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First order in Ludics

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In Girard (2001), J.-Y. Girard presents a new theory, *The Ludics*, which is a model of realisibility of logic that associates proofs with *designs*, and formulas with *behaviours*. In this article we study the interpretation in this semantics of formulas with first-order quantifications and their proofs. We extend to the first-order quantifiers the full completeness theorem obtained in Girard (2001) for $MALL_2$. A significant part of this article is devoted to the study of a uniformity property for the families of designs that represent proofs of formulas depending on a first-order free variable.

1. Introduction

In Girard (2001), J.-Y. Girard presents a new theory – *The Ludics*. This theory can be classified as amongst the semantics games theories (Abramsky *et al.* 2000; Hyland and Hong 2000), which have been greatly developed in recent years. It is a model of realisibility of logic that associates the proofs with *designs*, and formulas with *behaviours*. However, one of the motivations for Ludics was to take into account the real symmetry of logic and to go beyond the artificial syntax/semantics duality. The proofs are in fact strategies, but strategies of proof search; the designs are very close to formal proofs seen in the bottom to top direction. Moreover, having given a syntactic status to failure (in the proof search), one gains symmetry by introducing the concept of *counter-proof* (which is also represented by a design). The *interaction* between objects of the model (designs) is exactly the elimination of cuts.

Another innovation of Ludics is the importance given to the location (*locus*). Building on the 'linear logic' stage, which made it possible to take the use of resources into account, a new step is taken in the direction of the 'proofs *vs.* programs' paradigm, which makes it possible to take into account also the location where these resources are stored; in data processing the address of a resource is almost as significant as the resource itself. In the same way, in Ludics the objects cannot be defined independently of their location (locus). The links between Ludics and data-processing were clarified by P.-L. Curien in Curien (2001).

The objects in logic have until now been considered independently of their physical location, *up to identity*. It is possible to find this *delocalised* approach in Ludics, provided we place ourselves in the *spiritual* framework, that is, where all the occurrences of formulas and subformulas are placed in distinct locations (locus). In the context of 'spiritual logic', one can see Ludics as a model of realisibility that checks a completeness theorem for a linear second-order sequent calculus $MALL_2$, following the theorem of *full completeness* proved in Abramsky *et al.* (2000) and Hyland and Hong (2000). We can summarise this result by saying that if F is a closed Π_1-formula of $MALL_2$ and if $\mathfrak{D}$ is a design with 'good' properties (*winning* design) in the behaviour associated with this formula then there is a proof of F which is interpreted by $\mathfrak{D}$.

The main part of the work presented in this paper extends this completeness result for a linear predicate calculus. In Girard (2001), Girard presents a locative treatment of the first-order quantifiers for which one obtains surprising commutation results for prenex formulas with respect to the connectors. There is no hope of completeness for such an interpretation. However, a spiritual treatment is possible; it is within this framework that we have placed ourselves. Some specificities of a first-order language led us to modify slightly the basic definitions of the ludical objects (the locus, designs, behaviours. . .): the objects will be dependent on the elements of a countable set $\mathbb{D}$. In order to interpret the formula $F(x)$ depending on a first-order free variable x, we consider a family of behaviours indexed by $\mathbb{D}$; we associate the infinite additive conjunction of the *delocalised* behaviours of this family with the universal quantification of this formula. Finally, with the proof of such a quantified formula, we will associate a design, which can be seen as a 'union' of designs, each of which is in one of the components of the conjunction.

From a completeness point of view, the crucial point is to find the properties characterising the designs that are interpretations of proofs. In this context of first-order logic, we have concentrated on the study of the proofs of quantified formulas. Also, to take into account the correlation existing between the designs composing the design representing the proof of a universally quantified formula, we have described a *property of uniformity*, which expresses the fact that a *uniform* design must allow us *to pass* the universal quantification by representing the *same* proof for each possible assignment of the universally quantified variable. In other words, a uniform design must make it possible to find the *same* design in each behaviour $\mathbf{F}_d$ of the family associated with the formula $F(x)$. The uniformity of a design or a family of designs is defined using any functions of $\mathbb{D}$.

We then have the tools to study the completeness theorem for $MALL_2^1$ (the system obtained by adding the usual predicative rules to $MALL_2$). The concept of a 'winning design' representing a proof of a formula of $MALL_2$ is then replaced by the concept of a uniform family of winning designs. The result of completeness is stated *mutatis-mutandis*; one then shows it in two steps. First, one represents the linear predicate calculus $MALL_2^1$ in an infinite propositional calculus $MALL_2^\infty$ ($MALL_2$ with infinite additive propositional rules); the result of completeness shown in Girard (2001) transfers without problems. The second step consists of establishing the lemmas specific to the families of uniform designs representing the first-order quantified formulas.

The article is structured as follows. In Section 2 we outline the concepts and definitions of Ludics, extracted from Girard (2001), on which our definitions rest. Then we propose in Section 3 a definition of $\mathbb{D}$-uniformity. We compare $\mathbb{D}$-uniformity with uniformity of the second order (introduced in Girard (2001)), through some examples. In Section 4 we establish the 'full completeness theorem' for the system $MALL_2^1$.

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2. Ludics

It is beyond the scope of this paper to sum up the very new and extensive concepts that form the core of Ludics. Nevertheless, we shall give here a very brief survey of some of the ingredients of Ludics that we shall constantly use in this paper. You are, of course, strongly advised to refer to Girard's monograph (Girard 2001).

2.1. *Locus*

A very crucial notion in Ludics is the notion of *locus*. Considerable attention is paid to the relative places of objects: sub-formulas with respect to formulas, formulas with respect to sequents, different occurences with respect to each other, and so on. This consideration is taken into account and dealt with by the notions of *bias* and *address*:

- A *bias* is a natural number: we use i, j, k,... as notation for biases. A *ramification* is a finite set of biases.
- A *locus*, or *address*, is a sequence $< i_1, \ldots, i_n >$ of biases: we use σ, τ, ξ,... as notation for addresses. The parity of an address is defined as the parity of its length.
- A *pitchfork or base* is an expression $\Theta \vdash \Lambda$ where Θ and Λ are pairwise disjoint finite sets of loci. Θ contains at most one locus (this locus is *the handle*; the loci in Λ have the same parity, which may be opposite to the parity of Θ. The *polarity* of $\Theta \vdash \Lambda$ is positive when Θ is empty and negative when it is non-empty.

2.2. *Designs*

Designs are the central objects in Ludics. They are the 'concrete objects logic is made of' (Girard 2001). We can understand a design to be a proof seen from multiple points of view: a syntactical proof, a function, a clique, a λ-term...

A design is determined by the set of its successive actions (moves in the sense of both plays and changes of place). The plain definition of such objects is given by means of the notion of design as 'dessein'. The notion of design as 'dessin' is an alternative, simpler and more intuitive representation, but one that is less complete and general. Indeed, to one 'dessein' there are several corresponding 'dessins'.

Definition 2.1 (designs as dessins). A dessin is a proof-tree made of pitchforks. The last pitchfork (the root of the tree) of a 'dessin' is called the *conclusion* or *base*. Each pitchfork occurring in the 'dessin' is the conclusion of one of the rules given below. A branch cannot end with a negative rule.

Daemon:

$$\frac{}{\vdash \Lambda}\dagger$$

Positive rule: I is a ramification, the Λ_i are pairwise disjoint and included in Λ: one can apply the rule (which is finite, with one premise for each $i \in I$)

$$\frac{\cdots \qquad \xi * i \vdash \Lambda_i \qquad \cdots}{\vdash \xi, \Lambda}(\xi, I)$$

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Negative rule: $\mathcal{N}$ is a set of ramifications; for all $I \in \mathcal{N}$, $\Lambda_I \subset \Lambda$. One can apply the rule (which may be infinite, with one premise for each $I \in \mathcal{N}$)

$$\frac{\cdots \qquad \vdash \Lambda_I, \xi * I \qquad \cdots}{\xi \vdash \Lambda}\,(\xi, \mathcal{N})$$

Some examples: Let $\vdash \Lambda$ and $\xi \vdash \Lambda$ be two bases. We have the following designs:

$$\mathfrak{Dai}^{+} = \frac{}{\vdash \Lambda}\,\dagger \qquad \mathfrak{Dai}^{-} = \frac{\cdots \qquad \dfrac{}{\vdash \xi * I, \Lambda}\,\dagger \qquad \cdots}{\xi \vdash \Lambda}\,(\xi, \mathscr{P}_f^*(\mathbb{N}))$$

$$\mathfrak{Fax}_{\xi \vdash \xi'} = \frac{\cdots \qquad \dfrac{\cdots \qquad \begin{array}{c}\mathfrak{Fax}_{\xi' * i \vdash \xi * i} \\ \xi' * i \vdash \xi * i,\end{array} \qquad \cdots}{\vdash \xi * I, \xi'}\,(\xi', I) \qquad \cdots}{\xi \vdash \xi'}\,(\xi, \mathscr{P}_f^*(\mathbb{N}))$$

Definition 2.2 (designs as desseins). We recall here some notions needed for the definition of a design as 'dessein'.

An *action* is either a triple (ϵ, ξ, I) (where ϵ is the polarity $+$ or $-$ of the action, ξ is a locus and an I is a *ramification*) or the positive action ($+$, $\dagger$) which is said to be *the daemon*.

The locus of a proper action (that is, one that is not the daemon) $\kappa = (\epsilon, \xi, I)$ is said to be the *focus* of the action κ. (We shall use $\xi * I$ to denote the set of addresses $\xi * i$ for $i \in I$.)

A *chronicle* of base $\Theta \vdash \Lambda$ is a non-empty sequence of actions $\kappa_0, \ldots, \kappa_n$ such that:

— The sequence *alternates*: two successive actions are of opposite polarity.
— If Θ is non-empty, the first action of the chronicle is necessarily focused on the locus of Θ.
— Only the last action can be the daemon.
— A focus ξ_p of a negative action must be chosen either in θ or in $\xi_{p-1} * I_{p-1}$.
— A focus ξ_p of a positive action must be chosen either in Λ or in one of the $\xi_q * I_q$ where $(-, \xi_q, I_q)$ is one of the previous negative actions.

Two chronicles $\mathfrak{c}$, $\mathfrak{c}'$ are *coherent* when:

— Either one extends the other, or they first differ on negative actions.
— When $\mathfrak{c}$, $\mathfrak{c}'$ first differ on κ, κ' with distinct focuses, all ulterior focuses are distinct.

We use $\mathfrak{c}$, $\mathfrak{d}$, $\mathfrak{e}$... as notation for a chronicle.

A *design as dessein* $\mathfrak{D}$ of base $\Theta \vdash \Lambda$ is a set of pairwise coherent chronicles of base $\Theta \vdash \Lambda$ such that:

— $\mathfrak{D}$ is closed under restriction.
— If $\mathfrak{c} \in \mathfrak{D}$ has no extension in $\mathfrak{D}$, its last action is positive.
— If the base is positive, $\mathfrak{D}$ is non-empty.

Examples of desseins: We write the base as a subscript.

— $\mathfrak{Dai}^{+}_{\vdash\Lambda} = \{(+,\dagger)\}$
— $\mathfrak{Dai}^{-}_{\xi\vdash\Lambda} = \{(-,\xi,I)\ ;\ (-,\xi,I)*(+,\dagger); \forall I \in \mathscr{P}_f(\mathbb{N})\}$
— $\mathfrak{Fax}_{\xi\vdash\xi'} = \{(-,\xi,I);(-,\xi,I)*(+,\xi',I)\ ;\ (-,\xi,I)*(+,\xi',I)*(-,\xi'*i,J)\ ;\ (-,\xi,I)*(+,\xi',I)*(-,\xi'*i,J)*(+,\xi*i,J), \ldots$ for all $I,J,\ldots \in \mathscr{P}^*_f(\mathbb{N})$, for all $i \in I, \ldots\}$.

Remark: There is associated with a dessin $\mathfrak{D}$ based on $\Theta \vdash \Delta$ a unique dessein $\mathfrak{D}$. The converse is not true because it is not always possible to find and share the contexts Δ in a one to one way.

2.3. *Normalisation*

It is of course necessary to recover a notion equivalent to the notion of cut. Indeed, 'the designs have been constructed by imitation of cut-free proofs' (Girard 2001), but, nevertheless, the composition between the objects is the dynamic of logic. Ludics was born with the aim of fitting with both the dynamical and interactive aspect of logic.

