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RESEARCH ARTICLE

Finite Hypergraph Families with Rich Extremal Turán Constructions via Mixing Patterns

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Abstract

We prove that, for any finite set of minimal r-graph patterns, there is a finite family $\mathcal{F}$ of forbidden r-graphs such that the extremal Turán constructions for $\mathcal{F}$ are precisely the maximum r-graphs obtainable from mixing the given patterns in any way via blowups and recursion. This extends the result by the second author [30], where the above statement was established for a single pattern.

We present two applications of this result. First, we construct a finite family $\mathcal{F}$ of 3-graphs such that there are exponentially many maximum $\mathcal{F}$-free 3-graphs of each large order n and, moreover, the corresponding Turán problem is not finitely stable. Second, we show that there exists a finite family $\mathcal{F}$ of 3-graphs whose feasible region function attains its maximum on a Cantor-type set of positive Hausdorff dimension.

Contents

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1. Introduction

1.1. Turán problem

For an integer $r \geq 2$, an **r-uniform hypergraph** (henceforth an **r-graph**) H is a collection of r-subsets of some finite set V. Given a family $\mathcal{F}$ of r-graphs, we say that H is $\mathcal{F}$**-free** if it does not contain any member of $\mathcal{F}$ as a subgraph. The **Turán number** $\mathrm{ex}(n, \mathcal{F})$ of $\mathcal{F}$ is the maximum number of edges in an $\mathcal{F}$-free r-graph on n vertices. The **Turán density** $\pi(\mathcal{F})$ of $\mathcal{F}$ is defined as $\pi(\mathcal{F}) := \lim_{n\to\infty} \mathrm{ex}(n, \mathcal{F})/\binom{n}{r}$; the existence of the limit was established in [18]. The study of $\mathrm{ex}(n, \mathcal{F})$ is one of the central topics in extremal graph and hypergraph theory.

Much is known about $\mathrm{ex}(n, \mathcal{F})$ for graphs – that is, when $r = 2$. For example, Turán [36] determined $\mathrm{ex}(n, K_\ell^2)$ for all $n > \ell > 2$ (where, more generally, K_ℓ^r denotes the complete r-graph on ℓ vertices). Also, the Erdős–Stone–Simonovits theorem [8, 9] determines the Turán density for every family $\mathcal{F}$ of graphs; namely, it holds that $\pi(\mathcal{F}) = \min\{1 - 1/\chi(F) : F \in \mathcal{F}\}$, where $\chi(F)$ denotes the chromatic number of the graph F.

For $r \geq 3$, determining $\pi(\mathcal{F})$ for a given family $\mathcal{F}$ of r-graphs seems to be extremely difficult in general. For example, the problem of determining $\pi(K_\ell^r)$ raised already in the 1941 paper by Turán [36] is still open for all $\ell > r \geq 3$; thus, the \$500 prize of Erdős (see, for example, [7, Section III.1]) for determining $\pi(K_\ell^r)$ for at least one pair (ℓ, r) with $\ell > r \geq 3$ remains unclaimed.

The 'inverse' problem of understanding the sets

$$\Pi_{\mathrm{fin}}^{(r)} := \{\pi(\mathcal{F})\colon \mathcal{F} \text{ is a finite family of } r\text{-graphs}\}, \quad \text{and}$$
$$\Pi_{\infty}^{(r)} := \{\pi(\mathcal{F})\colon \mathcal{F} \text{ is a (possibly infinite) family of } r\text{-graphs}\}$$

of possible r-graph Turán densities is also very difficult for $r \geq 3$. (For $r = 2$, we have by the Erdős–Stone–Simonovits theorem [8, 9] that $\Pi_\infty^{(2)} = \Pi_{\mathrm{fin}}^{(2)} = \{1\} \cup \{1 - 1/k\colon \text{integer } k \geq 1\}$.)

One of the earliest results on this direction is the theorem of Erdős [5] from the 1960s that $\Pi_\infty^{(r)} \cap (0, r!/r^r) = \emptyset$ for every integer $r \geq 3$. However, our understanding of the locations and the lengths of other maximal intervals avoiding r-graph Turán densities and the right accumulation points of $\Pi_\infty^{(r)}$ (the so-called **jump problem**) is very limited; for some results in this direction, see, for example, [1, 12, 13, 31, 37].

It is known that the set $\Pi_\infty^{(r)}$ is the topological closure of $\Pi_{\mathrm{fin}}^{(r)}$ (see [30, Proposition 1]), and thus, the former set is determined by the latter. In order to show that the set $\Pi_{\mathrm{fin}}^{(r)} \subseteq [0, 1]$ has rich structure for each $r \geq 3$, the second author proved in [30, Theorem 3] that, for every minimal r-graph pattern P, there is a finite family $\mathcal{F}$ of r-graphs such that the maximum $\mathcal{F}$-free graphs are precisely the maximum r-graphs that can be obtained by taking blowups of P and using recursion. (See Section 1.2 for all formal definitions.) In particular, the maximum asymptotic edge density obtainable this way from the pattern P is an element of $\Pi_{\mathrm{fin}}^{(r)}$.

Another factor that makes the hypergraph Turán problem difficult is that some families may have many rather different (almost) extremal configurations. A series of recent papers [15, 26, 27] (discussed in more detail in Sections 1.3 and 1.4) concentrated on exhibiting examples for which the richness of extremal configurations can be proved.

Our paper contributes further to this line of research. The new results proved here are, informally speaking, as follows. Our main result, on which all new constructions are based, is Theorem 1.2. It extends [30, Theorem 3] to the case when there is a finite set $\{P_i : i \in I\}$ of minimal patterns (instead of just one), and we can mix them in any way when using recursion. By applying Theorem 1.2, we

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present two examples of a 3-graph family $\mathcal{F}$ with a rich set of (almost) extremal configurations. The first one (given by Theorem 1.3) has the property that the set of maximum $\mathcal{F}$-free 3-graphs on $[n]$ has exponentially many in n non-isomorphic hypergraphs and, moreover, the Turán problem for $\mathcal{F}$ is not finitely stable; that is, roughly speaking, there are no bounded number of constructions such that every almost maximum $\mathcal{F}$-free 3-graph is close to one of them. The second finite family $\mathcal{F}$ of 3-graphs (given by Corollary 1.6) satisfies the property that the limit set of possible densities of the shadows of asymptotically maximum $\mathcal{F}$-free 3-graphs is a Cantor-like set of positive Hausdorff dimension.

Let us now present the formal statements of the new results (together with some further definitions and background).

1.2. Patterns

In order to state our main result (Theorem 1.2), we need to give a number of definitions.

Let an **r-multiset** mean an unordered collection of r elements with repetitions allowed. Let E be a collection of r-multisets on $[m] := \{1, \dots, m\}$, let $V_1, \dots, V_m$ be disjoint sets and set $V := V_1 \cup \dots \cup V_m$. The **profile** of an r-set $X \subseteq V$ (with respect to $V_1, \dots, V_m$) is the r-multiset on $[m]$ that contains $i \in [m]$ with multiplicity $|X \cap V_i|$. For an r-multiset $Y \subseteq [m]$, let $Y((V_1, \dots, V_m))$ consist of all r-subsets of V whose profile is Y. We call this r-graph the **blowup** of Y (with respect to $V_1, \dots, V_m$), and the r-graph

$$E((V_1, \dots, V_m)) := \bigcup_{Y \in E} Y((V_1, \dots, V_m))$$

is called the **blowup** of E (with respect to $V_1, \dots, V_m$).

An **(r-graph) pattern** is a triple $P = (m, E, R)$ where m is a positive integer, E is a collection of r-multisets on $[m]$, and R is a subset of $[m]$ (we allow R to be the empty set). As a convention, we require that R does not contain i if E contains the multiset consisting of r copies of i. Suppose that I is a nonempty index set and

$$P_I := \{P_i : i \in I\}, \quad \text{where } P_i = (m_i, E_i, R_i) \text{ for } i \in I, \tag{1.1}$$

is a collection of r-graph patterns indexed by I.

Definition 1.1 (P_I-Mixed Constructions). For P_I as in (1.1), a $\mathbf{P_I}$**-mixing construction** on a set V is any r-graph with vertex set V which has no edges or can be recursively constructed as follows. Pick some $i \in I$ and take any partition $V = V_1 \cup \dots \cup V_{m_i}$ such that $V_j \neq V$ for each $j \in R_i$. Let G be obtained from the blowup $E_i((V_1, \dots, V_{m_i}))$ by adding, for each $j \in R_i$, an arbitrary P_I-mixing construction on V_j.

Informally speaking, we can start with a blowup $E_i((V_1, \dots, V_{m_i}))$ for some $i \in I$, then put a blowup of some $E_{i'}$ inside each **recursive part** (that is, V_j with $j \in R_i$), then put blowups into the new recursive parts, and so on. Note that there is no restriction on the choice of the index at any step. For example, on the second level of the recursion, different recursive parts V_j, $j \in R_i$ may choose different indices. See Section 2.4 for two examples illustrating this construction.

The family of all P_I-mixing constructions will be denoted by ΣP_I. We say G is a $\mathbf{P_I}$**-mixing subconstruction** if it is a subgraph of some P_I-mixing construction on $V(G)$.

Let $\Lambda_{P_I}(n)$ be the maximum number of edges that an r-graph in ΣP_I with n vertices can have:

$$\Lambda_{P_I}(n) := \max\left\{ |G| : G \text{ is a } P_I\text{-mixing construction on } [n] \right\}. \tag{1.2}$$

Using a simple averaging argument (see Lemma 3.3), one can show that the ratio $\Lambda_{P_I}(n)/\binom{n}{r}$ is non-increasing and therefore tends to a limit which we denote by λ_{P_I} and call it the **Lagrangian** of P_I:

$$\lambda_{P_I} := \lim_{n\to\infty} \frac{\Lambda_{P_I}(n)}{\binom{n}{r}}. \tag{1.3}$$

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If $P_I = \{P\}$ consists of a single pattern P, then we always have to use this pattern P and the definition of a P_I-mixing construction coincides with the definition of a **P-construction** from [30]. For brevity, we abbreviate $\Lambda_P := \Lambda_{\{P\}}$, $\lambda_P := \lambda_{\{P\}}$, etc.

For example, if $r = 2$ and $P = (2, \{\{1,2\}\}, \emptyset)$, then P-constructions (that is, $\{P\}$-mixing constructions) are exactly complete bipartite graphs, P-subconstructions are exactly graphs with chromatic number at most 2, $\Lambda_P(n) = \lfloor n^2/4 \rfloor$ for every integer $n \geq 0$, and $\lambda_P = 1/2$.

For a pattern $P = (m, E, R)$ and $j \in [m]$, let $P - j$ be the pattern obtained from P by **removing index j**; that is, we remove j from $[m]$ and delete all multisets containing j from E (and relabel the remaining indices to form the set $[m-1]$). In other words, $(P-j)$-constructions are precisely those P-constructions where we always let the j-th part be empty. We call P **minimal** if λ_{P-j} is strictly smaller than λ_P for every $j \in [m]$. For example, the 2-graph pattern $P := (3, \{\{1,2\},\{1,3\}\}, \emptyset)$ is not minimal as $\lambda_P = \lambda_{P-3} = 1/2$.

Let $\mathcal{F}_\infty$ be the family consisting of those r-graphs that are not P_I-mixing subconstructions; that is,

$$\mathcal{F}_\infty := \{r\text{-graph } F : \text{every } P_I\text{-mixing construction } G \text{ is } F\text{-free}\}, \tag{1.4}$$

and for every $M \in \mathbb{N}$, let $\mathcal{F}_M$ be the collection of members in $\mathcal{F}_\infty$ with at most M vertices; that is, for $v(F) := |V(F)|$, we have

$$\mathcal{F}_M := \{F \in \mathcal{F}_\infty : v(F) \leq M\}. \tag{1.5}$$

Our main result is as follows.

Theorem 1.2. *Let $r \geq 3$ and let $P_I = \{P_i : i \in I\}$ be an arbitrary collection of minimal r-graph patterns, where the index set I is finite. Then there exists $M \in \mathbb{N}$ such that the following statements hold.*

(a) *For every positive integer n, we have $\mathrm{ex}(n, \mathcal{F}_M) = \max\{|G| : v(G) = n \text{ and } G \in \Sigma P_I\}$. Moreover, every maximum n-vertex $\mathcal{F}_M$-free r-graph is a member in ΣP_I.*
(b) *For every $\varepsilon > 0$, there exist $\delta > 0$ and N_0 such that for every $\mathcal{F}_M$-free r-graph G on $n \geq N_0$ vertices with $|G| \geq (1-\delta)\mathrm{ex}(n, \mathcal{F}_M)$, there exists an r-graph $H \in \Sigma P_I$ on $V(G)$ such that $|G \triangle H| \leq \varepsilon n^r$.*

Note that Part (a) of Theorem 1.2 gives that the family of maximum $\mathcal{F}_M$-free r-graphs is exactly the family of maximum P_I-mixing constructions.

