C})$. However, this mission is generally too complicated to be accomplished. Recently, the Green rings were computed for some relatively less complicated tensor categories, for example the module categories of Taft algebras in [**2**], the module categories of generalized Taft algebras in [**11**], and pointed tensor categories of finite type [**8**]. A key feature of the tensor categories investigated in [**2**, **8**, **11**] is that there are only finitely many indecomposable objects, up to isomorphism, in them. In retrospect, this is the main reason that their Green rings are computable.

The aim of this paper is to compute the Green rings of some tame hereditary pointed tensor categories. From the viewpoint of the representation theory of algebras (see, for example, [**1**]), this is a further natural question that we may ask ourselves immediately after [**8**]. A tensor category $\mathcal{C}$ is said to be pointed if every simple object of $\mathcal{C}$ is invertible. By reconstruction theorem [**4**], a pointed tensor category with a fibre functor can be presented as the category of finite-dimensional right comodules over a pointed Hopf algebra. On the other hand, the tame hereditary condition is equivalent to saying that such a category is equivalent to the category of locally nilpotent finite-dimensional representations of a cyclic quiver or the infinite linear quiver (see [**7**,**12**]). Cyclic and infinite linear quivers are called minimal Hopf quivers as they are basic building blocks of general Hopf quivers [**3**,**7**]. Note that finite-dimensional indecomposable representations of cyclic and infinite linear quivers are explicitly classified in quiver representation theory [**1**] and the Hopf structures over such quivers are given in [**7**]. Now we are in a good position to compute the associated Green rings via a similar idea to that in [**8**]. We remark that the Green rings of the categories of quiver representations with the vertex-wise and arrow-wise tensor product were studied in [**5**, **6**, **9**, **10**]. Note that such tensor structures are generally not induced from a bialgebra, and hence are quite different from ours.

The paper is organized as follows. In § 2 we review some necessary facts. In §§ 3 and 4 we compute the Clebsch–Gordan formulae and Green rings of the tensor categories associated with cyclic and infinite linear quivers, respectively.

2. Preliminaries

In this section we recall some preliminary notions and facts about quivers, representations, path coalgebras, Hopf quivers and tensor categories.

2.1. Quivers and path coalgebras

A quiver is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is the set of vertices, Q_1 is the set of arrows and $s, t\colon Q_1 \to Q_0$ are two maps assigning, respectively, the source and the target for each arrow. For $a \in Q_1$, we write $a\colon s(a) \to t(a)$. A vertex is, by convention, said to be a trivial path of length 0. We also write $s(g) = g = t(g)$ for each $g \in Q_0$. The length of an arrow is set to be 1. In general, a non-trivial path of length n ($\geqslant 1$) is a sequence of concatenated arrows of the form $p = a_n \cdots a_1$ with $s(a_{i+1}) = t(a_i)$ for $i = 1, \dots, n-1$. By Q_n we denote the set of the paths of length n.

Let Q be a quiver and let $\Bbbk Q$ be the associated path space that is the $\Bbbk$-span of its paths. There is a natural coalgebra structure on $\Bbbk Q$ with comultiplication as the split of paths. Namely, for a trivial path g, set $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$; for a non-trivial path $p = a_n \cdots a_1$, set

$$\Delta(p) = t(a_n) \otimes p + \sum_{i=1}^{n-1} a_n \cdots a_{i+1} \otimes a_i \cdots a_1 + p \otimes s(a_1)$$

and $\varepsilon(p) = 0$. This is the so-called path coalgebra of the quiver Q.

There exists on $\Bbbk Q$ an intuitive length gradation $\Bbbk Q = \bigoplus_{n \geqslant 0} \Bbbk Q_n$, which is compatible with the comultiplication Δ just defined. It is clear that the path coalgebra $\Bbbk Q$ is pointed

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and the set of group-like elements $G(\Bbbk Q)$ is Q_0. Moreover, the coradical filtration of $\Bbbk Q$ is

$$\Bbbk Q_0 \subseteq \Bbbk Q_0 \oplus \Bbbk Q_1 \subseteq \Bbbk Q_0 \oplus \Bbbk Q_1 \oplus \Bbbk Q_2 \subseteq \cdots,$$

and therefore it is coradically graded.

2.2. Hopf quivers

A quiver Q is said to be a Hopf quiver if the corresponding path coalgebra kQ admits a graded Hopf algebra structure (see [**3**]). Hopf quivers can be determined by ramification data of groups. Let G be a group and let $\mathcal{C}$ be the set of conjugacy classes. A ramification datum R of the group G is a formal sum $\sum_{C\in\mathcal{C}} R_C C$ of conjugacy classes with coefficients in $\mathbb{N} = \{0, 1, 2, \dots\}$. The corresponding Hopf quiver $Q = Q(G, R)$ is defined as follows: the set of vertices Q_0 is G and, for each $x \in G$ and $c \in C$, there are R_C arrows going from x to cx. For a given Hopf quiver Q, the set of graded Hopf structures on kQ is in one-to-one correspondence with the set of kQ_0-Hopf bimodule structures on kQ_1.

A Hopf quiver $Q = Q(G, R)$ is connected if and only if the union of the conjugacy classes with non-zero coefficients in R generates G. We denote the unit element of G by e. If $R_{\{e\}} \neq 0$, then there are $R_{\{e\}}$-loops attached to each vertex; if the order of elements in a conjugacy class $C \neq e$ is n and $R_C \neq 0$, then corresponding to these data in Q there is a sub-quiver (n, R_C)-cycle (called an n-cyclic quiver if $R_C = 1$), i.e. the quiver having n vertices, indexed by the set $\mathbb{Z}_n$ of integers modulo n, and R_C arrows going from i to $i+1$ for each $i \in \mathbb{Z}_n$; if the order of elements in a conjugacy class C is ∞, then in Q there is a sub-quiver R_C-chain (called an infinite linear quiver if $R_C = 1$), i.e. a quiver having a set of vertices indexed by the set $\mathbb{Z}$ of integral numbers, and R_C arrows going from j to $j+1$ for each $j \in \mathbb{Z}$. Therefore, cyclic quivers and the infinite linear quiver are basic building blocks of general Hopf quivers and they are called minimal Hopf quivers.

2.3. Quiver representations

Let Q be a quiver. A representation of Q is a collection

$$V = (V_g, V_a)_{g\in Q_0, a\in Q_1}$$

consisting of a vector space V_g for each vertex g and a linear map $V_a\colon V_{s(a)} \to V_{t(a)}$ for each arrow a. A morphism of representations $\phi\colon V \to W$ is a collection $\phi = (\phi_g)_{g\in Q_0}$ of linear maps $\phi_g\colon V_g \to W_g$ for each vertex g such that $W_a\phi_{s(a)} = \phi_{t(a)}V_a$ for each arrow a. Given a representation V of Q and a path p, we define V_p as follows. If p is trivial, say $p = g \in Q_0$, then put $V_p = \mathrm{Id}_{V_g}$. For a non-trivial path $p = a_n \cdots a_2 a_1$, put $V_p = V_{a_n} \cdots V_{a_2} V_{a_1}$. A representation V of Q is said to be locally nilpotent if, for all $g \in Q_0$ and all $x \in V_g$, $V_p(x) = 0$ for all but finitely many paths p with source g. A representation V is said to be finite dimensional if $\sum_{g\in Q_0} \dim V_g < \infty$. Let $\mathrm{rep}^{\mathrm{ln}}(Q)$ denote the category of locally nilpotent finite-dimensional representations of Q. It is well known that the category of finite-dimensional right $\Bbbk Q$-comodules is equivalent to $\mathrm{rep}^{\mathrm{ln}}(Q)$ (see [**12**]).

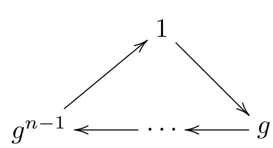
2.4. Tensor categories

A monoidal category is a sextuple $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$, where $\mathcal{C}$ is a category, $\otimes\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ is a bifunctor, $\mathbf{1}$ an object (to be called neutral), $\alpha\colon\ \otimes \circ (\otimes \times \mathrm{Id}) \to \otimes \circ (\mathrm{Id} \times \otimes)$, $\lambda\colon \mathbf{1} \otimes - \to \mathrm{Id}$, $\rho\colon - \otimes \mathbf{1} \to \mathrm{Id}$ are natural isomorphisms such that the associativity and unitarity constraints hold, or equivalently the pentagon and the triangle diagrams are commutative. A tensor category is a locally finite abelian rigid monoidal category in which the neutral object is simple. Natural examples of tensor categories are the categories of finite-dimensional H-modules and H-comodules, where H is a Hopf algebra equipped with an invertible antipode. Recall that if U and V are right H-comodules and we let $U \otimes V$ be the usual tensor product of $\Bbbk$-spaces, then the comodule structure is given by $u \otimes v \mapsto u_0 \otimes v_0 \otimes u_1 v_1$, where we use the Sweedler notation $u \mapsto u_0 \otimes u_1$ for comodule structure maps. The neutral object is the trivial comodule $\Bbbk$ with comodule structure map $k \mapsto k \otimes 1$. On the other hand, by the reconstruction formalism, a tensor category with a fibre functor is tensor equivalent to the category of H-modules for some Hopf algebra H. For more details on tensor categories, see [**4**].

3. The Green ring of a cyclic quiver

3.1. Hopf structures over a cyclic quiver

Let $G = \langle g | g^n = 1\rangle$ be a cyclic group of order n and let $\mathcal{Z}$ denote the Hopf quiver $Q(G, g)$. The quiver $\mathcal{Z}$ is a cyclic quiver of form



If $n = 1$, then $\mathcal{Z}$ is the one-loop quiver, that is, it consists of one vertex and one loop. It is easy to see that such a quiver provides only the familiar divided power Hopf algebra in one variable, which is isomorphic to the polynomial algebra in one variable [**7**].

From now on we assume that $n > 1$. For each integer $i \in \mathbb{Z}_n$, let a_i denote the arrow $g^i \to g^{i+1}$. Let p_i^l denote the path $a_{i+l-1} \cdots a_{i+1} a_i$ of length l. Then $\{p_i^l \mid i \in \mathbb{Z}_n,\ l \geqslant 0\}$ is a basis of $\Bbbk\mathcal{Z}$. We also need the notation of Gaussian binomials. For any $q \in \Bbbk$ and integers $l, m \geqslant 0$, let

$$l_q = 1 + q + \cdots + q^{l-1}, \qquad l!_q = 1_q \cdots l_q, \qquad \binom{l+m}{l}_q = \frac{(l+m)!_q}{l!_q m!_q}.$$

When $1 \neq q \in \Bbbk$ is an nth root of unity of multiplicative order d,

$$\binom{l+m}{l}_q = 0 \quad \text{if and only if} \quad \left[\frac{l+m}{d}\right] - \left[\frac{m}{d}\right] - \left[\frac{l}{d}\right] > 0, \tag{3.1}$$

where $[x]$ means the integer part of x.

We now recall the graded Hopf structures on $\Bbbk\mathcal{Z}$. By [**3**], they are in one-to-one correspondence with the $\Bbbk G$-module structures on $\Bbbk a_0$, and in turn with the set of nth roots of unity. For each $q \in \Bbbk$ with $q^n = 1$, let $g.a_0 = qa_0$ define a $\Bbbk G$-module. The corresponding $\Bbbk G$-Hopf bimodule is $\Bbbk G \otimes_{\Bbbk G} \Bbbk a_0 \otimes \Bbbk G = \Bbbk a_0 \otimes \Bbbk G$. We identify $a_i = a_0 \otimes g^i$; this is how we view $\Bbbk\mathcal{Z}_1$ as a $\Bbbk G$-Hopf bimodule. The path multiplication formula

$$p_i^l \cdot p_j^m = q^{im}\binom{l+m}{l}_q p_{i+j}^{l+m} \tag{3.2}$$

was given in [**3**] by induction. In particular,

$$g \cdot p_i^l = q^l p_{i+1}^l, \qquad p_i^l \cdot g = p_{i+1}^l, \qquad a_0^l = l_q! p_0^l. \tag{3.3}$$

For each q, the corresponding graded Hopf algebra is denoted by $\Bbbk\mathcal{Z}(q)$. The following lemma gives the algebra structure by generators and relations.

