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Optimal distribution of traffic flows in emergency cases

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The aim of this work is to present a technique for the optimisation of emergency vehicles travel times on assigned paths when critical situations, such as car accidents, occur. Using a fluid-dynamic model for the description of car density evolution, the attention is focused on a decentralised approach reducing to simple junctions with two incoming roads and two outgoing ones (junctions of 2×2 type). We assume the redirection of cars at junctions is possible and choose a cost functional that describes the asymptotic average velocity of emergency vehicles. Fixing an incoming road and an outgoing road for the emergency vehicle, we determine the local distribution coefficients that maximise such functional at a single junction. Then we use the local optimal coefficients at each node of the network. The overall traffic evolution is studied via simulations, both for simple junctions or cascade networks, evaluating global performances when optimal parameters on the network are used.

Key words: Conservation laws; traffic problems; optimal redistribution of flows.

1 Introduction

The exponentially increasing number of circulating cars in modern cities renders the problem of traffic control of paramount importance. Incidents (such as accidents or even a single car braking heavily in a previously smooth flow) may cause ripple effects (a cascading failure), which then spread out and create a sustained traffic jam. In particular, sudden decisions have to be taken in the case of an emergency. Fire, police, ambulance, repair crews, emergency and life-saving equipment, services and supplies must move quickly to places of emergency where there is the greatest need.

The problem can be solved with the identification of a network of dedicated municipal and provincial roads. Otherwise, one may choose a route for emergency vehicles (not dedicated, i.e. not limited only to emergency needs) and redistributing traffic flows at junctions on the basis of the current traffic load in such a way that emergency vehicles can travel at the maximum allowed speed along the assigned roads (and without blocking traffic on other roads). In this paper we focus on this second approach. In Figure 1 white arrows indicate the chosen path for the emergency vehicle; congested roads are marked in black. With this aim in mind, we choose a fluid-dynamic model for road

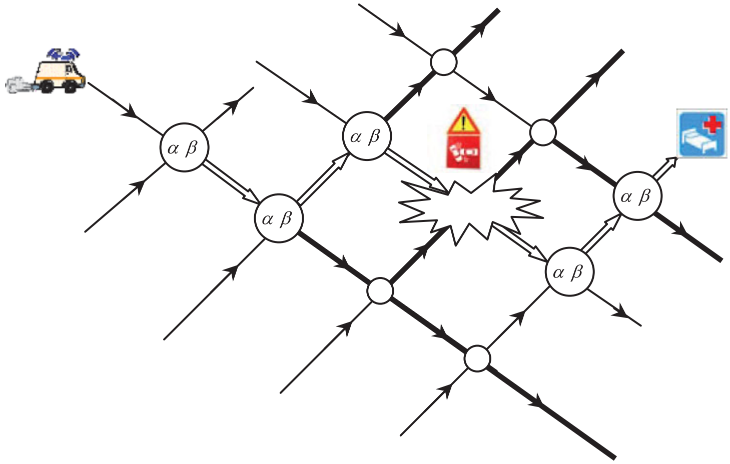

Figure 1. (Colour online) Car accident on a road network and flows redistribution.

networks [5, 8, 12] to find the optimal distribution of vehicles at junctions consisting of two incoming and two outgoing roads in order to maximise the velocity of emergency vehicles on an assigned path. In reality, such coefficients are determined by drivers' habits. However, drivers' preferences can be changed in presence of critical conditions in order to maximise the velocity of emergency vehicles on fixed paths.

Following the adopted model, the evolution of car densities is described by a conservation law [1]. In order to uniquely solve dynamics at junctions, the Riemann Problems (Cauchy problems with constant initial data on each road) are solved respecting the following rules:

(A) The incoming traffic at a node is distributed to outgoing roads according to some distribution coefficients.

(B) Drivers behave so as to maximise the flux through the junction.

If the road junction is of 2×2 type, namely it has two incoming roads, a and b, and two outgoing ones, c and d, rule (A) is expressed by two coefficients, α and β, that indicate the percentage of cars moving from roads a and b, respectively, to road c. Assigning the initial density for all incoming and outgoing roads of a node, we compute the asymptotic equilibrium as a function of α and β. Such equilibrium, belonging to the admissible region for final fluxes, is chosen according to rule (B).

Some optimisation problems for coefficients of fluid dynamic models have been already treated for car traffic in [3, 4], where three cost functionals, J_1, J_2 and J_3, indicating, respectively, cars average velocities, average travelling times and flux, have been introduced. For junctions of types 2×1 and 1×2, the optimisation has been done over right of way parameters and traffic distribution coefficients with the aim of maximising J_1 and J_3, and minimising J_2. Moreover, in [6], for junctions of 2×1 type, further cost functionals, measuring kinetic energy and average travelling time, weighted with the number of cars moving on the roads, have been considered. It was shown that only the velocity cost functional guarantees optimal global performances on urban networks.

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The goal of this paper is to extend these previous works to the case of 2×2 junctions. Here, assuming that emergency vehicles will cross fixed roads φ and ψ ($\varphi \in \{a, b\}$, $\psi \in \{c, d\}$), a cost functional $W_{\varphi,\psi}$, measuring the average velocities of such vehicles on the incoming road I_φ and the outgoing road I_ψ of 2×2 junctions, is considered. The optimisation results give the values of α and β, which maximise the functional, allowing a fast transit of emergency vehicles to reach car accident's place and hospital.

The analysis of complete functional $W_{\varphi,\psi}$ on a whole network is a very complex problem, hence we follow a decentralised approach. More precisely, we look at the asymptotic behaviour, i.e. for large times, at a single junction. It results possibly to find an exact solution for a single junction and an asymptotic expression of $W_{\varphi,\psi}$. Then we propose a global (sub)optimal solution for the whole network, simply obtained by applying at each junction the computed local optimal solution.

The correctness of analytical optimisation procedures is tested by simulations. For numerics, we refer to approximation methods described in [2, 10, 11, 13]. Simulations are run using two different choices of distribution coefficients: (locally) optimal and random. The first choice is given by the optimisation algorithm; the second one considers, at the beginning of the simulation process, a random choice of α and β, kept constant during all the simulation. Simulation results first refer on simple junctions of 2×2 type. Then we study the effects of the decentralised approach on the global performance of a network with cascade junctions. It is shown that, for the chosen initial data, either for simple junctions or networks, optimal parameters give better performances than other ones.

The paper is organised as follows. In Section 2, we describe briefly the basic model for road networks. In Section 3, we recall the construction of solutions to the Riemann Problems at junctions. Section 4 is devoted to the introduction of the cost functional $W_{\varphi,\psi}$ and its optimisation. Simulation results for simple junctions with different initial data and for a cascade network are reported in Section 5. The paper ends with conclusions in Section 6.

2 Road networks

A road network is described by a couple $(\mathcal{I}, \mathcal{J})$, where $\mathcal{I}$ represents the set of roads, and $\mathcal{J}$ is the collection of junctions. The roads are modelled by intervals $[a_i, b_i] \subset \mathbb{R}, i = 1, \ldots, N$.

The evolution of car traffic on each road is described by the Lighthill–Whitham–Richards model (see [14, 15]) given by the following equation:

$$\partial_t \rho + \partial_x f(\rho) = 0, \tag{2.1}$$

where $\rho = \rho(t, x) \in [0, \rho_{\max}]$ is the density of cars, $\rho_{\max}$ is the maximal density, $f(\rho) = \rho v(\rho)$ is the flux with $v(\rho)$ being the average velocity.

Setting $\rho_{\max} = 1$, we fix a velocity function,

$$v(\rho) = 1 - \rho. \tag{2.2}$$

The corresponding flux function,

$$f(\rho) = \rho(1 - \rho), \ \rho \in [0, 1], \tag{2.3}$$

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which presents a unique maximum $\sigma = \frac{1}{2}$, ensures the assumption (F):

(F) $f : [0, \rho_{\max}] \to [0, \sigma]$ is a strictly concave C^2 function such that $f(0) = f(\rho_{\max}) = 0$.

For a single conservation law (2.1) on a real line, the Riemann Problem (RP) is a Cauchy problem for a piecewise constant initial data with only one discontinuity. In an analogous way, we define an RP at a junction as a Cauchy problem with a constant initial datum for each incoming and outgoing road. We aim to solve RPs at junctions of a road network. Fix a junction J with n incoming roads I_φ, $\varphi = 1, \ldots, n$, and m outgoing roads, I_ψ, $\psi = n+1, \ldots, n+m$, and an initial data $\rho_0 = (\rho_{1,0}, \ldots, \rho_{n,0}, \rho_{n+1,0}, \ldots, \rho_{n+m,0})$.

Definition 1 *A Riemann Solver (RS) for the junction J is a map $RS : [0,1]^n \times [0,1]^m \to [0,1]^n \times [0,1]^m$ that associates to Riemann data $\rho_0 = (\rho_{\varphi,0}, \rho_{\psi,0})$ at J a vector $\widehat{\rho} = (\widehat{\rho}_\varphi, \widehat{\rho}_\psi)$ so that the solution on an incoming road $I_\varphi, \varphi = 1, \ldots, n$, is given by the wave $(\rho_{\varphi,0}, \widehat{\rho}_\varphi)$ and on an outgoing one I_ψ, $\psi = n+1, \ldots, n+m$ is given by the wave $(\widehat{\rho}_\psi, \rho_{\psi,0})$. We require the following conditions to hold true:*
(C1) $RS(RS(\rho_0)) = RS(\rho_0)$;
(C2) on each incoming road $I_\varphi, \varphi = 1, \ldots, n$, the wave $(\rho_{\varphi,0}, \widehat{\rho}_\varphi)$ has negative speed, while on each outgoing road I_ψ, $\psi = n+1, \ldots, n+m$, the wave $(\widehat{\rho}_\psi, \rho_{\psi,0})$ has positive speed.