A *cut net* is a set of designs that are linked to each other with cuts. More precisely, a cut-net is a non-empty finite set $\mathscr{R} = \{\mathfrak{D}_1,\ldots,\mathfrak{D}_n\}$ of designs of respective bases $\Theta_p \vdash \Lambda_p$ such that:

— The loci occurring in the bases are pairwise disjoint or equal.
— Every locus occurs at most in two bases, one in a Θ_p and the other in a Λ_q. Such a shared locus is called a *cut*.
— The graph whose vertices are the $\Theta_p \vdash \Lambda_p$ and whose edges are the cuts is connected and acyclic.

For example $<\mathfrak{D}_{\delta\vdash\xi}, \mathfrak{E}_{\xi\vdash\eta}, \mathfrak{F}_{\eta\vdash\delta}>$ is not a cut-net but $<\mathfrak{D}_{\vdash\xi}, \mathfrak{E}_{\xi\vdash\eta}, \mathfrak{F}_{\eta\vdash\delta}>$ is.

There is at most one handle that is not a cut, and we can form a pitchfork with the uncut loci, the conclusion or base of the cut-net.

The cut elimination procedure, which is called *normalisation*, is a strictly deterministic procedure that replaces a cut-net $\mathscr{R}$ with a design of the same base, its *normal form* $[\![\mathscr{R}]\!]$. Here we shall just describe the normalisation procedure for the closed case.

Closed normalisation procedure: Let $\mathscr{R}$ be a closed cut-net, $\mathfrak{D}$ be the unique positive design, κ be the main rule and ξ be the cut locus. Then three cases occur:

— *Daemon*: κ is the daemon. In this case the net normalises into a unique design with an empty base – the daemon.

$$\mathscr{R} = \left\{ \frac{}{\vdash \xi}\,\dagger, \frac{}{\xi \vdash}\,(\xi,\mathscr{N}) \right\};\ [\![\mathscr{R}]\!] = \frac{}{\vdash}\,\dagger$$

This case is the only case of termination for a closed net.

— *Immediate feature*: Let κ be (ξ,I). Hence ξ is a cut, and it occurs as the handle of another design $\mathfrak{E}$, the adjoint design of the net, whose first rule is necessarily of the form $(\xi,\mathscr{N})$. There are two cases:
  – $I \notin \mathscr{N}$: The normalisation fails.
  – $I \in \mathscr{N}$: For $i \in I$, let $\mathfrak{D}_i$ be the sub-design of $\mathfrak{D}$ whose conclusion is the premise of index i and let $\mathfrak{E}'_I$ be the sub-design of $\mathfrak{E}$ induced by the premise of index I. Define

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$\mathscr{F}$ by replacing $\mathfrak{D}$ by the $\mathfrak{D}_i$ and $\mathfrak{E}$ by $\mathfrak{E}'_I$. Since $\mathscr{F}$ is not necessarily connected, define $\mathscr{F}'$ to be the connected component of $\mathscr{F}$. Then $[\![\mathscr{R}]\!] = [\![\mathscr{F}']\!]$.

$$\mathscr{R} = \left\{ \frac{\begin{matrix}\mathfrak{D}_i \\ \xi * i \vdash\end{matrix} \quad \ldots \quad \begin{matrix}\mathfrak{D}_{i'} \\ \xi * i' \vdash\end{matrix}}{\vdash \xi}(\xi, I) ; \frac{\begin{matrix}\mathfrak{E}'_I \\ \vdash \xi * I\end{matrix} \quad \ldots \quad \begin{matrix}\vdots \\ \vdash \xi * J\end{matrix}}{\xi \vdash}(\xi, \mathscr{N}) \right\}$$

$$[\![\mathscr{R}]\!] = [\![\mathscr{F}']\!] = \left[\!\!\left[\frac{\mathfrak{E}'_I}{\vdash \xi * I}; \frac{\mathfrak{D}_i}{\xi * i \vdash}; \ldots ; \frac{\mathfrak{D}_{i'}}{\xi * i' \vdash} \right]\!\!\right]$$

Orthogonality: The designs $\mathfrak{D}$ and $\mathfrak{E}$ respectively based on $\xi \vdash$ and $\vdash \xi$ are said to be *orthogonal* when $[\![\mathfrak{D}, \ \mathfrak{E}]\!] = \mathfrak{Dai}$.

Order on designs: The set of designs of base $\Theta \vdash \Lambda$ is equipped with a partial order defined as follows:

$$\mathfrak{D} \preceq \mathfrak{D}' \quad \text{iff} \quad \mathfrak{D}' \in \mathfrak{D}^{\perp\perp}$$

This can be expressed in the tree representation: $\mathfrak{D}' \preceq \mathfrak{D}$ iff for every negative rule $(\xi, \mathscr{N}')$ of $\mathfrak{D}'$ there is a corresponding negative rule $(\xi, \mathscr{N})$ in $\mathfrak{D}$ such that $\mathscr{N}' \subset \mathscr{N}$ and for every positive rule (ξ, I) of $\mathfrak{D}'$ there is either the same positive rule in $\mathfrak{D}$ or the †-rule.

2.4. *Behaviours*

We have just recalled that a design can be understood as a proof seen from both syntactical and semantical points of view. In the same way, the concept of *behaviour* brings together the notions of logical formulas, types, Scott domains and coherent spaces.

A behaviour: This is a set of designs on a given base closed by bi-orthogonality. A behaviour is said to be positive or negative according to the polarity of its base.

Material (or incarnated) designs: The existence of a smallest subdesign in a given design $\mathfrak{D}$ relative to a given behaviour $\mathbf{G}$ is the counterpart to the normal form theorem in coherent semantics. This design is called the *incarnation* of $\mathfrak{D}$ and denoted $|\mathfrak{D}|_{\mathbf{G}}$. A design $\mathfrak{D} \in \mathbf{G}$ is *incarnated* or *material* when $\mathfrak{D} = |\mathfrak{D}|$. We define $|\mathbf{G}|$ to be the set of material designs in $\mathbf{G}$.

Here are some examples of behaviours and their incarnations:

$$\mathbf{0} = \{\mathfrak{Dai}\}^{\perp\perp}, \qquad |\mathbf{0}| = \{\mathfrak{Dai}\}.$$

$$\mathbf{T} = \left\{ \frac{\vdots}{<>\vdash}(<>, \mathscr{N}) ; \mathscr{N} \subset \mathscr{P}^*_f(\mathbb{N}) \right\}, \qquad |\mathbf{T}| = \left\{ \mathfrak{Skunk} = \frac{}{<>\vdash}(<>, \varnothing) \right\}$$

Let $\mathfrak{D}$ be a design. The behaviour $\{\mathfrak{D}\}^{\perp\perp}$ is said to be the **principal behaviour** generated by $\mathfrak{D}$. For example, if

$$\mathfrak{D} = \frac{\dfrac{\vdots}{1 \vdash}(1, \mathscr{N}_1) \qquad \dfrac{\vdots}{2 \vdash}(2, \mathscr{N}_2)}{\vdash <>}(<>, \{1, 2\}),$$

its principal behaviour is

$$\{\mathfrak{D}\}^{\perp\perp} = \left\{ \mathfrak{D},\ \dfrac{}{\vdash <>}\dagger,\ \dfrac{\dfrac{\vdots}{1\vdash}(1,\mathcal{N}'_1) \quad \dfrac{\vdots}{2\vdash}(2,\mathcal{N}'_2)}{\vdash <>}(<>,\{1,2\}),\ \mathcal{N}_1 \subset \mathcal{N}'_1,\ \mathcal{N}_2 \subset \mathcal{N}'_2 \right\}.$$

2.5. *The 'spiritual' connectives*

We can still find the usual logic within the ludic setting, provided we choose the 'spiritual' point of view, as opposed to the 'locative' point of view. Some general connectives are defined on behaviours; provided that these behaviours are in completely disjoint locations (by means of delocations) we recover the usual logical formulas.

Delocation:

— A *delocation* from locus ξ to locus ξ' is a partial injective map θ from the subloci of ξ to the subloci of ξ' such that $\theta(\xi) = \xi'$ and such that for all σ there is a function θ_σ satisfying $\theta(\sigma * i) = \theta(\sigma) * \theta_\sigma(i)$.

— It is straightforward to extend the delocation θ to chronicles and to designs.

— If $\mathbf{G}$ is a behaviour of base $\vdash \xi$ (respectively, $\xi \vdash$) and θ is a total delocation from ξ to ξ', we define the behaviour $\theta(\mathbf{G})$ of base $\vdash \xi'$ (respectively, $\xi' \vdash$) by: $\theta(\mathbf{G}) = \{\theta(\mathfrak{D})\ ;\ \mathfrak{D} \in \mathbf{G}\}^{\perp\perp}$.

In order to handle *polarity and focalisation* properties of formulas in the ludic framework, we need to introduce the notion of *shift*. The shift could be understood as a change of polarity: a step (a stop) between successive connectives rules that could *not* be performed simultaneously.

Shift:

— Let $\mathfrak{c}$ be a chronicle of base $\vdash \xi * i$ (respectively, $\xi * i \vdash$); the **shift** $\downarrow \mathfrak{c}$ is the chronicle $(\xi, \{i\}) * \mathfrak{c})$ of base $\xi \vdash$ (respectively, $\downarrow \mathfrak{c}$ is the chronicle based on $\vdash \xi$).

— For $\mathfrak{D}$ a positive (respectively, negative) design based on $\vdash \xi * i$ (respectively, $\xi * i \vdash$) we define $\downarrow \mathfrak{D}$ as $\{\downarrow \mathfrak{c}\ ; \mathfrak{c} \in \mathfrak{D}\} \cup \{< (\xi, \{\mathrm{i}\}) >\}$ of base $\xi \vdash$ (respectively, $\vdash \xi$).

— If $\mathbf{G}$ is a positive (respectively, negative) behaviour of base $\vdash \xi * i$ (respectively, $\xi * i \vdash$), then $\downarrow \mathbf{G}$ is defined by $\{\downarrow \mathfrak{D}\ ;\ \mathfrak{D} \in \mathbf{G}\}^{\perp\perp}$ (respectively, $\{\downarrow \mathfrak{D}\ ;\ \mathfrak{D} \in \mathbf{G}\}^{\perp}$) of base $\xi \vdash$ (respectively, $\vdash \xi$).

Linear spiritual connectives: Here we only give the definition of the linear connectives in the strict case that we are interested in here: those of the spiritual logic, that is, between some explicitly disconnected behaviours.

Let $\mathbf{G}$ and $\mathbf{H}$ be two disconnected behaviours of the same base $\vdash \xi$ (respectively, $\xi \vdash$), that is, the first actions of a design in $\mathbf{G}$ and of a design in $\mathbf{H}$ are disjoint.

— $\mathbf{G} \oplus \mathbf{H} = (\mathbf{G} \cup \mathbf{H})^{\perp\perp}$ provided $\mathbf{G}$ and $\mathbf{H}$ are positive disconnected behaviours.

— $\mathbf{G} \,\&\, \mathbf{H} = (\mathbf{G} \cap \mathbf{H})$ provided $\mathbf{G}$ and $\mathbf{H}$ are negative disconnected behaviours.

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— $\mathbf{G} \otimes \mathbf{H} = \{\mathfrak{D} \otimes \mathfrak{D}' \,;\, \mathfrak{D} \in \mathbf{G} \text{ and } \mathfrak{D}' \in \mathbf{H}\}^{\perp\perp}$ where $\mathfrak{D} \otimes \mathfrak{D}' = \{(\xi, I \cup J) * \mathfrak{c}' \,;\, (\xi, \mathfrak{I}) * \mathfrak{c}' \in \mathfrak{D}$ or $(\xi, \mathfrak{J}) * \mathfrak{c}' \in \mathfrak{D}'\}$ provided $\mathfrak{D} \in \mathbf{G}$ and $\mathfrak{D}' \in \mathbf{H}$ are disconnected positive designs distinct from the $\mathfrak{Dai}$.
— $\mathbf{G} \wp \mathbf{H}$ is defined by duality.
— One extends these definitions to behaviours of opposite polarities by adding shifts when necessary.