Lemma 3.1 (Huang *et al.* [7, Lemma 3.2]). *As an algebra, $\Bbbk\mathcal{Z}(q)$ can be presented by generators and relations as follows.*

(1) *If $q = 1$, then the generators are g and a_0, and the relations are $g^n = 1$ and $ga_0 = a_0 g$.*

(2) *If $\operatorname{ord}(q) = d > 1$, then the generators are g, a_0 and p_0^d, and the relations are $g^n = 1$, $ga_0 = qa_0g$, $a_0^d = 0$, $a_0p_0^d = p_0^d a_0$ and $gp_0^d = p_0^d g$.*

3.2. The tensor category associated with $\Bbbk\mathcal{Z}(q)$

The aim of this section is to compute the Green ring of the tensor category of finite-dimensional right $\Bbbk\mathcal{Z}(q)$-comodules. As mentioned in § 2.3, as a category it is equivalent to the category of locally nilpotent finite-dimensional representations of the quiver $\mathcal{Z}$. For each $i \in \mathbb{Z}_n$ and integer $l \geqslant 0$, let $V(i,l)$ be a vector space of dimension $l+1$ with a basis $\{v_m^i\}_{0 \leqslant m \leqslant l}$. $V(i,l)$ is made into a representation of $\mathcal{Z}$ by setting $V(i,l)_j$ as the $\Bbbk$-span of $\{v_m^i \mid i+m = j \text{ in } \mathbb{Z}_n\}$ and letting $V(i,l)_{a_j}$ map v_m^i to v_{m+1}^i if $i+m=j$ in $\mathbb{Z}_n$. Here, by convention v_k^i is understood as 0 if $k > l$. Note that $V(i,l)$ is viewed as a $\Bbbk\mathcal{Z}(q)$-comodule by

$$\begin{aligned} \delta\colon V(i,l) &\to V(i,l) \otimes \Bbbk\mathcal{Z}(q) \\ v_m^i &\mapsto \sum_{j=m}^{l} v_j^i \otimes p_{i+m}^{j-m}. \end{aligned} \tag{3.4}$$

Using (3.1), the comodule structure map of $V(i,l) \otimes V(j,m)$ is given by

$$\delta(v_s^i \otimes v_t^j) = \sum_{x=s}^{l}\sum_{y=t}^{m} q^{(i+s)(y-t)}\binom{x+y-s-t}{x-s}_q v_x^i \otimes v_y^j \otimes p_{i+j+s+t}^{x+y-s-t}. \tag{3.5}$$

It is well known that $\{V(i,l) \mid i \in \mathbb{Z}_n,\ l \geqslant 0\}$ is a complete set of indecomposable objects of $\operatorname{rep}^{\ln}(\mathcal{Z})$, and hence a complete set of finite-dimensional indecomposable $\Bbbk\mathcal{Z}(q)$-comodules.

For application in later computations, we also view $V(i,l) \otimes V(j,m)$ as a rational module of $\mathrm{gr}(\Bbbk\mathcal{Z}(q))^*$, the graded dual algebra of $\Bbbk\mathcal{Z}(q)$. The module structure map is given by

$$(p_e^f)^*(v_s^i \otimes v_t^j) = \delta_{e,i+j+s+t} \sum_{x+y=f} \sum_{x=0}^{l-s} \sum_{y=0}^{m-t} q^{(i+s)(f-x)} \binom{f}{x}_q v_{s+x}^i \otimes v_{t+y}^j. \tag{3.6}$$

Note that $\mathrm{gr}(\Bbbk\mathcal{Z}(q))^*$ is actually the path algebra associated with $\mathcal{Z}(q)$, so we have

$$(p_e^f)^*(v_s^i \otimes v_t^j) = (p_{e+k}^{f-k})^*(p_e^k)^*(v_s^i \otimes v_t^j) \tag{3.7}$$

for all $0 \leqslant k \leqslant f$.

From now on we denote by $\mathcal{C}_q$ the tensor category of $\Bbbk\mathcal{Z}(q)$-comodules for brevity and by $\mathcal{GR}(\mathcal{C}_q)$ its Green ring.

3.3. The case in which $q = 1$

As $q = 1$, the Gaussian binomial coefficients shrink to the usual ones, i.e. $\binom{f}{x}_1 = \binom{f}{x}$. We start with some useful lemmas.

Lemma 3.2.

$$V(1,0) \otimes V(i,l) = V(i+1,l) = V(i,l) \otimes V(1,0), \quad V(1,0)^{\otimes n} = V(0,0) \text{ in } \mathcal{C}_1.$$

Proof. Define $F\colon V(1,0) \otimes V(i,l) \to V(i+1,l)$ by $F(v_0^1 \otimes v_s^i) = v_s^{i+1}$. It is easy to verify that the map F is an isomorphism in $\mathcal{C}_1$. Similarly, one can prove the remaining equalities. □

Lemma 3.3.

$$V(0,1) \otimes V(0,l) = V(0,l+1) \oplus V(1,l-1) = V(0,l) \otimes V(0,1). \tag{3.8}$$

Proof. Define maps

$$\begin{aligned} \phi_1\colon V(0,l+1) &\to V(0,1) \otimes V(0,l) \\ v_0^0 &\mapsto v_0^0 \otimes v_0^0 \\ v_i^0 &\mapsto v_0^0 \otimes v_i^0 + i v_1^0 \otimes v_{i-1}^0 \\ v_{l+1}^0 &\mapsto (l+1) v_1^0 \otimes v_l^0, \end{aligned}$$

where $i = 1, 2, \ldots, l$ and

$$\begin{aligned} \phi_2\colon V(1,l-1) &\to V(0,1) \otimes V(0,l) \\ v_j^1 &\mapsto v_0^0 \otimes v_{j+1}^0 - (l-j) v_1^0 \otimes v_j^0, \end{aligned}$$

where $0 \leqslant j \leqslant l-1$. We now verify that the two maps are comodule monomorphisms. Obviously, the two maps are injective. Using (3.4) and the definition of ϕ_1, we have

$$\begin{aligned}
\delta(\phi_1(v_0^0)) &= \delta(v_0^0 \otimes v_0^0) \\
&= \sum_{y=0}^{l} v_0^0 \otimes v_y^0 \otimes p_0^y + \sum_{y=0}^{l} (y+1) v_1^0 \otimes v_y^0 \otimes p_0^{y+1} \\
&= (\phi_1 \otimes \mathrm{Id})\delta(v_0^0), \\
\delta(\phi_1(v_j^0)) &= \delta(v_0^0 \otimes v_j^0 + j v_1^0 \otimes v_{j-1}^0) \\
&= \sum_{x=j}^{l+1} [v_0^0 \otimes v_x^0 \otimes p_j^{x-j} + (x-j) v_1^0 \otimes v_{x-1}^0 \otimes p_j^{x-j}] + \sum_{y=j}^{l+1} j v_1^0 \otimes v_{y-1}^0 \otimes p_i^{y-j} \\
&= \sum_{x=j}^{l+1} [v_0^0 \otimes v_x^0 + x v_1^0 \otimes v_{x-1}^0] \otimes p_j^{x-j} \\
&= (\phi_1 \otimes \mathrm{Id})\delta(v_j^0)
\end{aligned}$$

for $1 \leqslant j \leqslant l$ and

$$\delta(\phi_1(v_{l+1}^0)) = \delta(v_1^0 \otimes v_l^0) = v_1^0 \otimes v_l^0 \otimes p_{l+1}^0 = (\phi_1 \otimes \mathrm{Id})\delta(v_{l+1}^0).$$

So, we have verified that ϕ_1 is indeed a comodule monomorphism. Similarly, one proves that ϕ_2 is also a comodule monomorphism. Considering the indices of the basis of $\phi_1(V(0,l+1))$ and $\phi_2(V(1,l-1))$, we can see that $\phi_1(V(0,l+1)) \cap \phi_2(V(1,l-1)) = \{0\}$. We then obtain that

$$V(0,l+1) \oplus V(1,l-1) \cong \phi_1(V(0,l+1)) \oplus \phi_2(V(1,l-1)) \subset V(0,1) \otimes V(0,l).$$

By comparing the dimensions we obtain the equality.

Define maps

$$\begin{aligned}
\psi_1 \colon V(0,l+1) &\to V(0,l) \otimes V(0,1) \\
v_0^0 &\mapsto v_0^0 \otimes v_0^0 \\
v_j^0 &\mapsto v_j^0 \otimes v_0^0 + j v_{j-1}^0 \otimes v_1^0 \\
v_{l+1}^0 &\mapsto (l+1) v_l^0 \otimes v_1^0
\end{aligned}$$

for $0 \leqslant j \leqslant l$ and

$$\begin{aligned}
\psi_2 \colon V(1,l-1) &\to V(0,1) \otimes V(0,l) \\
v_j^1 &\mapsto v_{j+1}^0 \otimes v_0^0 + (l-j) v_j^0 \otimes v_1^0
\end{aligned}$$

for $0 \leqslant j \leqslant l-1$. One can prove the second equality in a similar manner. □

For the convenience of later computations, we now make a convention. We will understand the identity in Lemma 3.3 as

$$V(0,l+1) = V(0,1) \otimes V(0,l) - V(1,l-1)$$

thanks to the Krull–Schmidt theorem. Similar identities that interpret the direct sum decomposition of indecomposable objects by suitable subtraction will be used freely in the rest of the paper.

Now we are ready to decompose $V(0,l)\otimes V(0,m)$.

Theorem 3.4.

$$V(0,l)\otimes V(0,m)=\begin{cases}\displaystyle\bigoplus_{i=0}^{l}V(i,l+m-2i), & l\leqslant m,\\ \displaystyle\bigoplus_{i=0}^{m}V(i,l+m-2i), & l>m.\end{cases}$$

Proof. We only prove the case in which $l\leqslant m$ as the other case is similar. Induct on l. For $l=1$, the claim is Lemma 3.3. Now assume that $l\geqslant 2$ and $V(0,j)\otimes V(0,m)=\bigoplus_{i=0}^{j}V(i,j+m-2i)$ if $j<l$. Then, by Lemma 3.3, we have

$$\begin{aligned}V(0,l)\otimes V(0,m)&=[V(0,1)\otimes V(0,l-1)-V(1,l-2)]\otimes V(0,m)\\&=V(0,1)\otimes\left[\bigoplus_{i=0}^{l-1}V(i,l+m-1-2i)\right]-\bigoplus_{i=0}^{l-2}V(1+i,l+m-2-2i)\\&=\bigoplus_{i=0}^{l}V(i,l+m-2i).\end{aligned}$$

The proof is complete. □

In order to present the ring $\mathcal{GR}(\mathcal{C}_1)$, we need the generalized Fibonacci polynomials used in [**2**].

Definition 3.5. For integer $k\geqslant 0$, define the polynomial $f_k(x,y)\in\mathbb{Z}[x,y]$ inductively by

(1) $f_0(x,y)=1$ and $f_1(x,y)=y$, and

(2) $f_k(x,y)=yf_{k-1}(x,y)-xf_{k-2}(x,y)$ if $k\geqslant 2$.

The general formula for $f_k(x,y)$ is given by [**2**, Lemma 3.11] as

$$f_k(x,y)=\sum_{i=0}^{[k/2]}(-1)^i\binom{k-i}{i}x^iy^{k-2i}. \tag{3.9}$$

We need the following fact about $f_k(x,y)$, the proof of which is easy and hence omitted.

Lemma 3.6. *$\{x^if_k(x,y)\mid k,i\geqslant 0\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[x,y]$.*

We are now in a position to present the Green ring $\mathcal{GR}(\mathcal{C}_1)$.

Theorem 3.7. $\mathcal{GR}(\mathcal{C}_1)\cong\mathbb{Z}[x,y]/\langle x^n-1\rangle$.