If $m \geqslant n$, a possible RS at a junction J is defined according to the following rules (see [5]):

(A) Preferences of drivers at J are represented by some coefficients, collected in a traffic distribution matrix $A = (\alpha_{\psi,\varphi})$, $\varphi \in \{1, \ldots, n\}$, $\psi \in \{n+1, \ldots, n+m\}$, $0 < \alpha_{\psi,\varphi} < 1$, $\sum_{\psi=n+1}^{n+m} \alpha_{\psi,\varphi} = 1$. The ψth column of A indicates the percentages of traffic that, from the incoming road I_φ, distribute to the outgoing roads.

(B) Fulfilling (A), drivers maximise the flux through J.

The distribution coefficients $\alpha_{\psi,\varphi}$ represent average values of statistical travel preferences. The latter may well change depending on the hour of the day, thus rendering the matrix A dependent on time. The case of time-varying coefficients was treated in [9]; however, here we focus on the simpler case of fixed coefficients.

Rule (B) describes the situation in which drivers, travelling on incoming roads, optimise the flow through the junction. Such assumption is reasonable but obviously may not be verified in practice because of the limitation in junction capacity and drivers' choices. We notice that the optimisation of velocity gives rise to the same solver for simple junctions. For a more deep discussion of the model and alternative ones, we refer the reader to [7].

The condition (C2) of Definition 1 imposes restrictions on possible values that $\hat{\rho} = RS(\rho_0)$ may attain. The following Proposition provides explicit expressions of sets where $\hat{\rho}$ may vary depending on the initial datum ρ_0 (see [3, 5, 8] for details).

Proposition 2 *Assume that the flux function is given by (2.3) and let $\hat{\rho} = RS(\rho_0)$. Then it holds:*

$$\widehat{\rho}_\varphi \in \begin{cases} \{\rho_{\varphi,0}\} \cup \left]\tau\left(\rho_{\varphi,0}\right), 1\right], & \text{if } 0 \leqslant \rho_{\varphi,0} \leqslant \frac{1}{2}, \\ \left[\frac{1}{2}, 1\right], & \text{if } \frac{1}{2} \leqslant \rho_{\varphi,0} \leqslant 1, \end{cases} \quad \varphi = 1, \ldots, n,$$

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and

$$\widehat{\rho}_\psi \in \begin{cases} \left[0, \frac{1}{2}\right], & \text{if } 0 \leqslant \rho_{\psi,0} \leqslant \frac{1}{2}, \\ \{\rho_{\psi,0}\} \cup \left[0, \tau\left(\rho_{\psi,0}\right)\right[, & \text{if } \frac{1}{2} \leqslant \rho_{\psi,0} \leqslant 1, \end{cases} \quad \psi = n+1, \dots, n+m,$$

where $\tau : [0,1] \to [0,1]$ *is the map such that* $f(\tau(\rho)) = f(\rho)$ *for every* $\rho \in [0,1]$, *and* $\tau(\rho) \neq \rho$ *for every* $\rho \in [0,1] \setminus \{\sigma\}$.

3 Choice of a Riemann Solver

We describe a Riemann Solver that satisfies rules (A) and (B) for a junction of 2×2 type, i.e. with two incoming roads, a and b, and two outgoing roads, c and d. For such a junction, the traffic distribution matrix A assumes the form

$$A = \begin{pmatrix} \alpha & \beta \\ 1-\alpha & 1-\beta \end{pmatrix},$$

where α is the probability that drivers go from road a to road c and β is the probability that drivers travel from road b to road c. Let us suppose that $\alpha \neq \beta$ in order to fulfill a technical condition for uniqueness of solutions, see [5] for details.

From Proposition 2, in order to obtain the solution on each road of the junction J, it is enough to specify the flux values $\widehat{\gamma}_\varphi = f(\widehat{\rho}_\varphi)$, $\varphi = a, b$, and $\widehat{\gamma}_\psi = f(\widehat{\rho}_\psi)$, $\psi = c, d$. In particular, from rule (A), it follows that

$$\begin{pmatrix} \widehat{\gamma}_c \\ \widehat{\gamma}_d \end{pmatrix} = A \begin{pmatrix} \widehat{\gamma}_a \\ \widehat{\gamma}_b \end{pmatrix}.$$

From rule (B), we have that $\widehat{\gamma}_\varphi$, $\varphi = a, b$ is found solving the linear programming problem:

$$\begin{aligned} &\max \ (\gamma_a + \gamma_b), \\ &0 \leqslant \alpha\gamma_a + \beta\gamma_b \leqslant \gamma_c^{\max}, \\ &0 \leqslant (1-\alpha)\gamma_a + (1-\beta)\gamma_b \leqslant \gamma_d^{\max}, \\ &0 \leqslant \gamma_\varphi \leqslant \gamma_\varphi^{\max}, \end{aligned} \tag{3.1}$$

where the maximum fluxes on roads are

$$\gamma_\varphi^{\max} = \begin{cases} f\left(\rho_{\varphi,0}\right), & \text{if } \rho_{\varphi,0} \in \left[0, \frac{1}{2}\right], \\ f\left(\frac{1}{2}\right), & \text{if } \rho_{\varphi,0} \in \left]\frac{1}{2}, 1\right], \end{cases} \quad \varphi = a, b, \tag{3.2}$$

$$\gamma_\psi^{\max} = \begin{cases} f\left(\frac{1}{2}\right), & \text{if } \rho_{\psi,0} \in \left[0, \frac{1}{2}\right], \\ f\left(\rho_{\psi,0}\right), & \text{if } \rho_{\psi,0} \in \left]\frac{1}{2}, 1\right], \end{cases} \quad \psi = c, d. \tag{3.3}$$

The solution of (3.1) is found as follows. Introduce the function $g(\gamma_1, \gamma_2, x, y)$ as

$$g(\gamma_1, \gamma_2, x, y) = \frac{\gamma_1}{x} - \frac{y}{x}\gamma_2.$$

Define the lines

$$l_1 = \left\{(\gamma_a, \gamma_b) \in \mathbb{R}^2 : \alpha\gamma_a + \beta\gamma_b = \gamma_c^{\max}\right\},$$

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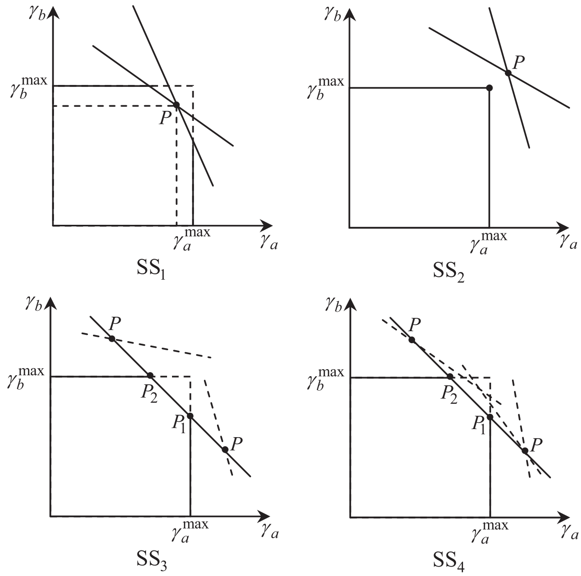

FIGURE 2. Different solution scenarios (SSs) for the problem (3.1).

$$l_2 = \left\{(\gamma_a, \gamma_b) \in \mathbb{R}^2 : (1-\alpha)\,\gamma_a + (1-\beta)\,\gamma_b = \gamma_d^{\max}\right\},$$

and set $P = l_1 \cap l_2 = (\widetilde{\gamma}_a, \widetilde{\gamma}_b)$. The fluxes $\widehat{\gamma}_a$ and $\widehat{\gamma}_b$ must belong to the region

$$\Omega = \left\{(\gamma_a, \gamma_b) \in \mathbb{R}^2 : 0 \leqslant \gamma_a \leqslant \gamma_a^{\max},\ 0 \leqslant \gamma_b \leqslant \gamma_b^{\max}\right\},$$

thus if P belongs to Ω, we set $(\widehat{\gamma}_a, \widehat{\gamma}_b) = (\widetilde{\gamma}_a, \widetilde{\gamma}_b)$, otherwise $(\widehat{\gamma}_a, \widehat{\gamma}_b) = proj_\Omega(P)$, where $proj$ is the projection on a convex set. Four different solution scenarios (SSs) are possible for (3.1) (see Figure 2). Various SSs are fully described by the conditions (A1)–(A12). More precisely, (A1) corresponds to SS$_1$ and (A2) to SS$_2$, while (A3)–(A12) distinguish various sub-cases for SS$_3$ and SS$_4$.