Sequent of behaviours: Let $\Theta \vdash \Lambda$ be a base and let $\mathbf{G}_\sigma$ be some positive behaviours of respective bases $\vdash \sigma$ for $\sigma \in \Theta \cup \Lambda$. We then define the *sequent of behaviours* $\mathbf{\Theta} \vdash \mathbf{\Lambda}$ of base $\Theta \vdash \Lambda$ to be the orthogonal of the set of families $(\mathfrak{D}_\sigma)$ of designs $\mathfrak{D}_\sigma$ where $\mathfrak{D}_\sigma \in \mathbf{G}_\sigma$ for $\sigma \in \Theta$ and $\mathfrak{D}_\sigma \in \mathbf{G}_\sigma^\perp$ for $\sigma \in \Lambda$.

2.6. *Bihaviours*

We know by means of the internal completeness (see the next subsection) how to decompose a connective, but what happens when we reach an atom?

In fact, atoms are propositional variables and can be interpreted by any positive behaviour. A design able to deal with this implicit second order quantification has to be *uniform* to be a *good* candidate to interpret a proof.

In order to express the notion of uniformity interactively, Girard in Girard (2001) added a PER (partial equivalence relation) structure to the behaviours. The key point of this equivalence is to separate the designs with respect to normalisation. The base of the equivalence is to identify the closed nets normalising into $\mathfrak{Dai}$ and those normalising into $\mathfrak{Fid}$.

We now briefly outline the main definitions relevant for the notion of bihaviours and uniformity:
$\mathfrak{Fid}$ is the empty set of chronicle represented by

$$\frac{}{\vdash \Lambda}\,\Omega.$$

$\mathfrak{Fid}$ is not a design, but a partial design that here represents the failure of a normalisation.

A **partial design** of a design $\mathfrak{D} \in \mathbf{G}$ is $\mathfrak{Fid}$ or a subdesign of $\mathfrak{D}$. $\mathbf{G}^p$ is the set of all partial designs in a design of $\mathbf{G}$.

The PER $\cong^\perp$: Suppose that $\cong$ is a PER (partial equivalence relation) on a behaviour, $\mathbf{G}$, $\cong^\perp$ is the PER on $(\mathbf{G}^\perp)^p$ such that: $\forall \mathfrak{D}, \mathfrak{D}' \in (\mathbf{G}^\perp)^p \quad \mathfrak{D} \cong^\perp \mathfrak{D}'$ iff $\forall \mathfrak{E}, \mathfrak{E}' \in \mathbf{G}^p$ if $\mathfrak{E} \cong \mathfrak{E}'$ then $[\![\mathfrak{D}, \mathfrak{E}]\!] = [\![\mathfrak{D}', \mathfrak{E}']\!]$.

Bihaviours: $(\mathbf{G}, \cong_\mathbf{G})$ is a bihaviour provided $\cong_\mathbf{G}$ is a PER on $\mathbf{G}$ such that:

$$\mathfrak{Dai} \cong_\mathbf{G} \mathfrak{Dai}; \mathfrak{Fid} \cong_\mathbf{G} \mathfrak{Fid} \quad \text{and} \quad \cong_\mathbf{G} \text{ is equal to its biorthogonal.}$$

PER on behaviours built by the ludic constructors $\otimes$, $\oplus$, &: The definitions are only given on behaviours with disjoint loci. In each of the following cases, we obtain a bihaviour by considering the partial equivalence defined below. Let $(\mathbf{G}_1, \cong_1)$ and $(\mathbf{G}_2, \cong_2)$ be two bihaviours with the same base.

(a) $\cong_{\mathbf{G}_1 \oplus \mathbf{G}_2}$ is defined as the biorthogonal of the following relation:
For all $\mathfrak{D}$, $\mathfrak{D}' \in (\mathbf{G}_1 \oplus \mathbf{G}_2)^p$,
$\mathfrak{D} \cong_{\mathbf{G}_1 \oplus \mathbf{G}_2} \mathfrak{D}'$ iff $\exists i \in \{1, 2\}$ such that $\mathfrak{D}$ and $\mathfrak{D}' \in \mathbf{G}_i$ and $\mathfrak{D} \cong_{\mathbf{G}_i} \mathfrak{D}'$.

(b) $\cong_{\mathbf{G}_1 \otimes \mathbf{G}_2}$ is defined as the biorthogonal of the following relation:
For all $\mathfrak{D}_1 \otimes \mathfrak{D}_2$, $\mathfrak{D}'_1 \otimes \mathfrak{D}'_2 \in (\mathbf{G}_1 \otimes \mathbf{G}_2)^p$,
$\mathfrak{D}_1 \otimes \mathfrak{D}_2 \cong_{\mathfrak{D}_1 \oplus \mathfrak{D}_2} \mathfrak{D}'_1 \otimes \mathfrak{D}'_2$ iff $\mathfrak{D}_1 \cong_{\mathbf{G}_1} \mathfrak{D}'_1$ and $\mathfrak{D}_2 \cong_{\mathbf{G}_2} \mathfrak{D}'_2$.

(c) $\cong_{\mathbf{G}_1 \& \mathbf{G}_2}$ is defined by:
For all $\mathfrak{D}$, $\mathfrak{D}' \in (\mathbf{G}_1 \cap \mathbf{G}_2)^p$ $\mathfrak{D} \cong_{\mathbf{G}_1 \& \mathbf{G}_2} \mathfrak{D}'$ iff $\forall i \in \{1, 2\}$ $\mathfrak{D} \cong_{\mathbf{G}_i} \mathfrak{D}'$.

PER on sequents of bihaviours: Consider the sequent $\mathbf{G}_0 \vdash \mathbf{G}_1, \ldots, \mathbf{G}_n$. We obtain a bihaviour by considering the following partial equivalence:

$\mathfrak{E} \cong_{\mathbf{G}_0 \vdash \mathbf{G}_1, \ldots, \mathbf{G}_n} \mathfrak{E}'$ iff $\forall \mathfrak{D}_0 \cong \mathfrak{D}'_0 \in \mathbf{G}_0^p$ $\forall \mathfrak{D}_i \cong \mathfrak{D}'_i \in \mathbf{G}_i^{\perp p}$,

$$[\![\mathfrak{E}, \mathfrak{D}_0, \mathfrak{D}_1, \ldots, \mathfrak{D}_n]\!] = [\![\mathfrak{E}', \mathfrak{D}'_0, \mathfrak{D}'_1, \ldots, \mathfrak{D}'_n]\!].$$

2.7. *The completeness results in $MALL_2$*

There are two levels of completeness. The first level is the internal completeness. Girard means by this the fact that behaviours built by means of connectives are easily decomposable. This is in some sense a ludic counterpart of the subformula property: no biorthogonal is needed in such a behaviour, so we have a complete description of the designs. For example, if $\mathfrak{D}$ is in $\mathbf{A} \otimes \mathbf{B}$ where $\mathbf{A}$ and $\mathbf{B}$ are disconnected behaviours, we are able to decompose it into a design in $\mathbf{A}$ and a design in $\mathbf{B}$ (projection commutes with the biorthogonal).

Moreover, Girard states a full completeness theorem in Girard (2001) by setting a correspondence between Ludics and a second-order propositional linear calculus $MALL_2$; $MALL_2$ is a linear sequent calculus that is only built on positive formulas, without exponential and with a stoup (which enables one to make the focalisation property explicit). Intuitively speaking, this correspondence consists of discriminating the ‘good’ designs, which are the candidates for representing the proofs of this calculus. Let us recall the properties that are needed for a ‘winning’ design.

Stubborn design: A design $\mathfrak{D} \in \mathbf{G}$ is *stubborn* when it does not use the daimon rule.

Exact design:

— *A positive rule is exact* when $\Lambda = \cup_i \Lambda_i$ and *a negative rule is exact* when $\Lambda_I = \Lambda$ for all $I \in \mathcal{N}$.

— *A design is exact* when it only contains exact rules.

Uniform design: The design $\mathfrak{D}$ is *uniform* in the bihaviour $\mathbf{G}$ if and only if $\mathfrak{D} \cong_{\mathbf{G}} \mathfrak{D}$.

Some examples of uniform designs and non-uniform designs will be given in Section 3.2.3.

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Winning design: A design $\mathfrak{D}$ is *winning* when it is stubborn, uniform and exact[†].

The full completeness result is stated in Girard (2001) in two cases: the affine and the non-affine version of the second-order linear calculus. We recall here the result in the non-affine case.

Soundness: *With every proof π of a closed sequent $\Gamma \vdash \Delta; \Sigma$ in $MALL_2$ one associates a design $\pi^* \in \mathbf{\Gamma} \vdash \mathbf{\Delta}; \mathbf{\Sigma}$. The design π^* is winning and material when the sequent is Π^1. Moreover, the interpretation is invariant under cut-elimination.*

Completeness: *Let $\Gamma \vdash \Delta;$ be a closed Π^1 $MALL_2$-sequent and let $\mathfrak{D} \in \mathbf{\Gamma} \vdash \mathbf{\Delta}$ be a material winning design. Then there is a $MALL_2$-proof π of $\Gamma \vdash \Delta$ such that $\mathfrak{D} = \pi^*$.*

2.8. *The quantifications*

Girard has stressed the fact that the ludic framework enables one to distinguish between two different approaches to the quantifiers and then to distinguish more radically between the first- and second-order quantifiers (Girard 2001). This distinction is relative to the duality spiritual/locative approach. In his monograph, Girard has given a full presentation of the second-order quantification as a locative intersection of a family of behaviours. He proposed a similar locative treatment of the first-order (but not the usual) quantification by using the intersection operation. The distinction between first and second order is then taken into account by the difference in cardinality of the index domains (which is countable for the first order, and $2^{\aleph_0}$ for the second). There are some interesting and surprising results of such a treatment. However, this does not enable us to deal with the usual first-order quantification in spiritual logic. Moreover, it is emphasised in Girard (2001) that no completeness result could be expected for a first-order universal quantification seen as a locative intersection of a family of behaviours indexed on a domain, even when it is countable.

3. Spiritual first-order quantification

Our purpose in this paper is to study the usual spiritual first-order quantifiers in the ludic framework. Our aim is to propose a ludic interpretation for predicate linear formulas and to obtain some completeness results in the spiritual case.

We begin this section by using examples to explain the expected properties of a design that is the representation of a proof in a predicate calculus. We propose the ludic objects convenient to the predicative formulas and the property that will assure the full completeness result: $\mathbb{D}$-uniformity. We then compare this $\mathbb{D}$-uniformity and the uniformity relative to the second order.

[†] The definition of winning design given in Girard (2001) uses parsimony instead of exactness. However, exactness is sufficient in our restricted framework (the non-affine case).

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3.1. *Expected properties*

We consider a given countable domain $\mathbb{D}$ for interpreting first-order terms (here these terms are variables), and then we associate with a first-order free variables formula $P(x)$ a family of behaviours indexed on $\mathbb{D}$: $(\mathbf{C}_d)_{d\in\mathbb{D}}$. As suggested in Girard (2001), we assume that a quantified formula $\forall x\, P(x)$ will be interpreted by $\&_{d\in\mathbb{D}}\mathbf{C}_d$, which is an infinite & of delocalised behaviours.

Let $\mathfrak{D}$ be a material design in $\&_{d\in\mathbb{D}}\mathbf{C}_d$; we are interested in the material part $\mathfrak{D}_d$ of $\mathfrak{D}$ in each $\mathbf{C}_d$.

So we represent a design $\mathfrak{D} \in \&_{d\in\mathbb{D}}\mathbf{C}_d$ by

$$\frac{\cdots \quad \mathfrak{D}_d \quad \cdots}{<>\vdash}\,\{d/\ d \in \mathbb{D}\},$$

which is an abbreviation for

$$\frac{\cdots \quad \mathfrak{D}_d \quad \cdots}{<>\vdash}\,(<>, \cup_{d\in\mathbb{D}}\mathcal{N}_d)$$

where for each $d \in \mathbb{D}$ $(<>, \mathcal{N}_d)$ is the first rule of $\mathfrak{D}_d$.

The idea that we want to develop in this section is that a design $\mathfrak{D} \in \&_{d\in\mathbb{D}}\mathbf{C}_d$ that is a candidate for representing a proof of $\forall x\, P(x)$ must verify the property

$$\forall d_1, d_2 \in \mathbb{D} \quad \mathfrak{D}_{d_1} \equiv \mathfrak{D}_{d_2}.$$

To make the meaning of this informal '$\equiv$' more precise, we are going to look at some examples and extract the necessary properties of such a design.

Notation: The examples we give here are set in a non-polarised linear sequent calculus.