Proof. Define

$$\begin{aligned} \Phi\colon \mathbb{Z}[x,y] &\to \mathcal{GR}(\mathcal{C}) \\ x &\mapsto [V(1,0)], \\ y &\mapsto [V(0,1)]. \end{aligned}$$

Obviously, Φ is surjective as $[V(1,0)]$ and $[V(0,1)]$ generate $\mathcal{GR}(\mathcal{C})$ by Lemmas 3.2 and 3.3 and Theorem 3.4. By the definition of $f_k(x,y)$ and Lemmas 3.2 and 3.3, we have

$$\Phi(x^i f_k(x,y)) = [V(1,0)]^i f_k([V(1,0)],[V(0,1)]) = [V(i,k)]. \qquad (*)$$

Note that $\Phi(x^n - 1) = 0$ as $V(1,0)^{\otimes n} = V(0,0) = \mathbf{1}$. It follows that $J = \langle x^n - 1\rangle \subset \ker \Phi$, so Φ induces an epimorphism

$$\begin{aligned} \bar{\Phi}\colon \mathbb{Z}[x,y]/J &\to \mathcal{GR}(\mathcal{C}) \\ \bar{x} &\mapsto [V(1,0)], \\ \bar{y} &\mapsto [V(0,1)]. \end{aligned}$$

Now, using Lemma 3.6 it is not hard to see that

$$\{\overline{x^i f_k(x,y)} \mid 0 \leqslant i \leqslant n-1,\ k \geqslant 0\}$$

makes a $\mathbb{Z}$-basis of $\mathbb{Z}[x,y]/J$, and hence $\bar{\Phi}$ is an isomorphism by $(*)$. □

3.4. The case of q being a non-trivial root of unity

In this section, q is set to be a root of unity with multiplicative order $d = \mathrm{ord}(q) \geqslant 2$. Similarly to Lemma 3.2, we have the following lemma.

Lemma 3.8.

$$V(1,0)\otimes V(i,l) = V(i+1,l) = V(i,l)\otimes V(1,0), \quad V(1,0)^n = V(0,0) \text{ in } \mathcal{C}_q.$$

Proof. Define $F_1\colon V(1,0)\otimes V(i,l) \to V(i+1,l)$ by $F_1(q^j v_0^1 \otimes v_j^i) = v_j^{i+1}$ and $F_2\colon V(i,l)\otimes V(1,0) \to V(i+1,l)$ by $F_2(v_0^1 \otimes v_j^i) = v_j^{i+1}$. It is easy to verify that F_1 and F_2 provide the desired equalities. □

The Clebsch–Gordan problem for $\mathcal{C}_q$ is much more complicated than that for $\mathcal{C}_1$. We split the problem into several cases.

Lemma 3.9.

$$\begin{aligned} V(0,1)\otimes V(0,md+l) &= \begin{cases} V(0,md+l+1)\oplus V(1,md+l-1), & 0\leqslant l\leqslant d-2, \\ V(0,(m+1)d-1)\oplus V(1,(m+1)d-1), & l = d-1. \end{cases} \\ &= V(0,md+l)\otimes V(0,1). \end{aligned}$$

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Proof. We only prove the first equality as the second one is similar. For the case in which $0 \leqslant l \leqslant d-2$, the proof is essentially the same as that of Lemma 3.3 and so is omitted. Now consider $l = d-1$. In this case we define

$$\begin{aligned}\phi_1\colon V(0, md+d-1) &\to V(0,1) \otimes V(0, md+d-1)\\ v_k^0 &\mapsto v_0^0 \otimes v_k^0 + k_q v_1^0 \otimes v_{k-1}^0\end{aligned}$$

and

$$\begin{aligned}\phi_2\colon V(1, md+d-1) &\to V(0,1) \otimes V(0, md+d-1)\\ v_k^1 &\mapsto q^k v_1^0 \otimes v_k^0\end{aligned}$$

for $0 \leqslant k \leqslant md+d-1$. The verification of ϕ_1 and ϕ_2 being comodule monomorphisms is similar to Lemma 3.3. It is also easy to see that

$$\phi_1(V(0,(m+1)d-1)) \cap \phi_2(V(1,(m+1)d-1)) = \{0\},$$

so we have

$$V(0,(m+1)d-1) \oplus V(1,(m+1)d-1) \subseteq V(0,1) \otimes V(0, md+l)$$

and the claimed equality is obtained by comparing the dimensions. □

Lemma 3.10. *Assume that $V = \bigoplus_{s,t} V(s,t)$ and $V(i,j) \subset V$ in $\mathcal{C}_q$. There then exists an inclusion map $\phi\colon V(i,j) \to V(s,t)$ for some indecomposable direct summand $V(s,t)$ of V with $t \geqslant j$ and $i+j = s+t$ in $\mathbb{Z}_n$.*

Proof. Suppose that $\psi\colon V(i,j) \to V$ is the comodule inclusion and that $\psi(v_0^i) = \sum_{s,t}\sum_{l=0}^{t} a_l^{s,t} v_l^s$. By (3.3) and the fact that ψ is a comodule map, we have

$$\delta \circ \psi(v_0^i) = \sum_{s,t}\sum_{l=0}^{t} a_l^{s,t} \sum_{m=l}^{t} v_m^s \otimes p_{s+l}^{m-l} = \sum_{k=0}^{j} \psi(v_k^i) \otimes p_i^k = (\psi \otimes \mathrm{Id}) \circ \delta(v_0^i).$$

Comparing the third tensor factors of the two terms in the middle, it follows that there must be some s, t such that $p_{s+l}^{t-l} = p_i^j$. This obviously leads to $t \geqslant j$ and $i+j = s+t$ in $\mathbb{Z}_n$. For the s, t we chose, it is not hard to verify that $\phi\colon V(i,j) \to V(s,t)$ with $\phi(v_k^i) = v_{t-j+k}^s$ for $0 \leqslant k \leqslant j$ is a comodule inclusion. We are done. □

Lemma 3.11. *For all $0 < l \leqslant d-1$, we have*

$$V(0,l) \otimes V(0,md) = V(0,md+l) \oplus \bigoplus_{i=1}^{l} V(i, md-1) = V(0,md) \otimes V(0,l).$$

Proof. As before we only prove the first identity. By (3.6) we have

$$(p_{i+j}^{md})^*(v_i^0 \otimes v_j^0) = \sum_{x+y=md}\sum_{x=0}^{l-i}\sum_{y=0}^{md-j} q^{i(md-x)} \binom{md}{x}_q v_{i+x}^0 \otimes v_{j+y}^0.$$

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According to (3.1), $\binom{md}{x}_q \neq 0$ if and only if $x = kd$ for some k. Note that $x \leqslant l-i \leqslant l \leqslant d-1$, which implies that $\binom{md}{x}_q \neq 0$ if and only if $x = 0$. So we have $(p_{i+j}^{md})^*(v_i^0 \otimes v_j^0) \neq 0$ if and only if $j = 0$. As

$$(p_0^{md})^*(v_0^0 \otimes v_0^0) = v_0^0 \otimes v_{md}^0 \quad \text{and} \quad (p_{md}^l)^*(v_0^0 \otimes v_{md}^0) = v_l^0 \otimes v_{md}^0,$$

it follows that $\{(p_0^k)^*(v_0^0 \otimes v_0^0)\}_{0 \leqslant k \leqslant md+l}$ spans a subcomodule isomorphic to $V(0, md+l)$.

If we assume that $i \geqslant 0$, $j > 0$ and $i + j \leqslant l$, then $(p_{i+j}^{md})^*(v_i^0 \otimes v_j^0) = 0$, while by (3.7)

$$\begin{aligned}
(p_{i+j}^{md-1})^*(v_i^0 \otimes v_j^0) &= (p_{i+j+(m-1)d}^{d-1})^*[(p_{i+j}^{(m-1)d})^*(v_i^0 \otimes v_j^0)] \\
&= (p_{i+j+(m-1)d}^{d-1})^*(v_i^0 \otimes v_{j+(m-1)d}^0) \\
&= \sum_{x+y=d-1} \sum_{x=0}^{l-i} \sum_{y=0}^{d-j} q^{i(d-1-x)} \binom{d-1}{x}_q v_{i+x}^0 \otimes v_{j+(m-1)d+y}^0 \\
&\neq 0.
\end{aligned}$$

It follows that $\{(p_{i+j}^k)^*(v_i^0 \otimes v_j^0) \mid 0 \leqslant k \leqslant md-1\}$ spans a subcomodule isomorphic to $V(i+j, md-1)$. We have hence proved that $V(0, md+l)$ and $V(k, md-1)$ for $0 < k < l$ are subcomodules of $V(0, l) \otimes V(0, md)$. Assume that $V(0, l) \otimes V(0, md) = \bigoplus_{s,t} V(s, t)$. Then, by the previous lemma, $V(0, md+l)$ or $V(k, md-1)$ must be included in some $V(s, t)$ with $md + l = s + t$ or $k + md - 1 = s + t$ in $\mathbb{Z}_n$. It is clear that $V(0, md + l)$ and $V(k, md - 1)$ for $0 < k < l$ are in different $V(s, t)$, and therefore

$$V(0, l + md) \oplus \bigoplus_{i=1}^{l} V(i, md - 1) \subseteq V(0, l) \otimes V(0, md).$$

Now the identity is obtained by comparing the dimensions. □

Lemma 3.12. *For all $m \geqslant 1$ we have*

$$\begin{aligned}
&V(0, d) \otimes V(0, md) \\
&\quad = V(0, (m+1)d) \oplus V(1, (m+1)d - 2) \oplus \bigoplus_{i=2}^{d-1} V(i, md - 1) \oplus V(d, (m-1)d) \\
&\quad = V(0, md) \otimes V(0, d).
\end{aligned}$$

Proof. We only prove the first identity. First we verify that each indecomposable summand of the second formula is included in the first term.

It is obvious that $V(0, (m+1)d) \subset V(0, d) \otimes V(0, md)$ since

$$(p_0^{(m+1)d})^*(v_0^0 \otimes v_0^0) = m v_d^0 \otimes v_{md}^0, \qquad (p_0^{(m+1)d+1})^*(v_0^0 \otimes v_0^0) = 0.$$

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Furthermore, by (3.7) we have

$$\begin{aligned}(p_1^{(m+1)d-1})^*\left(v_1^0\otimes v_0^0-\frac{1}{m}v_0^0\otimes v_1^0\right) &= (p_{md+1}^{d-1})^*\left[(p_1^{md})^*\left(v_1^0\otimes v_0^0-\frac{1}{m}v_0^0\otimes v_1^0\right)\right]\\ &= (p_{md+1}^{d-1})^*(v_1^0\otimes v_{md}^0-v_d^0\otimes v_{(m-1)d+1}^0)\\ &= v_d^0\otimes v_{md}^0-v_d^0\otimes v_{md}^0\\ &= 0\end{aligned}$$

and

$$\begin{aligned}(p_1^{(m+1)d-2})^*\left(v_1^0\otimes v_0^0-\frac{1}{m}v_0^0\otimes v_1^0\right) &= (p_{md+1}^{d-2})^*\left[(p_1^{md})^*\left(v_1^0\otimes v_0^0-\frac{1}{m}v_0^0\otimes v_1^0\right)\right]\\ &= (p_{md+1}^{d-2})^*(v_1^0\otimes v_{md}^0-v_d^0\otimes v_{(m-1)d+1}^0)\\ &= v_{d-1}^0\otimes v_{md}^0-v_d^0\otimes v_{md-1}^0\\ &\neq 0,\end{aligned}$$

and hence $V(1,(m+1)d-2)\subset V(0,d)\otimes V(0,md)$.

Now, for all $i>0$, $j>0$ with $i+j<d$, one may verify that $\{(p_{i+j}^k)^*(v_i^0\otimes v_j^0)\mid 0\leqslant k\leqslant md-1\}$ spans a subcomodule isomorphic to $V(i+j,md-1)$ as in the preceding lemma.