(A1) $\widetilde{\gamma}_a < \gamma_a^{\max}, \widetilde{\gamma}_b < \gamma_b^{\max}$, $g\left(\gamma_c^{\max}, \gamma_b^{\max}, \alpha, \beta\right) < g\left(\gamma_d^{\max}, \gamma_b^{\max}, 1-\alpha, 1-\beta\right) < \gamma_a^{\max}$, $g\left(\gamma_d^{\max}, \gamma_a^{\max}, 1-\beta, 1-\alpha\right) < g\left(\gamma_c^{\max}, \gamma_a^{\max}, \beta, \alpha\right) < \gamma_b^{\max}$;

(A2) $\widetilde{\gamma}_a \geqslant \gamma_a^{\max}$, $\widetilde{\gamma}_b \geqslant \gamma_b^{\max}$;

(A3) $\widetilde{\gamma}_a < \gamma_a^{\max}$, $\widetilde{\gamma}_b > \gamma_b^{\max}$, $g\left(\gamma_c^{\max}, \gamma_b^{\max}, \alpha, \beta\right) < \gamma_a^{\max} < g\left(\gamma_d^{\max}, \gamma_b^{\max}, 1-\alpha, 1-\beta\right)$;

(A4) $\widetilde{\gamma}_a > \gamma_a^{\max}$, $\widetilde{\gamma}_b < \gamma_b^{\max}$, $g\left(\gamma_c^{\max}, \gamma_a^{\max}, \beta, \alpha\right) < \gamma_b^{\max} < g\left(\gamma_d^{\max}, \gamma_a^{\max}, 1-\beta, 1-\alpha\right)$;

(A5) $\widetilde{\gamma}_a < \gamma_a^{\max}$, $\widetilde{\gamma}_b > \gamma_b^{\max}$, $g\left(\gamma_d^{\max}, \gamma_b^{\max}, 1-\alpha, 1-\beta\right) < \gamma_a^{\max} < g\left(\gamma_c^{\max}, \gamma_b^{\max}, \alpha, \beta\right)$;

(A6) $\widetilde{\gamma}_a > \gamma_a^{\max}$, $\widetilde{\gamma}_b < \gamma_b^{\max}$, $g\left(\gamma_d^{\max}, \gamma_a^{\max}, 1-\beta, 1-\alpha\right) < \gamma_b^{\max} < g\left(\gamma_c^{\max}, \gamma_a^{\max}, \beta, \alpha\right)$;

(A7) $\widetilde{\gamma}_a < \gamma_a^{\max}$, $\widetilde{\gamma}_b > \gamma_b^{\max}$, $g\left(\gamma_c^{\max}, \gamma_b^{\max}, \alpha, \beta\right) < g\left(\gamma_d^{\max}, \gamma_b^{\max}, 1-\alpha, 1-\beta\right) < \gamma_a^{\max}$;

(A8) $\widetilde{\gamma}_a > \gamma_a^{\max}$, $\widetilde{\gamma}_b < \gamma_b^{\max}$, $g\left(\gamma_c^{\max}, \gamma_a^{\max}, \beta, \alpha\right) < g\left(\gamma_d^{\max}, \gamma_a^{\max}, 1-\beta, 1-\alpha\right) < \gamma_b^{\max}$;

(A9) $\widetilde{\gamma}_a < \gamma_a^{\max}$, $\widetilde{\gamma}_b > \gamma_b^{\max}$, $\gamma_a^{\max} > g\left(\gamma_c^{\max}, \gamma_b^{\max}, \alpha, \beta\right) > g\left(\gamma_d^{\max}, \gamma_b^{\max}, 1-\alpha, 1-\beta\right)$;

(A10) $\widetilde{\gamma}_a > \gamma_a^{\max}$, $\widetilde{\gamma}_b < \gamma_b^{\max}$, $\gamma_b^{\max} > g\left(\gamma_c^{\max}, \gamma_a^{\max}, \beta, \alpha\right) > g\left(\gamma_d^{\max}, \gamma_a^{\max}, 1-\beta, 1-\alpha\right)$;

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(A11) $\widetilde{\gamma}_a < \gamma_a^{\max}$, $\widetilde{\gamma}_b > \gamma_b^{\max}$, $g\left(\gamma_c^{\max}, \gamma_b^{\max}, \alpha, \beta\right) > \gamma_a^{\max}$, $g\left(\gamma_d^{\max}, \gamma_b^{\max}, 1-\alpha, 1-\beta\right) > \gamma_a^{\max}$;

(A12) $\widetilde{\gamma}_a > \gamma_a^{\max}$, $\widetilde{\gamma}_b < \gamma_b^{\max}$, $g\left(\gamma_c^{\max}, \gamma_a^{\max}, \beta, \alpha\right) > \gamma_b^{\max}$, $g\left(\gamma_d^{\max}, \gamma_a^{\max}, 1-\beta, 1-\alpha\right) > \gamma_b^{\max}$.

Following are the solutions $\widehat{\gamma}_a$ and $\widehat{\gamma}_b$ of the RP:

- If A_1 holds, then $(\widehat{\gamma}_a, \widehat{\gamma}_b) = (\widetilde{\gamma}_a, \widetilde{\gamma}_b)$.
- If A_2 or A_{11} or A_{12} hold, then $(\widehat{\gamma}_a, \widehat{\gamma}_b) = (\gamma_a^{\max}, \gamma_b^{\max})$.
- If A_3 or A_7 are satisfied, then $(\widehat{\gamma}_a, \widehat{\gamma}_b) = (\check{\gamma}_a, \check{\gamma}_b)$, where

$$(\check{\gamma}_a, \check{\gamma}_b) = \begin{cases} \left(g\left(\gamma_c^{\max}, \gamma_b^{\max}, \alpha, \beta\right), \gamma_b^{\max}\right), & \text{if } g\left(\gamma_c^{\max}, \gamma_b^{\max}, \alpha, \beta\right) \geqslant 0, \\ \left(0, \frac{\gamma_c^{\max}}{\beta}\right), & \text{otherwise.} \end{cases}$$

- If A_4 or A_8 hold, then $(\widehat{\gamma}_a, \widehat{\gamma}_b) = (\breve{\gamma}_a, \breve{\gamma}_b)$, where

$$(\breve{\gamma}_a, \breve{\gamma}_b) = \begin{cases} \left(\gamma_a^{\max}, g\left(\gamma_c^{\max}, \gamma_a^{\max}, \beta, \alpha\right)\right), & \text{if } g\left(\gamma_c^{\max}, \gamma_a^{\max}, \beta, \alpha\right) \geqslant 0, \\ \left(\frac{\gamma_c^{\max}}{\alpha}, 0\right), & \text{otherwise.} \end{cases}$$

- If A_5 or A_9 are satisfied, then $(\widehat{\gamma}_a, \widehat{\gamma}_b) = (\bar{\gamma}_a, \bar{\gamma}_b)$, where

$$(\bar{\gamma}_a, \bar{\gamma}_b) = \begin{cases} \left(g\left(\gamma_d^{\max}, \gamma_b^{\max}, 1-\alpha, 1-\beta\right), \gamma_b^{\max}\right), & \text{if } g\left(\gamma_d^{\max}, \gamma_b^{\max}, 1-\alpha, 1-\beta\right) \geqslant 0, \\ \left(0, \frac{\gamma_d^{\max}}{1-\beta}\right), & \text{otherwise.} \end{cases}$$

- If A_6 or A_{10} hold, then $(\widehat{\gamma}_a, \widehat{\gamma}_b) = (\mathring{\gamma}_a, \mathring{\gamma}_b)$, where

$$(\mathring{\gamma}_a, \mathring{\gamma}_b) = \begin{cases} \left(\gamma_a^{\max}, g\left(\gamma_d^{\max}, \gamma_a^{\max}, 1-\beta, 1-\alpha\right)\right), & \text{if } g\left(\gamma_d^{\max}, \gamma_a^{\max}, 1-\beta, 1-\alpha\right) \geqslant 0, \\ \left(\frac{\gamma_d^{\max}}{1-\alpha}, 0\right), & \text{otherwise.} \end{cases}$$

4 Optimisation of distribution coefficients

Our aim is to find the values of traffic distribution parameters at a junction in order to manage critical situations, such as car accidents. In this case, beside the ordinary cars flows, other traffic sources, due to emergency vehicles, are present. More precisely, assume that a car accident occurs on a road of an urban network and that some emergency vehicles have to reach the place of the accident, or hospital.

We define the velocity function for such vehicles as

$$\omega(\rho) = 1 - \delta + \delta v(\rho), \tag{4.1}$$

with $0 < \delta < 1$ and $v(\rho)$ as in (2.2). Since $\omega(\rho_{\max}) = 1 - \delta > 0$, it follows that the emergency vehicles travel with a higher velocity with respect to cars. Notice that (4.1) coincides with the velocity of the ordinary traffic for $\delta = 1$.

Consider a junction J with n incoming roads and m outgoing roads. Fix an incoming road I_φ, $\varphi = 1,\dots,n$, and an outgoing road I_ψ, $\psi = n+1,\dots,n+m$. Given an initial data $(\rho_{\varphi,0}, \rho_{\psi,0})$, we define the cost functional $W_{\varphi,\psi}(t)$, which indicates the average velocity of emergency vehicles crossing I_φ and I_ψ:

$$W_{\varphi,\psi}(t) = \int_{I_\varphi} \omega\left(\rho_\varphi(t,x)\right) dx + \int_{I_\psi} \omega\left(\rho_\psi(t,x)\right) dx.$$

As maximising $W_{\varphi,\psi}(t)$ with respect to the traffic distribution parameters $\alpha_{\psi,\varphi}$ is a huge task, we find a solution for the optimisation problem in the asymptotic regime, i.e. after a long time has elapsed, using $\hat{\rho} = (\hat{\rho}_\varphi, \hat{\rho}_\psi)$ as densities. So we fix a time horizon $[0, T]$ and formulate the problem in the following way:

(P) Consider a junction J with n incoming roads and m outgoing roads, the traffic distribution coefficients $\alpha_{\psi,\varphi}$ as controls and the functional $W_{\varphi,\psi}(t)$. We want to maximise $W_{\varphi,\psi}(T)$ for T sufficiently big.