In the case of a single pattern (when $|I| = 1$), Theorem 1.2 gives [30, Theorem 3]. As we will see in Lemma 3.4, it holds that $\lambda_{P_I} = \max\{\lambda_{P_i} : i \in I\}$, and thus, we do not increase the set of obtainable Turán densities by mixing patterns. The main purpose of Theorem 1.2 is to show that some finite Turán problems have rich sets of (almost) extremal graphs. In this paper, we present two applications of Theorem 1.2 of this kind as follows.

1.3. Finite families with exponentially many extremal Turán r-graphs

Let us call a family $\mathcal{F}$ of r-graphs **t-stable** if for every $n \in \mathbb{N}$, there are r-graphs $G_1(n), \ldots, G_t(n)$ on $[n]$ such that for every $\varepsilon > 0$, there are $\delta > 0$ and n_0 so that if G is an $\mathcal{F}$-free r-graph with $n \geq n_0$ vertices and least $(1-\delta)\mathrm{ex}(n, \mathcal{F})$ edges, then G is within edit distance εn^r to some $G_i(n)$. The **stability number** $\xi(\mathcal{F})$ of $\mathcal{F}$ is the smallest $t \in \mathbb{N}$ such that $\mathcal{F}$ is t-stable; we set $\xi(\mathcal{F}) := \infty$ if no such t exists. We call $\mathcal{F}$ **stable** if $\xi(\mathcal{F}) = 1$ (that is, if $\mathcal{F}$ is 1-stable). According to our definition, every family $\mathcal{F}$ of r-graphs with $\pi(\mathcal{F}) = 0$ is stable: take $G_1(n)$ to be the edgeless r-graph on $[n]$.

The first stability theorem, which says that K_ℓ^2 is stable for all integers $\ell \geq 3$, was established independently by Erdős [6] and Simonovits [35]. In fact, the classical Erdős–Stone–Simonovits theorem [9, 8] and Erdős–Simonovits stability theorem [6, 35] imply that every family of graphs is stable.

For hypergraphs, there are some conjectures on the Turán density of various concrete families which, if true, imply that these families are not stable. One of the most famous examples in this regard is the tetrahedron K_4^3 whose conjectured Turán density is 5/9. If this conjecture is true, then the constructions by Brown [3] (see also [10, 14, 22]) show that $\xi(K_4^3) = \infty$. A similar statement applies to some other complete 3-graphs K_ℓ^3; we refer the reader to [19, 34] for details. Another natural example of conjectured infinite stability number is the Erdős–Sós Conjecture on triple systems with bipartite links; we refer the reader to [11] for details.

Despite these old conjectures, no finite family with more than one asymptotic Turán extremal construction was known until recently. In [26], Mubayi and the first author constructed the first finite non-stable family $\mathcal{F}$ of 3-graphs; in fact, their family satisfies $\xi(\mathcal{F}) = 2$. Further, in [27], Mubayi, Reiher, and the first author found, for every integer $t \geq 3$, a finite family $\mathcal{F}_t$ of 3-graphs with $\xi(\mathcal{F}_t) = t$. In [15], Hou, Li, Mubayi, Zhang, and the first author constructed a finite family $\mathcal{F}$ of 3-graphs such that $\xi(\mathcal{F}) = \infty$.

Note that it is possible that $\xi(\mathcal{F}) = 1$, but there are many maximum $\mathcal{F}$-free r-graphs of order n. For example, if $k \geq 5$ is odd and we forbid the star $K_{1,k}^2$ (where, more generally, $K_{k_1,\ldots,k_\ell}^2$ denotes the complete ℓ-partite graph with part sizes $k_1, \ldots, k_\ell$), then the extremal graphs on $n \geq r$ vertices are precisely $(k-1)$-regular graphs, and, as it is easy to see, there are exponentially many in n such non-isomorphic graphs. For r-graphs with $r \geq 3$, a similar conclusion for an infinite sequence of n can be achieved by forbidding, for example, two r-edges intersecting in $r-1$ vertices: if a sufficiently large n satisfies the obvious divisibility conditions, then by the result of Keevash [20], there are $\exp(\Omega(n^{r-1} \log n))$ extremal r-graphs on $[n]$ – namely, designs where each $(r-1)$-set is covered exactly once. While the above Turán problems are **degenerate** (i.e., have the Turán density 0), a nondegenerate example for graphs can be obtained by invoking a result of Simonovits [35], a special case of which is that every maximum graph of order $n \to \infty$ without $K_{1,t,t}^2$ can be obtained from a complete bipartite graph $K_{a,n-a}$ with $a = (1/2+o(1))n$ by adding a maximum $K_{1,t}^2$-free graph into each part. Very recently, Balogh, Clemen and Luo [2] found a single 3-graph F with $\pi(F) > 0$ and with $\exp(\Omega(n^2 \log n))$ non-isomorphic extremal constructions on n vertices for an infinite sequence of n. Note that all families in this paragraph are 1-stable.

In the other direction, the relation $\xi(\mathcal{F}) = \infty$ does not generally imply that there are many maximum $\mathcal{F}$-free r-graphs, since one of asymptotically optimal constructions may produce strictly better bounds (in lower order terms) on $\mathrm{ex}(n, \mathcal{F})$ than any other. So it is of interest to produce $\mathcal{F}$ with many extremal graphs and with $\xi(\mathcal{F}) = \infty$. The above-mentioned paper [15] made a substantial progress towards this problem, by giving a finite familly $\mathcal{F}$ of 3-graphs such that, in addition to $\xi(\mathcal{F}) = \infty$, there are $\Omega(n)$ non-isomorphic maximum $\mathcal{F}$-free for infinitely many n (e.g., for all large n divisible by 3).

As an application of Theorem 1.2, we provide a finite family of 3-graphs with infinite stability number and exponentially many extremal constructions for **all** large integers n.

Theorem 1.3. *There is a finite family $\mathcal{F}$ of 3-graphs such that, for some $C > 0$ and for every $n \in \mathbb{N}$, the number of non-isomorphic maximum $\mathcal{F}$-free 3-graphs on n vertices is at least $\exp(Cn)$. Additionally, $\xi(\mathcal{F}) = \infty$.*

1.4. Feasible region

Theorem 1.2 has also applications to the so-called feasible region problem of hypergraphs. To state our results, we need more definitions.

Given an r-graph G, its **s-shadow** is defined as

$$\partial_s G := \left\{ A \in \binom{V(G)}{r-s} : \exists\, B \in G \text{ such that } A \subseteq B \right\}.$$

We use ∂G to represent $\partial_1 G$ and call it the **shadow** of G. The **edge density** of G is defined as $\rho(G) := |G| / \binom{v(G)}{r}$, and the **shadow density** of G is defined as $\rho(\partial G) := |\partial G| / \binom{v(G)}{r-1}$.

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For a family $\mathcal{F}$, the **feasible region** $\Omega(\mathcal{F})$ of $\mathcal{F}$ is the set of points $(x, y) \in [0,1]^2$ such that there exists a sequence of $\mathcal{F}$-free r-graphs $(G_n)_{n=1}^{\infty}$ with

$$\lim_{n\to\infty} v(G_n) = \infty, \quad \lim_{n\to\infty} \rho(\partial G_n) = x, \quad \text{and} \quad \lim_{n\to\infty} \rho(G_n) = y.$$

The feasible region unifies and generalizes asymptotic versions of some classical problems such as the Kruskal–Katona theorem [17, 23] and the Turán problem. It was introduced in [25] to understand the extremal properties of $\mathcal{F}$-free hypergraphs beyond just the determination of $\pi(\mathcal{F})$.

The following general results about $\Omega(\mathcal{F})$ were proved in [25]. The projection of $\Omega(\mathcal{F})$ to the first coordinate,

$$\operatorname{proj}\Omega(\mathcal{F}) := \{x\colon \text{there is } y \in [0,1] \text{ such that } (x,y) \in \Omega(\mathcal{F})\},$$

is an interval $[0, c(\mathcal{F})]$ for some $c(\mathcal{F}) \in [0,1]$. Moreover, there is a left-continuous almost everywhere differentiable function $g(\mathcal{F})\colon \operatorname{proj}\Omega(\mathcal{F}) \to [0,1]$ such that

$$\Omega(\mathcal{F}) = \left\{(x,y) \in [0, c(\mathcal{F})] \times [0,1]\colon 0 \le y \le g(\mathcal{F})(x)\right\}.$$

The function $g(\mathcal{F})$ is called the **feasible region function** of $\mathcal{F}$. It was showed in [25] that $g(\mathcal{F})$ is not necessarily continuous, and it was showed in [24] that $g(\mathcal{F})$ can have infinitely many local maxima even for some simple and natural families $\mathcal{F}$.

For every r-graph family $\mathcal{F}$, define the set $M(\mathcal{F}) \subseteq \operatorname{proj}\Omega(\mathcal{F})$ as the collection of x such that $g(\mathcal{F})$ attains its global maximum at x; that is,

$$M(\mathcal{F}) := \{x \in \operatorname{proj}\Omega(\mathcal{F})\colon g(\mathcal{F})(x) = \pi(\mathcal{F})\}.$$

For most families $\mathcal{F}$ that were studied before, the set $M(\mathcal{F})$ has size one (i.e., the function $g(\mathcal{F})$ attains its maximum at only one point). In general, the set $M(\mathcal{F})$ is not necessarily a single point, and, in fact, trying to understand how complicated $M(\mathcal{F})$ can be is one of the motivations for constructions in [15, 26, 27]. Indeed, the construction in [26] shows that there exists a finite family $\mathcal{F}$ of 3-graphs for which $M(\mathcal{F})$ has size exactly two, the constructions in [27] show that for every positive integer t, there exists a finite family $\mathcal{F}$ of 3-graphs for which $M(\mathcal{F})$ has size exactly t, and the constructions in [15] show that there exists a finite family $\mathcal{F}$ of 3-graphs for which $M(\mathcal{F})$ is a nontrivial interval.

We show that $M(\mathcal{F})$ can be even more complicated than the examples above. More specifically, we show that there exists a finite family $\mathcal{F}$ of 3-graphs for which $M(\mathcal{F})$ is a **Cantor-type set** (i.e., is topologically homeomorphic to the standard Cantor set $\left\{\sum_{i=1}^{\infty} t_i 3^{-i}\colon \forall\, i\ t_i \in \{0,2\}\right\}$). For this, we need some further preliminaries.

Suppose that $P_i = (m_i, E_i, R_i)$ for $i \in I$ is a collection of patterns with the same Lagrangian, say λ for some $\lambda \in [0,1]$. Recall that $\mathcal{F}_\infty$ is defined by (1.4); thus $\mathcal{F}_\infty$-free graphs are exactly subgraphs of P_I-mixing constructions. It follows that $M(\mathcal{F}_\infty)$ can be equivalently described as the set of all points $x \in [0,1]$ such that there exists a sequence $(H_n)_{n=1}^{\infty}$ of r-graphs such that H_n is a P_I-mixing subconstruction for all $n \ge 1$ and

$$\lim_{n\to\infty} v(H_n) = \infty, \quad \lim_{n\to\infty} \rho(H_n) = \lambda, \quad \text{and} \quad \lim_{n\to\infty} \rho(\partial H_n) = x.$$

Using a standard diagonalization argument in analysis, we obtain the following observation.

Observation 1.4. The set $M(\mathcal{F}_\infty)$ is a closed subset of $[0,1]$.

Using Theorem 1.2 and some further arguments, we obtain the following result for 3-graphs.

Theorem 1.5. *Suppose that I is a finite set, $P_i = (m_i, E_i, R_i)$ is a minimal 3-graph pattern for each $i \in I$, and there exists $\lambda \in (0, 1)$ such that $\lambda_{P_i} = \lambda$ for all $i \in I$. Then there exists a finite family $\mathcal{F} \subseteq \mathcal{F}_\infty$ such that $M(\mathcal{F}) = M(\mathcal{F}_\infty)$.*

Later, we will give specific patterns P_1 and P_2 with the same Lagrangian such that, for $P_I = \{P_1, P_2\}$, the set $M(\mathcal{F}_\infty)$ is a Cantor-type set with Hausdorff dimension $\log 2/\log\left(4\sqrt{7} + 11\right) \approx 0.225641$. Thus, by Theorem 1.5, we obtain the following corollary.