Next we prove that $V(d,(m-1)d)$ is a subcomodule of $V(0,d)\otimes V(0,md)$. Set $\alpha_i=(-1)^iq^{-(1+i)i/2}$ for $0\leqslant i\leqslant d$, then one has

$$(p_{md}^1)^*\left(\sum_{i=0}^d \alpha_i v_i^0\otimes v_{md-i}^0\right)=0$$

by direct computation. Since we have

$$\begin{aligned}(p_d^{(m-1)d})^*(v_0^0\otimes v_d^0)&=(m-1)v_d^0\otimes v_{(m-1)d}^0+v_0^0\otimes v_{(m)d}^0,\\ (p_d^{(m-1)d})^*(v_d^0\otimes v_0^0)&=v_d^0\otimes v_{(m-1)d}^0\end{aligned}$$

and

$$(p_d^{(m-1)d})^*(v_i^0\otimes v_{d-i}^0)=v_i^0\otimes v_{md-i}^0,\quad i\neq 0,d,$$

it follows that

$$(p_d^{(m-1)d})^*\left(\sum_{i=0}^d \alpha_i v_i^0\otimes v_{d-i}^0-(m-1)v_d^0\otimes v_0^0\right)=\sum_{i=0}^d \alpha_i v_i^0\otimes v_{md-i}^0\neq 0$$

and

$$(p_d^{(m-1)d+1})^*\left(\sum_{i=0}^d \alpha_i v_i^0\otimes v_{d-i}^0-(m-1)v_d^0\otimes v_0^0\right)=p_{md}^1\left(\sum_{i=0}^d \alpha_i v_i^0\otimes v_{md-i}^0\right)=0.$$

This implies that

$$\left\{(p_d^k)^*\left(\sum_{i=0}^{d}\alpha_i v_i^0\otimes v_{d-i}^0-(m-1)v_d^0\otimes v_0^0\right)\,\middle|\, 0\leqslant k\leqslant (m-1)d\right\}$$

spans a subcomodule isomorphic to $V(d,(m-1)d)$, and hence $V(d,(m-1)d)$ is a subcomodule of $V(0,d)\otimes V(0,md)$.

Again, using Lemma 3.10 one can show that

$$V(0,(m+1)d)\oplus V(1,(m+1)d-2)\oplus\bigoplus_{i=2}^{d-1}V(i,md-1)\oplus V(d,(m-1)d)\subseteq V(0,d)\otimes V(0,md)$$

and the identity is obtained by comparing the dimensions. □

Corollary 3.13. *The Green ring $\mathcal{GR}(\mathcal{C}_q)$ is commutative and is generated by $[V(1,0)]$, $[V(0,1)]$ and $[V(0,d)]$.*

Proof. This is a direct consequence of Lemmas 3.8, 3.9, 3.11 and 3.12. □

Remark 3.14. Thanks to Lemmas 3.8, 3.9, 3.11 and 3.12, we have

$$\begin{aligned}V(i,l)\otimes V(j,m)=V(j,m)\otimes V(i,l)&=V(i+j,0)\otimes V(0,l)\otimes V(0,m)\\&=V(i+j,0)\otimes V(0,m)\otimes V(0,l).\end{aligned}$$

Hence, it is enough to compute the decomposition of $V(0,l)\otimes V(0,m)$ with $l\leqslant m$.

Lemma 3.15. *For $0\leqslant l\leqslant m\leqslant d-1$, set $\gamma=l+m-d+1$ and we have*

$$V(0,l)\otimes V(0,m)=\begin{cases}\bigoplus_{i=0}^{l}V(i,l+m-2i), & l+m\leqslant d-1,\\ \bigoplus_{i=0}^{\gamma}V(i,d-1)\oplus\bigoplus_{j=\gamma+1}^{l}V(j,l+m-2j), & l+m>d-1.\end{cases}$$

Proof. For the case in which $l+m\leqslant d-1$, the proof is similar to that of Theorem 3.4 and so is omitted. Now, let $l+m\geqslant d$ and we will prove the lemma by induction on l. If $l=1$, then by assumption $m=d-1$, and in this case the claim has been proved in Lemma 3.8. If $l>1$ and $l+m=d$, then we have

$$\begin{aligned}&V(0,l)\otimes V(0,m)\\&\quad=[V(0,1)\otimes V(0,l-1)-V(1,l-2)]\otimes V(0,m)\\&\quad=\bigoplus_{i=0}^{l-1}[V(i,l+m-2i)\oplus V(i+1,l-2+m-2i)]-\bigoplus_{j=0}^{l-2}V(i+1,l-2+m-2j)\\&\quad=V(0,d-1)\oplus V(1,d-1)\oplus\bigoplus_{j=2}^{l}V(j,l+m-2j),\end{aligned}$$

where in the second equality the case with $l+m \leqslant d-1$ is applied. Similarly, one can prove the case in which $l>1$ and $l+m=d+1$. Now let $l+m>d+1$ and by the induction hypothesis we have

$$\begin{aligned} V(0,l)\otimes V(0,m) &= [V(0,1)\otimes V(0,l-1)-V(1,l-2)]\otimes V(0,m)\\ &= V(0,1)\otimes\left[\bigoplus_{i=0}^{\gamma-1}V(i,d-1)\oplus\bigoplus_{j=\gamma}^{l}V(j,l-1+m-2j)\right]\\ &\quad-\left[\bigoplus_{i=0}^{\gamma-2}V(i+1,d-1)\oplus\bigoplus_{j=\gamma-1}^{l}V(j+1,l-2+m-2j)\right]\\ &= \bigoplus_{i=0}^{\gamma}V(i,d-1)\oplus\bigoplus_{j=\gamma+1}^{l}V(j,l+m-2j). \end{aligned}$$

The proof is finished. □

Lemma 3.16.

$$V(0,l)\otimes V(0,md+h) = \begin{cases} \bigoplus_{i=0}^{l}V(i,md+h+l-2i), \quad 0<l\leqslant h<d-1,\ l+h\leqslant d-1,\\ \bigoplus_{i=0}^{h}V(i,md+h+l-2i)\oplus\bigoplus_{j=h+1}^{l}V(j,md-1),\\ \qquad\qquad 0<h<l<d-1,\ l+h\leqslant d-1,\\ \bigoplus_{i=0}^{\gamma}V(i,(m+1)d-1)\oplus\bigoplus_{j=\gamma+1}^{l}V(j,md+h+l-2j),\\ \qquad\qquad 0<l\leqslant h\leqslant d-1,\ l+h\geqslant d,\\ \bigoplus_{i=0}^{\gamma}V(i,(m+1)d-1)\oplus\bigoplus_{j=\gamma+1}^{h}V(j,md+h+l-2j)\oplus\bigoplus_{k=h+1}^{l}V(k,md-1),\\ \qquad\qquad 0<h<l\leqslant d-1,\ l+h\geqslant d. \end{cases}$$

Proof. The case in which $0<l\leqslant h<d-1$, $l+h\leqslant d-1$ can be proved in the same manner as Theorem 3.4. The proofs for the remaining three cases are similar, so we only provide the proof for the case in which $0<h<l<d-1$, $l+h\leqslant d-1$.

If $l-h=1$, we have

$$\begin{aligned} V(0,l)\otimes V(0,md+h) &= [V(0,1)\otimes V(0,l-1)-V(1,l-2)]\otimes V(0,md+h)\\ &= \bigoplus_{i=0}^{l-1}[V(i,md+h+l-2i)\oplus V(i+1,md+h+l-2-2i)] \end{aligned}$$

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$$- \bigoplus_{i=0}^{l-2} V(i+1, md+h+l-2-2i)$$
$$= \bigoplus_{i=0}^{h} V(i, md+h+l-2i) \oplus V(l, md-1).$$

Similarly, one can prove the formula for the case in which $l = h+2$. Next assume that $l - h > 2$. In this situation we have

$$\begin{aligned}
&V(0,l) \otimes V(0, md+h) \\
&\quad = [V(0,1) \otimes V(0,l-1) - V(1,l-2)] \otimes V(0,md+h) \\
&\quad = \bigoplus_{i=0}^{h} [V(i, md+h+l-2i) \oplus V(i+1, md+h+l-2-2i)] \\
&\qquad \oplus \bigoplus_{j=h+1}^{l-1} [V(j, md-1) \oplus V(j+1, md-1)] \\
&\qquad - \left[\bigoplus_{i=0}^{h} V(i+1, md+h+l-2-2i) \oplus \bigoplus_{j=h+1}^{l-2} V(j+1, md-1) \right] \\
&\quad = \bigoplus_{i=0}^{h} V(i, md+h+l-2i) \oplus \bigoplus_{j=h+1}^{l} V(j, md-1).
\end{aligned}$$

We are done. □

Lemma 3.17. *For all $h > 0$ we have*

$$V(0,d) \otimes V(0, md+h)$$
$$= \begin{cases} V(0, (m+1)d+h) \oplus \bigoplus_{i=1}^{h} V(i, (m+1)d-1) \oplus V(h+1, (m+1)d-h-2) \\ \qquad \oplus \bigoplus_{j=h+2}^{d-1} V(j, md-1) \oplus V(d, (m-1)d+h), \quad h \leqslant d-2, \\ V(0, (m+1)d+d-1) \oplus \bigoplus_{i=1}^{d-1} V(i, (m+1)d-1) \oplus V(d, md-1), \quad h = d-1. \end{cases}$$

Proof. Only the proof for the case in which $h < d-2$ is provided as the proofs for other cases are similar and much easier. We prove by induction on m. When $m = 1$ we have

$$V(0,d) \otimes V(0, d+h)$$
$$= V(0,d) \otimes \left[V(0,h) \otimes V(0,d) - \bigoplus_{i=1}^{h} V(i, d-1) \right]$$

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$$= V(0,h) \otimes \left[V(0,2d) \oplus V(1,2d-2) \oplus \bigoplus_{i=2}^{d-1} V(i,d-1) \oplus V(d,0)\right]$$
$$- V(0,d) \otimes \left[\bigoplus_{i=0}^{h} V(i,d-1)\right]$$
$$= V(0,2d+h) \oplus \bigoplus_{i=1}^{h} V(i,2d-1) \oplus \bigoplus_{i=0}^{h-1} V(1+i,2d-1) \oplus V(h+1,2d-2-h)$$
$$\oplus \bigoplus_{i=2}^{d-1} \bigoplus_{j=0}^{h} V(i+j,d-1) \oplus V(d,h) - \bigoplus_{i=1}^{h} \left[V(i,2d-1) \oplus \bigoplus_{j=1}^{d-1} V(i+j,d-1)\right]$$
$$= V(0,2d+h) \oplus \bigoplus_{i=1}^{h} V(i,2d-1) \oplus V(h+1,2d-h-2)$$
$$\oplus \bigoplus_{j=h+2}^{d-1} V(j,d-1) \oplus V(d,h).$$

Note that for the first equality Lemma 3.11 is applied, for the second Lemma 3.12 and for the third Lemma 3.11. Similarly, one can verify the formula for $m=2$. In the following computation we assume that $m \geqslant 3$:

$$V(0,d) \otimes V(0,md+h)$$
$$= V(0,d) \otimes \left[V(0,h) \otimes V(0,md) - \bigoplus_{i=1}^{h} V(i,md-1)\right]$$
$$= V(0,h) \otimes \Big[V(0,(m+1)d) \oplus V(1,(m+1)d-2)$$
$$\oplus \bigoplus_{i=2}^{d-1} V(i,md-1) \oplus V(d,(m-1)d)\Big]$$
$$- V(0,d) \otimes \left[\bigoplus_{i=0}^{h} V(i,md-1)\right]$$
$$= V(0,(m+1)d+h) \oplus \bigoplus_{i=1}^{h} V(i,(m+1)d-1) \oplus \bigoplus_{i=0}^{h-1} V(1+i,(m+1)d-1)$$
$$\oplus V(h+1,(m+1)d-2-h) \oplus \bigoplus_{i=2}^{d-1} \bigoplus_{j=0}^{h} V(i+j,md-1) \oplus V(d,(m-1)d+h)$$
$$\oplus \bigoplus_{i=1}^{h} V(i+d,(m-1)d-1)$$
$$- \bigoplus_{i=1}^{h} \left[V(i,(m+1)d-1) \oplus \bigoplus_{j=1}^{d-1} V(i+j,md-1) \oplus V(i+d,(m-1)d-1)\right]$$

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$$= V(0,(m+1)d+h) \oplus \bigoplus_{i=1}^{h} V(i,(m+1)d-1) \oplus V(h+1,(e+1)d-h-2)$$
$$\oplus \bigoplus_{j=h+2}^{d-1} V(j,ed-1) \oplus V(d,(m-1)d+h).$$

We note that in the third equality the inductive assumption is applied. The proof is complete. □

Now we are ready to give the Clebsch–Gordan formulae for the tensor category $\mathcal{C}_q$. The aim is to decompose $V(0,ed+f) \otimes V(0,md+h)$ into a direct sum of indecomposable summands for all e, m and $0 \leqslant f,\ h \leqslant d-1$. According to Remark 3.14, in the following we may assume that $ed+f \leqslant md+h$. In this situation, $e \leqslant m$ and if $f > h$, then $e < m$. Set $\gamma = f+h-d+1$.