— Our first example is based on the sequent $\vdash A(x) \otimes A(x) \multimap A(x) \otimes A(x)$.
  It is well known that this sequent has two distinct proofs: the first is just the η-proof, and will be denoted by Id, the second uses the exchange rule, and will be denoted by Tr.
  We consider the two designs $\mathfrak{I}d$ and $\mathfrak{T}r$ associated with these proofs. We can then build two designs in the behaviour $\&_{d\in\mathbb{D}}\mathbf{C}_d$. The first one, $\mathfrak{M}$, is such that $\mathfrak{D}_{d_1}$ is a delocation of $\mathfrak{I}d$ and $\mathfrak{D}_{d_2}$ is a delocation of $\mathfrak{T}r$:

$$\mathfrak{M} = \frac{\mathfrak{D}_{d_1} = \mathfrak{I}d \qquad \cdots \qquad \mathfrak{D}_{d_2} = \mathfrak{T}r}{<>\vdash}\,\{d/\ d \in \mathbb{D}\}$$

  In the second design, $\mathfrak{N}$, we set for every $d \in \mathbb{D}$, $\mathfrak{E}_d = \varphi_d(\mathfrak{I}d)$ where the φ_d's are delocations mapping $\mathfrak{I}d$ on delocalised designs:

$$\mathfrak{N} = \frac{\mathfrak{E}_{d_1} \qquad \cdots \qquad \mathfrak{E}_{d_2}}{\vdash<>}\,\{d; d \in \mathbb{D}\}.$$

  It is clear that the design $\mathfrak{M}$ must be disqualified as a representation of a proof of the formula $\forall x\, (A(x) \otimes A(x) \multimap A(x) \otimes A(x))$, while $\mathfrak{N}$ is a good candidate for representing such a proof.
  If we consider a design $\mathfrak{D}$ associated with a proof of $\vdash \forall x P(x)$, every $\mathfrak{D}_d$ within it must be distinct delocations of the 'same' design. The delocations depend on the indexes

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d, so we shall consider that the biases must be $\mathbb{D}$-dependent in order to handle such delocations φ_d. Moreover, we have to build on this technical material the 'equivalence' relation previously mentioned, which we shall call $\mathbb{D}$-uniformity.

— In our second example we will focus on the sequent $\vdash \exists y\ (R(x) \multimap R(y))$ where R is a predicate variable. This sequent can be proved in linear logic as follows:

$$\pi = \cfrac{\cfrac{\cfrac{R(x) \vdash R(x)}{\vdash R(x)^{\perp},\ R(x)}}{\vdash R(x) \multimap R(x)}}{\vdash \exists y\ (R(x) \multimap R(y))}$$

A family $(\mathfrak{D}_d)_d$ associated with the proof π is such that for all $d \in \mathbb{D}$, $\mathfrak{D}_d \in \bigoplus_e \mathbf{C}_{d,e}$ where $(\mathbf{C}_{d,e})_{d,e}$ is a family of disconnected behaviours interpreting the formula $R(x) \multimap R(y)$. Our intuition is that the 'good' choice will be to take for all $d \in \mathbb{D}$, $\mathfrak{D}_d \in \mathbf{C}_{d,d}$. This must be taken into account by the uniformity property.

Our proposition: In order to define a $\mathbb{D}$-uniform family $(\mathfrak{D}_d)_d$, we work with maps ϕ from $\mathbb{D}$ to $\mathbb{D}$, which can be extended on dependent biases and also to designs; we then compare $\phi(\mathfrak{D}_d)$ and $\mathfrak{D}_{\phi(d)}$. We shall say that a $\mathbb{D}$-uniform family is obtained when for every map ϕ, we have $\phi(\mathfrak{D}_d)$ and $\mathfrak{D}_{\phi(d)}$ are almost the same.

3.2. $\mathbb{D}$-*uniformity*

3.2.1. *Preliminary definitions*

Definition 3.1 (Domain and domain morphism).

— A *domain* is a countable set $\mathbb{D}$.

— A $\mathbb{D}$-*morphism* ϕ is a function from $\mathbb{D}$ into $\mathbb{D}$.

In the following definitions and theorems, we assume a domain $\mathbb{D}$ is given.

Definition 3.2 (Dependent bias). Consider a countable set of injective functions $(f_n^k)_{n,k}$, from $\mathbb{D}^k$ in $\mathbb{N}$ such that if $(k,n) \neq (k',n')$, then $Im(f_n^k) \cap Im(f_{n'}^{k'}) = \emptyset$. For all integers k, n and all $\vec{d} \in \mathbb{D}^k$,

— the integer $f_n^k(\vec{d})$ is said to be *a dependent bias*;

— the functions f_n^k are said to be the *bias functions*.

Remark: The dependent biases set is a countable subset of $\mathbb{N}$ and will be denoted by $\mathbb{N}_{\mathbb{D}}$. We should emphasise that for every integer $r \in \mathbb{N}_{\mathbb{D}}$ there is a unique bias function (saying f_n^k) and a unique k-uple $\vec{d} \in \mathbb{D}^k$ such that $r = f_n^k(\vec{d})$. The bias functions give an explicit embedding of $\mathbb{N}_{\mathbb{D}}$ inside $\mathbb{N}$.

Definition 3.3 (The $\mathbb{D}$-designs). The notions of address, ramification, chronicle, base and design are extended in the obvious way in order to get the notions of $\mathbb{D}$-address, $\mathbb{D}$-ramification, $\mathbb{D}$-chronicle, $\mathbb{D}$-base and $\mathbb{D}$-design.

Notation and terminology:

— By a constant bias b we mean a bias that is the image of a 0-ary function f_r^0 (which is said to be 'constant function') and, as usual, we confuse b and f_r^0.
— An address $\xi = b_1.\dots.b_n$ where the b_j's are constant biases is said to be a constant address. In the same way, any $\mathbb{D}$-base $\xi \vdash \gamma$ containing only constant addresses is said to be a constant base.

Definition 3.4. Let ϕ be a $\mathbb{D}$-morphism.

— We define a function $\overline{\phi}$ (which we shall still write ϕ) from $\mathbb{N}_{\mathbb{D}}$ to $\mathbb{N}_{\mathbb{D}}$ by setting

$$\overline{\phi}\big(f_p^k(d_1\dots,d_k)\big) = f_p^k(\phi(d_1),\dots,\phi(d_k)).$$

— We extend this function to $\mathbb{D}$-addresses, $\mathbb{D}$-bases and $\mathbb{D}$-chronicles.
— If $\mathfrak{D}$ is a $\mathbb{D}$-design, we set $\overline{\phi}(\mathfrak{D}) = \{\overline{\phi}(\mathfrak{c});\mathfrak{c}\in\mathfrak{D}\}$.

Remark: The set of chronicles $\overline{\phi}(\mathfrak{D})$ is not always a design. Consider the following example:
Let $\mathbb{D}$ be a domain and $\{a,b,c,d\}$ be four pairwise distincts elements; let f be a unary-function f_n^1; let ϕ be a non-injective $\mathbb{D}$-morphism such that $\phi(a)=\phi(c)=a,\quad \phi(b)=b$, $\phi(d)=d$; and let $\mathfrak{D}$ be the following $\mathbb{D}$-design based on $<>\vdash$:

$$\dfrac{\dfrac{f(a).f(b)\vdash}{\vdash f(a)}\,(f(a),\{f(b)\}) \qquad \dfrac{f(c).f(d)\vdash}{\vdash f(c)}\,(f(c),\{f(d)\})}{<>\vdash}\,\{\{f(a)\};\{f(c)\}\}$$

The set of chronicles $\phi(\mathfrak{D})$ contains the chronicles

$$(<>,\{f(a)\}).(<f(a)>,\{f(b)\})$$

and

$$(<>,\{f(a)\}).(<f(a)>,\{f(d)\}),$$

which are not coherent, since they differ on a positive action.

Remark: When the function ϕ is injective, the $\overline{\phi}$-image of a $\mathbb{D}$-design is a $\mathbb{D}$-design.

Terminology: Let $(\mathbf{C}_{\vec{d}})_{\vec{d}\in\mathbb{D}^n}$ be a family of behaviours of the same base. We shall say that the family of $\mathbb{D}$-designs $(\mathfrak{D}_{\vec{d}})_{\vec{d}}$ is in the family of behaviours $(\mathbf{C}_{\vec{d}})_{\vec{d}\in\mathbb{D}^n}$ when for every $\vec{d}\in\mathbb{D}^n$ we have $\mathfrak{D}_{\vec{d}}\in\mathbf{C}_{\vec{d}}$.

3.2.2. $\mathbb{D}$-*uniformity*

Definition 3.5 ($\mathbb{D}$-uniform designs family). Let $(\mathfrak{D}_{d_1,\dots,d_n})_{d_1,\dots,d_n}$ be a family of $\mathbb{D}$-designs of the same base. Such a family is said to be $\mathbb{D}$-*uniform* when for every $\mathbb{D}$-morphism ϕ, and for every $d_1,\dots,d_n\in\mathbb{D}^n$, we have $\phi(\mathfrak{D}_{d_1,\dots,d_n})$ is a subdesign of $\mathfrak{D}_{\phi(d_1),\dots\phi(d_n)}$.

Remark: The $\mathbb{D}$-uniformity forces $\phi(\mathfrak{D}_{d_1,\dots,d_n})$ to be a design.

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Definition 3.6 ($\mathbb{D}$-uniform design). A $\mathbb{D}$-design $\mathfrak{D}$ in a behaviour **C** is said to be $\mathbb{D}$-*uniform* when for every $\mathbb{D}$-morphism ϕ, we have $\phi(\mathfrak{D})$ is a subdesign of $\mathfrak{D}$.

Example 1: Let f, g, h and k be bias functions, and i be a dependent bias. For every $(d_1, d_2) \in \mathbb{D}^2$ we consider the following $\mathbb{D}$-design where $\xi = f(d_1, d_2).g(d_1)$, $\xi' = h(d_1, d_2)$:

$$\mathfrak{F}_{d_1,d_2} = \cfrac{\cfrac{\cfrac{\mathfrak{Fax}_{\xi.i,\xi'.i}}{\ldots \xi.i \vdash \xi'.i \ldots}}{\vdash \xi.I, \xi'}\,(\xi', I)}{\xi \vdash \xi'}\,(\xi, \mathscr{P}_f^*(\mathbb{N}_D))$$

Note that this definition of the design $\mathfrak{F}ax_{\xi,\xi'}$ is very slightly different from the one given in Girard (2001). We have only replaced $\mathscr{P}_f^*(\mathbb{N})$ by $\mathscr{P}_f^*(\mathbb{N}_{\mathbb{D}})$.

To check that the family $(\mathfrak{F}_{d_1,d_2})_{d_1,d_2}$ is $\mathbb{D}$-uniform, we just need to note that:

— For any $\mathbb{D}$-morphism ϕ, the designs $\phi(\mathfrak{F}_{d_1,d_2})$ and $\mathfrak{F}_{\phi(d_1),\phi(d_2)}$ are both based on $\phi(\xi) \vdash \phi(\xi')$, where $\phi(\xi) = f(\phi(d_1), \phi(d_2)).g(\phi(d_1))$ and $\phi(\xi') = h(\phi(d_1), \phi(d_2))$.

— Moreover, let $\mathfrak{c} = (\xi, I)(\xi', I)(\xi'.k(\vec{e}), J)(\xi.k(\vec{e}), J)\ldots$ be a chronicle in $\mathfrak{F}_{d_1,d_2}$. Then in $\phi(\mathfrak{F}_{d_1,d_2})$ it becomes

$$\phi(\mathfrak{c}) = (\phi(\xi), \phi(I)), (\phi(\xi'), \phi(I))(\phi(\xi').k(\phi(\vec{e})), \phi(J))(\phi(\xi).k(\phi(\vec{e})), \phi(J)).$$

Since $\phi(I)$ and $\phi(J)$ are in $\mathscr{P}_f^*(\mathbb{N}_D)$, and $f_n^k(\phi(\vec{e})) \in \phi(I)$, the chronicle $\phi(\mathfrak{c})$ is in $\mathfrak{F}ax_{\phi(\xi),\phi(\xi')}$.

Note that the non-injectivity of ϕ is no longer the problem it was in the previous counter-example. If $\vec{d}$ and $\vec{e}$ are identified in some negative action, they are also identified in the following positive actions.