Corollary 1.6. *There exists a finite family $\mathcal{F}$ of 3-graphs such that the set $M(\mathcal{F})$ is a Cantor-type set with Hausdorff dimension $\log 2/\log\left(4\sqrt{7} + 11\right)$.*

Organisation of the paper. The rest of the paper is organized as follows. In Section 2, we define some further notation and present two examples. Theorems 1.2, 1.3 and 1.5 are proved in Sections 3, 4 and 5, respectively. Corollary 1.6 is proved in Section 6. Some concluding remarks are contained in Section 7.

2. Notation

Let us introduce some further notation complementing and expanding that from the Introduction. Some other (infrequently used) definitions are given shortly before they are needed for the first time in this paper.

Let $\mathbb{N} := \{1, 2, \ldots\}$ be the set of all positive integers. Also, recall that $[m]$ denotes the set $\{1, \ldots, m\}$.

Recall that an r-**multiset** D is an unordered collection of r elements $x_1, \ldots, x_r$ with repetitions allowed. Let us denote this as $D = \{\!\{ x_1, \ldots, x_r \}\!\}$. The **multiplicity** $\mathbf{D(x)}$ of x in D is the number of times that x appears. If the underlying set is understood to be $[m]$, then we can represent D as the ordered m-tuple $(D(1), \ldots, D(m)) \in \{0, \ldots, r\}^m$ of multiplicities. Thus, for example, the profile of $X \subseteq V_1 \cup \cdots \cup V_m$ is the multiset on $[m]$ whose multiplicities are $(|X \cap V_1|, \ldots, |X \cap V_m|)$. Also, let $x^{(r)}$ denote the sequence consisting of r copies of x; thus, the multiset consisting of r copies of x is denoted by $\{\!\{ x^{(r)} \}\!\}$. If we need to emphasise that a multiset is in fact a set (that is, no element has multiplicity more than 1), we call it a **simple set**.

For $D \subseteq [m]$ and sets $U_1, \ldots, U_m$, denote $U_D := \bigcup_{i \in D} U_i$. Given a set X let $\binom{X}{r}$ and $\left(\!\binom{X}{r}\!\right)$ denote the collections of all r-subsets of X and all r-multisets of X, respectively.

2.1. *Hypergraphs*

Let G be an r-graph. The **complement** of G is $\overline{G} := \{E \subseteq V(G) : |E| = r,\ E \notin G\}$. For a vertex $v \in V(G)$, its **link** is the $(r-1)$-hypergraph

$$L_G(v) := \{E \subseteq V(G) : v \notin E,\ E \cup \{v\} \in G\}.$$

For $U \subseteq V(G)$, its **induced subgraph** is $G[U] := \{E \in G : E \subseteq U\}$. The vertex sets of $\overline{G}$, $L_G(v)$, and $G[U]$ are by default $V(G)$, $V(G) \setminus \{v\}$, and U, respectively. The **degree** of $v \in V(G)$ is $d_G(v) := |L_G(v)|$. Let $\Delta(G)$ and $\delta(G)$ denote respectively the maximum and minimum degrees of the r-graph G.

An **embedding** of an r-graph F into G is an injection $f : V(F) \to V(G)$ such that $f(E) \in G$ for every $E \in F$. Thus, F is a subgraph of G if and only if F admits an embedding into G. An embedding is **induced** if non-edges are mapped to non-edges.

The **edit distance** between two r-graphs G and H with the same number of vertices is the smallest number of **edits** (i.e., removal and addition of edges) that have to be applied to G to make it isomorphic to H; in other words, this is the minimum of $|G \triangle \sigma(H)|$ over all bijections $\sigma : V(H) \to V(G)$. We say that G and H are **s-close** if they are at edit distance at most s.

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2.2. Further definitions and results for a single pattern

In this section, we focus on a single pattern $P = (m, E, R)$. Let the **Lagrange polynomial** of E be

$$\lambda_E(x_1, \ldots, x_m) := r! \sum_{D \in E} \prod_{i=1}^{m} \frac{x_i^{D(i)}}{D(i)!}. \tag{2.1}$$

The special case of (2.1) when E is an r-graph (i.e., E consists of simple sets) gives the well-known **hypergraph Lagrangian** (see, for example, [1, 13]) that has been successfully applied to Turán-type problems, with the basic idea going back to Motzkin and Straus [28].

For $i \in [m]$, let the **link** $L_E(i)$ consist of all $(r-1)$-multisets A such that if we increase the multiplicity of i in A by one, then the obtained r-multiset belongs to E.

The following simple fact follows easily from the definitions.

Fact 2.1. The following statements hold for every r-graph pattern $P = (m, E, R)$.

(a) For every partition $[n] = V_1 \cup \cdots \cup V_m$, we have that

$$\lambda_E\left(\frac{|V_1|}{n}, \ldots, \frac{|V_m|}{n}\right) = \rho(E((V_1, \ldots, V_m))) + o(1), \qquad \text{as } n \to \infty. \tag{2.2}$$

(b) For every $i \in [m]$, we have $\frac{\partial \lambda_E}{\partial_i}(\mathbf{x}) = r \cdot \lambda_{L_E(i)}(\mathbf{x})$.

See also Lemma 2.4 that relates λ_E and λ_P.

We call a pattern P **proper** if it is minimal and $0 < \lambda_P < 1$. Trivially, every minimal pattern $P = (m, E, R)$ satisfies that

$$L_E(i) \neq \emptyset, \qquad \text{for every } i \in [m]. \tag{2.3}$$

Lemma 2.2 [30, Lemma 16]. *Let $P = (m, E, R)$ be a minimal pattern. If distinct $j, k \in [m]$ satisfy $L_E(j) \subseteq L_E(k)$, then $j \in R$, $k \notin R$, and $L_E(j) \neq L_E(k)$. In particular, no two vertices in $[m]$ have the same links in E.*

We will also need the following result from [30], which characterizes those patterns whose Lagrangian is 1.

Lemma 2.3 [30, Lemma 12]. *An r-graph pattern $P = (m, E, R)$ satisfies $\lambda_P = 1$ if and only if at least one of the following holds:*

(a) *there is $i \in [m]$ such that $\{\!\{ i^{(r)} \}\!\} \in E$, or*
(b) *there are $i \in R$ and $j \in [m] \setminus \{i\}$ such that $\{\!\{ i^{(r-1)}, j \}\!\} \in E$.*

The **standard $(\mathbf{m}-\mathbf{1})$-dimensional simplex** is

$$\mathbb{S}_m := \{\mathbf{x} \in \mathbb{R}^m : x_1 + \cdots + x_m = 1,\ \forall i \in [m]\ x_i \geq 0\}. \tag{2.4}$$

Also, let

$$\mathbb{S}_m^* := \{\mathbf{x} \in \mathbb{R}^m : x_1 + \cdots + x_m = 1,\ \forall i \in [m]\ 0 \leq x_i < 1\}$$

be obtained from the simplex $\mathbb{S}_m$ by excluding all its vertices (i.e., the standard basis vectors).

For a pattern $P = (m, E, R)$, we say that a vector $\mathbf{x} \in \mathbb{R}^m$ is **P-optimal** if $\mathbf{x} \in \mathbb{S}_m^*$ and

$$\lambda_E(\mathbf{x}) + \lambda_P \sum_{j \in R} x_j^r = \lambda_P. \tag{2.5}$$

Note that when we define P-optimal vectors we restrict ourselves to $\mathbb{S}_m^*$ (i.e., we do not allow any standard basis vector to be included).

We will need the following result from [30], which extends some classical results (see e.g. [13, Theorem 2.1]) about the Lagrangian of hypergraphs.

Lemma 2.4 ([30, Lemma 14]). *Let $P = (m, E, R)$ be a proper r-graph pattern and let*

$$f(\mathbf{x}) \coloneqq \lambda_E(\mathbf{x}) + \lambda_P \sum_{j\in R} x_j^r$$

be the left-hand side of (2.5). Let $\mathcal{X}$ be the set of all P-optimal vectors. Then the following statements hold.

(a) *The set $\mathcal{X}$ is nonempty.*
(b) *We have $f(\mathbf{x}) \le \lambda_P$ for all $\mathbf{x} \in \mathbb{S}_m$.*
(c) *The set $\mathcal{X}$ does not intersect the boundary of $\mathbb{S}_m$ (i.e., no vector in $\mathcal{X}$ has a zero coordinate).*
(d) *For every $\mathbf{x} \in \mathcal{X}$ and $j \in [m]$, we have $\frac{\partial f}{\partial_j}(\mathbf{x}) = r \cdot \lambda_P$.*
(e) *The set $\mathcal{X}$, as a subset of $\mathbb{S}_m$, is closed. (In particular, no sequence of P-optimal vectors can converge to a standard basis vector.)*
(f) *For every $\varepsilon > 0$, there is $\alpha > 0$ such that for every $\mathbf{y} \in \mathbb{S}_m$ with $\max(y_1, \ldots, y_m) \le 1 - \varepsilon$ and $f(\mathbf{y}) \ge \lambda_P - \alpha$, there is $\mathbf{x} \in \mathcal{X}$ with $\|\mathbf{x} - \mathbf{y}\|_\infty \le \varepsilon$.*
(g) *There is a constant $\beta > 0$ such that for every $\mathbf{x} \in \mathcal{X}$ and every $j \in [m]$, we have $x_j \ge \beta$.*

Observe that, in the above lemma, Parts (a) and (b) imply that $\mathcal{X}$ is precisely the set of elements in $\mathbb{S}_m^*$ that maximise $\lambda_E(\mathbf{x}) + \lambda_P \sum_{j\in R} x_j^r$. Also, note that (c) is a consequence of (g).

We will also need the following easy inequality.

Lemma 2.5. *Suppose that E is a collection of r-multisets on $[m]$ and $\mathbf{u}, \mathbf{x} \in \mathbb{S}_m$. Then for every $j \in [m]$, we have*

$$\left|\frac{\partial \lambda_E}{\partial_j}(\mathbf{u}) - \frac{\partial \lambda_E}{\partial_j}(\mathbf{x})\right| \le r^2 m \cdot \|\mathbf{u} - \mathbf{x}\|_\infty.$$

Proof. Fix $j_* \in [m]$. By Fact 2.1(b), we have $\frac{\partial \lambda_E}{\partial_{j_*}}(\mathbf{y}) = r \cdot \lambda_{L_E(j_*)}(\mathbf{y})$ for every $\mathbf{y} \in \mathbb{S}_m$. Let $\hat{E} \coloneqq L_E(j_*)$. Similarly, for every $i \in [m]$ and $\mathbf{y} \in \mathbb{S}_m$, we have $\frac{\partial \lambda_{\hat{E}}}{\partial_i}(\mathbf{y}) = (r-1) \cdot \lambda_{L_{\hat{E}}(i)}(\mathbf{y})$, and hence,

$$\begin{aligned}\left|\frac{\partial^2 \lambda_E}{\partial_i \partial_{j_*}}(\mathbf{y})\right| &= r \cdot \frac{\partial \lambda_{\hat{E}}}{\partial_i}(\mathbf{y}) = r(r-1) \cdot \lambda_{L_{\hat{E}}(i)}(\mathbf{y}) \\ &\le r^2 \sum_{S \in \left(\!\binom{[m]}{r-2}\!\right)} \prod_{i\in S} y_i = r^2 \left(\sum_{i=1}^m y_i\right)^{r-2} = r^2.\end{aligned}$$

Given $\mathbf{u}, \mathbf{x} \in \mathbb{S}_m$, it follows from the Mean Value Theorem that there exists $\mathbf{y} = \alpha\mathbf{u} + (1-\alpha)\mathbf{x} \in \mathbb{S}_m$ for some $\alpha \in [0, 1]$ such that

$$\begin{aligned}\left|\frac{\partial \lambda_E}{\partial_{j_*}}(\mathbf{u}) - \frac{\partial \lambda_E}{\partial_{j_*}}(\mathbf{x})\right| &= \left|\sum_{i\in[m]} \frac{\partial^2 \lambda_E}{\partial_i \partial_{j_*}}(\mathbf{y}) \cdot (u_i - x_i)\right| \le \max_{i\in[m]} \left\{\left|\frac{\partial^2 \lambda_E}{\partial_i \partial_{j_*}}(\mathbf{y})\right|\right\} \cdot \sum_{i=1}^m |u_i - x_i| \\ &\le r^2 \sum_{i=1}^m |u_i - x_i| \le r^2 m \cdot \|\mathbf{u} - \mathbf{x}\|_\infty.\end{aligned}$$

This proves Lemma 2.5. □

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2.3. Mixing patterns

Let (1.1) apply; that is, $P_I = \{P_i : i \in I\}$, where $P_i = (m_i, E_i, R_i)$ is an r-graph pattern for each $i \in I$.