In what follows, we focus the attention on a junction J of type 2×2, fixing an incoming road I_φ, $\varphi = a, b$, and an outgoing road I_ψ, $\psi = c, d$. For T sufficiently big we have

$$W_{\varphi,\psi}(T) = \omega\left(\widehat{\rho}_\varphi\right) + \omega\left(\widehat{\rho}_\psi\right) = 2 - \delta - \frac{\delta}{2}\left(s_\varphi\sqrt{1-4\widehat{\gamma}_\varphi} + s_\psi\sqrt{1-4\widehat{\gamma}_\psi}\right), \tag{4.2}$$

where s_φ and s_ψ are defined as

$$s_\varphi = \begin{cases} +1, & \text{if } \rho_{\varphi,0} \geqslant \frac{1}{2}, \text{or } \rho_{\varphi,0} < \frac{1}{2} \text{ and } \gamma_\varphi^{\max} > \widehat{\gamma}_\varphi, \\ -1, & \text{if } \rho_{\varphi,0} < \frac{1}{2} \text{ and } \gamma_\varphi^{\max} = \widehat{\gamma}_\varphi, \end{cases}$$

$$s_\psi = \begin{cases} +1, & \text{if } \rho_{\psi,0} > \frac{1}{2} \text{ and } \gamma_\psi^{\max} = \widehat{\gamma}_\psi, \\ -1, & \text{if } \rho_{\psi,0} \leqslant \frac{1}{2}, \text{or } \rho_{\psi,0} > \frac{1}{2} \text{ and } \gamma_\psi^{\max} > \widehat{\gamma}_\psi. \end{cases}$$

Without loss of generality, choosing $\varphi = a$ and $\psi = c$, we have that (4.2) becomes

$$W_{a,c}(T) = \omega\left(\widehat{\rho}_a\right) + \omega\left(\widehat{\rho}_c\right) = 2 - \delta - \frac{\delta}{2}\left(s_a\sqrt{1-4\widehat{\gamma}_a} + s_c\sqrt{1-4\widehat{\gamma}_c}\right). \tag{4.3}$$

Notice that $\widehat{\gamma}_a$ and $\widehat{\gamma}_c$ in (4.3) depend on traffic coefficients α and β, which have to be determined in order to maximise the velocity of emergency vehicles on roads a and c.

The cost functional $W_{a,c}(T)$ is optimised choosing the distribution coefficients according to the following theorem.

Theorem 3 *Consider a junction J with two incoming roads, a and b, and two outgoing roads, c and d. For T sufficiently big, the values of α and β, which optimise the cost functional $W_{a,c}(T)$, are $\alpha_{opt} = 1 - \frac{\gamma_d^{\max}}{\gamma_a^{\max}}$, $0 \leqslant \beta_{opt} < 1 - \frac{\gamma_d^{\max}}{\gamma_a^{\max}}$, with the exception of the following cases, where the optimal controls do not exist but the optimal values are approximated by*

- $\alpha_{opt} = \varepsilon_1$, $\beta_{opt} = \varepsilon_2$, *if* $\gamma_a^{\max} \leqslant \gamma_d^{\max}$;
- $\alpha_{opt} = \frac{\gamma_c^{\max}}{\gamma_c^{\max}+\gamma_d^{\max}} - \varepsilon_1$, $\beta_{opt} = \frac{\gamma_c^{\max}}{\gamma_c^{\max}+\gamma_d^{\max}} - \varepsilon_2$, *if* $\gamma_a^{\max} > \gamma_c^{\max} + \gamma_d^{\max}$, *for* ε_1 *and* ε_2 *small, positive and such that* $\varepsilon_1 \neq \varepsilon_2$.

Proof For simplicity, from now on we drop the dependence on T from $W_{a,c}$. Fix a junction J and an initial datum $\rho_0 = (\rho_{a,0}, \rho_{b,0}, \rho_{c,0}, \rho_{d,0})$.

The proof is organised in the following several steps:

(1) Divide the rectangular region $\Lambda = \{(\alpha, \beta) \in \mathbb{R}^2 : 0 \leqslant \alpha \leqslant 1,\ 0 \leqslant \beta \leqslant 1\}$ into subregions $\Lambda_k \subset \Lambda$, $k = 1, \dots, N$, for which the solution to the RP obeys the same SS.

(2) Compute the explicit expression of $W_{a,c}(\alpha, \beta)$ for every Λ_k, $k = 1, \dots, N$.

(3) Compute $(\alpha_k, \beta_k) \in \Lambda_k\ \forall\ \Lambda_k$, $k = 1, \dots, N$, such that $W_{a,c}(\alpha_k, \beta_k) = M_{\Lambda_k} = \max_{(\alpha,\beta)\in\Lambda_k} W_{a,c}(\alpha, \beta)$.

(4) Find $(\alpha_{opt}, \beta_{opt}) \in \Lambda$ such that $M_\Lambda = \max_{(\alpha,\beta)\in\Lambda} W_{a,c}(\alpha, \beta) = \max\{M_{\Lambda_1},\ M_{\Lambda_2}, \dots,\ M_{\Lambda_N}\}$.

Notice that,

- N is at most equal to six, depending on the chosen ρ_0 at J, namely different initial conditions ρ_0 imply different subdivision of Λ in terms of the number of regions;
- optimal values α_{opt} and β_{opt} are not always well defined due to strict inequalities that define some subregions Λ_k.

We proceed now with the details of the proof. Denoted by $\Gamma_{in}^{\max}$ and $\Gamma_{out}^{\max}$, the sum of maximum fluxes on incoming and outgoing roads, respectively are

$$\Gamma_{in}^{\max} = \gamma_a^{\max} + \gamma_b^{\max}, \quad \Gamma_{out}^{\max} = \gamma_c^{\max} + \gamma_d^{\max}.$$

In what follows, we make the following assumptions on initial data (for all the other cases, the proof is similar):

(H1) $\rho_{a,0} < \frac{1}{2}, \quad \rho_{c,0} > \frac{1}{2};$

(H2) $\gamma_d^{\max} < \gamma_b^{\max} < \gamma_c^{\max} < \gamma_a^{\max} < \Gamma_{out}^{\max} < \Gamma_{in}^{\max}.$

Define the lines:

$$r = \left\{(\alpha, \beta) \in \mathbb{R}^2 : \beta = \frac{\gamma_c^{\max} - \alpha\gamma_a^{\max}}{\Gamma_{out}^{\max} - \gamma_a^{\max}}\right\},$$

$$s = \left\{(\alpha, \beta) \in \mathbb{R}^2 : \beta = \frac{\gamma_c^{\max} - \alpha(\Gamma_{out}^{\max} - \gamma_b^{\max})}{\gamma_b^{\max}}\right\},$$

$$t = \left\{(\alpha, \beta) \in \mathbb{R}^2 : \beta = \alpha\right\},$$

and the regions into which r, s and t divide the plane (α, β):

$$r^+ = \left\{(\alpha, \beta) \in \mathbb{R}^2 : \beta \geqslant \frac{\gamma_c^{\max} - \alpha\gamma_a^{\max}}{\Gamma_{out}^{\max} - \gamma_a^{\max}}\right\},$$

$$r^- = \left\{(\alpha, \beta) \in \mathbb{R}^2 : \beta \leqslant \frac{\gamma_c^{\max} - \alpha\gamma_a^{\max}}{\Gamma_{out}^{\max} - \gamma_a^{\max}}\right\},$$

$$s^+ = \left\{(\alpha, \beta) \in \mathbb{R}^2 : \beta \geqslant \frac{\gamma_c^{\max} - \alpha(\Gamma_{out}^{\max} - \gamma_b^{\max})}{\gamma_b^{\max}}\right\},$$

$$s^- = \left\{(\alpha, \beta) \in \mathbb{R}^2 : \beta \leqslant \frac{\gamma_c^{\max} - \alpha(\Gamma_{out}^{\max} - \gamma_b^{\max})}{\gamma_b^{\max}}\right\},$$

$$t^+ = \{(\alpha,\beta) \in \mathbb{R}^2 : \beta > \alpha\}, \quad t^- = \{(\alpha,\beta) \in \mathbb{R}^2 : \beta < \alpha\}.$$

The open set

$$\Lambda = \{(\alpha,\beta) \in \mathbb{R}^2 : 0 \leqslant \alpha \leqslant 1, \ 0 \leqslant \beta \leqslant 1\}$$

is decomposed as $\Lambda = \bigcup_{k=1}^{5} \Lambda_k$, where:

$$\Lambda_1 = \Lambda \cap r^+ \cap t^+, \quad \Lambda_2 = \Lambda \cap s^+ \cap t^-, \quad \Lambda_3 = \Lambda \cap \left[\left(r^- \cap s^+ \cap t^+\right) \cup \left(r^+ \cap s^- \cap t^-\right)\right],$$

$$\Lambda_4 = \Lambda \cap s^- \cap t^+, \quad \Lambda_5 = \Lambda \cap r^- \cap t^-.$$

A unique RS is associated with each region Λ_m, $m = 1,\dots,5$, on the basis of conditions A_j, $j = 1,\dots,12$. Precisely, we have that given a couple (α, β):

- if $(\alpha, \beta) \in \Lambda_1$, A_4 or A_8 are satisfied;
- if $(\alpha, \beta) \in \Lambda_2$, A_3 or A_7 are satisfied;
- if $(\alpha, \beta) \in \Lambda_3$, A_1 holds;
- if $(\alpha, \beta) \in \Lambda_4$, A_5 or A_9 are satisfied;
- if $(\alpha, \beta) \in \Lambda_5$, A_6 or A_{10} hold.