Theorem 3.18.

(1) *If* $0 \leqslant f \leqslant h$ *and* $f+h < d-1$, *then*

$$V(0,ed+f) \otimes V(0,md+h) = \bigoplus_{k=0}^{e-1} \Bigg[\bigoplus_{i=0}^{f} V(kd+i,(e+m-2k)d+f+h-2i)$$
$$\oplus \bigoplus_{j=f+1}^{h} V(kd+j,(e+m-2k)d-1)$$
$$\oplus \bigoplus_{r=h+1}^{f+h+1} V(kd+r,(e+m-2k)d+h+f-2r)$$
$$\oplus \bigoplus_{s=f+h+2}^{d-1} V(kd+s,(e+m-1-2k)d-1) \Bigg]$$
$$\oplus \bigoplus_{i=0}^{f} V(ed+i,(m-e)d+h+f-2i);$$

(2) *if* $0 < f \leqslant h < d$ *and* $f+h \geqslant d$, *then*

$$V(0,ed+f) \otimes V(0,md+h)$$
$$= \bigoplus_{k=0}^{e-1} \Bigg[\bigoplus_{i=0}^{\gamma} V(kd+i,(e+m-2k)d+d-1)$$
$$\oplus \bigoplus_{j=\gamma+1}^{f} V(kd+j,(e+m-2k)d+f+h-2i)$$
$$\oplus \bigoplus_{r=f+1}^{h} V(kd+r,(e+m-2k)d-1)$$

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$$\oplus \bigoplus_{s=h+1}^{d-1} V(kd+s,(e+m-2k)d+h+f-2s)\Bigg]$$

$$\oplus \bigoplus_{i=0}^{\gamma} V(ed+i,(m-e)d+d-1)$$

$$\oplus \bigoplus_{i=\gamma+1}^{f} V(ed+i,(m-e)d+f+h-2i);$$

(3) *if* $0 \leqslant h < f < d$ and $f+h < d-1$, *then*

$$V(0,ed+f)\otimes V(0,md+h) = \bigoplus_{k=0}^{e-1}\Bigg[\bigoplus_{i=0}^{h} V(kd+i,(e+m-2k)d+h+f-2i)$$

$$\oplus \bigoplus_{j=h+1}^{f} V(kd+j,(e+m-2k)d-1)$$

$$\oplus \bigoplus_{r=f+1}^{h+f+1} V(kd+r,(e+m-2k)d+h+f-2r)$$

$$\oplus \bigoplus_{s=f+h+2}^{d-1} V(kd+s,(e+m-1-2k)d-1)\Bigg]$$

$$\oplus \bigoplus_{i=0}^{h} V(ed+i,(m-e)d+h+f-2i)$$

$$\oplus \bigoplus_{i=h+1}^{f} V(ed+i,(m-e)d-1);$$

(4) *if* $0 < h < f < d$ and $f+h \geqslant d$, *then*

$$V(0,ed+f)\otimes V(0,md+h) = \bigoplus_{k=0}^{e-1}\Bigg[\bigoplus_{i=0}^{\gamma} V(kd+i,(e+m-2k)d+d-1)$$

$$\oplus \bigoplus_{j=\gamma+1}^{h} V(kd+j,(e+m-2k)d+f+h-2j)$$

$$\oplus \bigoplus_{r=h+1}^{f} V(kd+r,(e+m-2k)d-1)$$

$$\oplus \bigoplus_{s=f+1}^{d-1} V(kd+s,(e+m-2k)d+h+f-2s)\Bigg]$$

$$\oplus \bigoplus_{i=0}^{\gamma} V(ed+i,(m-e)d+d-1)$$

$$\oplus \bigoplus_{i=\gamma+1}^{h} V(ed+i,(m-e)d+f+h-2i)$$
$$\oplus \bigoplus_{i=h+1}^{f} V(ed+i,(m-e)d-1).$$

Proof. The rules of the decomposition of $V(0,\alpha)\otimes V(0,\beta)$ are divided into four cases. We prove the theorem by induction on α. The claim has been proved for the situation in which $1 \leqslant \alpha \leqslant d$, thanks to Lemmas 3.11, 3.12, 3.16 and 3.17. Now assume that the claim holds when $1 \leqslant \alpha \leqslant (e-1)d$ and we will prove that it still does when $(e-1)d+1 \leqslant \alpha \leqslant ed$.

Because the proofs of the four cases are similar, we only focus on case (1) and the proofs of other cases are omitted.

If $f = 1$, then

$$V(0,(e-1)d+1)\otimes V(0,md+h) = [V(0,1)\otimes V(0,(e-1)d)-V(1,(e-1)d-1)]\otimes V(0,md+h).$$

By the inductive assumption of case (1), we have

$$\begin{aligned} &V(0,(e-1)d)\otimes V(0,md+h)\\ &\quad= \bigoplus_{k=0}^{e-2}\Bigg[V(kd,(e-1+m-2k)d+h)\\ &\qquad\qquad \oplus \bigoplus_{i=1}^{h} V(kd+i,(e-1+m-2k)d-1)\\ &\qquad\qquad\qquad \oplus V(kd+h+1,(e-1+m-2k)d-h-2)\\ &\qquad\qquad\qquad\qquad \oplus \bigoplus_{j=h+2}^{d-1} V(kd+j,(e+m-2-2k)d-1)\Bigg]\\ &\qquad \oplus V((e-1)d,(m-e+1)d+h). \end{aligned}$$

It follows that

$$\begin{aligned} &V(0,1)\otimes V(0,(e-1)d)\otimes V(0,md+h)\\ &\quad= \bigoplus_{k=0}^{e-2}\Bigg\{V(kd,(e-1+m-2k)d+h+1)\oplus V(kd+1,(e-1+m-2k)d+h-1)\\ &\qquad\qquad \oplus \bigoplus_{i=1}^{h}[V(kd+i,(e-1+m-2k)d-1)\\ &\qquad\qquad\qquad \oplus V(kd+i+1,(e-1+m-2k)d-1)]\\ &\qquad\qquad \oplus V(kd+h+1,(e-1+m-2k)d-h-1)\\ &\qquad\qquad \oplus V(kd+h+2,(e-1+m-2k)d-h-3) \end{aligned}$$

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$$\oplus \bigoplus_{j=h+2}^{d-1} [V(kd+j,(e+m-2-2k)d-1) \\ \oplus V(kd+j+1,(e+m-2-2k)d-1)]\Big\} \\ \oplus V((e-1)d,(m-e+1)d+h+1) \oplus V((e-1)d+1,(m-e+1)d+h-1).$$

By the inductive assumption of case (4), we have

$$V(1,(e-1)d-1) \otimes V(0,md+h) \\ = \bigoplus_{k=0}^{e-2} \Bigg[\bigoplus_{i=0}^{h} V(kd+i+1,(e-1+m-2k)d-1) \\ \oplus \bigoplus_{j=h+1}^{d-1} V(kd+j+1,(e-2+m-2k)d-1) \Bigg].$$

Subtracting the preceding two identities, we obtain (1) with $\alpha = (e-1)d+1$.

Next, assume that $2 \leqslant f \leqslant d-1$. First note that

$$V(0,(e-1)d+f) \otimes V(0,md+h) = V(0,1) \otimes V(0,(e-1)d+f-1) \otimes V(0,md+h) \\ - V(1,(e-1)d+f-2) \otimes V(0,md+h).$$

Then, by the inductive hypothesis of case (1), we have

$$V(0,1) \otimes V(0,(e-1)d+f-1) \otimes V(0,md+h) \\ = \bigoplus_{k=0}^{e-2} \Bigg\{ \bigoplus_{i=0}^{f-1} [V(kd+i,(e-1+m-2k)d+f+h-2i) \\ \oplus V(kd+i+1,(e-1+m-2k)d+f-2+h-2i)] \\ \oplus \bigoplus_{j=f}^{h} [V(kd+j,(e-1+m-2k)d-1) \\ \oplus V(kd+j+1,(e-1+m-2k)d-1)] \\ \oplus \bigoplus_{r=h+1}^{f+h} [V(kd+r,(e-1+m-2k)d+h+f-2r) \\ \oplus V(kd+r+1,(e-1+m-2k)d+h+f-2r-2)] \\ \oplus \bigoplus_{s=f+h+1}^{(d-1)} [V(kd+s,(e+m-2-2k)d-1) \\ \oplus V(kd+s+1,(e+m-2-2k)d-1)] \Bigg\}$$

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$$\oplus \bigoplus_{i=0}^{f-1} [V((e-1)d+i,(m-e+1)d+h+f-2i) \oplus V((e-1)d+i+1,(m-e+1)d+h+f-2-2i)].$$

Again applying the inductive hypothesis of case (1), we obtain

$$\begin{aligned} &V(1,(e-1)d+f-2) \otimes V(0,md+h) \\ &= \bigoplus_{k=0}^{e-2} \Bigg[\bigoplus_{i=0}^{f-2} V(kd+i+1,(e-1+m-2k)d+f-2+h-2i) \\ &\quad \oplus \bigoplus_{j=f-1}^{h} V(kd+j+1,(e-1+m-2k)d-1) \\ &\quad \oplus \bigoplus_{r=h+1}^{f+h-1} V(kd+r+1,(e-1+m-2k)d+h+f-2-2r) \\ &\quad \oplus \bigoplus_{s=f+h}^{d-1} V(kd+s+1,(e+m-2-2k)d-1) \Bigg] \\ &\quad \oplus \bigoplus_{i=0}^{f-2} V(i+1,(m-e+1)d+h+f-2-2i). \end{aligned}$$

Now subtracting the foregoing two identities, we obtain (1) for $\alpha = (e-1)d+f$ with $1 \leqslant f \leqslant d-1$.

Finally, we prove (1) for $f = d$, i.e. for $\alpha = ed$. In the following we assume that $2 \leqslant h \leqslant d-2$. The proof for the situation with $h = 1$ or $h = d-1$ can be given in a similar and much easier way, and we thus omit it.