First, to each P_I-mixing construction G, we will associate its **base value** $b(G)$. For this, we need to designate some special element not in I, say, $\emptyset \notin I$. If G has no edges, then we set $b(G) := \emptyset$; otherwise, let $b(G)$ be the element of I such that the first step when making the P_I-mixing construction G was to take a blowup of E_j. It may in principle be possible that some different choices of j lead to another representation of the same r-graph G as a P_I-mixing construction. We fix one choice for $b(G)$, using it consistently. We say that G has **base** $P_{b(G)}$ and call b the **base function**.

Next, to every $G \in \Sigma P_I$, we will associate a pair $(\mathbf{V}_G, \mathbf{T}_G)$ which encodes how the P_I-mixing construction G is built. In brief, the **tree** $\mathbf{T}_G$ of G records the information how the patterns are mixed while the **partition structure** $\mathbf{V}_G$ specifies which vertex partitions were used when making each blowup.

Let us formally define $\mathbf{V}_G$ and $\mathbf{T}_G$, also providing some related terminology. We start with $\mathbf{V}_G$ having only one set $V_\emptyset := V(G)$ and with $\mathbf{T}_G$ consisting of the single root $((), \emptyset)$. If G has no edges, then this finishes the definition of $(\mathbf{V}_G, \mathbf{T}_G)$. So suppose that $|G| > 0$. By the definition of $j := b(G)$, there exists a partition $V(G) = V_1 \cup \cdots \cup V_{m_j}$ with $V_i \neq V$ for $i \in R_j$ such that $G \setminus \left(\bigcup_{k \in R_j} G[V_k]\right) = E_j((V_1, \ldots, V_{m_j}))$. (Again, it may be possible that some different choices of the partition lead to another representation of $G \in \Sigma P_I$, so fix one choice and use it consistently.) This initial partition $V(G) = V_1 \cup \cdots \cup V_{m_j}$ is called the **level-1** partition and $V_1, \ldots, V_{m_j}$ are called the **level-1** (or **bottom**) parts. We add the parts $V_1, \ldots, V_{m_j}$ to $\mathbf{V}_G$ and add $((1), j), \ldots, ((m_j), j)$ as the children of the root $((), \emptyset)$ of $\mathbf{T}_G$. Note that we add all m_j children, even if some of the corresponding parts are empty. Next, for every $i \in R_j$ with $|G[V_i]| > 0$, we apply recursion to $G[V_i]$ to define the descendants of $((i), j)$ as follows. Let P_t be the base of $G[V_i]$ (that is, $t := b(G[V_i])$). Add $((i, 1), t), \ldots, (i, m_t), t)$ as the children of $((i), j)$ in $\mathbf{T}_G$ and add to $\mathbf{V}_G$ the sets $V_{i,1}, \ldots, V_{i,m_t}$ (called **level-2** parts) forming the partition of V_i that was used for the blowup of E_t. In other words, the set of level-2 parts of G is the union over $i \in R_j$ of level-1 parts of $G[V_i]$. We repeat this process. Namely, a node $(\mathbf{i}, j)$ in $\mathbf{T}_G$, where $\mathbf{i} = (i_1, \ldots, i_s)$, has children if and only if $|G[V_\mathbf{i}]| > 0$ (then necessarily, $i_s \in R_j$), in which case the children are $((\mathbf{i}, k), t)$ for $k \in [m_t]$, where $t := b(G[V_\mathbf{i}])$; then we also add to $\mathbf{V}_G$ the **level-$(\mathbf{s}+\mathbf{1})$** parts $V_{\mathbf{i},1}, \ldots, V_{\mathbf{i},m_t}$ which we used for the blowup of E_t. Note that we are slightly sloppy with our bracket notation, with $\mathbf{i}, k$ meaning $i_1, \ldots, i_s, k$ (and with $\emptyset$ sometimes meaning the empty sequence). Thus, level-$(s+1)$ parts of G can be defined as the level-s parts of $G[V_i]$ for all $i \in R_{b(G)}$. This finishes the definition of $(\mathbf{V}_G, \mathbf{T}_G)$. For some examples, see Section 2.4.

Clearly, the pair $(\mathbf{V}_G, \mathbf{T}_G)$ determines G (although this representation has some redundancies). We often work with just $\mathbf{V}_G$ (without explicitly referring to $\mathbf{T}_G$). When G is understood, we may write $\mathbf{V}$ for $\mathbf{V}_G$.

A sequence $(i_1, \ldots, i_s)$ is called **legal** (with respect to G) if $V_{i_1,\ldots,i_s}$ is defined during the above process. This includes the empty sequence, which is always legal. We view $\mathbf{V}_G$ and $\mathbf{T}_G$ as vertical (see, for example, Figure 3) with the index sequence growing as we go up in level. This motivates calling the level-1 parts **bottom** (even though we regard $V_\emptyset$ as level-0). By default, the profile of $X \subseteq V(G)$ is taken with respect to the bottom parts; that is, its multiplicities are $(|X \cap V_1|, \ldots, |X \cap V_m|)$. The **height** of G is the maximum length of a legal sequence (equivalently, the maximum number of edges in a directed path in the tree $\mathbf{T}_G$). Call a part $V_{i_1,\ldots,i_s}$ **recursive** if $s = 0$ or the (unique) node $((i_1, \ldots, i_s), t)$ of $\mathbf{T}_G$ satisfies $i_s \in R_t$. (Note that we do not require that $|G[V_{i_1,\ldots,i_s}]| > 0$ in this definition.) Otherwise, we call $V_{i_1,\ldots,i_s}$ **non-recursive**. The **branch** $\mathrm{br}_\mathbf{G}(\mathbf{v})$ of a vertex $v \in V(G)$ is the (unique) maximal sequence $\mathbf{i}$ such that $v \in V_\mathbf{i}$. If $b(G) = \emptyset$, then every branch is the empty sequence; otherwise, every branch is nonempty. Let $\mathbf{T}_G^{\text{level} \le \ell}$ be obtained from $\mathbf{T}_G$ by removing all nodes at level higher than ℓ.

Note that the values of the base function b are incorporated into the tree $\mathbf{T}_G$ by 'shifting' them one level up: namely, if a part $V_\mathbf{i}$ is recursive and $j := b(G[V_\mathbf{i}])$ is not $\emptyset$, then j appears as the second coordinate on all children of the unique node of $\mathbf{T}_G$ with the first coordinate $\mathbf{i}$. This is notationally convenient: for example, if G' is obtained from $G \in \Sigma P_I$ by deleting all edges inside some part $V_\mathbf{i}$ then $\mathbf{T}_{G'}$ is a **subtree** of $\mathbf{T}_G$; that is, it can be obtained from $\mathbf{T}_G$ by iteratively deleting leaves. (In fact, the

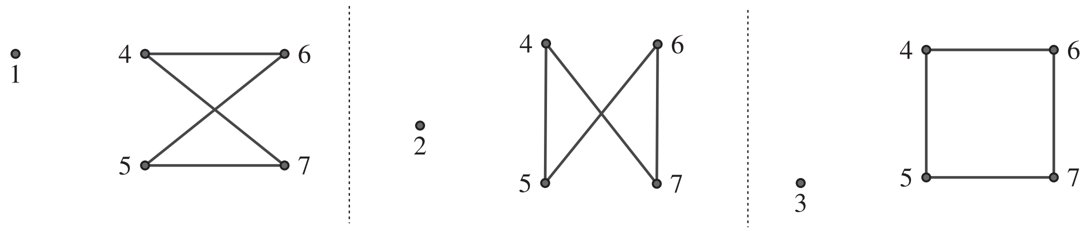

Figure 1. *The induced subgraph of $L_{B_{5,3}}(i)$ on vertex set $\{4,5,6,7\}$ is a copy of $K_{2,2}$ for $i \in \{1,2,3\}$.*

subtree $\mathbf{T}_{G'}$ is **full**, that is, every non-leaf node of $\mathbf{T}_{G'}$ has the same set of children in $\mathbf{T}_{G'}$ as in $\mathbf{T}_G$.)

We say that $\mathbf{T}$ is a **feasible tree** if there exists a P_I-mixing construction G with $\mathbf{T}_G = \mathbf{T}$. We say that a feasible tree $\mathbf{T}$ is **non-extendable** if it has positive height and every leaf $v = ((i_1, \ldots, i_s), t)$ of $\mathbf{T}$ satisfies $i_s \notin R_t$. Equivalently, $\mathbf{T}$ is non-extendable if it is not a subtree of a strictly larger feasible tree. Otherwise, we say that $\mathbf{T}$ is **extendable**.

Given a family $P_I = \{P_i : i \in I\}$ of r-graph patterns, recall that $\mathcal{F}_\infty$ consists of those r-graphs F that do not embed into a P_I-mixing construction. For an integer s, let $\mathcal{F}_s$ consist of all members of $\mathcal{F}_\infty$ with at most s vertices.

Note that we do not require here (nor in Theorem 1.2) that all patterns P_i have the same Lagrangian (although this will be the case in all concrete applications of Theorem 1.2 that we present in this paper). As we will see in Lemma 3.4, patterns P_i with $\lambda_{P_i} < \max\{\lambda_{P_j} : j \in I\}$ will not affect the largest asymptotic density of P_I-mixing constructions (however, they may appear in maximum constructions at the recursion level when we consider parts of bounded size).

2.4. Examples

In this section, we give some examples to illustrate some of the above definitions. The set P_I of the first example will be later used in our proof of Theorem 1.5.

Recall that K_5^3 is the complete 3-graph on 5 vertices (let us assume that its vertex set is $[5]$). Let $B_{5,3}$ be the 3-graph on 7 vertices (let us assume that its vertex set is $[7]$) whose edge set is the union of all triples that have at least two vertices in $\{1,2,3\}$ and all triples (with their ordering ignored) from the following set:

$$\Big(\{1\} \times \{4,5\} \times \{6,7\}\Big) \cup \Big(\{2\} \times \{4,6\} \times \{5,7\}\Big) \cup \Big(\{3\} \times \{4,7\} \times \{5,6\}\Big).$$

In particular, for $i \in \{1,2,3\}$, the induced subgraph of $L_{B_{5,3}}(i)$ on $\{4,5,6,7\}$ is a copy of $K_{2,2}$ (where $K_{m,n}$ denotes the complete bipartite graph with parts of size m and n), and their union covers each pair in $\{4,5,6,7\}$ exactly twice. See Figure 1 for an illustration.

The motivation for defining $B_{5,3}$ comes from the so-called **crossed blowup** defined in [15] so, in fact, $B_{5,3}$ is a 3-crossed blowup of K_5^3.

Let $P_1 := (5, K_5^3, \{1\})$ and $P_2 := (7, B_{5,3}, \{1\})$. Suppose that G is a $\{P_1, P_2\}$-mixing construction with exactly three levels: the base for G is P_1, the base for $G[V_1]$ is P_2, and the base for $G[V_{1,1}]$ is P_1 (see Figure 2). It is clear that every feasible tree (see, for example, Figure 3) is extendable since every pattern P_i has nonempty R_i.

For the second example, we take $P_1 = (3, \{123\}, \{1\})$ and $P_2 = (4, K_4^3, \emptyset)$. Let G be a $\{P_1, P_2\}$-mixing construction with three levels: the base for G is P_1, the base for $G[V_1]$ is P_1, and the base for $G[V_{1,1}]$ is P_2; see Figure 4. Note that the tree $\mathbf{T}_G$ of G (see Figure 5) of G is non-extendable, since every leaf in $\mathbf{T}_G$ is non-recursive.