Example 2: Let $(\mathfrak{D}_d)_d$ be the family such that for all $d \in \mathbb{D}$ the design $\mathfrak{D}_d$ is defined by

$$\mathfrak{D}_d = \cfrac{}{\vdash f_1^0}\,\dagger\,.$$

The family $(\mathfrak{D}_d)_d$ is, of course, $\mathbb{D}$-uniform.

Example 3: Let d and e_d be elements of $\mathbb{D}$, and $\mathfrak{D}_d$ be the design defined for each $d \in \mathbb{D}$ by

$$\mathfrak{D}_d = \cfrac{\cfrac{\cfrac{\mathfrak{Fax}_{\xi,\xi'}}{f(d).h(e_d) \vdash g(d)}}{\vdash f(d), g(d)}}{<>\vdash}$$

The family $(\mathfrak{D}_d)_d$ is $\mathbb{D}$-uniform only if for all $d \in \mathbb{D}$, we have $e_d = d$. It suffices to apply a $\mathbb{D}$-morphism ϕ such that $\phi(e_d) \neq e_d$ and $\phi(d) = d$ to discriminate the other cases.

3.2.3. *Second-order uniformity vs* $\mathbb{D}$*-uniformity*

What are the common properties between a uniform design as defined in Ludics (Girard 2001) for the propositional second-order sequent calculus and a $\mathbb{D}$-uniform design as defined above? Why are these properties needed? Do they interfere? Is one of them a consequence of the other, or are they totally independent?

Let us first recall that uniform properties appear as soon as we deal with the full completeness theorem when we are searching to extract a proof from a design with 'good' properties (Faggian *et al.* 2003). We meet the problem of uniformity in a $MALL_2$ sequent $A \vdash \Gamma$ containing atom(s) in the left part. Let us consider, for example, a design in $\bigcap_{\mathbf{A}} \mathbf{A} \vdash \mathbf{B} \oplus \mathbf{C}$. It must have the shape

$$\mathfrak{D} = \dfrac{\cdots \qquad \overset{\vdots}{\vdash 1 * I, \gamma} \qquad \cdots}{1 \vdash \gamma}\,(1, \mathscr{P}_f^*(\mathbb{N})).$$

Let us write

$$\mathfrak{D}_I = \dfrac{\overset{\vdots}{\vdash 1 * I, \gamma}}{1 \vdash \gamma}\,I$$

for every I.

It is necessary that for every I the subdesigns $\mathfrak{D}_I$ of $\mathfrak{D}$ are 'equivalent' in the sense that we would be able to extract the *same proof* from any of the others. In particular, the choice between a proof of B and a proof of C has to be the same. The family $(\mathfrak{D}_I)_{I \in \mathscr{P}_f^*(\mathbb{N})}$ is said to be *uniform* or, equivalently, the design $\mathfrak{D}$ is said to be *uniform.*

Now let us consider a design $\mathfrak{D}$ in the interpretation of a quantified formula $\forall x\, \Phi(x)$, so $\mathfrak{D} \in \bigcap_{d \in \mathbb{D}} \mathbf{\Phi_d}$. Let us consider the subdesigns $\mathfrak{D}_d$ that are the material part of each $\mathfrak{D}$ in $\mathbf{\Phi_d}$. Since we want to extract a proof of $\forall x\; \Phi(x)$ from $\mathfrak{D}$, the same proof must be extracted from $\mathfrak{D}_d$. We then consider that the designs $\mathfrak{D}_d$ are pairwise 'equivalent'; we shall say that the family $\mathfrak{D}_d$ is uniform or, equivalently, that $\mathfrak{D}$ is *uniform.*

These remarks fully justify our using $\mathbb{D}$*-uniformity* as the name for the property we require for a design to be a candidate for representing a proof of a quantified formula.

We will now use some examples to show that these uniformity properties are independent.

All our previous examples are variants of the two designs $\mathfrak{D}_1$ and $\mathfrak{D}_2$, which represent two proofs of $A \vdash A \oplus A$[†]. The first proof is the copy of a proof π of A in a proof of $A \oplus A$ obtained by a left $\oplus$-rule applied to π. Similarly, the second proof is obtained by a right $\oplus$-rule applied to π.

[†] If we were being precise, we would write $A \vdash \uparrow\downarrow A \oplus \uparrow\downarrow A$.

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Consider the sequent of behaviours $\mathbf{A} \vdash \mathbf{A} \oplus \mathbf{A}^{\ddagger}$ based on $3 \vdash \xi$ and its designs $\mathfrak{D}_1$ and $\mathfrak{D}_2$ below:

$$\mathfrak{D}_1 = \cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\mathfrak{Fax}}{\cdots \xi * 1 * 4 * i \vdash 3 * i \cdots}}{\vdash 3 * I, \xi * 1 * 4}\,(\xi * 1 * 4, I)}{\xi * 1 \vdash 3 * I}}{\vdash 3 * I, \xi} \qquad I \in \mathscr{P}_f^*(\mathbb{N})}{3 \vdash \xi}$$

$$\mathfrak{D}_2 = \cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\mathfrak{Fax}}{\cdots \xi * 2 * 4 * i \vdash 3 * i \cdots}}{\vdash 3 * I, \xi * 2 * 4}\,(\xi * 2 * 4, I)}{\xi * 2 \vdash 3 * I}}{\vdash 3 * I, \xi} \qquad I \in \mathscr{P}_f^*(\mathbb{N})}{3 \vdash \xi}$$

Note that $\mathfrak{D}_2$ is obtained by replacing 1 with 2 in $\mathfrak{D}_1$.

The foregoing examples of a designs family are built on a domain $\mathbb{D}$ with a given element d_1; the integers that appear as biases in $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are assimilated with constant $\mathfrak{D}$-biases.

Example 1: *A family of uniform designs that is not $\mathbb{D}$-uniform.* Let $(\mathfrak{D}_d)_d$ be the family defined by $\mathfrak{D}_{d_1} = \mathfrak{D}_1$ and $\mathfrak{D}_d = \mathfrak{D}_2$ for all $d \neq d_1$. Without any doubt we could conclude that all the $\mathfrak{D}_d$ are uniform in $MALL_2$. (Informally, we find the same 'subdesign' above every I) but the family is not $\mathbb{D}$-uniform because, by applying the map ϕ defined by $\phi(d) = d_1$, we do not obtain for a $d \neq d_1$, $\phi(\mathfrak{D}_d)$ as a subdesign of $\mathfrak{D}_{\phi(d)}$.

Example 2: *A $\mathbb{D}$-uniform family of uniform designs.* Let $(\mathfrak{E}_d)_d$ be the family defined by $\mathfrak{E}_d = \mathfrak{D}_1$ for all $d \in \mathbb{D}$. Then the condition of $\mathbb{D}$-uniformity is trivially verified. Moreover, $\mathfrak{D}_1$ is a uniform design.

Example 3: *A $\mathbb{D}$-uniform family of non uniform designs.* Let I and J be two distinct subsets of $\mathscr{P}_f^*(\mathbb{N})$, and let $\mathfrak{D}'$ be the following design:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\mathfrak{Fax}}{\cdots 0 * 1 * 4 * i \vdash 3 * i \cdots}}{\vdash 3 * I, 0 * 1 * 4}\,\mathrm{I}}{0 * 1 \vdash 3 * I}}{\vdash 3 * I, 0} \quad \cdots \quad \cfrac{\cfrac{\cfrac{\cfrac{\mathfrak{Fax}}{\cdots 0 * 2 * 4 * j \vdash 3 * j \cdots}}{\vdash 3 * J, 0 * 2 * 4}\,\mathrm{J}}{0 * 2 \vdash 3 * J}}{\vdash 3 * J, 0}\ \mathscr{P}_f^*(\mathbb{N})}{3 \vdash 0}\ .$$

Then consider the family $\mathfrak{G}_d = \mathfrak{D}'$ for all $d \in \mathbb{D}$. We can check without difficulty that every $\mathfrak{G}_d$ is not uniform, but the family $(\mathfrak{G}_d)_d$ is $\mathbb{D}$-uniform since for every ϕ we get $\phi(\mathfrak{G}_d) = \mathfrak{G}_{\phi(d)} = \mathfrak{D}'$.

‡ Similarly, we would write $\mathbf{A} \vdash \uparrow_1 \downarrow_4 \mathbf{A} \oplus \uparrow_2 \downarrow_4 \mathbf{A}$. The indices are the biases used by the shifts.

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3.3. *Ludic representations of formulas and proofs*

3.3.1. *Some tools* As before, $\mathbb{D}$ is a given domain and all the notions are related to $\mathbb{D}$ ($\mathbb{D}$-behaviours, $\mathbb{D}$-designs, and so on ...).

According to the idea that quantifiers are dealt with by infinite additive *spiritual* connectives, we define for all $\vec{d} \in \mathbb{D}$ a delocation $\varphi_{\vec{d}}$, which can act on every design $\mathfrak{D}$ of constant base.

Definition 3.7 (The delocations $\varphi_{\vec{d}}$).

— Let ξ be an address and $\vec{d}$ be a sequence $d_1, \ldots, d_n$ of elements in $\mathbb{D}$. The delocation $\varphi^{\xi}_{\vec{d}}$ is defined by

$$\varphi^{\xi}_{\vec{d}}\left(\xi * i^k_p(\vec{e}) * \sigma\right) = \xi * i^{k+n}_p(\vec{d}, \vec{e}) * \sigma.$$

— Let $\mathbf{C}_\xi$ be a behaviour,

$$\varphi^{\xi}_{\vec{d}}(\mathbf{C}) = \left\{\varphi^{\xi}_{\vec{d}}(\mathfrak{D}) \ ; \ \mathfrak{D} \in \mathbf{C}\right\}^{\perp\perp}.$$

Remark: The delocations $\varphi^{\xi}_{\vec{d}}$ only act on biases with an occurrence in a ramification I of an action of $\mathfrak{D}$ focusing on ξ, so a constant base remains constant.

Proposition 3.8. Let $(\mathfrak{D}_{\vec{e},\vec{d}})_{\vec{e},\vec{d}}$ be a $\mathbb{D}$-uniform family of designs. Then $(\varphi_{\vec{e}}(\mathfrak{D}_{\vec{e},\vec{d}}))_{\vec{e},\vec{d}}$ is a $\mathbb{D}$-uniform family of designs.

Proof. The proof is straightforward from the definitions. □

Proposition 3.9. Let $\mathbf{C}$ be a behaviour based on a constant base ξ.

— $\varphi^{\xi}_{\vec{d}}(|\mathbf{C}|) = |\varphi^{\xi}_{\vec{d}}(\mathbf{C})|$.
— $|\varphi^{\xi}_{\vec{d}}(\mathbf{C}^{\perp})| = |(\varphi^{\xi}_{\vec{d}}(\mathbf{C}))^{\perp}|$.

Proof. The proof follows the same scheme as in Girard (2001) since the delocations $\varphi^{\xi}_{\vec{d}}$ are total and injective. Moreover, we use the properties of bias functions: injectivity and disconnected images. □

Definition 3.10 (Quantifiers on behaviours).

— Assume that $(\mathbf{A}_{d,\vec{e}})_{d,\vec{e}}$ is a family of negative behaviours based on $<>\vdash$.
$(\forall d \in \mathbb{D}\ \mathbf{A}_{d,\vec{e}})_{\vec{e}}$ is the bihaviours family based on $<>\vdash$ defined by

$$\forall d \in \mathbb{D}\ \mathbf{A}_{d,\vec{e}} = \bigcap_{d \in \mathbb{D}} \varphi_d(\mathbf{A}_{d,\vec{e}}).$$

— Assume that $(\mathbf{A}_{d,\vec{e}})_{d,\vec{e}}$ is a family of positive behaviours based on $\vdash<>$.
$(\exists d \in \mathbb{D}\ \mathbf{A}_{d,\vec{e}})_{\vec{e}}$ is the behaviours family based on $\vdash<>$ defined by

$$\exists d \in \mathbb{D}\ \mathbf{A}_{d,\vec{e}} = \left(\bigcup_{d \in \mathbb{D}} \varphi_d(\mathbf{A}_{d,\vec{e}})\right)^{\perp\perp}.$$

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Proposition 3.11.

$$\left(\bigcap_{d\in\mathbb{D}} \varphi_d(\mathbf{A}_{d,\vec{e}})\right)^{\perp} = \left(\bigcup_{d\in\mathbb{D}} \varphi_d\left(\mathbf{A}^{\perp}_{d,\vec{e}}\right)\right).$$

Proof. We can apply the equivalent result proved in Girard (2001). Indeed, the delocations φ_d separate all the behaviours occurring in the intersection. □

Remark: The above proposition directly implies the internal completeness:

$$\left(\bigcup_{d\in\mathbb{D}} \varphi_d(\mathbf{A}_{d,\vec{e}})\right)^{\perp\perp} = \bigcup_{d\in\mathbb{D}} \varphi_d(\mathbf{A}_{d,\vec{e}}).$$

The next definition is only set for families of disconnected bihaviours. For the definition of the PER on an infinite additive conjunction of behaviours we only need to generalise the finite case.