By Lemma 3.12, we have

$$\begin{aligned} &V(0,ed) \otimes V(0,md+h) \\ &= \Bigg\{ V(0,d) \otimes V(0,(e-1)d) \\ &\quad - \Bigg[V(1,ed-2) \oplus \bigoplus_{i=2}^{d-1} V(i,(e-1)d-1) \oplus V(d,(e-2)d) \Bigg] \Bigg\} \\ &\quad \otimes V(0,md+h). \end{aligned}$$

We then apply the inductive assumption to obtain

$$\begin{aligned} &V(0,(e-1)d) \otimes V(0,md+h) \\ &= \bigoplus_{k=0}^{e-2} \Bigg[V(kd,(e-1+m-2k)d+h) \\ &\quad \oplus \bigoplus_{i=1}^{h} V(kd+i,(e-1+m-2k)d-1) \end{aligned}$$

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$$\oplus V(kd+h+1,(e-1+m-2k)d-h-2) \oplus \bigoplus_{j=h+2}^{d-1} V(kd+j,(e+m-2-2k)d-1)\Big] \oplus V((e-1)d,(m-e+1)d+h)$$

and apply Lemma 3.17 to obtain

$$\begin{aligned}
&V(0,d)\otimes V(0,(e-1)d)\otimes V(0,md+h)\\
&=\bigoplus_{k=0}^{e-2}\Bigg\{\Big[V(kd,(e+m-2k)d+h)\oplus\bigoplus_{i=1}^{h}V(kd+i,(e+m-2k)d-1)\\
&\quad\oplus V(kd+h+1,(e+m-2k)d-h-2)\\
&\quad\oplus\bigoplus_{j=h+2}^{d-1}V(kd+j,(e+m-1-2k)d-1)\\
&\quad\oplus V((k+1)d,(e+m-2-2k)d+h)\Big]\\
&\quad\oplus\bigoplus_{i=1}^{h}\Big[V(kd+i,(e+m-2k)d-1)\\
&\quad\oplus\bigoplus_{j=1}^{d-1}V(kd+i+j,(e-1+m-2k)d-1)\\
&\quad\oplus V((k+1)d+i,(e-2+m-2k)d-1)\Big]\\
&\quad\oplus\Big[V(kd+h+1,(e+m-2k)d-h-2)\\
&\quad\oplus\bigoplus_{i=1}^{d-h-2}V(kd+h+1+i,(e+m-1-2k)d-1)\\
&\quad\oplus V((k+1)d,(e+m-2-2k)d+h)\\
&\quad\oplus\bigoplus_{j=d-h}^{d-1}V(kd+h+1+j,(e+m-2-2k)d-1)\\
&\quad\oplus V((k+1)d+h+1,(e+m-2-2k)d-h-2)\Big]\\
&\quad\oplus\bigoplus_{i=h+2}^{d-1}\Big[V(kd+i,(e+m-1-2k)d-1)\\
&\quad\oplus\bigoplus_{j=1}^{d-1}V(kd+i+j,(e+m-2-2k)d-1)\\
&\quad\oplus V((k+1)d+i,(e+m-3-2k)d-1)\Big]\Bigg\}
\end{aligned}$$

$$\oplus \Big[V((e-1)d, (m-e+2)d+h) \oplus \bigoplus_{i=1}^{h} V((e-1)d+i, (m-e+2)d-1) \oplus V((e-1)d+h+1, (m-e+2)d-h-2) \oplus \bigoplus_{j=h+2}^{d-1} V((e-1)d+j, (m-e+1)d-1) \oplus V(ed, (m-e)d+h) \Big].$$

Next we apply the inductive assumptions of formulae in (1) and (4) to obtain

$$\Big[V(1, ed-2) \oplus \bigoplus_{i=2}^{d-1} V(i, (e-1)d-1) \oplus V(d, (e-2)d) \Big] \otimes V(0, md+h)$$
$$= \Big\{ \bigoplus_{k=0}^{e-2} \Big[\bigoplus_{i=0}^{h-1} V(kd+i+1, (e+m-2k)d-1) \oplus V(kd+h+1, (e+m-2k)d-2-h) \oplus \bigoplus_{r=h+1}^{d-2} V(kd+r+1, (e-1+m-2k)d-1) \oplus V((k+1)d, (e+m-2-2k)d+h) \Big]$$
$$\oplus \bigoplus_{i=0}^{h-1} V((e-1)d+i+1, (m-e+2)d-1) \oplus V((e-1)d+h+1, (m-e+2)d-h-2) \oplus \bigoplus_{i=h+1}^{d-2} V((e-1)d+i+1, (m-e+1)d-1) \Big\}$$
$$\oplus \bigoplus_{i=2}^{d-1} \Big\{ \bigoplus_{k=0}^{e-3} \Big[\bigoplus_{l=0}^{h} V(kd+l+i, (e-1+m-2k)d-1) \oplus \bigoplus_{r=h+1}^{d-1} V(kd+r+i, (e-2+m-2k)d-1) \Big]$$
$$\oplus \bigoplus_{l=0}^{h} V((e-2)d+l+i, (m-e+3)d-1) \oplus \bigoplus_{l=h+1}^{d-1} V((e-2)d+l+i, (m-e+2)d-1) \Big\}$$

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$$\oplus \left\{ \bigoplus_{k=0}^{e-3} \left[V((k+1)d, (e-2+m-2k)d+h) \right.\right.$$
$$\oplus \bigoplus_{i=1}^{h} V((k+1)d+i, (e-2+m-2k)d-1)$$
$$\oplus V((k+1)d+h+1, (e-2+m-2k)d-h-2)$$
$$\left. \oplus \bigoplus_{j=h+2}^{d-1} V((k+1)d+j, (e+m-3-2k)d-1) \right]$$
$$\left. \oplus V((e-1)d, (m-e+2)d+h) \right\}.$$

Now, by subtracting the foregoing two identities we get the formula in (1) for $\alpha = ed$. We are done. □

Next we determine the ring structure of $\mathcal{GR}(\mathcal{C}_q)$. We will need to prepare some polynomials.

Definition 3.19. The series $\{f_k(x,y,z)\}_{k\geqslant 0}$ in $\mathbb{Z}[x,y,z]$ is defined inductively as:

(1) $f_0(x,y,z) = 1$, $f_1(x,y,z) = y$ and $f_d(x,y,z) = z$;

(2) $f_{md+l}(x,y,z) = yf_{md+l-1}(x,y,z) - xf_{md+l-2}(x,y,z)$ if $m \geqslant 0$ and $0 \leqslant l \leqslant d-1$;

(3) $$f_{(m+1)d}(x,y,z) = zf_{md}(x,y,z) - xf_{(m+1)d-2}(x,y,z) \sum_{i=2}^{d-1} x^i f_{md-1}(x,y,z) \\ -x^d f_{(m-1)d}(x,y,z).$$

Remark 3.20. If $k \leqslant d-1$, then $f_k(x,y,z)$ is independent of z and is essentially the generalized Fibonacci polynomials defined in Definition 3.5.

In order to use $f_k(x,y,z)$ to construct a basis of $\mathbb{Z}[x,y,z]$, we need to define an order of the polynomials as follows.

Definition 3.21. For monomials, define

$$\operatorname{ord}(x^i y^l z^m) = (m,l)$$

and we say $(m,l) \geqslant (m',l')$ if and only if either $m > m'$, or $m = m'$ and $l \geqslant l'$. Say $(m,l) = (m',l')$ if and only if $m = m'$ and $l = l'$. Define the order of a polynomial to be the order of the highest-order term.

Lemma 3.22. *The highest-order term of $f_{md+l}(x,y,z)$ is $y^l z^m$.*

Proof. It is easy to see that the highest-order term of $f_l(x,y,z)$ is y^l if $l \leqslant d-1$ and that of $f_d(x,y,z)$ is z. By induction on m one can easily prove the lemma. □

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Lemma 3.23. *We have that*

$$\{x^i f_k(x,y,z) \mid i,k \geqslant 0\} \cup \left\{ x^j \left(y^l - \sum_{s=0}^{l} \binom{l}{s} x^s \right) f_{md-1}(x,y,z) \,\middle|\, j \geqslant 0,\ l,m \geqslant 1 \right\}$$

is a basis of $\mathbb{Z}[x,y,z]$.

Proof. Let $g(x,y,z) \in \mathbb{Z}[x,y,z]$ and the highest-order term of $g(x,y,z)$ is $h(x)y^l z^m$, where $h(x) \in \mathbb{Z}[x]$. If $l > d-1$, set

$$g_1(x,y,z) = g(x) - h(x)\left(y^{l-d+1} - \sum_{s=0}^{l-d+1} \binom{l-d+1}{s} x^s \right) f_{(m+1)d-1}(x,y,z)$$

and if $l \leqslant d-1$, set

$$g_1(x,y,z) = g(x) - h(x) f_{md+l}(x,y,z).$$

Then the order of $g_1(x,y,z)$ is less than that of $g(x,y,z)$. Repeat the process enough times and we eventually arrive at some $g_k(x,y,z) \in \mathbb{Z}[x]$. This implies that $g(x,y,z)$ is a $\mathbb{Z}$-combination of

$$\{x^i f_k(x,y,z) \mid i,k \geqslant 0\} \quad \text{and} \quad \left\{ x^j \left(y^l - \sum_{s=0}^{l} \binom{l}{s} x^s f_{md-1}(x,y,z) \right) \,\middle|\, j \geqslant 0,\ l,m \geqslant 1 \right\}.$$

By considering the order of the highest-order term of each polynomial we obtain the linear independence. □

Lemma 3.24. $f_{d-1}(x,y,z)$ *is a factor of* $f_{md-1}(x,y,z)$ *for all* $m \geqslant 1$.

Proof. Note first that for any $1 \leqslant i \leqslant d-1$,

$$f_i(x,y,z) f_{md}(x,y,z) = f_{md+i}(x,y,z) + \sum_{j=1}^{i} x^j f_{md-1}(x,y,z). \tag{3.10}$$

Indeed, for $i = 1$,

$$\begin{aligned} f_1(x,y,z) f_{md}(x,y,z) &= y f_{md}(x,y,z) \\ &= f_{md+1}(x,y,z) + x f_{md-1}(x,y,z). \end{aligned}$$

Assume that

$$f_i(x,y,z) f_{md}(x,y,z) = f_{md+i}(x,y,z) + \sum_{j=1}^{i} x^j f_{md-1}(x,y,z)$$

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for $1 \leqslant i \leqslant l-1 < d-1$, then we have

$$\begin{aligned} f_l(x,y,z)f_{md}(x,y,z) &= [yf_{l-1}(x,y,z) - xf_{l-2}(x,y,z)]f_{md}(x,y,z) \\ &= y\left[f_{md+l-1}(x,y,z) + \sum_{j=1}^{l-1} x^j f_{md-1}(x,y,z)\right] \\ &\quad - x\left[f_{md+l-2}(x,y,z) + \sum_{j=1}^{l-2} x^j f_{md-1}(x,y,z)\right] \\ &= f_{md+l}(x,y,z) + \sum_{j=1}^{l} x^j f_{md-1}(x,y,z). \end{aligned}$$

So by induction we have proved that

$$f_i(x,y,z)f_{md}(x,y,z) = f_{md+i}(x,y,z) + \sum_{j=1}^{i} x^j f_{md-1}(x,y,z).$$

In particular, we have

$$f_{(m+1)d-1}(x,y,z) = f_{d-1}(x,y,z)f_{md}(x,y,z) - \sum_{j=1}^{d-1} x^j f_{md-1}(x,y,z).$$

This equation leads easily to $f_{d-1}(x,y,z) | f_{(m+1)d-1}(x,y,z)$ with induction on m. □

Theorem 3.25. *$\mathcal{GR}(\mathcal{C}_q)$ is isomorphic to $\mathbb{Z}[x,y,z]/J$, where J is the ideal of $\mathbb{Z}[x,y,z]$ generated by*

$$\left\{x^n - 1, (y-x-1)\left[\sum_{i=0}^{[(d-1)/2]} (-1)^i \binom{d-1-i}{i} x^i y^{d-1-2i}\right]\right\}.$$

Proof. Define

$$\begin{aligned} \varPhi\colon \mathbb{Z}[x,y,z] &\to \mathcal{GR}(\mathcal{C}_q) \\ x &\mapsto [V(1,0)], \\ y &\mapsto [V(0,1)], \\ z &\mapsto [V(0,d)]. \end{aligned}$$

By Corollary 3.13, the ring $\mathcal{GR}(\mathcal{C}_q)$ is generated by $[V(1,0)]$, $[V(0,1)]$ and $[V(0,d)]$. Hence, $\varPhi$ is surjective. From the definition of $f_k(x,y,z)$ and Theorem 3.18 we can see that

$$\varPhi(x^i f_k(x,y,z)) = [V(1,0)]^i f_k([V(1,0)],[V(0,1)],[V(0,d)]) = [V(i,k)].$$

Because

$$V(0,1)^{\otimes l} \otimes V(0,md-1) = \sum_{j=0}^{l} \binom{l}{j} V(j,md-1) = \sum_{j=0}^{l} \binom{l}{j} V(1,0)^{\otimes j} \otimes V(0,md-1),$$

we have

$$\Phi\biggl(\biggl(y^l - \sum_{j=0}^{l} \binom{l}{j} x^j\biggr) f_{md-1}(x,y,z)\biggr) = 0.$$