Hence, the cost functional $W_{a,c}$ is written as follows:

$$W_{a,c} = \begin{cases} 2-\delta-\frac{\delta}{2}\left(s_a\sqrt{1-4\breve{\gamma}_a}+s_c\sqrt{1-4\left(\alpha\breve{\gamma}_a+\beta\breve{\gamma}_b\right)}\right), & \text{if } (\alpha,\beta)\in\Lambda_1, \\ 2-\delta-\frac{\delta}{2}\left(\sqrt{1-4\left(\frac{\gamma_c^{\max}-\beta\gamma_b^{\max}}{\alpha}\right)}+\sqrt{1-4\gamma_c^{\max}}\right), & \text{if } (\alpha,\beta)\in\Lambda_2, \\ 2-\delta-\frac{\delta}{2}\left(\sqrt{1-4\widetilde{\gamma}_a}+\sqrt{1-4\gamma_c^{\max}}\right), & \text{if } (\alpha,\beta)\in\Lambda_3, \\ 2-\delta-\frac{\delta}{2}\left(s_a\sqrt{1-4\bar{\gamma}_a}+s_c\sqrt{1-4\left(\alpha\bar{\gamma}_a+\beta\bar{\gamma}_b\right)}\right), & \text{if } (\alpha,\beta)\in\Lambda_4, \\ 2-\delta-\frac{\delta}{2}\left(s_a\sqrt{1-4\mathring{\gamma}_a}+s_c\sqrt{1-4\left(\alpha\mathring{\gamma}_a+\beta\mathring{\gamma}_b\right)}\right), & \text{if } (\alpha,\beta)\in\Lambda_5. \end{cases}$$

Notice that (H1) establishes the values of s_a and s_c and the functional $W_{a,c}$ assumes different expressions in regions Λ_1, Λ_4 and Λ_5, as the values of $(\breve{\gamma}_a, \breve{\gamma}_b)$, $(\bar{\gamma}_a, \bar{\gamma}_b)$ and $(\mathring{\gamma}_a, \mathring{\gamma}_b)$ depend, respectively, on the sign of $g(\gamma_c^{\max}, \gamma_a^{\max}, \beta, \alpha)$, $g(\gamma_d^{\max}, \gamma_b^{\max}, 1-\alpha, 1-\beta)$ and $g(\gamma_d^{\max}, \gamma_a^{\max}, 1-\beta, 1-\alpha)$. In particular, we have that

$$g\left(\gamma_c^{\max}, \gamma_a^{\max}, \beta, \alpha\right) \geqslant 0 \Leftrightarrow \alpha \leqslant \frac{\gamma_c^{\max}}{\gamma_a^{\max}},$$

$$g\left(\gamma_d^{\max}, \gamma_b^{\max}, 1-\alpha, 1-\beta\right) \geqslant 0 \Leftrightarrow \beta \geqslant 1 - \frac{\gamma_d^{\max}}{\gamma_b^{\max}},$$

$$g\left(\gamma_d^{\max}, \gamma_a^{\max}, 1-\beta, 1-\alpha\right) \geqslant 0 \Leftrightarrow \alpha \geqslant 1 - \frac{\gamma_d^{\max}}{\gamma_a^{\max}}.$$

Such inequalities allow further divisions of the regions $\Lambda_i, i = 1,4,5$; see Figure 3, where

$$O = (0,0), \quad A = \left(\frac{\gamma_c^{\max}}{\gamma_a^{\max}}, 0\right), \quad B = (1,0),$$

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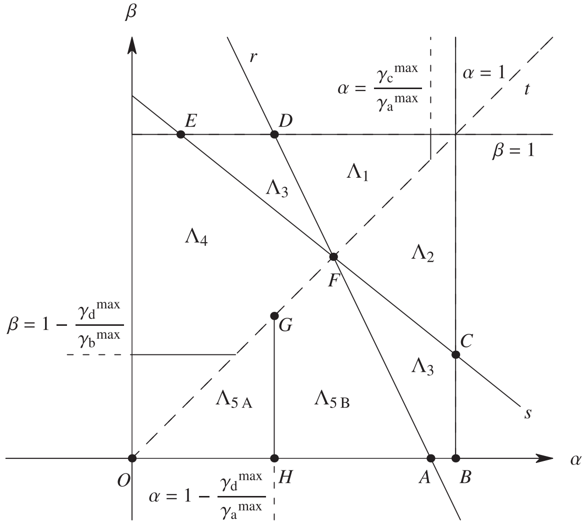

FIGURE 3. Division of Λ in regions and subregions.

$$C=\left(1,1-\frac{\gamma_d^{\max}}{\gamma_b^{\max}}\right),\quad D=\left(1-\frac{\gamma_d^{\max}}{\gamma_a^{\max}},1\right),\quad E=\left(\frac{\gamma_b^{\max}-\gamma_c^{\max}}{\gamma_b^{\max}-\Gamma_{out}^{\max}},1\right),$$

$$F=\left(\frac{\gamma_c^{\max}}{\Gamma_{out}^{\max}},\frac{\gamma_c^{\max}}{\Gamma_{out}^{\max}}\right),\quad G=\left(1-\frac{\gamma_d^{\max}}{\gamma_a^{\max}},1-\frac{\gamma_d^{\max}}{\gamma_a^{\max}}\right),\quad H=\left(1-\frac{\gamma_d^{\max}}{\gamma_a^{\max}},0\right).$$

Consider the region Λ_3. According to the RS, the cost functional $W_{a,c}$ is

$$J(\alpha,\beta)=2-\delta-\frac{\delta}{2}\sqrt{1-4\gamma_c^{\max}}-\frac{\delta}{2}\sqrt{\frac{\alpha+\beta\left(4\Gamma_{out}^{\max}-1\right)-4\gamma_c^{\max}}{\alpha-\beta}}.$$

We have that

$$\frac{\partial J(\alpha,\beta)}{\partial\alpha}=\frac{\delta\left(\beta\Gamma_{out}^{\max}-\gamma_c^{\max}\right)}{(\alpha-\beta)^2\sqrt{\frac{\alpha+\beta\left(4\Gamma_{out}^{\max}-1\right)-4\gamma_c^{\max}}{\alpha-\beta}}},$$

$$\frac{\partial J(\alpha,\beta)}{\partial\beta}=-\frac{\delta\left(\alpha\Gamma_{out}^{\max}-\gamma_c^{\max}\right)}{(\alpha-\beta)^2\sqrt{\frac{\alpha+\beta\left(4\Gamma_{out}^{\max}-1\right)-4\gamma_c^{\max}}{\alpha-\beta}}},$$

and we conclude that there are no critical points inside Λ_3 such that $\beta\neq\alpha$.

Now we study the behaviour of $J(\alpha,\beta)$ on boundaries. On segments $\overline{DF}\cup\overline{AF}$ and $\overline{EF}\cup\overline{CF}$, $J(\alpha,\beta)$ is constant and, in particular, its values are, respectively,

$$J\left(\alpha,\frac{\gamma_c^{\max}-\alpha\gamma_a^{\max}}{\Gamma_{out}^{\max}-\gamma_a^{\max}}\right)=2-\delta-\frac{\delta}{2}\left(\sqrt{1-4\gamma_a^{\max}}+\sqrt{1-4\gamma_c^{\max}}\right),$$

$$J\left(\alpha,\frac{\gamma_c^{\max}-\alpha(\Gamma_{out}^{\max}-\gamma_b^{\max})}{\gamma_b^{\max}}\right)=2-\delta-\frac{\delta}{2}\left(\sqrt{1-4\gamma_c^{\max}}+\sqrt{1+4\gamma_b^{\max}-4\Gamma_{out}^{\max}}\right).$$

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On the segment $\overline{DE}$, $W_{a,c}$ is equal to

$$J(\alpha,1) = 2-\delta-\frac{\delta}{2}\left(\sqrt{1-4\gamma_c^{max}}+\sqrt{\frac{1-\alpha-4\gamma_d^{max}}{1-\alpha}}\right), \quad \frac{\gamma_b^{max}-\gamma_c^{max}}{\gamma_b^{max}-\Gamma_{out}^{max}} \leqslant \alpha \leqslant 1-\frac{\gamma_d^{max}}{\gamma_a^{max}}.$$

Since

$$J'(\alpha,1) = \frac{\delta\gamma_d^{max}}{(1-\alpha)^2\sqrt{\frac{1-\alpha-4\gamma_d^{max}}{1-\alpha}}} > 0,$$

we conclude that $W_{a,c}(E) < W_{a,c}(D)$. Hence, the maximum is given by the point D for which

$$W_{a,c}(D) = 2-\delta-\frac{\delta}{2}\left(\sqrt{1-4\gamma_a^{max}}+\sqrt{1-4\gamma_c^{max}}\right) = J\left(\alpha, \frac{\gamma_c^{max}-\alpha\gamma_a^{max}}{\Gamma_{out}^{max}-\gamma_a^{max}}\right).$$

As for the analysis on the segment $\overline{BC}$, $W_{a,c}$ becomes

$$J(1,\beta) = 2-\delta-\frac{\delta}{2}\left(\sqrt{1-4\gamma_c^{max}}+\sqrt{\frac{1+\beta(4\Gamma_{out}^{max}-1)-4\gamma_c^{max}}{1-\beta}}\right), \quad 0\leqslant\beta\leqslant 1-\frac{\gamma_d^{max}}{\gamma_b^{max}},$$

whose derivative is

$$J'(1,\beta) = -\frac{\delta\gamma_d^{max}}{(1-\beta)^2\sqrt{\frac{1+\beta\left(4\Gamma_{out}^{max}-1\right)-4\gamma_c^{max}}{1-\beta}}} < 0.$$

Hence, $W_{a,c}(C) < W_{a,c}(B)$ and the maximum point is attained in B with

$$W_{a,c}(E) = 2-\delta\left(1+\sqrt{1-4\gamma_c^{max}}\right).$$

Finally, on the segment $\overline{AB}$, $W_{a,c}$ is equal to

$$J(\alpha,0) = 2-\delta-\frac{\delta}{2}\left(\sqrt{1-4\gamma_c^{max}}+\sqrt{1-4\frac{\gamma_c^{max}}{\alpha}}\right), \quad \frac{\gamma_c^{max}}{\gamma_a^{max}}\leqslant\alpha\leqslant 1,$$

and

$$J'(\alpha,0) = -\frac{\delta\gamma_c^{max}}{\alpha^2\sqrt{\frac{\alpha-4\gamma_c^{max}}{\alpha}}} < 0.$$