Definition 3.12 (PER on the behaviours $\mathbf{Q} = \forall d \in \mathbb{D}\ \mathbf{A}_d$ and $\mathbf{E} = \exists d \in \mathbb{D}\ \mathbf{A}_d$). We obtain a bihaviour on $\mathbf{Q}$ and $\mathbf{E}$ by considering the relations $\cong_{\mathbf{Q}}$ and $\cong_{\mathbf{E}}$ defined by: $\forall \mathfrak{D}, \mathfrak{D}' \in \mathbf{Q}^p$ (respectively, $\forall \mathfrak{E}, \mathfrak{E}' \in \mathbf{E}^p$)

$$\mathfrak{D} \cong_{\mathbf{Q}} \mathfrak{D}' \quad \text{iff} \quad \forall d \in \mathbb{D}\ \mathfrak{D} \cong_{\mathbf{A}_d} \mathfrak{D}' \quad \text{and} \quad \mathfrak{E} \cong_{\mathbf{E}} \mathfrak{E}' \quad \text{iff} \quad \exists d \in \mathbb{D}\ \mathfrak{E} \cong_{\mathbf{A}_d} \mathfrak{E}'.$$

3.3.2. *Definitions of ludic representations of $MALL_2^1$ formulae, sequents and proofs* In this section we extend the second-order propositional calculus defined in Girard (2001) in order to obtain a second-order predicate calculus, which we call $MALL_2^1$. Like $MALL_2$, it is a two-sided linear sequent calculus with stoup, built on positive formulas only.

We add the following two first-order rules to $MALL_2$:

— $\dfrac{\Gamma(\vec{u}) \vdash \Delta(\vec{u}); P[z/x, \vec{u}]}{\Gamma(\vec{u}) \vdash \Delta(\vec{u}); \exists x P(x, \vec{u})}\ \exists_r$ where z belongs to $\{\vec{u}\}$.

— $\dfrac{\Gamma(\vec{u}), P(x, \vec{u}) \vdash \Delta(\vec{u})\ ;}{\Gamma(\vec{u}), \exists x P(x, \vec{u}) \vdash \Delta(\vec{u})\ ;}\ \exists_l$ where $x \notin \{\vec{u}\}$.

The interpretations of formulas and sequents will be families of bihaviours indexed by $\vec{d}$ (families of designs for proofs). In the following definitions, α is given data associated with the predicate symbols: for every k-ary predicate symbol A we consider a given family $(\mathbf{C}_{\vec{d}})_{\vec{d}}$ of positive bihaviours based on $\vdash$<>. We give the interpretations relative to the data α, then the interpretations of formulas and sequents are obtained by generalisation on α ($\mathbf{G} = (\bigcap_\alpha \mathbf{G}^{\alpha}_{\vec{d}})_{\vec{d}\in\mathbb{D}^k}$).

Ludic representation of a formula (with free variables among $\vec{x}$)**:** As in Girard (2001), we suppose that formulas are labelled with given delocations in every propositional step. The delocations we use are totally independent of all the possible bihaviours. For example, in propositional calculus if we consider the positive bihaviours **A** and **B** (based on $\vdash$<>) associated with the formulas A and B, respectively, then with the formula $A \oplus B$ is

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associated the bihaviour $\varphi(\mathbf{A}) \oplus \psi(\mathbf{B})$ where φ and ψ are delocations mapping the address $i * \sigma$ on $2i * \sigma$ and $i * \sigma$ on $(2i+1) * \sigma$, respectively. In the same way, in predicate calculus with the formula $A(x) \oplus B(x)$ is associated the family of bihaviours $(\varphi(\mathbf{A}_d) \oplus \psi(\mathbf{B}_d))_{d \in \mathbb{D}}$ where the delocations φ and ψ maps the addresses $f_i^k(\vec{d}) * \sigma$ on $f_{2i}^k(\vec{d}) * \sigma$ and $f_i^k(\vec{d}) * \sigma$ on $f_{2i+1}^k(\vec{d}) * \sigma$, respectively.

Let $P(x_1, \ldots, x_k)$ be a positive $MALL_2^1$-formula. We shall denote the bihaviour (or the family of bihaviours) associated with the formula P by $\mathbf{P}^\alpha$. The inductive definition of $\mathbf{P}^\alpha$ follows (Girard 2001) for the propositional steps. We only describe the atomic case and the definitions that are relative to first-order quantifiers.

— Let $A(t_1, \ldots, t_n)$ be an atomic formula where the terms t_i's are built over variables $x_1, \ldots, x_k$. We set $\mathbf{A}(\mathbf{t_1}, \ldots, \mathbf{t_n})^\alpha = (\mathbf{C}_{c_1,\ldots,c_n})_{d_1,\ldots,d_k}$ where the c_i's are the interpretation of the t_i's in $\mathbb{D}$.
— A formula $P = \exists x\, Q(x, \vec{y})$ is associated with a family $(\mathbf{P}^\alpha_{\vec{\mathbf{d}}})_{\vec{\mathbf{d}}}$ of bihaviours as follows:

$$\mathbf{P}^\alpha_{\vec{\mathbf{d}}} = \bigcup_{e \in \mathbb{D}} \varphi_e \left[\mathbf{Q}^\alpha_{\mathbf{e},\vec{\mathbf{d}}}\right]$$

with its PER defined as in Definition 3.12.

The ludic representation $\mathbf{\Sigma}$ of a $MALL_2^1$-sequent Σ (with free variables among $\vec{x}$)**:** This is a family of sequents of bihaviours defined as follows. For each $\vec{d} \in \mathbb{D}^k$, $\mathbf{\Sigma}^\alpha_{\vec{d}}$ is based on the same *constant* base. It is built in the usual way from some convenient delocalisations of the bihaviours associated with the formulas occurring in the sequent. The left part of the sequent, localised in 1, is the tensor of the bihaviours associated with the left formulas. The associated partial relation is defined in the usual way for a sequent of bihaviours.

The ludic representation of a $MALL_2^1$-proof: This is given by the following proposition.

Proposition 3.13. Let π be a $MALL_2^1$-proof with conclusion Σ. Suppose that the free variables of Σ are among $\vec{x} = x_1, \ldots, x_k$ (pairwise distinct). We associate with π a $\mathbb{D}$-*uniform* family of *material* and *winning* designs $(\pi^*_{\vec{d}})_{\vec{d} \in \mathbb{D}^k}$, denoted $\pi^* \in \mathbf{\Sigma}$. The interpretation is invariant under cut elimination.

Proof. The inductive definition given in Girard (2001) enables us to build each $\pi^*_{\vec{d}}$ with the expected properties in every step where a propositional rule is performed. The $\mathbb{D}$-uniformity of the family $(\pi^*_{\vec{d}})_{\vec{d}}$ is trivially preserved at each step. We just focus on the axiom and on the first-order rules.

— If π is an axiom $\overline{A(\vec{x}) \vdash \;\; ; A(\vec{x})}$, we associate with π the family $(\mathfrak{D}_{\vec{d}})_{\vec{d}}$ of base $1 \vdash \xi$ where for each $\vec{d} \in \mathbb{D}^k$ $\mathfrak{D}_{\vec{d}} = \mathscr{F}ax_{1,\xi}$, which is a family of material and winning designs. The $\mathbb{D}$-uniformity of such a family is verified in Example 1 of Section 3.2.2.
— If π is obtained from the proof λ by the $\exists_r$-rule,

$$\dfrac{\begin{array}{c}\lambda \vdots \\ \Sigma = \Gamma(\vec{u}) \vdash \Delta(\vec{u}); P[z/x, \vec{u}]\end{array}}{\Sigma' = \Gamma(\vec{u}) \vdash \Delta(\vec{u}); \exists x P(x, \vec{u})}\ \exists_r$$

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The free variables of P are among $\{x, \vec{u}\}$, and z belongs to $\{\vec{u}\}$ (let us say $z = u_i$). We set $\pi^*_{\vec{d}} = \varphi^{\xi}_{d_i}(\lambda^*_{\vec{d}})$, where ξ is the address of $\mathbf{P[z/x, \vec{u}]}$ in $\mathbf{\Sigma}$. The result $\pi^*_{\vec{d}} \in \mathbf{\Sigma'}_{\vec{d}}$ is an immediate consequence of $(\exists\mathbf{x}\mathbf{P}(\mathbf{x}, \vec{\mathbf{u}})^{\alpha}_{\vec{d}}) = \bigcup_{e\in\mathbb{D}} \varphi^{\xi}_{e}(\mathbf{P}^{\alpha}_{\vec{d}})$ and $\bigcup_{e\in\mathbb{D}} \varphi^{\xi}_{e}(\mathbf{P}^{\alpha}_{\vec{d}}) \supset \varphi^{\xi}_{d_i}(\mathbf{P}^{\alpha}_{\vec{d}})$.

— If π is obtained from the proof λ by the $\exists_l$-rule,

$$\dfrac{\begin{matrix}\lambda\vdots \\ \Sigma = \Gamma(\vec{u}), P(x, \vec{u}) \vdash \Delta(\vec{u})\end{matrix}}{\Sigma' = \Gamma(\vec{u}), \exists x P(x, \vec{u}) \vdash \Delta(\vec{u})}\ \exists_l \quad \text{where } x \text{ is not free in } \Gamma, \Delta$$

Let us set $\pi^*_{\vec{d}} = \bigcup_{e\in\mathbb{D}} \varphi^{\xi}_{e}(\lambda^*_{e,\vec{d}})$. It is easy to verify that $\pi^*_{\vec{d}} \in \mathbf{\Sigma}'_{\vec{d}}$, as it follows immediately from the definition of $(\forall \mathbf{x}\, \mathbf{P}^{\perp,\alpha})_{\vec{d}} = \bigcap_{e\in\mathbb{D}} \varphi^{\xi}_{e}(\mathbf{P}^{\perp,\alpha}_{e,\vec{d}})$. As previously, the uniformity of $(\pi^*_{\vec{d}})_{\vec{d}}$ is straightforward due to the uniformity of $(\lambda^*_{e\vec{d}})$ and Proposition 3.8. □

4. Full completeness theorem

We have seen that 'internal completeness' is still true for the behaviours associated with the first-order quantified formulas. Indeed, we have $(\exists\mathbf{x}\mathbf{P}(\mathbf{x}))^{\perp\perp} = \exists\mathbf{x}\mathbf{P}(\mathbf{x})$.

We are here interested in the full correspondence between designs and proofs in the case of the predicate calculus $MALL^1_2$. We are going to set a 'full completeness theorem' that is a generalisation of the theorem recalled in Section 2.7.

Theorem 4.1 (The full completeness theorem). Let Σ be a Π^1 $MALL^1_2$-sequent whose free variables are among $x_1 \ldots, x_k$. Let $(\mathfrak{D}_{\vec{d}})_{\vec{d}\in D^k}$ be a $\mathbb{D}$-uniform family of designs in $(\mathbf{\Sigma}_{\vec{\mathbf{d}}})_{\vec{\mathbf{d}}}$. If for every $\vec{d} \in \mathbb{D}^k$ the design $\mathfrak{D}_{\vec{d}}$ is *material* and *winning* in $\mathbf{\Sigma}_{\vec{\mathbf{d}}}$, then there exists a $MALL^1_2$-proof π of Σ such that for all $\vec{d} \in \mathbb{D}^k$, $\pi^*_{\vec{d}} = \mathfrak{D}_{\vec{d}}$.

Proof overview. We do not intend to fill pages reproducing the proof provided by Girard in Girard (2001), but we base our proof on it. As in the propositional case, the proof of the theorem is by induction on an obvious definition of sequent-complexity. The point is 'to find the last rule' in the current design that is performed by finding the premises and checking that the designs in premises still have good properties.

The new point here is that we deal with $\mathbb{D}$-uniform families of winning designs indexed on $\vec{d} \in \mathbb{D}^k$. We therefore have to make sure that for each $\vec{d}$ it is possible to find the last rule. Then, we have to verify that this last rule is the same for every term of the family. Moreover, we have to check that the premises are still $\mathbb{D}$-uniform families.