We also have $\Phi(x^n - 1) = 0$ since $V(1,0)^{\otimes n} = V(0,0)$. This implies that

$$J' = \biggl\langle x^n - 1, \biggl\{ \biggl(y^l - \sum_{j=0}^{l} \binom{l}{i} x^j\biggr) f_{md-1}(x,y,z) \biggm| l, m \geqslant 1 \biggr\} \biggr\rangle \subseteq \ker \Phi,$$

and so Φ induces an epimorphism

$$\begin{aligned} \bar{\Phi}\colon \mathbb{Z}[x,y,z]/J' &\to \mathcal{GR}(\mathcal{C}_q) \\ \bar{x} &\mapsto [V(1,0)], \\ \bar{y} &\mapsto [V(0,1)], \\ \bar{z} &\mapsto [V(0,d)]. \end{aligned}$$

Next we prove that $\bar{\Phi}$ is in fact a ring isomorphism. Write $\overline{g(x)} = g(x) + J' \in \mathbb{Z}[x,y,z]/J'$ and by Lemma 3.20 we have

$$g(x) = \sum c_{i,k} x^i f_k(x,y,z) + \sum c_{j,l,m} x^j \biggl(y^l - \sum_{j=0}^{l} \binom{l}{j} x^j\biggr) f_{md-1}(x,y,z),$$

where $c_{i,k}, c_{j,l,m} \in \mathbb{Z}$. So

$$\overline{g(x)} = \sum c_{i,k} \overline{x^i f_k(x,y,z)}.$$

It follows that $\{\overline{x^i f_k(x,y,z)} \mid i,k \geqslant 0\}$ spans $\mathbb{Z}[x,y,z]/J'$. Note that $\bar{\Phi}(\overline{x^i f_k(x,y,z)}) = [V(i,k)]$ and $\{[V(i,k)] \mid i,k \geqslant 0\}$ is a basis of $\mathcal{GR}(\mathcal{C}_q)$. We can see that $\{\overline{x^i f_k(x,y,z)} \mid i,k \geqslant 0\}$ is linearly independent, and hence a basis of $\mathbb{Z}[x,y,z]/J'$. Thus, $\bar{\Phi}$ is an isomorphism.

Finally, we prove that $J' = \langle x^n - 1, (y-x-1) f_{d-1}(x,y,z) \rangle = J$. It is enough to show that $(y-x-1) f_{d-1}(x,y,z)$ is a factor of

$$\biggl(y^l - \sum_{j=0}^{l} \binom{l}{j} x^j\biggr) f_{md-1}(x,y,z).$$

This follows from Lemma 3.22 and the fact that

$$y^l - \sum_{j=0}^{l} \binom{l}{j} x^j = y^l - (1+x)^l \quad \text{and} \quad f_{d-1}(x,y,z) = \sum_{i=0}^{[(d-1)/2]} (-1)^i \binom{d-1-i}{i} x^i y^{d-1-2i}.$$

The proof is complete. □

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4. The Green ring of the infinite linear quiver

4.1. Hopf structures over the infinite linear quiver

Let $G = \langle g \rangle$ be the infinite cyclic group and let $\mathcal{A}$ denote the Hopf quiver $Q(G, g)$. Then $\mathcal{A}$ is the infinite linear quiver. Let e_i denote the arrow $g^i \to g^{i+1}$ and p_i^l the path $e_{i+l-1} \cdots e_i$ of length $l \geqslant 1$ for each $i \in \mathbb{Z}$. The notation p_i^0 is understood as g^i.

We collect in this section some useful results on graded Hopf structures on $\Bbbk\mathcal{A}$. The graded Hopf structures are in one-to-one correspondence with the left $\Bbbk G$-module structures on $\Bbbk e_0$, and thus in one-to-one correspondence with non-zero elements of $\Bbbk$. Assume that $g.e_0 = qe_0$ for some $0 \neq q \in \Bbbk$. The corresponding kG-Hopf bimodule is $\Bbbk e_0 \otimes \Bbbk G$. We identify e_i and $e_0 \otimes g^i$, and in this way we have a $\Bbbk G$-Hopf bimodule structure on $\Bbbk\mathcal{A}_1$. We denote the corresponding graded Hopf algebra by $\Bbbk\mathcal{A}(q)$. Recall that the path multiplication formula of $\Bbbk\mathcal{A}(q)$ is as follows:

$$p_i^l \cdot p_j^m = q^{im} \binom{l+m}{l}_q p_{i+j}^{l+m}. \tag{4.1}$$

In particular, we have

$$g \cdot p_i^l = q^l p_{i+1}^l, \qquad p_i^l \cdot g = p_{i+1}^l, \qquad a_0^l = l_q! p_0^l.$$

The following lemma presents $\Bbbk\mathcal{A}(q)$ via generators with relations.

Lemma 4.1 (Huang *et al.* [7, Lemma 4.2]). *The algebra $\Bbbk\mathcal{A}(q)$ can be presented via generators with relations as follows.*

(1) *When $q = 1$: generators, g, g^{-1}, e_0; relations, $gg^{-1} = 1 = g^{-1}g$, $ge_0 = e_0 g$.*

(2) *When $q \neq 1$ is not a root of unity: generators, g, g^{-1}, e_0; relations, $gg^{-1} = 1 = g^{-1}g$, $ge_0 = qe_0 g$.*

(3) *When $q \neq 1$ is a root of unity of order d: generators, g, g^{-1}, e_0, p_0^d; relations, $gg^{-1} = 1 = g^{-1}g$, $e_0^d = 0$, $ge_0 = qe_0 g$, $gp_0^d = p_0^d g$, $e_0 p_0^d = p_0^d e_0$.*

4.2. The tensor category associated with $\Bbbk\mathcal{A}(q)$

For each $i \in \mathbb{Z}$ and $l \geqslant 0$, let $V(i, l)$ be a linear space with $\Bbbk$-basis $\{v_j^i\}_{0 \leqslant j \leqslant l}$. We endow on $V(i, l)$ an $\mathcal{A}$-representation structure by letting

$$V(i,l)_j = \begin{cases} \Bbbk v_{j-i}^i, & i \leqslant j \leqslant i+l, \\ 0 & \text{otherwise,} \end{cases}$$

and letting $V(i,l)_{e_j}$ map v_{j-i}^i to v_{j-i+1}^i for all $i \leqslant j \leqslant i+l$. Here, by convention, v_k^i is understood as 0 if $k > l$. Note also that $V(i, l)$ can be viewed as a right $\Bbbk\mathcal{A}(q)$-comodule via

$$\begin{aligned} \delta\colon V(i,l) &\to V(i,l) \otimes \Bbbk\mathcal{A}(q) \\ v_m^i &\mapsto \sum_{j=m}^{l} v_j^i \otimes p_{i+m}^{j-m}. \end{aligned}$$

By (4.1), the comodule structure of $V(i,l)\otimes V(j,m)$ is given by

$$\delta(v_s^i\otimes v_t^j)=\sum_{x=s}^{l}\sum_{y=t}^{m}q^{(i+s)(y-t)}\binom{x+y-s-t}{x-s}_q v_x^i\otimes v_y^j\otimes p_{i+j+s+t}^{x+y-s-t}.$$

It is well known that $\{V(i,l)\mid i\in\mathbb{Z},\ l\geqslant 0\}$ is a complete set of finite-dimensional indecomposable representations of the quiver $\mathcal{A}$, and thus a complete set of finite-dimensional indecomposable comodules of $\Bbbk\mathcal{A}(q)$. Similarly to § 3.2, we will also view $V(i,l)\otimes V(j,m)$ as a rational module of $(\Bbbk\mathcal{A}(q))^*$, the dual algebra of $\Bbbk\mathcal{A}(q)$. The module structure map is given by

$$(p_e^f)^*(v_s^i\otimes v_t^j)=\sum_{x=0}^{f}q^{(i+s)(f-x)}\binom{f}{x}_q v_{s+x}^i\otimes v_{t+f-x}^j.$$

In this section we let $\mathcal{D}_q$ denote the tensor category of finite-dimensional $\Bbbk\mathcal{A}(q)$-comodules and let $\mathcal{GR}(\mathcal{D}_q)$ denote the Green ring of $\mathcal{D}_q$.

Recall that Lemma 3.10 plays an important role in the computation of the Clebsch–Gordan formulae of $\mathcal{C}_q$. Here, for $\mathcal{D}_q$, we have a similar lemma. As the proof is similar, we do not repeat it.

Lemma 4.2. *Assume that $V=\bigoplus_{s,t}V(s,t)$ and $V(i,j)\subset V$ in $\mathcal{D}_q$. There then exists an inclusion map $\phi\colon V(i,j)\to V(s,t)$ for some indecomposable direct summand $V(s,t)$ of V with $t\geqslant j$ and $i+j=s+t$.*

4.3. The case in which $q=1$ or q is not a root of unity

In this section we compute the Clebsch–Gordan formulae and Green ring of $\mathcal{D}_q$ if $q=1$ or q is not a root of unity.

Similarly to § 3, we have the following lemmas.

Lemma 4.3.

$$\begin{gathered}V(0,0)\otimes V(i,l)=V(i,l)\otimes V(0,0)=V(i,l),\\ V(1,0)\otimes V(i,l)=V(i,l)\otimes V(1,0)=V(i+1,l),\\ V(-1,0)\otimes V(i,l)=V(i,l)\otimes V(-1,0)=V(i-1,l),\\ V(1,0)^{\otimes m}=V(m,0),\qquad V(-1,0)^{\otimes m}=V(-m,0),\\ V(m,0)\otimes V(-m,0)=V(-m,0)\otimes V(m,0)=V(0,0).\end{gathered}$$

Lemma 4.4.

$$V(0,1)\otimes V(0,l)=V(0,l+1)\oplus V(1,l-1)=V(0,l)\otimes V(0,1). \tag{4.2}$$

Proof. As before, we only prove the first equality. Consider the maps

$$\begin{aligned}\phi_1\colon V(0,l+1)&\to V(0,1)\otimes V(0,l)\\ v_0^0&\mapsto v_0^0\otimes v_0^0,\\ v_i^0&\mapsto v_0^0\otimes v_i^0+(i)_q v_1^0\otimes v_{i-1}^0,\quad i=1,2,\dots,l,\\ v_{l+1}^0&\mapsto (l+1)_q v_1^0\otimes v_l^0\end{aligned}$$

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and

$$\begin{aligned}\phi_2\colon V(1,l-1) &\to V(0,1)\otimes V(0,l)\\ v_i^1 &\mapsto v_0^0\otimes v_{i+1}^0 + q^{-1}(l-i)_{q^{-1}} v_1^0\otimes v_i^0, \quad i=1,2,\ldots,l-1.\end{aligned}$$

It is not difficult to verify that ϕ_1 and ϕ_2 are comodule monomorphisms. Then, by Lemma 4.2, we have

$$V(0,l+1)\oplus V(1,l-1)\subset V(0,1)\otimes V(0,l).$$

The claimed equality now follows by comparing the dimensions. □

With the help of Lemma 4.4, one may prove the following identity easily by induction on l.

Proposition 4.5. $V(0,l)\otimes V(0,m)=\bigoplus_{i=0}^{l} V(i,l+m-2i)=V(0,m)\otimes V(0,l).$

Combining Lemma 4.2 and Proposition 4.4, we get the Clebsch–Gordan formulae for $\mathcal{D}_q$ as follows.

Corollary 4.6. $V(s,l)\otimes V(t,m)=\bigoplus_{i=0}^{l} V(s+t+i,l+m-2i)=V(t,m)\otimes V(s,l).$

Now we are ready to give the Green ring structure of $\mathcal{D}_q$ when $q=1$ or q is not a root of unity. As before, we need some facts about polynomials. Let $f_k(x,y)$ be the generalized Fibonacci polynomial as defined in Definition 3.5. We then have the following lemma.