So $W_{a,c}(B) < W_{a,c}(A)$, and the maximum point is A with $W_{a,c}(A) = W_{a,c}(D)$. Notice that

$$W_{a,c}(D) = W_{a,c}(A) > J\left(\alpha, \frac{\gamma_c^{max}-\alpha(\Gamma_{out}^{max}-\gamma_b^{max})}{\gamma_b^{max}}\right) \Leftrightarrow \Gamma_{in}^{max} > \Gamma_{out}^{max},$$

$$J\left(\alpha, \frac{\gamma_c^{max}-\alpha\gamma_a^{max}}{\Gamma_{out}^{max}-\gamma_a^{max}}\right) > W_{a,c}(B) \Leftrightarrow \gamma_c^{max} < \gamma_a^{max},$$

$$J\left(\alpha, \frac{\gamma_c^{max}-\alpha(\Gamma_{out}^{max}-\gamma_b^{max})}{\gamma_b^{max}}\right) < W_{a,c}(B) \Leftrightarrow \gamma_d^{max} < \gamma_b^{max},$$

which satisfy (H2). Hence, we get

$$W_{a,c}(D) = W_{a,c}(A) > W_{a,c}(B) > J\left(\alpha, \frac{\gamma_c^{\max} - \alpha(\Gamma_{out}^{\max} - \gamma_b^{\max})}{\gamma_b^{\max}}\right),$$

and the absolute maximum in Λ_3 is achieved in all points of the set

$$\Lambda \cap r \cap \left\{(\alpha, \beta) \in \mathbb{R}^2 : \beta \neq \alpha\right\},$$

for which the value of the cost functional is

$$M_{\Lambda_3} = 2 - \delta - \frac{\delta}{2}\left(\sqrt{1 - 4\gamma_a^{\max}} + \sqrt{1 - 4\gamma_c^{\max}}\right).$$

Now focus the attention on Λ_5. The line $\alpha = 1 - \frac{\gamma_d^{\max}}{\gamma_a^{\max}}$ divides Λ_5 into two subregions, $\Lambda_{5,-}$ and $\Lambda_{5,+}$ (see Figure 3). Precisely,

$$\Lambda_{5,-} = \Lambda_5 \cap u^-, \quad \Lambda_{5,+} = \Lambda_5 \cap u^+,$$

where

$$u^+ = \left\{(\alpha, \beta) \in \mathbb{R}^2 : \alpha \geqslant 1 - \frac{\gamma_d^{\max}}{\gamma_a^{\max}}\right\}, \quad u^- = \left\{(\alpha, \beta) \in \mathbb{R}^2 : \alpha \leqslant 1 - \frac{\gamma_d^{\max}}{\gamma_a^{\max}}\right\}.$$

The cost functional $W_{a,c}$ is given by

$$W_{a,c} = \begin{cases} J_1(\alpha), & \text{if } (\alpha, \beta) \in \Lambda_{5,-}, \\ J_2(\alpha, \beta), & \text{if } (\alpha, \beta) \in \Lambda_{5,+}, \end{cases}$$

with:

$$J_1(\alpha) = 2 - \delta + \frac{\delta}{2}\left(\sqrt{1 - \frac{4\alpha\gamma_d^{\max}}{1-\alpha}} - \sqrt{1 - \frac{4\gamma_d^{\max}}{1-\alpha}}\right),$$

$$J_2(\alpha, \beta) = 2 - \delta + \frac{\delta}{2}\left(\sqrt{\frac{1 - 4\alpha\gamma_a^{\max} - \beta\left(1 + 4\gamma_d^{\max} - \gamma_a^{\max}\right)}{1-\beta}} + \sqrt{1 - 4\gamma_a^{\max}}\right).$$

We have,

$$J_1'(\alpha) = \frac{\delta\gamma_d^{\max}}{(1-\alpha)^2}\left(\frac{1}{\sqrt{1 - \frac{4\gamma_d^{\max}}{1-\alpha}}} - \frac{1}{\sqrt{1 - \frac{4\alpha\gamma_d^{\max}}{1-\alpha}}}\right),$$

which does not vanish for any value of α. Moreover,

$$\frac{\partial J_2(\alpha, \beta)}{\partial \alpha} = -\frac{\delta\gamma_a^{\max}}{(1-\beta)\sqrt{\frac{1 - 4\alpha\gamma_a^{\max} - \beta\left(1 + 4\gamma_d^{\max} - \gamma_a^{\max}\right)}{1-\beta}}},$$

$$\frac{\partial J_2(\alpha, \beta)}{\partial \beta} = -\frac{\delta\left[\gamma_d^{\max} + \gamma_a^{\max}(1+\alpha)\right]}{(1-\beta)^2\sqrt{\frac{1 - 4\alpha\gamma_a^{\max} - \beta\left(1 + 4\gamma_d^{\max} - \gamma_a^{\max}\right)}{1-\beta}}},$$

and we get that there are no critical points inside $\Lambda_{5,+}$.

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Now we study the behaviour of $W_{a,c}$ on boundaries. First we consider $\Lambda_{5,-}$. Since $J_1'(\alpha) \geqslant 0$ for $0 \leqslant \alpha \leqslant 1-\frac{\gamma_d^{\max}}{\gamma_a^{\max}}$, the function $J_1(\alpha)$ is increasing with respect to α. It follows that $W_{a,c}(O) < W_{a,c}(G)$, and the maximum on the segment $\overline{OG}$ is attained at point G where the functional assumes the value,

$$W_{a,c}(G) = 2 - \delta + \frac{\delta}{2}\left(\sqrt{1-4\gamma_a^{\max}+4\gamma_d^{\max}} - \sqrt{1-4\gamma_a^{\max}}\right).$$

Then, as $W_{a,c}(O) < W_{a,c}(H)$, the velocity functional assumes the maximum value on the segment $\overline{OH}$ at point H, and $W_{a,c}(G) = W_{a,c}(H)$. Finally, on the segment $\overline{GH}$, the functional is constant and equals to

$$J_1\left(1-\frac{\gamma_d^{\max}}{\gamma_a^{\max}}\right) = W_{a,c}(G).$$

As $W_{a,c}(G) = W_{a,c}(H) = J_1\big(1-\frac{\gamma_d^{\max}}{\gamma_a^{\max}}\big)$, we conclude that in $\Lambda_{5,-}$, $W_{a,c}$ assumes the absolute maximum at all points of the segment $\overline{GH}$, and the value of the cost functional is

$$M_{\Lambda_{5,-}} = 2 - \delta + \frac{\delta}{2}\left(\sqrt{1-4\gamma_a^{\max}+4\gamma_d^{\max}} - \sqrt{1-4\gamma_a^{\max}}\right).$$

Now consider the subregion $\Lambda_{5,+}$. On the segment $\overline{AH}$, $W_{a,c}$ is equal to

$$J_2(\alpha,0) = 2 - \delta + \frac{\delta}{2}\left(\sqrt{1-4\gamma_a^{\max}} + \sqrt{1-4\alpha\gamma_d^{\max}}\right), \quad 1-\frac{\gamma_d^{\max}}{\gamma_a^{\max}} \leqslant \alpha \leqslant \frac{\gamma_c^{\max}}{\gamma_a^{\max}},$$

whose derivative is

$$J_2'(\alpha,0) = -\frac{\delta\gamma_a^{\max}}{\sqrt{1-4\alpha\gamma_d^{\max}}} < 0, \quad 1-\frac{\gamma_d^{\max}}{\gamma_a^{\max}} \leqslant \alpha \leqslant \frac{\gamma_c^{\max}}{\gamma_a^{\max}}.$$

Hence, $W_{a,c}(A) < W_{a,c}(H)$, and the maximum is achieved at point H, where we have

$$W_{a,c}(H) = 2 - \delta + \frac{\delta}{2}\left(\sqrt{1-4\gamma_a^{\max}} + \sqrt{1-4\gamma_a^{\max}+4\gamma_d^{\max}}\right).$$

The cost functional is constant on the segment $\overline{GH}$, where it is given by

$$J_2\left(1-\frac{\gamma_d^{\max}}{\gamma_a^{\max}},\beta\right) = W_{a,c}(H), \quad 0 \leqslant \beta < 1-\frac{\gamma_d^{\max}}{\gamma_a^{\max}}.$$

As for the analysis of $W_{a,c}$ on the segment $\overline{FG}$, we have to consider the following function:

$$\widetilde{J}_2(\alpha) = \lim_{\beta\to\alpha} J_2(\alpha,\beta) = 2 - \delta + \frac{\delta}{2}\left(\sqrt{1-4\gamma_a^{\max}} + \sqrt{\frac{1-\alpha(1+4\gamma_d^{\max})}{1-\alpha}}\right),$$

$1-\frac{\gamma_d^{\max}}{\gamma_a^{\max}} \leqslant \alpha \leqslant \frac{\gamma_c^{\max}}{\Gamma_{out}^{\max}}$. Since

$$\widetilde{J}_2'(\alpha) = -\frac{\delta\gamma_d^{\max}}{(1-\alpha)^2\sqrt{1-\frac{4\alpha\gamma_d^{\max}}{1-\alpha}}} < 0, \quad 1-\frac{\gamma_d^{\max}}{\gamma_a^{\max}} \leqslant \alpha \leqslant \frac{\gamma_c^{\max}}{\Gamma_{out}^{\max}},$$

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it follows that $W_{a,c}(F) < W_{a,c}(G)$, and the maximum is attained at point G, where we have $W_{a,c}(G) = W_{a,c}(H)$. Finally, on the segment $\overline{AF}$, $W_{a,c}$ is a constant function:

$$J_2\left(\alpha, \frac{\gamma_c^{\max} - \alpha\gamma_a^{\max}}{\Gamma_{out}^{\max} - \gamma_a^{\max}}\right) = 2 - \delta + \frac{\delta}{2}\left(\sqrt{1 - 4\gamma_a^{\max}} + \sqrt{1 - 4\gamma_c^{\max}}\right), \quad \frac{\gamma_c^{\max}}{\Gamma_{out}^{\max}} \leqslant \alpha \leqslant \frac{\gamma_c^{\max}}{\gamma_a^{\max}}.$$

Notice that

$$W_{a,c}(G) = W_{a,c}(H) = J_2\left(1 - \frac{\gamma_d^{\max}}{\gamma_a^{\max}}, \beta\right) > J_2\left(\alpha, \frac{\gamma_c^{\max} - \alpha\gamma_a^{\max}}{\Gamma_{out}^{\max} - \gamma_a^{\max}}\right),$$

hence the absolute maximum in $\Lambda_{5,+}$ is attained at all points of the segment $\overline{GH}$, for which the value of the cost functional is

$$M_{\Lambda_{5,+}} = 2 - \delta + \frac{\delta}{2}\left(\sqrt{1 - 4\gamma_a^{\max}} + \sqrt{1 - 4\gamma_a^{\max} + 4\gamma_d^{\max}}\right).$$

Finally, as $M_{\Lambda_{5,+}} > M_{\Lambda_{5,-}}$, we get that the maximum in Λ_5 is achieved at all points:

$$\left(1 - \frac{\gamma_d^{\max}}{\gamma_a^{\max}}, \beta\right), \quad 0 \leqslant \beta < 1 - \frac{\gamma_d^{\max}}{\gamma_a^{\max}},$$

and the value of the cost functional is $M_{\Lambda_{5,+}}$.