In order to obtain these results, we use two arguments:

— First we use a transposition of the $MALL_2$-results by way of a translation between $MALL^1_2$ and an infinitary propositional second-order system, which we denote $MALL^{\infty}_2$. This system is obtained by adding a countable positive disjunction ($\oplus_{d\in\mathbb{D}}$) to $MALL_2$ (this is performed in Section 4.1); it is easy to check that in this extended system the full completeness theorem of $MALL_2$ remains available. This translation enables one to say that the result is true for each element of the family.

— We then continue by checking that $\mathbb{D}$-uniformity is preserved in the premises of the last rule. This is done in the propositional cases in Lemmas 4.4, 4.5 and 4.6. Finally, in

Lemmas 4.7 and 4.8 we focus on the specific issues arising when the first-order rules are applied. □

4.1. $\oplus_{d\in\mathbb{D}}$ *disjunction*

We begin by briefly describing $MALL_2^\infty$ and outlining the translation we use. In this section, we assume we are given a countable set $\mathbb{D} = \{d_1, d_2, \dots, d_n, \dots\}$.

The sequent calculus $MALL_2^\infty$: The formulas are built as previously in $MALL_2$, with the addition of the new possibility of considering formulas built by an infinite constructor $\oplus_{d\in\mathbb{D}}$ from a family of formulas indexed by $d \in \mathbb{D}$.

The rules of $MALL_2^\infty$ are derived from those of $MALL_2$ by generalising the $\oplus$-rules (where $(F_d)_d$ is a family of formulas indexed by d):

$$\frac{\Gamma \vdash \Delta; F_d}{\Gamma \vdash \Delta; \oplus_d F_d}\oplus_r^\infty \qquad\qquad \frac{\Gamma, F_{d_1} \vdash \Delta; \quad \cdots \quad \Gamma, F_{d_2} \vdash \Delta;}{\Gamma, \oplus_d F_d \vdash \Delta;}\oplus_l^\infty$$

No extra conditions are required for the first rule; in the second one, Γ and Δ do not depend on d_i.

Theorem 4.2 (Completeness for $MALL_2^\infty$). Let $\Gamma \vdash \Delta$; be a Π^1 $MALL_2^\infty$-sequent and let $\mathfrak{D} \in \mathbf{\Gamma} \vdash \mathbf{\Delta}$ be a material winning design. Then there is a $MALL_2^\infty$-proof π of $\Gamma \vdash \Delta$ such that $\mathfrak{D} = \pi^*$.

Girard's proof in Girard (2001) does not depend on whether the disjunctions $\oplus$ are finite or not. Therefore, we conclude that the theorem in $MALL_2^\infty$ is an easy generalisation of the result in $MALL_2$.

Translation between $MALL_2^1$ and $MALL_2^\infty$: First, let us make our notation precise. Suppose that a countable set of variables $\{x_1, x_2, \dots, x_n, \dots\}$ is given. The atomic formula $R(x_{i_1}, x_{i_2}, \dots, x_{i_n})$ and the quantified formulas are built on it.

The translation of a formula of $MALL_2^1$ into a formula of $MALL_2^\infty$ is performed as follows:

— The atomic formula $R(x_{i_1}, x_{i_2}, \dots, x_{i_n})$ is associated with a propositional atom $A = R_{d_{i_1}, d_{i_2}, \dots, d_{i_n}}$.
— The translation of composed formulas is done as usual:

$$(F \oplus G)^* = F^* \oplus G^*; (\downarrow F)^* = \downarrow F^*; (\exists x\, F)^* = \oplus_{d\in\mathbb{D}} F^*.$$

— The translation of a sequent of formulas is naturally the sequent of the translated formulas.

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4.2. *Some lemmas*

The following lemma emphasises the fact that all the elements of a $\mathbb{D}$-uniform family of positive designs have the same last rule.

Lemma 4.3. Let $(\mathfrak{D}_{\vec{d}})_{\vec{d}}$ be a $\mathbb{D}$-uniform family of winning positive designs of constant base $\vdash \vec{\delta}$.
All the $\mathfrak{D}_{\vec{d}}$'s have the same first focus (saying δ_i). Moreover, if we denote the ramification such that $(\delta_i, I_{\vec{d}})$ is the first rule of $\mathfrak{D}_{\vec{d}}$ by $I_{\vec{d}}$, then, for every $\vec{d}$, the ramifications $I_{\vec{d}}$ contain the same bias functions and the arguments of these functions are only among $\{\vec{d}\}$.

Proof. Let $\vec{d}$ be a given element of $\mathbb{D}^k$ and suppose that there is a dependant bias $f(d_1, \vec{d_2})$ in $I_{\vec{d}}$ such that $d_1 \notin \{\vec{d}\}$. It is then possible to consider a $\mathbb{D}$-morphism ϕ such that for all $e \neq d_1$, $\phi(e) = e$ and $\phi(d_1) \in \vec{d}$. The $\mathbb{D}$-uniformity implies that $\phi(I_{\vec{d}}) = I_{\phi(\vec{d})}$ (indeed $(\lambda_i, I_{\vec{d}})$ is a first positive action). So we obtain a contradiction.

Let $\vec{d}$ and $\vec{d'}$ be two distinct elements of $\mathbb{D}^k$. Suppose $\vec{d}$ is such that $f(d_{i_1}, \ldots, d_{i_n}) \in I_{\vec{d}}$ and $\vec{d'}$ is such that no bias in $I_{\vec{d'}}$ is in the image of the bias function f. Then it is always possible to choose an element $\vec{e}$ in $\mathbb{D}^k$ and a domain morphism ϕ such that $\phi(\vec{d}) = \phi(\vec{d'}) = \vec{e}$. By the $\mathbb{D}$-uniformity and by the fact that the image of the bias functions are disjoint, we have $f(e_{i_1}, \ldots, e_{i_n}) \in I_{\vec{e}}$ and $f(e_{i_1}, \ldots, e_{i_n}) \notin I_{\vec{e}}$ simutaneously. This is a contradiction. □

4.2.1. *About the generalisation of propositional steps*

Lemma 4.4. Let $\Sigma = A(\vec{x}) \vdash; A(\vec{x})$ be a sequent where A is a predicate variable. The only designs family in which the designs are winning and material in $(\mathbf{A}_{\vec{\mathbf{d}}} \vdash \mathbf{A}_{\vec{\mathbf{d}}})_d$ is the family $(\mathfrak{D}_{\vec{d}} = \mathfrak{F}ax_{\ 1,\xi})_{\vec{d}}$. This family is obviously $\mathbb{D}$-uniform.

Proof. One can show without major difficulties that this result is an extension of the propositional case obtained without supplementary requirements (even the $\mathbb{D}$-uniformity is unnecessary). □

The following lemma is the key to the generalisation of propositional steps when exploring a negative sequent for 'finding the last rule'; it shows how the properties of the families are preserved during the operation, in particular, the $\mathbb{D}$-uniformity property.

Lemma 4.5. Let $R \vdash \Delta$; be a Π^1 $MALL_2$-sequent such that its free variables belong to $\{x_1, \ldots x_k\}$, and such that R is a propositionally composed formula. Let $(\mathfrak{D}_{\vec{d}})_{\vec{d}}$ be a $\mathbb{D}$-uniform family of material and winning designs in the bihaviours sequent $(\mathbf{R} \vdash \mathbf{\Delta})_{\vec{d}}$ based on the constant base $1 \vdash \gamma$. We then have:

- If $R = P \otimes Q$, then for every $\vec{d} \in \mathbb{D}^k$ the bihaviours sequent $(\mathbf{P} \otimes \mathbf{Q} \vdash \mathbf{\Delta})_{\vec{d}}$ is equal to the bihaviour sequent $(\mathbf{P}, \mathbf{Q} \vdash \mathbf{\Delta})_{\vec{d}}$.
- If $R = P \oplus Q$, then there exist two $\mathbb{D}$-uniform families of material and winning designs $(\mathfrak{D}'_{\vec{d}})_{\vec{d}}$ and $(\mathfrak{D}''_{\vec{d}})_{\vec{d}}$ such that the first belongs to the bihaviours sequent $\mathbf{P} \vdash \mathbf{\Delta}$ and the second belongs to the bihaviours sequent $\mathbf{Q} \vdash \mathbf{\Delta}$ and such that for all $\vec{d}$, $\mathfrak{D}_{\vec{d}} = \varphi(\mathfrak{D}'_{\vec{d}}) \bigcup \psi(\mathfrak{D}''_{\vec{d}})$.

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Proof. The first assertion immediately follows from the definition of the interpretation of a bihaviours sequent. The $\mathbb{D}$-uniformity of the family is obviously preserved.

Before checking the second assertion, let us recall the situation in propositional calculus. Let φ, ψ and $\vec{\theta}$ be some delocations for the formulas of the sequent such that $\mathbf{P} \oplus \mathbf{Q} \vdash \mathbf{\Delta}$ is based on $1 \vdash \vec{\delta}$. Assuming that $\mathfrak{D}$ is a material and winning design in $\varphi(\mathbf{P}) \oplus \psi(\mathbf{Q}) \vdash \vec{\theta}(\mathbf{\Delta})$, we have

$$\mathfrak{D} = \cfrac{\vdots}{1 \vdash \vec{\delta}}\ \mathscr{N} = \{\varphi(I)/\ I \in \mathscr{N}'\} \cup \{\psi(J)/\ J \in \mathscr{N}''\}.$$

We can decompose $\mathfrak{D}$ into two designs $\mathfrak{D}'$ and $\mathfrak{D}''$ such that

$$\mathfrak{D}' = \cfrac{\vdots}{1 \vdash \vec{\delta}}\ \mathscr{N}' \ \in (\mathbf{P} \vdash \mathbf{\Delta}) \quad \text{and} \quad \mathfrak{D}'' = \cfrac{\vdots}{1 \vdash \vec{\delta}}\ \mathscr{N}'' \ \in (\mathbf{Q} \vdash \mathbf{\Delta}).$$

This situation can be generalised to the predicate case:

— First we apply the foregoing propositional result to each $\mathfrak{D}_{\vec{d}}$. For this we use the translation previously defined: we consider that $\mathfrak{D}_{\vec{d}}$ is a winning and material design in the ludical interpretation of the $MALL_2^{\infty}$-translation of the sequent $(R \vdash \Delta;)^*_{\vec{d}}$. We built the decomposition of $\mathfrak{D}_{\vec{d}}$ into $\mathfrak{D}'_{\vec{d}}$ and $\mathfrak{D}''_{\vec{d}}$ from $\mathscr{N}_{\vec{d}} = \{\varphi(I)/I \in \mathscr{N}'_{\vec{d}}\} \cup \{\psi(J)/J \in \mathscr{N}''_{\vec{d}}\}$. Note that the delocations φ and ψ do not depend on $\vec{d}$, as explained in Section 3.3.2.

— Then we use the $\mathbb{D}$-uniformity of $(\mathfrak{D}_{\vec{d}})_{\vec{d}}$ to conclude to the $\mathbb{D}$-uniformity of the families $(\mathfrak{D}'_{\vec{d}})_{\vec{d}}$ and $(\mathfrak{D}''_{\vec{d}})_{\vec{d}}$. □

Lemma 4.6. Let $\vdash \Delta; P$ be a Π^1 $MALL_2$-sequent that has its free variables among $\vec{x} = \{x_1, \ldots, x_n\}$ and where P is a propositionally composed formula.

Let $\vdash \mathbf{\Delta}, \mathbf{P}$ based on $\vdash \vec{\delta}, \xi$ be the interpretation of the sequent $\vdash \Delta; P$. Let $(\mathfrak{D}_{\vec{d}})_{\vec{d}}$ be a $\mathbb{D}$-uniform-family of material and winning designs in $\vdash \mathbf{\Delta}, \mathbf{P}$ such that their first focus is ξ. We then have:

— If $P = P_1 \oplus P_2$, then either for every $\vec{d}$ the design $\mathfrak{D}_{\vec{d}}$ is in $\vdash \mathbf{\Delta}, \mathbf{P_1}$ or for every $\vec{d}$ the design $\mathfrak{D}_{\vec{d}}$ is in $\vdash \mathbf{\Delta}, \mathbf{P_2}$ (according to some implicit delocations).