Lemma 4.7. *$\{x^i f_k(x,y)\}_{i\in\mathbb{Z},k\geqslant 0}$ is a basis of $\mathbb{Z}[x,x^{-1},y]$.*

Proof. The proof is similar to that of Lemma 3.6. □

Theorem 4.8. *$\mathcal{GR}(\mathcal{D}_q)$ is isomorphic to $\mathbb{Z}[x,x^{-1},y]$ when $q=1$ or q is not a root of unity.*

Proof. Define a ring morphism

$$\begin{aligned}\varPhi\colon \mathbb{Z}[x,x^{-1},y] &\to \mathcal{GR}(\mathcal{D}_q)\\ x &\mapsto [V(1,0)],\\ x^{-1} &\mapsto [V(-1,0)],\\ y &\mapsto [V(0,1)].\end{aligned}$$

By the definition of $f_k(x,y)$ and (4.2), it is not hard to verify that

$$\varPhi(x^i f_k(x,y))=[V(i,k)].$$

This expression says that $\varPhi$ maps the basis $\{x^i f_k x,y\}_{i\in\mathbb{Z},k\geqslant 0}$ of $\mathbb{Z}[x,x^{-1},y]$ to the basis $\{[V(i,k)]\}_{i\in\mathbb{Z},k\geqslant 0}$ of $\mathcal{GR}(\mathcal{D}_q)$, and thus it is an isomorphism. □

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4.4. The case in which q is a root of unity

In this section q is assumed to be a root of unity with multiplicative order $d \geqslant 2$. First we list some facts without proofs, as they can be obtained in a similar way as before.

Lemma 4.9.

$$\begin{aligned} V(0,0)\otimes V(i,l) &= V(i,l)\otimes V(0,0) = V(i,l),\\ V(1,0)\otimes V(i,l) &= V(i,l)\otimes V(1,0) = V(i+1,l),\\ V(-1,0)\otimes V(i,l) &= V(i,l)\otimes V(-1,0) = V(i-1,l). \end{aligned}$$

Lemma 4.10.

$$\begin{aligned} V(0,1)\otimes V(0,md+l) &= V(0,md+l)\otimes V(0,1)\\ &= \begin{cases} V(0,md+l+1)\oplus V(1,md+l-1), & 0\leqslant l\leqslant d-2,\\ V(0,(m+1)d-1)\oplus V(1,(m+1)d-1), & l=d-1. \end{cases} \end{aligned}$$

Lemma 4.11. *For all $m \geqslant 1$, one has*

$$\begin{aligned} &V(0,d)\otimes V(0,md)\\ &\quad = V(0,md)\otimes V(0,d)\\ &\quad = V(0,(m+1)d)\oplus V(1,(m+1)d-2)\oplus \bigoplus_{i=2}^{d-1} V(i,md-1)\oplus V(d,(m-1)d). \end{aligned}$$

Obviously, Lemmas 4.9–4.11 imply that the Green ring $\mathcal{GR}(\mathcal{D}_q)$ is a commutative ring and is generated by $[V(-1,0)]$, $[V(1,0)]$, $[V(0,1)]$ and $[V(0,d)]$. The preceding three lemmas also help to compute the decomposition of $V(0,ed+f)\otimes V(0,md+h)$ for all $e,m \geqslant 0$ and $0 \leqslant f,\ h \leqslant d-1$. In the following, let $\gamma = f+h-d+1$.

Theorem 4.12.

(1) *If $0 \leqslant f \leqslant h$ and $f+h < d-1$, then*

$$\begin{aligned} V(0,ed+f)\otimes V(0,md+h) = \bigoplus_{k=0}^{e-1}\Bigg[&\bigoplus_{i=0}^{f} V(kd+i,(e+m-2k)d+f+h-2i)\\ &\oplus \bigoplus_{j=f+1}^{h} V(kd+j,(e+m-2k)d-1)\\ &\oplus \bigoplus_{r=h+1}^{f+h+1} V(kd+r,(e+m-2k)d+h+f-2r)\\ &\oplus \bigoplus_{s=f+h+2}^{d-1} V(kd+s,(e+m-1-2k)d-1)\Bigg]\\ \oplus \bigoplus_{i=0}^{f} V(ed+i,&(m-e)d+h+f-2i); \end{aligned}$$

(2) *if* $0 < f \leqslant h < d$ and $f + h \geqslant d$, *then*

$$V(0, ed+f) \otimes V(0, md+h) = \bigoplus_{k=0}^{e-1} \Bigg[\bigoplus_{i=0}^{\gamma} V(kd+i, (e+m-2k)d+d-1) \oplus \bigoplus_{j=\gamma+1}^{f} V(kd+j, (e+m-2k)d+f+h-2i) \oplus \bigoplus_{r=f+1}^{h} V(kd+r, (e+m-2k)d-1) \oplus \bigoplus_{s=h+1}^{d-1} V(kd+s, (e+m-2k)d+h+f-2s) \Bigg] \oplus \bigoplus_{i=0}^{\gamma} V(ed+i, (m-e)d+d-1) \oplus \bigoplus_{i=\gamma+1}^{f} V(ed+i, (m-e)d+f+h-2i);$$

(3) *if* $0 \leqslant h < f < d$ and $f + h < d - 1$, *then*

$$V(0, ed+f) \otimes V(0, md+h) = \bigoplus_{k=0}^{e-1} \Bigg[\bigoplus_{i=0}^{h} V(kd+i, (e+m-2k)d+h+f-2i) \oplus \bigoplus_{j=h+1}^{f} V(kd+j, (e+m-2k)d-1) \oplus \bigoplus_{r=f+1}^{h+f+1} V(kd+r, (e+m-2k)d+h+f-2r) \oplus \bigoplus_{s=f+h+2}^{d-1} V(kd+s, (e+m-1-2k)d-1) \Bigg] \oplus \bigoplus_{i=0}^{h} V(ed+i, (m-e)d+h+f-2i) \oplus \bigoplus_{i=h+1}^{f} V(ed+i, (m-e)d-1);$$

(4) *if* $0 < h < f < d$ and $f + h \geqslant d$, *then*

$$V(0, ed+f) \otimes V(0, md+h) = \bigoplus_{k=0}^{e-1} \Bigg[\bigoplus_{i=0}^{\gamma} V(kd+i, (e+m-2k)d+d-1) \oplus \bigoplus_{j=\gamma+1}^{h} V(kd+j, (e+m-2k)d+f+h-2j)$$

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$$\oplus \bigoplus_{r=h+1}^{f} V(kd+r,(e+m-2k)d-1)$$
$$\left.\oplus \bigoplus_{s=f+1}^{d-1} V(kd+s,(e+m-2k)d+h+f-2s)\right]$$
$$\oplus \bigoplus_{i=0}^{\gamma} V(ed+i,(m-e)d+d-1)$$
$$\oplus \bigoplus_{i=\gamma+1}^{h} V(ed+i,(m-e)d+f+h-2i)$$
$$\oplus \bigoplus_{i=h+1}^{f} V(ed+i,(m-e)d-1).$$

Proof. The proof is similar to that of Theorem 3.18. □

Let $\{f_k(x,y,z)\}_{k\geqslant 0}$ be polynomials defined as in Definition 3.19 and we have the following easy fact.

Lemma 4.13. *We have that*

$$\{x^i f_k(x,y,z) \mid i\in\mathbb{Z}, k\geqslant 0\} \bigcup \left\{ x^j\left(y^l - \sum_{s=0}^{l}\binom{l}{s} x^s f_{md-1}(x,y,z)\right) \,\middle|\, j\in\mathbb{Z},\ l,m\geqslant 1\right\}$$

is a basis of $\mathbb{Z}[x,x^{-1},y,z]$.

Now we are in a position to determine the Green ring $\mathcal{GR}(\mathcal{D}_q)$.

Theorem 4.14. $\mathcal{GR}(\mathcal{D}_q)$ *is isomorphic to* $\mathbb{Z}[x,x^{-1},y,z]/J$, *where* J *is the ideal of* $\mathbb{Z}[x,x^{-1},y,z]$ *generated by*

$$(y-x-1)\left(\sum_{i=0}^{[(d-1)/2]}(-1)^i\binom{d-1-i}{i}x^i y^{d-1-2i}\right).$$

Proof. The proof is similar to that of Theorem 3.25. Define a ring map

$$\begin{aligned}\Psi\colon \mathbb{Z}[x,x^{-1},y,z] &\to \mathcal{GR}(\mathcal{D}_q)\\ x &\mapsto [V(1,0)],\\ x^{-1} &\mapsto [V(-1,0)],\\ y &\mapsto [V(0,1)],\\ z &\mapsto [V(0,d)].\end{aligned}$$

Obviously, Ψ is surjective as x, x^{-1}, y and z map to a set of generators of $\mathcal{GR}(\mathcal{D}_q)$. According to the definition of $f_k(x,y,z)$ and Lemmas 4.10 and 4.11, we have

$$\Psi(x^i f_k(x,y,z)) = [V(1,0)]^i f_k([V(1,0)],[V(0,1)],[V(0,d)]) = [V(i,k)],$$
$$\Psi(x^{-i} f_k(x,y,z)) = [V(-1,0)]^i f_k([V(1,0)],[V(0,1)],[V(0,d)]) = [V(-i,k)]$$

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for all $i \geqslant 0$. Because

$$V(0,1)^{\otimes l} \otimes V(0, md-1) = \sum_{j=0}^{l} \binom{l}{j} V(j, md-1)$$
$$= \sum_{j=0}^{l} \binom{l}{j} V(1,0)^{\otimes j} V(0, md-1),$$

we have

$$\Psi\left(\left(y^l - \sum_{j=0}^{l} \binom{l}{j} x^j\right) f_{md-1}(x,y,z)\right) = 0.$$

This implies that

$$J' = \left\langle \left\{ \left(y^l - \sum_{j=1}^{l} \binom{l}{i} x^j\right) f_{md-1}(x,y,z) \right\}_{l,m \geqslant 1} \right\rangle \subseteq \ker \Psi.$$

Thus, Ψ induces an epimorphism

$$\begin{aligned} \bar{\Psi}\colon \mathbb{Z}[x, x^{-1}, y, z]/J' &\to \mathcal{GR}(\mathcal{D}_q) \\ \bar{x} &\mapsto [V(1,0)], \\ \overline{x^{-1}} &\mapsto [V(-1,0)], \\ \bar{y} &\mapsto [V(0,1)], \\ \bar{z} &\mapsto [V(0,d)]. \end{aligned}$$

Next we prove that $\bar{\Psi}$ is a ring isomorphism. Let

$$\overline{g(x,x^{-1},y,z)} = g(x,x^{-1},y,z) + J' \in \mathbb{Z}[x,x^{-1},y,z]/J'.$$

By Lemma 4.11 we have

$$g(x) = \sum_{i \in \mathbb{Z}, k \geqslant 0} c_{i,k} x^i f_k(x,y,z) + \sum_{j \in \mathbb{Z}, m \geqslant 0, l \geqslant 1} c_{j,l,m} x^j \left(y^l - \sum_{j=1}^{l} \binom{l}{i} x^j\right) f_{md-1}(x,y,z),$$

where $c_{i,k}, c_{j,l,m} \in \mathbb{Z}$. So

$$\overline{g(x,x^{-1},y,z)} = \sum_{i \in \mathbb{Z}, k \geqslant 0} c_{i,k} \overline{x^i f_k(x,y,z)}.$$

In other words, $\{\overline{x^i f_k(x,y,z)} \mid i \in \mathbb{Z},\ k \geqslant 0\}$ spans $\mathbb{Z}[x,x^{-1},y,z]/J'$. Note that $\bar{\Psi}(\overline{x^i f_k(x,y,z)}) = [V(i,k)]$, and $\{[V(i,k)] \mid i \in \mathbb{Z},\ k \geqslant 0\}$ is a basis of $\mathcal{GR}(\mathcal{D}_q)$. It follows that $\{\overline{x^i f_k(x,y,z)} \mid i \in \mathbb{Z},\ k \geqslant 0\}$ is linearly independent, and hence a basis of $\mathbb{Z}[x,x^{-1},y,z]/J'$. So $\bar{\Psi}$ is an isomorphism. Now, by Lemma 3.24 we have $J' = \langle (y-x-1) f_{d-1}(x,y,z) \rangle = J$.

We are done. □

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