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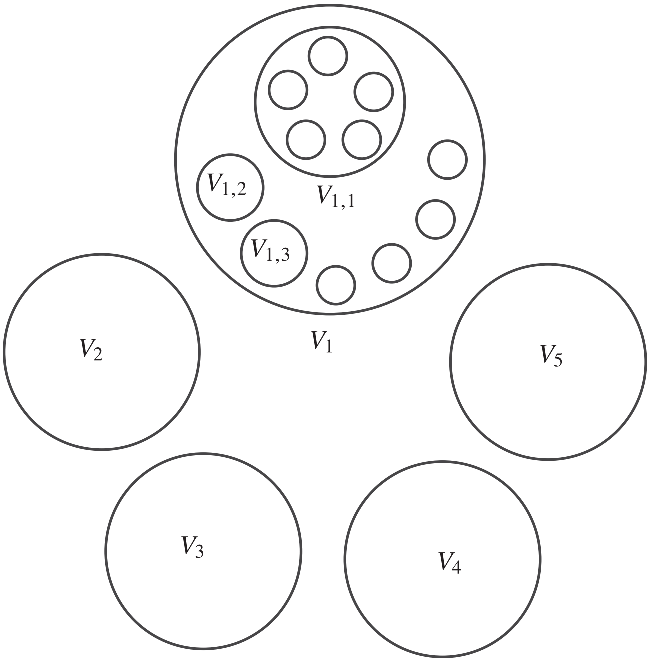

Figure 2. *The partition structure of a $\{P_1, P_2\}$-mixing construction G with exactly three levels: the base for level-1 is P_1, while the bases for the (unique) recursive parts at levels 2 and 3 are, respectively, P_2 and P_1.*

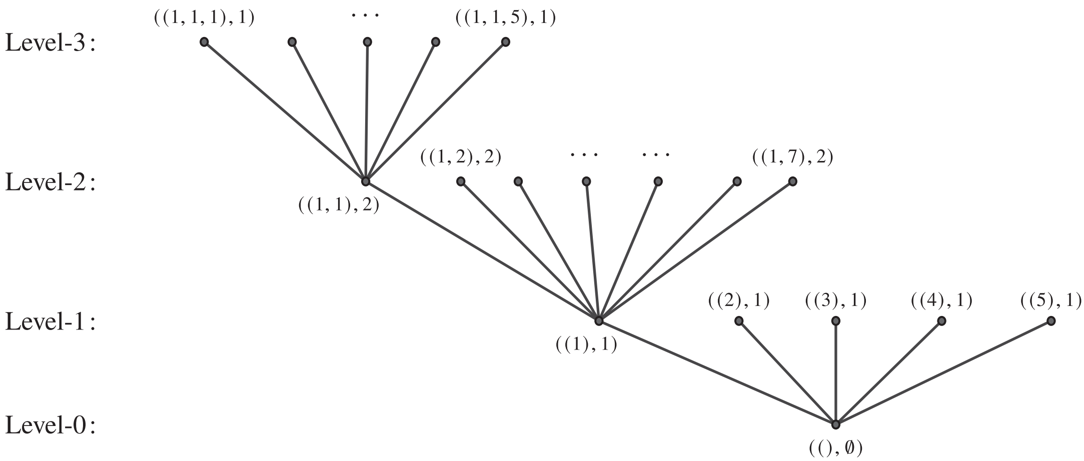

Figure 3. *The tree $\mathbf{T}_G$ of G.*

3. Proof of Theorem 1.2

The main idea of the proof of Theorem 1.2(a) is similar to the proof of Theorem 3 in [30]. The starting point is the easy observation (Lemma 3.2) that by forbidding $\mathcal{F}_\infty$, we restrict ourselves to subgraphs of P_I-mixing constructions; thus, Part (a) of Theorem 1.2 would trivially hold if infinite forbidden families were allowed. Our task is to show that, for some large M, the finite subfamily $\mathcal{F}_M$ of $\mathcal{F}_\infty$ still has the above properties. The Strong Removal Lemma of Rödl and Schacht [33] (stated as Lemma 3.17 here) implies that for every $\varepsilon > 0$, there is M such that every $\mathcal{F}_M$-free r-graph with $n \geq M$ vertices can be made $\mathcal{F}_\infty$-free by removing at most $\frac{c_0}{2}\binom{n}{r}$ edges. It follows that every maximum $\mathcal{F}_M$-free r-graph G on

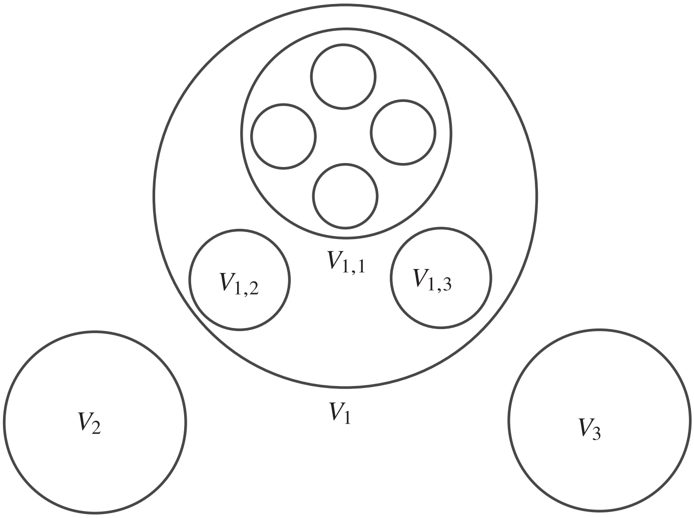

Figure 4. *The partition structure of a $\{P_1, P_2\}$-mixing construction G with exactly three levels: the base for level-1 is P_1, while the bases for the unique recursive parts at level 2 and 3 are P_1 and P_2, respectively.*

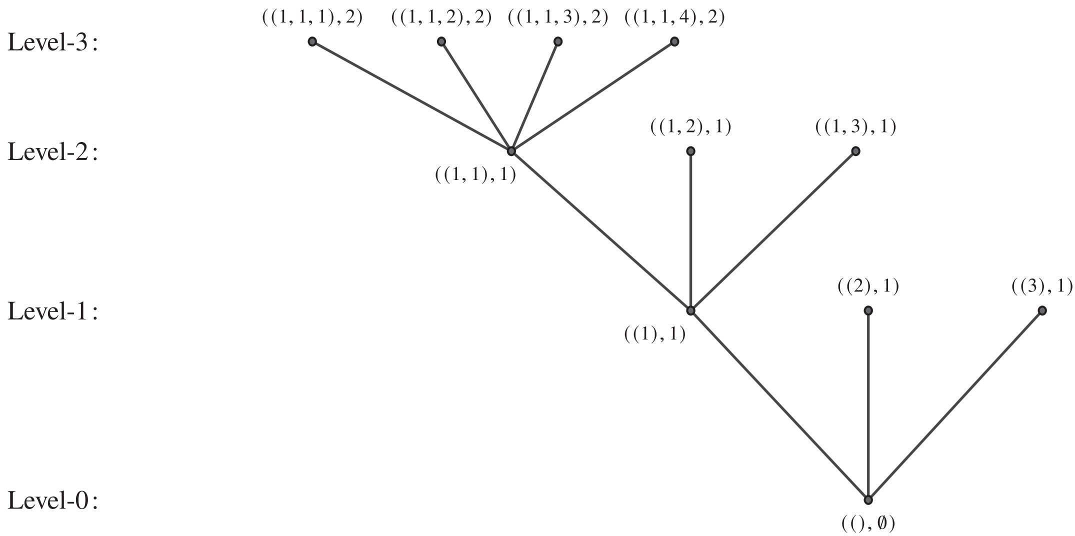

Figure 5. *The tree $\mathbf{T}_G$ of G.*

$[n]$ is $c_0\binom{n}{r}$-close in the edit distance to a P_I-mixing construction (see Lemma 3.18), where $c_0 > 0$ can be made arbitrarily small by choosing M large. Then our key Lemma 3.16 (which heavily relies on another important result, the existence of a 'rigid' $F \in \Sigma P_I$ as proved in Lemma 3.13) shows via stability-type arguments that some small constant $c_0 > 0$ (independent of n) suffices to ensure that there is a partition $V(G) = V_1 \cup \cdots \cup V_{m_i}$ for some $i \in I$ such that $G \setminus (\bigcup_{j \in R_i} G[V_j]) = E((V_1, \ldots, V_{m_i}))$; that is, G follows exactly the bottom level of some P_i-construction (but nothing is stipulated about what happens inside the recursive parts V_j). The maximality of G implies that each $G[V_j]$ with $j \in R_i$ is maximum $\mathcal{F}_M$-free (see Lemma 3.6), allowing us to apply induction.

Part (b) of Theorem 1.2 (which has no direct analogue in [30]) is needed in those applications where we have to analyse almost extremal constructions. It does not directly follow from Lemma 3.17 (i.e., from the Removal Lemma), since the same constant M in Theorem 1.2(b) has to work for **every** $\varepsilon > 0$. Similarly to Part (a), the key idea here that, once we forced our $\mathcal{F}_M$-free graph G on $[n]$ to be $c_0\binom{n}{r}$-close to a P_I-mixing construction for some sufficiently small $c_0 > 0$ (but independent of ε), then we can further bootstrap this to the required $\varepsilon\binom{n}{r}$-closeness by stability-type arguments.

Many lemmas that are needed for our proof are borrowed from [30], verbatim or with some minor modifications. However, new challenges arise to accommodate our situation $|I| \geq 2$ and some new ideas are required here.

3.1. Some additional assumptions and definitions related to P_I

Recall that I is finite, $P_I = \{P_i : i \in I\}$ with each pattern $P_i = (m_i, E_i, R_i)$ for $i \in I$ being minimal. Let us define

$$\lambda := \max\{\lambda_{P_i} : i \in I\} \qquad \text{and} \qquad I' := \{i \in I : \lambda_{P_i} = \lambda\}. \tag{3.1}$$

We can assume that $\lambda > 0$: if $\lambda = 0$, then every P_I-mixing construction is edgeless, and we can satisfy Theorem 1.2 by letting $M = r$, with $\mathcal{F}_M = \{K_r^r\}$ consisting of a single edge.

Furthermore, we can assume that $\lambda_{P_i} > 0$ for every $i \in I$ by removing all patterns with zero Lagrangian. (This does not change the family of all ΣP_i-mixing constructions, since $E_i = \emptyset$ for every $i \in I$ with $\lambda_{P_i} = 0$ and we always have the option to put the empty r-graph into a part.)

Note that if $\lambda = 1$ (i.e., $\lambda_{P_t} = 1$ for some $t \in I$), then every complete r-graph is a P_t-construction by Lemma 2.3. Indeed, for example, if the second statement of Lemma 2.3 holds – that is, $\{\!\{ i^{(r-1)}, j \}\!\} \in E_t$ for some $i \in R_t$ – then this is attained by making all partitions to consist of only two nonempty parts, the j-th part consisting of a single vertex and the i-th (recursive) part being the rest. So, if $\lambda = 1$, then $\Lambda_{P_I}(n) = \binom{n}{r}$, and we can satisfy Theorem 1.2 by letting $M = 0$ with $\mathcal{F}_M = \emptyset$ being the empty forbidden family. Therefore, let us assume that every P_i, in addition to being minimal, is also proper (that is, that $0 < \lambda_{P_i} < 1$ for each $i \in I$).

We can additionally assume that, for all distinct $i, j \in [m]$, the patterns P_i and P_j are **non-isomorphic** (that is, there is no bijection $f : [m_i] \to [m_j]$ with $f(R_i) = R_j$ and $f(E_i) = E_j$), by just keeping one representative from each isomorphism class.

Also, let us state here some definitions that apply in Section 3 (that is, for the rest of the proof of Theorem 1.2). Since the set I is finite, we can fix a constant $\beta > 0$ that satisfies Part (g) of Lemma 2.4 for each pattern P_i with $i \in I$. Also, for $i \in I$, let us define

$$f_i(\mathbf{x}) := \lambda_{E_i}(\mathbf{x}) + \lambda_{P_i} \sum_{j \in R_i} x_j^r, \qquad \mathbf{x} \in \mathbb{S}_{m_i}, \tag{3.2}$$

and let $\mathcal{X}_i \subseteq \mathbb{S}_{m_i}^*$ be the set of P_i-optimal vectors. Similar to Fact 2.1(a), for every P_I-mixing construction G on $[n]$ with base P_i and bottom partition $[n] = V_1 \cup \cdots \cup V_{m_i}$, we have

$$\lambda \geq f_i\big(|V_1|/n, \ldots, |V_{m_i}|/n\big) = \rho(G) + o(1). \tag{3.3}$$

3.2. Basic properties of P_I-mixing constructions

In this section, we present some simple properties of P_I-mixing constructions.

The following two lemmas follow easily from the definitions. We refer the reader to [30, Lemmas 6 and 7], where the proofs for Lemmas 3.1 and 3.2 are provided for P-constructions.