— If $P = P_1 \otimes P_2$, then there are two $\mathbb{D}$-uniform families of material and winning designs $(\mathfrak{D}'_{\vec{d}})_{\vec{d}}$ and $(\mathfrak{D}''_{\vec{d}})_{\vec{d}}$ such that the first belongs to the bihaviours sequent $[\vdash \mathbf{\Delta}', \mathbf{P_1}]$ and the second belongs to the bihaviours sequent $[\vdash \mathbf{\Delta}'', \mathbf{P_2}]$, where $\Delta' \cup \Delta'' = \Delta$ independently of $\vec{d}$.

  Moreover, $\forall \mathfrak{E}_1 \in \mathbf{\Delta}^{\perp}_{\mathbf{1},\vec{\mathbf{d}}}$ $\forall \mathfrak{E}_2 \in \mathbf{\Delta}^{\perp}_{\mathbf{2},\vec{\mathbf{d}}}$ $[\![\mathfrak{D}_{\vec{\mathbf{d}}}, \mathfrak{E}_1 \otimes \mathfrak{E}_2]\!] = [\![\mathfrak{D}'_{\vec{\mathbf{d}}}, \mathfrak{E}_1]\!] \otimes [\![\mathfrak{D}''_{\vec{\mathbf{d}}}, \mathfrak{E}_2]\!]$ (according to some implicit delocations).

Proof. If $P = P_1 \oplus P_2$, the propositional results (Girard 2001) generalised by means of our translation show that for all $\vec{d} \in \mathbb{D}^k$ the designs $\mathfrak{D}_{\vec{d}}$ are in $[\vdash \mathbf{\Delta}, \mathbf{P_1}]_{\vec{d}}$ or in $[\vdash \mathbf{\Delta}, \mathbf{P_2}]_{\vec{d}}$. Because of the $\mathbb{D}$-uniformity of the family $(\mathfrak{D}_{\vec{d}})_{\vec{d}}$ and to the delocations implicitly applied to P_1 and P_2, this splitting is independent of $\vec{d}$.

If $P = P_1 \otimes P_2$, then, as in the propositional case (Girard 2001), for each $\vec{d}$ we can decompose $\mathfrak{D}_{\vec{d}}$ in $\mathfrak{D}'_{\vec{d}} \odot \mathfrak{D}''_{\vec{d}}$ and Δ in Δ' and Δ'' in such a way that for every $\mathfrak{E}_{\delta'} \in \mathbf{\Delta}'^{\perp}$

and $\mathfrak{E}_{\delta''} \in \mathbf{\Delta}''^{\perp}$ we have $[\![\mathfrak{D}'_{\vec{d}}, \mathfrak{E}_{\delta'}]\!] \in (\vdash \mathbf{P_1})_{\vec{\mathbf{d}}}$ and $[\![\mathfrak{D}''_{\vec{d}}, \mathfrak{E}_{\delta''})]\!] \in (\vdash \mathbf{P_2})_{\vec{\mathbf{d}}}$. By the same remark as in the previous case, the splitting of $\mathfrak{D}_{\vec{d}}$ into $\mathfrak{D}'_{\vec{d}}$ and $\mathfrak{D}''_{\vec{d}}$ is independent of $\vec{d}$, as is also the splitting of $\mathbf{\Delta}_{\vec{\mathbf{d}}}$ between $\mathbf{\Delta}'_{\vec{\mathbf{d}}}$ and $\mathbf{\Delta}''_{\vec{\mathbf{d}}}$. Moreover, $(\mathfrak{D}'_{\vec{d}})_{\vec{d}}$ is a $\mathbb{D}$-uniform family of winning and material designs in $\vdash \mathbf{\Delta}', \mathbf{P_1}$, and $(\mathfrak{D}''_{\vec{d}})_{\vec{d}}$ is a $\mathbb{D}$-uniform family of winning and material designs in $\vdash \mathbf{\Delta}'', \mathbf{P_2}$.

The last property of the splitting is a direct consequence of the construction. □

4.2.2. *The first-order quantifier steps*

Lemma 4.7. Let $\exists y P(y,x) \vdash \Delta(x);$ be a Π^1 $MALL_2$-sequent where $P(y,x)$ is a positive formula with free variables among $\{x,y\}$ with $y \neq x$. Let $(\mathfrak{D}_d)_{d\in\mathbb{D}}$ be a $\mathbb{D}$-uniform designs family of material and winning designs in $(\exists \mathbf{y}\mathbf{P}(\mathbf{y},\mathbf{x}) \vdash \mathbf{\Delta})_d$ based on $1 \vdash \delta$.

Then, for every $d \in \mathbb{D}$, $\mathfrak{D}_d = \bigcup_{e\in\mathbb{D}} \varphi_e(\mathfrak{E}_{e,d})$ and for every e,d, the design $\mathfrak{E}_{e,d}$ is material and winning in $[\mathbf{P}(\mathbf{y},\mathbf{x}) \vdash \mathbf{\Delta}(\mathbf{x})]_{\mathbf{e,d}}$. Moreover, the designs family $(\mathfrak{E}_{e,d})_{e,d}$ is $\mathbb{D}$-uniform.

Proof. Without loss of generality, we deal with a sequent where $\Delta(x)$ only contains a formula $R(x)$.

In a first step, let d be a given element in $\mathbb{D}$. Let $\mathfrak{F}_d$ be a design in $\mathbf{R}^{\perp}_d$. The design $[\![\mathfrak{D}_d, \mathfrak{F}_d]\!]$ contains a material design of $[\exists \mathbf{y}\mathbf{P}(\mathbf{y},\mathbf{x})]^{\perp}_{\mathbf{d}}$ that looks like the following:

$$\dfrac{\cdots \qquad \begin{matrix}\vdots \\ \vdash 1 * \varphi_e(I_{e,d})\end{matrix} \qquad \cdots}{1 \vdash} \cup_e \varphi_e(\mathcal{N}_{e,d}).$$

So we can build for every $d \in \mathbb{D}$ and every $e \in \mathbb{D}$ the design $\mathfrak{E}_{e,d}$ as the set of chronicles $(\xi, I_{e,d}) * \mathfrak{c}$ provided $(\xi, \varphi_e(I_{e,d}) * \mathfrak{c})$ is in $\mathfrak{D}_d$.

$$\mathfrak{E}_{e,d} = \dfrac{\cdots \qquad \begin{matrix}\vdots \\ \vdash 1 * I_{e,d}, \delta\end{matrix} \qquad \cdots}{1 \vdash \delta} \{\mathcal{N}_{e,d}\}.$$

The design $\mathfrak{E}_{e,d}$ is in $[\mathbf{P}(\mathbf{y},\mathbf{x}) \vdash \mathbf{\Delta}(\mathbf{x})]_{\mathbf{e,d}}$. Indeed, for some $\mathfrak{M}_{e,d} \in \mathbf{P_{e,d}}$, the design $\varphi_e(\mathfrak{M}_{e,d})$ is in $[\exists \mathbf{y}\mathbf{P}(\mathbf{y},\mathbf{x})]_{\mathbf{d}}$. Moreover, $[\![\mathfrak{E}_{e,d}, \mathfrak{M}_{e,d}]\!] = [\![\mathfrak{D}_d, \varphi_e(\mathfrak{M}_{e,d})]\!]$, and this is in $[\vdash \mathbf{\Delta}(\mathbf{x})]_{\mathbf{d}}$.

From the fact that every $\mathfrak{D}_d$ is material and winning, we also deduce that $\mathfrak{D}_d = \bigcup_{e\in\mathbb{D}} \varphi_e(\mathfrak{E}_{e,d})$ and that for every $d \in \mathbb{D}$, $e \in \mathbb{D}$ the design $\mathfrak{E}_{e,d}$ is material and winning in $[\mathbf{P}(\mathbf{y},\mathbf{x}) \vdash \mathbf{\Delta}(\mathbf{x})]_{\mathbf{e,d}}$.

The $\mathbb{D}$-uniformity of the family $(\mathfrak{E}_{e,d})_{e,d}$ directly follows from that of $(\mathfrak{D}_d)_d$. □

Lemma 4.8. Let $\vdash \Delta(\vec{x}); \exists y P(y,\vec{x})$ be a Π^1 $MALL_2$-sequent where $P(y,\vec{x})$ is a positive formula with free variables among $\{y\} \cup \{x_1,\ldots,x_k\}$ $(\forall i\ y \neq x_i)$. Let $(\mathfrak{D}_{\vec{d}})_{\vec{d}\in\mathbb{D}^k}$ be a $\mathbb{D}$-uniform designs family of material and winning designs in $\vdash \mathbf{\Delta}(\vec{\mathbf{x}}); \exists \mathbf{y}\mathbf{P}(\mathbf{y},\vec{\mathbf{x}})$ of base $\vdash \delta, \xi$.

For every $\vec{d} \in \mathbb{D}^k$ there is an element $e_{\vec{d}}$ of $\mathbb{D}$ and a winning and material design $\mathfrak{F}_{\vec{d}}$ in $(\vdash \mathbf{\Delta}(\vec{\mathbf{x}}); \mathbf{P}(\mathbf{y},\vec{\mathbf{x}}))_{e_{\vec{d}},\vec{d}}$ such that $\mathfrak{D}_{\vec{d}} = \varphi_{e_{\vec{d}}}(\mathfrak{F}_{\vec{d}})$, where e_d is one of the d_i's. Furthermore, the family $(\mathfrak{F}_{\vec{d}})_{\vec{d}}$ is $\mathbb{D}$-uniform.

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Proof. As before, we deal with a sequent $\Delta(x)$ containing only one formula $R(x)$. For all $\vec{d} \in \mathbb{D}^k$, let $(\xi, I_{\vec{d}})$ be the first action of $\mathfrak{D}_{\vec{d}}$.

1 By definition, for every $\vec{d} \in \mathbb{D}^k$, the normalisation between $\mathfrak{D}_{\vec{d}}$ and a material design of $\forall \mathbf{y} \mathbf{P}^{\perp}(\mathbf{y}, \vec{\mathbf{x}})$ converges. This implies that there exists $e_{\vec{d}} \in \mathbb{D}$ such that $I_{\vec{d}} = \varphi_{e_{\vec{d}}}(I_{e_{\vec{d}},\vec{d}})$ where $(\xi, I_{e_{\vec{d}},\vec{d}})$ is a first action of a design of $\mathbf{P}^{\perp}(\mathbf{y}, \vec{\mathbf{x}})_{e_{\vec{d}},\vec{d}}$.

2 By Lemma 4.3, we know that all the first actions $I_{\vec{d}}$'s contain the same bias functions. Moreover, for each $\vec{d}$ these bias functions are only applied to some $d \in \vec{d}$, so there is a j such that $e_{\vec{d}} = d_j$.

3 Let $\mathfrak{E}_{e_{\vec{j}},\vec{d}}$ be the design obtained from $\mathfrak{D}_{\vec{d}}$ by exchanging the first action $(\xi, I_{\vec{d}})$ with $(\xi, I_{e_{\vec{j}},\vec{d}})$. From the above result, we can rewrite the family $\mathfrak{E}_{e_{\vec{d}},\vec{d}}$ as a family $(\mathfrak{F}_{\vec{d}})_{\vec{d}}$ indexed by $\vec{d}$ only, and we see without difficulty that for every $\vec{d}$ we have $\mathfrak{D}_{\vec{d}} = \varphi_{e_{\vec{j}}}(\mathfrak{F}_{\vec{d}})$ and $\mathfrak{F}_{\vec{d}} \in (\vdash \mathbf{P}(\mathbf{y}, \vec{\mathbf{x}}), \mathbf{R}(\vec{\mathbf{x}}))_{\vec{\mathbf{d}}}$ where y is instantiated by e_d.

4 We then check the $\mathbb{D}$-uniformity of the family $(\mathfrak{F}_{\vec{d}})_{\vec{d}}$, all the designs of which are winning and material in $(\vdash \mathbf{P}(\mathbf{y}, \vec{\mathbf{x}}), \mathbf{R}(\vec{\mathbf{x}}))_{\vec{d}}$. □

5. Work in progress

In order to give an interactive definition of uniformity, Girard extends the notion of behaviours by putting on them a partial equivalence relation (PER). Two PER-equivalent partial designs react in the same manner during the process of normalisation against all PER-equivalent partial designs. In this paper, the $\mathbb{D}$-uniformity is not presented in such an interactive way. Our definition could be said to be external. Our aim now is to combine the two notions of uniformity to give a global and interactive definition.

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