In a similar way, we compute the absolute maxima in regions Λ_1, Λ_2 and Λ_4. We obtain that

- the absolute maximum in Λ_1 is represented by all points of the set

$$\Lambda_1 \cap \left\{(\alpha, \beta) \in \mathbb{R}^2 : \alpha \leqslant \frac{\gamma_c^{\max}}{\gamma_a^{\max}}\right\},$$

  and the corresponding value of the cost functional is

$$M_{\Lambda_1} = 2 - \delta + \frac{\delta}{2}\sqrt{1 - 4\gamma_a^{\max}} - \frac{\delta}{2}\sqrt{1 - 4\gamma_c^{\max}};$$

- the absolute maximum in Λ_2 is given by all points of the set $\Lambda \cap s \cap t^-$, and the value of $W_{a,c}$ is

$$M_{\Lambda_2} = 2 - \delta - \frac{\delta}{2}\sqrt{1 - 4\gamma_c^{\max}} - \frac{\delta}{2}\sqrt{1 + 4\gamma_b^{\max} - 4\gamma_c^{\max} - 4\gamma_d^{\max}};$$

- the absolute maximum in Λ_4 is attained at point O, and the value of the cost functional is $M_{\Lambda_4} = 2 - \delta$.

Since

$$M_{\Lambda_5} > M_{\Lambda_4} > M_{\Lambda_1} > M_{\Lambda_3} > M_{\Lambda_2},$$

the values of α and β that optimise $W_{a,c}$ in Λ are same as those that maximise the cost functional in Λ_5. This concludes the proof. □

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Table 1. *Initial conditions for cases A, B and C*

	$\rho_{a,0}$	$\rho_{b,0}$	$\rho_{c,0}$	$\rho_{d,0}$
Case A	0.15	0.6	0.8	0.9
Case B	0.15	0.6	0.9	0.8
Case C	0.25	0.1	0.85	0.95

5 Simulations

In this section, we present some simulation results in order to test the optimisation algorithm for the cost functional $W_{a,c}$ both for single junctions or networks. In particular, we analyse the effects of different control procedures, applied locally at each junction, on the global performances of networks.

5.1 Single junctions

We consider single junctions of 2×2 type. Again, the incoming roads are labelled with a and b, and the outgoing ones with c and d. We compare the cost functional behaviour using random coefficients (*random case*), i.e. parameters taken randomly when the simulation starts and then kept constant; optimal distribution coefficients (*optimal case*).

We analyse three different situations, denoted by A, B and C, with initial data reported in Table 1, and chosen in such way as to test all possible optimal solutions reported in Theorem 3. Boundary data are assumed equal to initial conditions. Initial densities on outgoing roads c and d are chosen very high (close to $\rho_{\max} = 1$) to test how optimal choices of distribution parameters can create a decongestion effect in critical condition for the network.

Indicating by α_{opt} and β_{opt} the values of optimal distribution coefficients α and β, we have for case A, $\alpha_{opt} = 0.294118$ and $0 \leqslant \beta_{opt} < \alpha_{opt}$ (we choose β_{opt} equals to 0.2); for case B, $\alpha_{opt} = \varepsilon_1$, $\beta_{opt} = \varepsilon_2$; for case C, $\alpha_{opt} = 0.708571 + \varepsilon_1$, $\beta_{opt} = 0.708571 + \varepsilon_2$ with ε_1 and ε_2 being small and positive such that $\varepsilon_1 \neq \varepsilon_2$.

The traffic evolution is simulated using the Godunov scheme with space step $\Delta x = 0.0125$, time step Δt satisfying the Courant–Friedrichs–Lewy (CFL) condition (see [11]) and the flux function (2.3) in a time interval $[0, T]$, where T is 30 min for cases A and B and 100 min for the case C.

Figures 4–6 sketch $W_{a,c}(t)$ and the 3D behaviour of $W_{a,c}(T)$ in cases A, B and C, respectively, with $\delta = 0.5$. We notice that the optimal simulations, in accordance with the theoretical results of Theorem 3, are always highest, indicating that optimal parameters allow to maximise the velocity of emergency vehicles with respect to random cases. This is also confirmed by 3D plots of $W_{a,c}(T)$ in the plane (α, β): the maximum values are in accordance with those obtained analytically.

Indeed, some random simulations approach the optimal one. This occurs when values of α and β are such that the ordinary traffic does not fill the outgoing road c that interests the paths of emergency vehicles. In particular, for cases A and B, random choices of

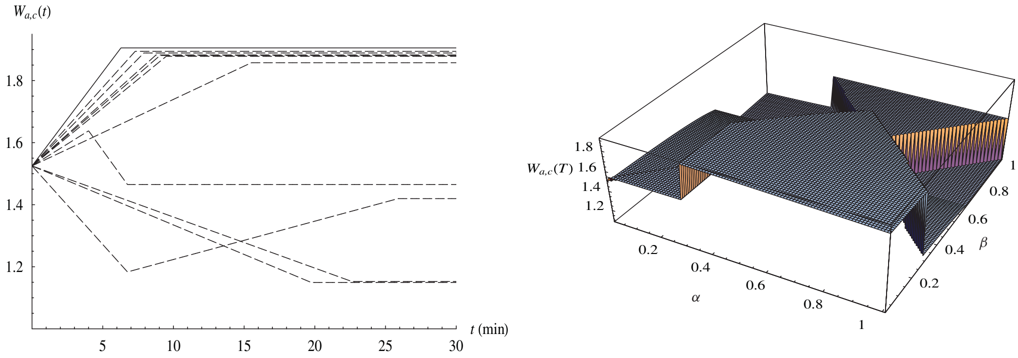

FIGURE 4. (Colour online) Case A, evolution of $W_{a,c}(t)$; (left) choice of optimal distribution coefficients (continuous line) and random parameters (dashed lines); (right) 3D plots of $W_{a,c}(T)$.

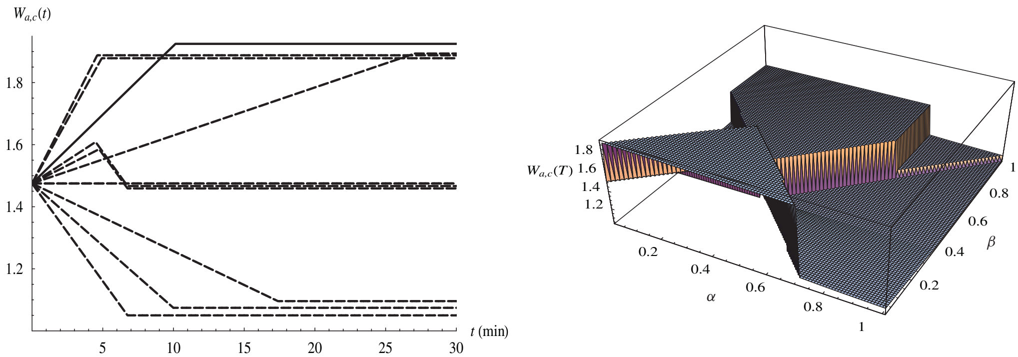

FIGURE 5. (Colour online) Case B, evolution of $W_{a,c}(t)$; (left) choice of optimal distribution coefficients (continuous line) and random parameters (dashed lines); (right) 3D plots of $W_{a,c}(T)$.

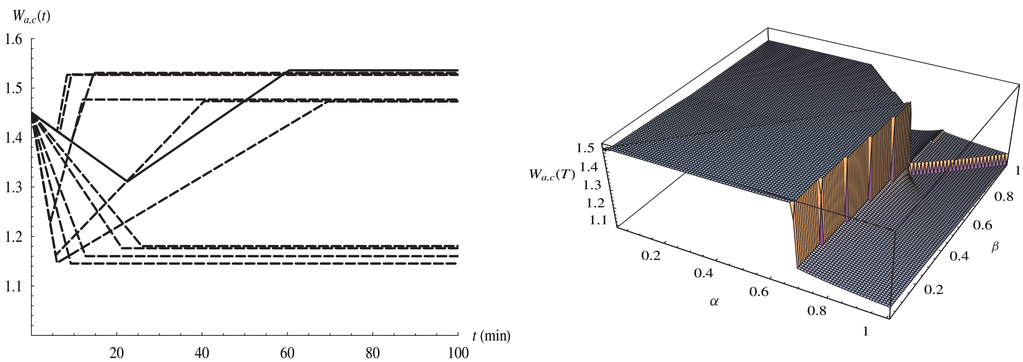

FIGURE 6. (Colour online) Case C, evolution of $W_{a,c}(t)$; (left) choice of optimal distribution coefficients (continuous line) and random parameters (dashed lines); (right): 3D plots of $W_{a,c}(T)$.