Lemma 3.1. *Take any $G \in \Sigma P_I$. Then*

(a) *every induced subgraph of G is contained in ΣP_I, and*
(b) *every blowup of G is a P_I-mixing subconstruction (that is, a subgraph in some element of ΣP_I).*

Lemma 3.2. *The following statements are equivalent for an arbitrary r-graph G on at most n vertices:*

(a) *G is $\mathcal{F}_n$-free;*
(b) *G is $\mathcal{F}_\infty$-free;*
(c) *G is a P_I-mixing subconstruction.*

It follows from Lemma 3.2 that $\mathrm{ex}(n, \mathcal{F}_n) = \mathrm{ex}(n, \mathcal{F}_\infty) = \Lambda_{P_I}(n)$.

Lemma 3.3. *For all $t \ge s \ge r$, it holds that $\Lambda_{P_I}(s)/\binom{s}{r} \ge \Lambda_{P_I}(t)/\binom{t}{r}$.*

Proof. Take a maximum P_I-mixing construction G on $[t]$. By Lemma 3.1(a), each s-set spans at most $\Lambda_{P_I}(s)$ edges. However, the average number of the edges spanned by a uniformly random s-subset of $[t]$ is $\Lambda_{P_I}(t)\binom{t-r}{s-r}/\binom{t}{s}$, giving the required. □

The following result states that the maximum asymptotic edge density of a P_I-mixing construction is the same as the largest one attained by a single pattern P_i.

Lemma 3.4. *We have $\lim_{n\to\infty} \Lambda_{P_I}(n)/\binom{n}{r} = \lambda$.*

Proof. By fixing some i in $I' \neq \emptyset$, we trivially have that $\Lambda_{P_I}(n) \ge \Lambda_{P_i}(n)$ for each n. This implies that

$$\lim_{n\to\infty} \Lambda_{P_I}(n)/\binom{n}{r} \ge \lim_{n\to\infty} \Lambda_{P_i}(n)/\binom{n}{r} = \lambda.$$

Let us show the converse inequality. Let $\Sigma_h P_I$ consist of all P_I-mixing constructions whose partition structure has height at most h. For example, $\Sigma_1 P_I$ consists of all possible blowups of E_i, for $i \in I$, without putting any edges into their recursive parts.

Let us show by induction on $h \ge 1$ that $\lambda_h \le \lambda$, where we let

$$\lambda_h := \lim_{n\to\infty} \frac{\max\{|G|\colon v(G) = n \text{ and } G \in \Sigma_h P_I\}}{\binom{n}{r}}.$$

(The limit exists since the ratios are non-increasing in n by the same argument as in Lemma 3.3.)

If $h = 1$, then every r-graph in $\Sigma_h P_I$ is a P_i-construction for some $i \in I$ and has asymptotic density at most $\max\{\lambda_{P_i} : i \in I\} = \lambda$, as desired. Let $h \ge 2$. Take any maximum $G_n \in \Sigma_h P_I$ of order $n \to \infty$. By passing to a subsequence assume that there is $i \in I$ such that each G_n has base pattern P_i. Let $V(G_n) = V_{n,1} \cup \cdots \cup V_{n,m_i}$ be the base partition of G_n. Let $x_{n,j} := |V_{n,j}|/n$ for $j \in [m_i]$. By passing to a subsequence again, assume that $x_{n,j}$ tends to some limit x_j for each $j \in [m_i]$. Then it follows from the definition of $\Sigma_h P_I$, continuity of λ_{E_i}, and induction that

$$\begin{aligned}\rho(G_n) &\le \lambda_{E_i}(x_{n,1}, \ldots, x_{n,m_i}) + \lambda_{h-1} \sum_{j\in R_i} x_{n,j}^r + o(1)\\ &\le \lambda_{E_i}(x_1, \ldots, x_{m_i}) + \lambda \sum_{j\in R_i} x_j^r + o(1).\end{aligned}$$

Furthermore, by Part (b) of Lemma 2.4 and by $\lambda_{P_i} \le \lambda$, we have

$$\lambda_{E_i}(x_1, \ldots, x_{m_i}) \le \lambda_{P_i}\left(1 - \sum_{j\in R_i} x_j^r\right) \le \lambda\left(1 - \sum_{j\in R_i} x_j^r\right).$$

By putting these together, we conclude that $\rho(G_n) \le \lambda + o(1)$, giving the required. This finishes the proof of the lemma. □

Given a P_I-mixing construction G and a vertex v in G, a newly added vertex v' is called a **clone** of v if the link of v' in the new r-graph is identical to the link of v in G. Note that adding a clone of a vertex to a P_I-mixing construction results a P_I-mixing subconstruction.

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Lemma 3.5. *For every $s \in \mathbb{N} \cup \{\infty\}$, if an r-graph G is $\mathcal{F}_s$-free, then every blowup of G is $\mathcal{F}_s$-free.*

Proof. By the definition of blowup, it suffices to show that cloning a vertex of G will not create a copy of any member in $\mathcal{F}_s$. So, let G' be obtained from G by adding a clone x' of some vertex x of G. Take any $U \subseteq V(G')$ with $|U| \le s$. If at least one of x and x' is not in U, then $G'[U]$ is isomorphic to a subgraph of G and cannot be in $\mathcal{F}_s$; so suppose otherwise. Since G is $\mathcal{F}_s$-free, we have that $G'[U \setminus \{x'\}] = G[U \setminus \{x'\}]$ is a P_I-mixing subconstruction. By Lemma 3.1(b), $G'[U]$ is a P_I-mixing subconstruction. So it follows that G' is $\mathcal{F}_s$-free. □

Lemma 3.6. *Let $s \in \mathbb{N} \cup \{\infty\}$ and $i \in I$. Let G be an r-graph on $V = V_1 \cup \cdots \cup V_{m_i}$ obtained by taking $E((V_1, \ldots, V_{m_i}))$ and putting arbitrary $\mathcal{F}_s$-free r-graphs into parts V_j for each $j \in R_i$. Then G is $\mathcal{F}_s$-free.*

Proof. Take an arbitrary $U \subseteq V(G)$ with $|U| \le s$. Let $U_k := V_k \cap U$ for all $k \in [m_i]$. Notice that $G[U_k]$ has no edges for $k \in [m_i] \setminus R_i$. However, for every $k \in R_i$ we have that $G[U_k]$ is a P_I-mixing subconstruction, since $|U_k| \le s$ and $G[U_k] \subseteq G[V_k]$ is $\mathcal{F}_s$-free. By combining the partition structure of each $G[U_k]$ together with the level-1 decomposition $U = U_1 \cup \cdots \cup U_{m_i}$, we see that $G[U]$ is a P_I-mixing subconstruction. Therefore, G is $\mathcal{F}_s$-free. □

The following lemma says that the minimum degree of a maximum $\mathcal{F}_s$-free r-graph is close to its average degree.

Lemma 3.7. *For every $\varepsilon > 0$, there is n_0 such that for every $s \in \mathbb{N} \cup \{\infty\}$, every maximum $\mathcal{F}_s$-free r-graph G with $n \ge n_0$ vertices has minimum degree at least $(\lambda - \varepsilon)\binom{n-1}{r-1}$.*

Proof. Fix $s \in \mathbb{N} \cup \{\infty\}$. The difference between any two vertex degrees in a maximum $\mathcal{F}_s$-free r-graph G on $n \to \infty$ vertices is at most $\binom{n-2}{r-2}$, as otherwise by deleting one vertex and cloning the other we can strictly increase the number of edges, while the r-graph remains $\mathcal{F}_s$-free by Lemma 3.5, a contradiction. Thus, each degree is within $O(n^{r-2})$ of the average degree which in turn is at least $(\lambda + o(1))\binom{n-1}{r-1}$ by $\rho(G) + o(1) = \pi(\mathcal{F}_s) \ge \pi(\mathcal{F}_\infty) = \lambda$ (the last equality follows from Lemma 3.2). □

3.3. Properties of proper patterns

Recall that all assumptions made in Section 3.1 continue to apply to the fixed family P_I; in particular, we have $0 < \lambda < 1$. The following lemma shows that no bottom part of a P_I-mixing construction G can contain almost all vertices if G has large minimum degree.

Lemma 3.8. *For every $c' > 0$, there is n_0 such that for every $c > c'$ and every r-graph $G \in \Sigma P_I$ on $n \ge n_0$ vertices with $\delta(G) \ge c\binom{n-1}{r-1}$, each bottom part V_j of G has at most $(1 - c/r)n$ vertices.*

Proof. Given $c' > 0$, let $n \to \infty$ and take any c and G as in the lemma. Let the base of G be P_i. If $j \in [m_i] \setminus R_i$, then it follows from $\delta(G)n/r \le |G| \le \binom{n}{r} - \binom{|V_j|}{r}$ that

$$\left(\frac{|V_j| - r}{n}\right)^r \le \frac{\binom{|V_j|}{r}}{\binom{n}{r}} \le 1 - c.$$

Simplifying this inequality, we obtain

$$|V_j| \le (1-c)^{1/r} n + r \le \left(1 - \frac{c}{r} - \frac{(r-1)c^2}{2r^2}\right)n + r \le \left(1 - \frac{c}{r}\right)n,$$

as desired. Here, we used the inequality $(1-x)^{1/r} \le 1 - \frac{x}{r} - \frac{(r-1)x^2}{2r^2}$ for all $r \ge 1$ and $x \in [0, 1]$.

Now suppose that $j \in R_i$. Since $V_j \ne V(G)$, pick any vertex $v \in V_k$ with $k \ne j$. Since $\{\{ j^{(r-1)}, k \}\} \notin E_i$ by Lemma 2.3, every edge through v contains at least one other vertex outside of V_j. Thus, $c\binom{n-1}{r-1} \le d_G(v) \le (n - |V_j|)\binom{n-2}{r-2}$, implying $|V_j| \le (1 - c/(r-1) + o(1))n \le (1 - c/r)n$. □

Informally speaking, the following lemma (which is a routine generalization of [30, Lemma 15]) implies, among other things, that all part ratios of bounded height in a P_I-mixing construction with large minimum degree approximately follow some optimal vectors. In particular, for each $i \in I'$, the set $\mathcal{X}_i$ consists precisely of optimal limiting bottom ratios that lead to asymptotically maximum P_I-mixing constructions with base pattern P_i. Recall that $\beta > 0$ is the constant that satisfies Part (g) of Lemma 2.4 for every $i \in I$ while $\mathcal{X}_i$ is the set of P_i-optimal vectors.

Lemma 3.9. *For every $\varepsilon > 0$ and $\ell \in \mathbb{N}$, there are constants $\alpha_0 < \varepsilon_0 < \cdots < \alpha_\ell < \varepsilon_\ell < \alpha_{\ell+1}$ in $(0, \varepsilon)$ such that the following holds for all sufficiently large n. Let G be an arbitrary P_I-mixing construction on n vertices with the partition structure $\mathbf{V}$ such that $\delta(G) \geq (\lambda - \alpha_0)\binom{n-1}{r-1}$. Suppose that $\mathbf{i} = (i_1, \ldots, i_s)$ is a legal sequence of length s with $0 \leq s \leq \ell$, and the induced subgraph $G[V_{\mathbf{i}}]$ has base P_j for some $j \in I$. Let $\mathbf{v}_{\mathbf{i}} := (|V_{\mathbf{i},1}|/|V_{\mathbf{i}}|, \ldots, |V_{\mathbf{i},m_j}|/|V_{\mathbf{i}}|)$, where $V_{\mathbf{i}} = V_{\mathbf{i},1} \cup \cdots \cup V_{\mathbf{i},m_j}$ is the bottom partition of $G[V_{\mathbf{i}}]$. Then:*

(a) *$j \in I'$ (that is, $\lambda_{P_j} = \lambda$);*
(b) *$\|\mathbf{v}_{\mathbf{i}} - \mathbf{x}\|_\infty \leq \varepsilon_s$ for some $\mathbf{x} \in \mathcal{X}_j$;*
(c) *$|V_{\mathbf{i},k}| \geq \left(\frac{\beta}{2}\right)^{s+1} n$ and $|V_{\mathbf{i},k}| \leq \left(1 - \frac{\lambda}{2r}\right)^{s+1} n$ for all $k \in [m_j]$;*
(d) *$\delta(G[V_{\mathbf{i},k}]) \geq (\lambda - \alpha_{s+1})\binom{|V_{\mathbf{i},k}|-1}{r-1}$ for all $k \in R_j$.*

Proof. We choose positive constants in this order:

$$\alpha_{\ell+1} \gg \varepsilon_\ell \gg \alpha_\ell \gg \cdots \gg \varepsilon_0 \gg \alpha_0,$$

each being sufficiently small depending on P_I, ε, β, and the previous constants. We use induction on $s = 0, 1, \ldots, \ell$, with the induction step also working for the base case $s = 0$, in which $\mathbf{i}$ is the empty sequence. Take any $s \geq 0$ and suppose that the lemma holds for all smaller values of s.

Let n be large and let G and $\mathbf{i}$ be as in the lemma. Let $U_k := V_{\mathbf{i},k}$ for $k \in [m_j]$ and $U := V_{\mathbf{i}}$. Thus, $U = U_1 \cup \cdots \cup U_{m_j}$, and $\mathbf{v}_{\mathbf{i}} = (|U_1|/|U|, \ldots, |U_{m_j}|/|U|)$. Note that

$$\delta(G[U]) \geq (\lambda - \alpha_s)\binom{|U| - 1}{r - 1}, \tag{3.4}$$

which holds by Part (d) of the inductive assumption applied to G if $s > 0$, and by the assumption of the lemma if $s = 0$ (when $U = V_\emptyset = V(G)$). Thus, we have that

$$|G[U]| \geq \delta(G[U])|U|/r \geq (\lambda - \alpha_s)\binom{|U|}{r}. \tag{3.5}$$

Note that $|U|$ can be made arbitrarily large by choosing n large. Indeed, this follows from the induction assumption for $s - 1$ (namely, Part (c)) applied to G if $s \geq 1$, while $U = V(G)$ has n elements if $s = 0$.

We claim that $\lambda_{P_j} = \lambda$. Suppose on the contrary that the base P_j of $G[U]$ satisfies $j \in I \setminus I'$. Using the asymptotic notation with respect to $|U| \to \infty$, we have by Lemma 2.4(b) that

$$\begin{aligned}\rho(G[U]) &\leq \lambda_{E_j}(\mathbf{v}_{\mathbf{i}}) + \lambda \sum_{i \in R_j} \mathbf{v}_{\mathbf{i},i}^r + o(1) \\ &= f_j(\mathbf{v}_{\mathbf{i}}) + (\lambda - \lambda_{P_j}) \sum_{i \in R_j} \mathbf{v}_{\mathbf{i},i}^r + o(1) \\ &\leq \lambda_{P_j} + (\lambda - \lambda_{P_j}) \sum_{i \in R_j} \mathbf{v}_{\mathbf{i},i}^r + o(1).\end{aligned}$$

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This is strictly smaller than $\lambda - \alpha_s$ since, by (3.4) and Lemma 3.8, each $\mathbf{v}_{\mathbf{i},i}$ is at most, say, $1 - \lambda/2r$, and thus,

$$\sum_{i \in R_j} \mathbf{v}_{\mathbf{i},i}^r \le \left(1 - \frac{\lambda}{2r}\right) \cdot \sum_{i \in R_j} \mathbf{v}_{\mathbf{i},i}^{r-1} \le \left(1 - \frac{\lambda}{2r}\right) \cdot \left(\sum_{i \in R_j} \mathbf{v}_{\mathbf{i},i}\right)^{r-1} \le 1 - \frac{\lambda}{2r}$$

is bounded away from 1. This contradiction shows that $j \in I'$, proving Part (a).

Since $\alpha_s \ll \varepsilon_s$, Equation (3.5), Lemma 3.8 and Part (f) of Lemma 2.4 (when applied to $P = P_j$ and $\mathbf{y} = \mathbf{v}_\mathbf{i}$) give the desired P_j-optimal vector $\mathbf{x}$, proving Part (b). Fix one such $\mathbf{x} = (x_1, \ldots, x_{m_j})$.

For all $k \in [m_j]$, we have $|U_k| \ge (x_k - \varepsilon_s)|U| \ge (\beta/2)|U|$. This is at least $(\beta/2)^{s+1} n$ by the inductive assumption on $|U|$ if $s \ge 1$ (and is trivial if $s = 0$). Likewise, $|U| \le \left(1 - \frac{\lambda}{2r}\right)^s n$. Therefore, it follows from Lemma 3.8 (with $c' := \lambda - \alpha_s$) and (3.4) that

$$|U_k| \le \left(1 - \frac{\lambda}{2r}\right)|U| \le \left(1 - \frac{\lambda}{2r}\right)^{s+1} n.$$

This proves Part (c).

Finally, take arbitrary $k \in R_j$ and $y \in U_k$. By the definition of Lagrange polynomials and Fact 2.1(b), the degree of y in $E_j((U_1, \ldots, U_{m_j}))$ is

$$d_{E_j((U_1,\ldots,U_{m_j}))}(y) = \left(\lambda_{L_{E_j}(k)}(\mathbf{v}_\mathbf{i}) + o(1)\right)\binom{|U|-1}{r-1} = \left(\frac{1}{r} \cdot \frac{\partial \lambda_{E_j}}{\partial_k}(\mathbf{v}_\mathbf{i}) + o(1)\right)\binom{|U|-1}{r-1}.$$

Since $\|\mathbf{v}_\mathbf{i} - \mathbf{x}\|_\infty \le \varepsilon_s \ll \alpha_{s+1}$, we have by Part (d) of Lemma 2.4 and Lemma 2.5 that, for example,

$$\frac{\partial \lambda_{E_j}}{\partial_k}(\mathbf{v}_\mathbf{i}) - \alpha_{s+1}^2 \le \frac{\partial \lambda_{E_j}}{\partial_k}(\mathbf{x}) = \frac{\partial f}{\partial_k}(\mathbf{x}) - r\lambda x_k^{r-1} = r\lambda - r\lambda x_k^{r-1}.$$

Combining this with the previous equality, we obtain

$$d_{E_j((U_1,\ldots,U_{m_j}))}(y) \le \frac{1}{r}\left(r\lambda - r\lambda x_k^{r-1} + \alpha_{s+1}^2 + o(1)\right)\binom{|U|-1}{r-1} \le \left(\lambda - \lambda x_k^{r-1} + \frac{2\alpha_{s+1}^2}{r}\right)\binom{|U|-1}{r-1}.$$

Thus, by (3.4) and the fact $\|\mathbf{v}_\mathbf{i} - \mathbf{x}\|_\infty \le \varepsilon_s$, we have

$$\begin{aligned} d_{G[U_k]}(y) &= d_{G[U]}(y) - d_{E_j((U_1,\ldots,U_{m_j}))}(y) \\ &\ge \left((\lambda - \alpha_s) - \left(\lambda - \lambda x_k^{r-1} + \frac{2\alpha_{s+1}^2}{r}\right)\right)\binom{|U|-1}{r-1} \\ &= \left(\lambda x_k^{r-1} - \alpha_s - \frac{2\alpha_{s+1}^2}{r}\right)\binom{|U|-1}{r-1} \\ &\ge \left(\lambda(\mathbf{v}_{\mathbf{i},k} - \varepsilon_s)^{r-1} - \alpha_s - \frac{2\alpha_{s+1}^2}{r}\right)\binom{|U|-1}{r-1} \\ &\ge \left(\lambda \mathbf{v}_{\mathbf{i},k}^{r-1} - \lambda(r-1)\mathbf{v}_{\mathbf{i},k}^{r-2}\varepsilon_s - \alpha_s - \frac{2\alpha_{s+1}^2}{r}\right)\binom{|U|-1}{r-1}. \end{aligned}$$

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Since $\mathbf{v}_{\mathbf{i},k} \gg \alpha_{s+1} \gg \varepsilon_s \gg \alpha_s$, we have $\lambda(r-1)\mathbf{v}_{\mathbf{i},k}^{r-2}\varepsilon_s + \alpha_s + \frac{2\alpha_{s+1}^2}{r} \le \alpha_{s+1}\mathbf{v}_{\mathbf{i},k}^{r-1}$. Therefore, the above inequality on $d_{G[U_k]}(y)$ continues as

$$d_{G[U_k]}(y) \ge \left(\lambda \mathbf{v}_{\mathbf{i},k}^{r-1} - \alpha_{s+1}\mathbf{v}_{\mathbf{i},k}^{r-1}\right)\binom{|U|-1}{r-1} \ge (\lambda - \alpha_{s+1})\binom{\mathbf{v}_{\mathbf{i},k}|U|-1}{r-1} = (\lambda - \alpha_{s+1})\binom{|U_k|-1}{r-1}.$$

This finishes the proof of Part (d). □

For every P_I-mixing construction G with base P_i, the following lemma gives a bound for the degree of a vertex in G in terms of the distance between the vector of the part ratios and the set $\mathcal{X}_i$.

Lemma 3.10. *Let $G \in \Sigma P_I$ be an r-graph on n vertices with base P_i and bottom partition $V := V(G) = V_1 \cup \cdots \cup V_{m_i}$ for some $i \in I'$. Let $\mathbf{u} := \left(|V_1|/|V|, \ldots, |V_{m_i}|/|V|\right)$ and $\mathbf{x} \in \mathcal{X}_i$. Then for every $j \in [m_i]$ and for every $v \in V_j$ we have*

$$d_G(v) \le \begin{cases} \left(\lambda - \lambda x_j^{r-1} + r m_i \cdot \|\mathbf{u}-\mathbf{x}\|_\infty + o(1)\right)\binom{n-1}{r-1} + d_{G[V_j]}(v), & \text{if } j \in R_i,\\ \left(\lambda + r m_i \cdot \|\mathbf{u}-\mathbf{x}\|_\infty + o(1)\right)\binom{n-1}{r-1}, & \text{if } j \in [m_i]\setminus R_i. \end{cases}$$

Proof. First, it is easy to see from the definition of ΣP_I that

$$d_G(v) = d_{E_i((V_1,\cdots,V_{m_i}))}(v) + d_{G[V_j]}(v) = \left(\frac{1}{r}\cdot\frac{\partial \lambda_{E_i}}{\partial_j}(\mathbf{u}) + o(1)\right)\binom{n-1}{r-1} + d_{G[V_j]}(v).$$

So it follows from Lemma 2.5 that

$$d_G(v) \le \left(\frac{1}{r}\cdot\frac{\partial \lambda_{E_i}}{\partial_j}(\mathbf{x}) + r m_i \cdot \|\mathbf{u}-\mathbf{x}\|_\infty + o(1)\right)\binom{n-1}{r-1} + d_{G[V_j]}(v).$$

If $j \in R_i$, then it follows from Lemma 2.4(d) that

$$\frac{\partial \lambda_{E_i}}{\partial_j}(\mathbf{x}) = \frac{\partial\left(\lambda_{E_i} + \lambda\sum_{k\in R_i} x_k^r\right)}{\partial_j}(\mathbf{x}) - \frac{\partial\left(\lambda\sum_{k\in R_i} x_k^r\right)}{\partial_j}(\mathbf{x}) = r\lambda - r\lambda x_j^{r-1},$$

and hence,

$$d_G(v) \le \left(\lambda - \lambda x_j^{r-1} + r m_i \cdot \|\mathbf{u}-\mathbf{x}\|_\infty + o(1)\right)\binom{n-1}{r-1} + d_{G[V_j]}(v).$$

If $j \in [m_i]\setminus R_i$, then $d_{G[V_j]}(u) = 0$ and we have by Lemma 2.4(d) that $\partial\lambda_{E_i}(\mathbf{x})/\partial_j = r\lambda$, and hence,

$$d_G(v) \le \left(\frac{1}{r}\cdot r\lambda + r m_i \cdot \|\mathbf{u}-\mathbf{x}\|_\infty + o(1)\right)\binom{n-1}{r-1} = \left(\lambda + r m_i \cdot \|\mathbf{u}-\mathbf{x}\|_\infty + o(1)\right)\binom{n-1}{r-1}.$$

This proves Lemma 3.10. □

The next lemma shows that every P_I-mixing subconstruction with large minimum degree is almost regular.

Lemma 3.11. *For every $\varepsilon > 0$, there exist $\delta > 0$ and n_0 such that every P_I-mixing subconstruction G on $n \ge n_0$ vertices with $\delta(G) \ge (\lambda-\delta)\binom{n-1}{r-1}$ satisfies $\Delta(G) \le (\lambda+\varepsilon)\binom{n-1}{r-1}$.*

Proof. Fix any $\varepsilon > 0$. Choose a large constant $\ell \in \mathbb{N}$. Let $\alpha_0 < \varepsilon_0 < \cdots < \alpha_\ell < \varepsilon_\ell < \alpha_{\ell+1}$ in $(0, \varepsilon)$ be the constants given by Lemma 3.9, where ℓ and ε corresponds to the same constants in both lemmas. Next, choose sufficiently small constants $\delta \gg 1/n_0$. Let us show that they satisfy the lemma. So pick any G as in the lemma.