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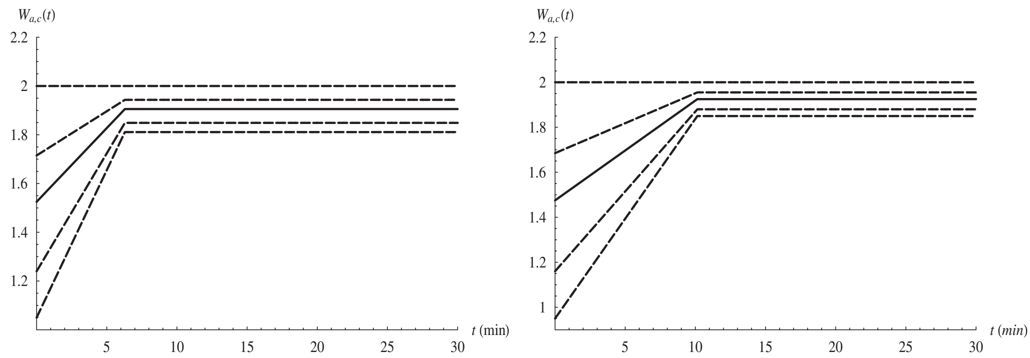

FIGURE 7. Evolution of the optimal behaviour of $W_{a,c}(t)$ in cases A (left) and B (right), computed for different values of δ.

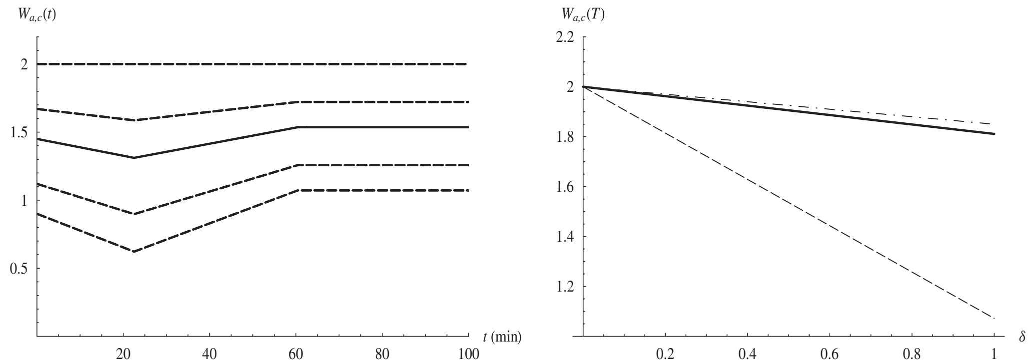

FIGURE 8. (Left) Evolution of optimal behaviour of $W_{a,c}(t)$ in case C, computed for different values of δ. (Right) $W_{a,c}(T)$ vs. δ in cases A (dot dashed line), B (continuous line) and C (dashed line).

parameters $\alpha = 0.26$, $\beta = 0.85$, and $\alpha = 0.81$, $\beta = 0.58$, respectively, assure the lowest behaviours of $W_{a,c}(T)$: The values of β indicate that a high amount of ordinary traffic crosses the outgoing road c, coming from the incoming road b, with consequent difficulties for emergency vehicles to reach the final destination. For case C, a similar phenomenon happens for $\alpha = 0.93$ and $\beta = 0.28$, since the greatest percentage of traffic crossing the outgoing road c is due to road a, which has a higher initial data with respect to the incoming road b. Figures 7 and 8 show the behaviour of functional $W_{a,c}(t)$ with optimal α and β parameters in cases A, B and C for various values of δ. The continuous line refers to the case $\delta = 0.5$, used during all simulations. When δ increases, $W_{a,c}(t)$ decreases. In particular, notice that when $\delta = 0$, $W_{a,c}(t)$ assumes the maximal value and is trivially constant; when $\delta = 1$, $W_{a,c}(t)$ is only influenced by the ordinary car traffic and achieves the lowest value. Finally, Figure 11 (right) shows the behaviour of the optimal asymptotic value $W_{a,c}(T)$ versus δ. Unlike cases A and B, the asymptotic value $W_{a,c}(T)$ in case C is strongly influenced by the choice of δ.

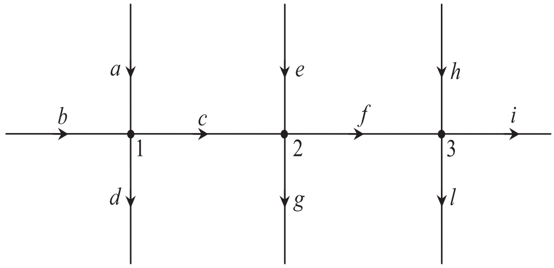

FIGURE 9. Topology of the cascade junction network.

Table 2. *Initial conditions and boundary data for roads of the cascade junction network*

Road	Initial condition	Boundary data
a	0.1	0.1
b	0.65	0.65
c	0.75	/
d	0.95	0.95
e	0.2	0.2
f	0.65	/
g	0.95	0.95
h	0.25	0.25
i	0.55	0.55
l	0.95	0.95

5.2 A network with cascade junctions

This subsection is devoted to a cascade junction network consisting of consecutive junctions. The aim is to understand the effects of the 'local type' optimal algorithm on the whole network.

The topology of the network (see Figure 9) is described by ten roads, divided into two subsets, $R_1 = \{a, d, e, g, h, l\}$ and $R_2 = \{b, c, f, i\}$, that are, respectively, the set of inner and external roads. All junctions are of 2×2 type and labelled by numbers 1, 2 and 3.

Assuming that the emergency vehicles have an assigned path, we analyse the behaviour of the functional

$$W(t) = W_{ac}(t) + W_{ef}(t) + W_{hi}(t).$$

The evolution of traffic flows is simulated using the Godunov scheme with $\Delta x = 0.0125$, and $\Delta t = \frac{\Delta x}{2}$ in a time interval $[0, T]$, where $T = 100$ min. Initial conditions and boundary data for densities are given in Table 2.

Also in this case, initial and boundary data are chosen to simulate a network with critical conditions on some roads, as congestion is due to the presence of accidents. We consider again two different types of simulation cases: (locally) optimal distribution coefficients applied at each node (*optimal case*); a *random case*, whose characteristics have already been explained in previous subsection.

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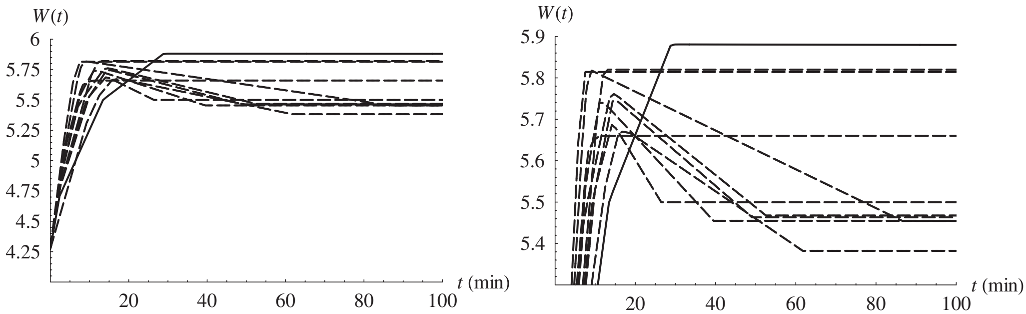

FIGURE 10. Evolution of $W(t)$ for optimal choices (continuous line) and random parameters (dashed line); (left) behaviour in $[0, T]$; (right) zoom around the asymptotic values.

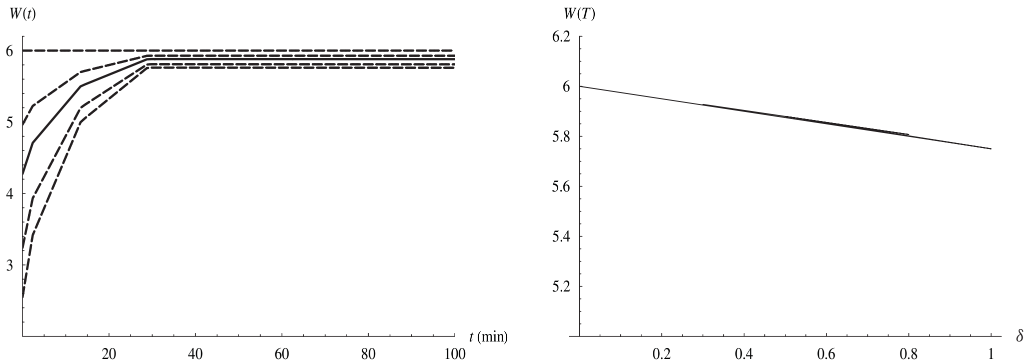

FIGURE 11. (Left) Evolution of optimal behaviour of $W(t)$, computed for different values of δ. (Right) $W(T)$ vs. δ.

Figure 10 shows the temporal behaviour of $W(t)$ measured on the whole network. As we can see, the optimal cost functional is higher than the random ones, hence the principal aim is achieved for the chosen data set. Notice that in general optimal global performances on networks could not be achieved, as the traffic state is strictly dependent on initial and boundary data. In Figure 11, we show the simulation of $W(t)$ for different values of δ and optimal value parameters at junctions. The behaviour is exactly the same as for single junctions, hence $\delta = 0$ corresponds to the highest curve and $\delta = 1$ to the lowest one. Notice that the continuous line corresponds to the case $\delta = 0.5$. Moreover, there are no meaningful changes in the asymptotic value $W(T)$ when δ varies.

6 Conclusions

In this paper, an optimisation technique is presented for the maximisation of the velocity of emergency vehicles on assigned paths when emergency occurs.

The optimisation is made over traffic distribution coefficients at junctions, considered fixed, using a cost functional that describes the average velocity of emergency vehicles. An exact analytical solution is found for simple junctions with two incoming roads and two outgoing ones, in steady state, i.e. after a long time has passed.

Then a sub-optimal strategy, consisting of using the local optimal coefficients at every junction, is tested through simulations. In particular, for a cascade network, it is shown that such strategy is outperforming random choices.

Future investigations may encompass the following extensions:

- The case treated in this paper refers to fixed traffic distribution coefficient. In reality such coefficients may vary during the day and for this case an existence theory is already available, see [9].
- Besides redirecting traffic, a stronger measure is the closure of roads. This is modelled by a problem in which the network topology varies.
- The present approach is focused on optimising a single junction. Even if the optimisation of a whole network may be out of reach, the selection of a simple path could be addressed.

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