By adding edges to the given r-graph G, we see that it is enough to prove the lemma when $G \in \Sigma P_I$. So suppose that G is a P_I-mixing construction on $n \geq n_0$ vertices with partition structure $\mathbf{V}$ such that $\delta(G) \geq (\lambda - \delta)\binom{n-1}{r-1}$. Let $u \in V$ and let $\mathbf{i} := (i_1, \ldots, i_s) = \mathrm{br}_{\mathbf{V}}(u)$ denote the branch of u, where s is an integer depending on u. Let $b_j := b(G[V_{i_1,\ldots,i_j}])$ and $\mathbf{i}_j := (i_1, \ldots, i_j)$ for $0 \leq j \leq s$. Notice that $i_{j+1} \in R_{b_j}$ for $0 \leq j \leq s-1$ and $d_{G[V_{\mathbf{i}_s}]}(u) = 0$. Additionally, note that $\mathbf{i}_0 = ()$ is the empty sequence, $\mathbf{i}_s = \mathbf{i}$, and $V_{\mathbf{i}_0} = V(G)$. Let $\mathbf{u}_j := \left(|V_{\mathbf{i}_{j-1},1}|/|V_{\mathbf{i}_{j-1}}|, \ldots, |V_{\mathbf{i}_{j-1},m_{b_{j-1}}}|/|V_{\mathbf{i}_{j-1}}|\right)$ for $1 \leq j \leq s$. Notice from Lemma 3.9(a) that $b_j \in I'$ for every $1 \leq j \leq \ell$. Additionally, by Lemma 3.9(b) and our choice of constants, for each $j \in [\ell]$, the vector $\mathbf{u}_j$ is within distance ε_j from some $P_{b_{j-1}}$-optimal vector $\mathbf{x}_j$. Also, let $m := r \cdot \max\{m_i : i \in I\}$.

By Lemma 3.10, we obtain

$$
\begin{aligned}
d_G(u) &\leq \left(\lambda - \lambda \mathbf{x}_{1,i_1}^{r-1} + rm \cdot \|\mathbf{u}_1 - \mathbf{x}_1\|_\infty\right)\binom{|V_{\mathbf{i}_0}| - 1}{r-1} + d_{G[V_{\mathbf{i}_1}]}(u) \\
&\leq \left(\lambda - \lambda\left(\mathbf{u}_{1,i_1} - \varepsilon_1\right)^{r-1} + rm\varepsilon_1\right)\binom{|V_{\mathbf{i}_0}| - 1}{r-1} + d_{G[V_{\mathbf{i}_1}]}(u) \\
&\leq \left(\lambda - \lambda \mathbf{u}_{1,i_1}^{r-1} + (r-1)\mathbf{u}_{1,i_1}^{r-2}\varepsilon_1 + rm\varepsilon_1\right)\binom{|V_{\mathbf{i}_0}| - 1}{r-1} + d_{G[V_{\mathbf{i}_1}]}(u) \\
&= \lambda\binom{|V_{\mathbf{i}_0}| - 1}{r-1} - \lambda \mathbf{u}_{1,i_1}^{r-1}\binom{|V_{\mathbf{i}_0}| - 1}{r-1} + \left((r-1)\mathbf{u}_{1,i_1}^{r-2}\varepsilon_1 + rm\varepsilon_1\right)\binom{|V_{\mathbf{i}_0}| - 1}{r-1} + d_{G[V_{\mathbf{i}_1}]}(u) \\
&\leq \lambda\binom{|V_{\mathbf{i}_0}| - 1}{r-1} - \lambda\binom{\mathbf{u}_{1,i_1}|V_{\mathbf{i}_0}| - 1}{r-1} + 2rm\varepsilon_1\binom{n-1}{r-1} + d_{G[V_{\mathbf{i}_1}]}(u) \\
&= \lambda\binom{|V_{\mathbf{i}_0}| - 1}{r-1} - \lambda\binom{|V_{\mathbf{i}_1}| - 1}{r-1} + 2rm\varepsilon_1\binom{n-1}{r-1} + d_{G[V_{\mathbf{i}_1}]}(u).
\end{aligned}
$$

If $s < \ell$, then by repeating the argument above for s times, we obtain

$$
\begin{aligned}
d_G(u) &\leq \sum_{j=0}^{s-1}\left(\lambda\binom{|V_{\mathbf{i}_j}| - 1}{r-1} - \lambda\binom{|V_{\mathbf{i}_{j+1}}| - 1}{r-1} + 2rm\varepsilon_{j+1}\binom{n-1}{r-1}\right) + d_{G[V_{\mathbf{i}_s}]}(u) \\
&= \lambda\binom{|V_{\mathbf{i}_0}| - 1}{r-1} - \lambda\binom{|V_{\mathbf{i}_s}| - 1}{r-1} + \sum_{j=0}^{s-1}\varepsilon_{j+1} \cdot 2rm\binom{n-1}{r-1} \\
&\leq \lambda\binom{n-1}{r-1} + \sum_{j=0}^{s-1}\varepsilon_{j+1} \cdot 2rm\binom{n-1}{r-1} \leq (\lambda + \varepsilon)\binom{n-1}{r-1}.
\end{aligned}
$$

If $s \geq \ell$, then by repeating the argument above for ℓ times and applying Lemma 3.9(c) to $V_{\mathbf{i}_\ell}$, we obtain

$$
\begin{aligned}
d_G(u) &\leq \sum_{j=0}^{\ell-1}\left(\lambda\binom{|V_{\mathbf{i}_j}| - 1}{r-1} - \lambda\binom{|V_{\mathbf{i}_{j+1}}| - 1}{r-1} + 2rm\varepsilon_{j+1}\binom{n-1}{r-1}\right) + d_{G[V_{\mathbf{i}_\ell}]}(u) \\
&= \lambda\binom{|V_{\mathbf{i}_0}| - 1}{r-1} - \lambda\binom{|V_{\mathbf{i}_\ell}| - 1}{r-1} + \sum_{j=0}^{\ell-1}\varepsilon_{j+1} \cdot 2rm\binom{n-1}{r-1} + \binom{|V_{\mathbf{i}_\ell}| - 1}{r-1}
\end{aligned}
$$

$$\leq \lambda\binom{|V_{\mathbf{i}_0}|-1}{r-1} + \sum_{j=0}^{\ell-1}\varepsilon_{j+1}\cdot 2rm\binom{n-1}{r-1} + \binom{\left(1-\frac{\lambda}{2r}\right)^{\ell} n-1}{r-1}$$
$$\leq \left(\lambda + \sum_{j=0}^{\ell-1}\varepsilon_{j+1}\cdot 2rm + \left(1-\frac{\lambda}{2r}\right)^{\ell}\right)\binom{n-1}{r-1} \leq (\lambda+\varepsilon)\binom{n-1}{r-1}.$$

This proves Lemma 3.11. □

Given a collection E of r-multisets on a set V, a subset $U \subseteq V$ is called **externally E-homogeneous** if every permutation σ of V that fixes every vertex outside of U is a symmetry of the set of multisets in E that intersect the complement of U, that is, $\sigma\left(E \setminus \left(\!\binom{U}{r}\!\right)\right) = E \setminus \left(\!\binom{U}{r}\!\right)$. Equivalently, every permutation of V that moves only the elements of U preserves the set of multisets from E that intersect both U and $V \setminus U$.

It is clear from the definition that if $|U| = 1$, then U is always externally E-homogeneous. In addition, we have the following simple fact.

Fact 3.12. If E is an r-graph (that is, all multisets in E are simple sets), then a set $U \subseteq V(E)$ is externally E-homogeneous if and only if for every $s \in [r-1]$ and for every $S \in \binom{V\setminus U}{s}$

$$\text{either} \quad L_E(S) \cap \binom{U}{r-s} = \emptyset \quad \text{or} \quad L_E(S) \cap \binom{U}{r-s} = \binom{U}{r-s}.$$

Given a pattern $P = (m, E, R)$, a map $h : [m] \to [m]$ is an **automorphism** of the pattern P if h is bijective, $h(R) = R$, and h is an automorphism of E (that is, $h(E) = E$). Let us call a P_I-mixing construction G with the base P_i and the bottom partition $V_1 \cup \cdots \cup V_{m_i}$ for some $i \in I'$ **rigid** if, for every embedding f of G into a P_I-mixing construction H with some base P_j and bottom partition $U_1 \cup \cdots \cup U_{m_j}$ such that $f(V(G))$ intersects at least two different parts U_k, we have $j = i$ and there is an automorphism h of P_i such that $f(V_k) \subseteq U_{h(k)}$ for every $k \in [m_i]$.

The following key lemma generalizes [30, Lemma 17] by allowing more than one pattern. Its proof requires some new ideas. For example, a trick that was used a number of times in [30], in particular when proving Lemma 17 there, is that any embedding of a maximum P-construction G into another P-construction is induced (that is, non-edges are mapped to non-edges). However, a maximum P_I-mixing construction whose base has to be P_i for given $i \in I'$ need not be maximum (nor even maximal) among all P_I-mixing constructions and some different arguments are required.

Lemma 3.13. *There exist constants $\varepsilon_0 > 0$ and n_0 such that every P_I-mixing construction G on $n \geq n_0$ vertices with $\delta(G) \geq (\lambda - \varepsilon_0)\binom{n-1}{r-1}$ is rigid.*

Proof. For every $i \in I$, define

$$\eta_i \coloneqq \min\{\lambda - \lambda_{P_i - j} : j \in [m_i]\}.$$

Since P_i is minimal, we have $\eta_i > 0$. Let $\eta \coloneqq \min\{\eta_i : i \in I\}$. Clearly, $\eta > 0$.

Recall that $\beta > 0$ is a constant satisfying Lemma 2.4(g) for each pattern P_i. Choose sufficiently small positive constants in this order $\varepsilon_2 \gg \varepsilon_1 \gg \varepsilon_0$. Let n be a sufficiently large integer and G be a P_I-mixing construction on $[n]$ that satisfies the assumptions in Lemma 3.13. Let $\mathbf{V}$ denote the partition structure of G. Assume that the bottom partition is $V(G) = V_1 \cup \cdots \cup V_{m_i}$ for some $i \in I$.

By our choice of the constants (and by Lemmas 3.9 and 3.11), we have that

(A) $|V_k| \geq \beta n/2$ for all $k \in [m_i]$,
(B) $\delta(G[V_k]) \geq (\lambda - \varepsilon_1)\binom{|V_k|-1}{r-1}$ for all $k \in R_i$, and
(C) every P_I-mixing subconstruction G' on n vertices with $\delta(G') \geq (\lambda - \varepsilon_0)\binom{n-1}{r-1}$ satisfies $\Delta(G') \leq (\lambda + \varepsilon_1)\binom{n-1}{r-1}$.

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Take any embedding f of G into some P_I-mixing construction H with the base P_j and the bottom partition $V(H) = U_1 \cup \cdots \cup U_{m_j}$ for some $j \in I$ such that $f(V(G))$ intersects at least two different parts U_k.

Since $|V_k| \geq \beta n/2 > \varepsilon_2 m_j n$ for all $k \in [m_i]$, by the Pigeonhole Principle, there is a function $h\colon [m_i] \to [m_j]$ such that

$$|f(V_k) \cap U_{h(k)}| \geq \varepsilon_2 n, \quad \text{for all } k \in [m_i]. \tag{3.6}$$

Claim 3.13.1. We can choose h in (3.6) so that, additionally, $h(R_i) \subseteq R_j$ and h assumes at least two different values.

Proof of Claim. First, we prove that we can choose h so that $h(R_i) \subseteq R_j$. This is trivially true if R_i is empty, so suppose otherwise.

We claim that R_j is also nonempty. Suppose to the contrary that $R_j = \emptyset$. Let $k \in R_i$. If there exists $j_0 \in [m_j]$ such that $|f(V_k) \cap U_{j_0}| \leq \varepsilon_2 n$, then there exists a subset $V_k' \subseteq V_k$ of size at least $|V_k| - \varepsilon_2 n$ such that $f(V_k') \subseteq U \setminus U_{j_0}$. Since, by (A),

$$\begin{aligned}\rho(G[V_k']) = |G[V_k']|/\binom{|V_k'|}{r} &\geq \left(|G[V_k]| - \varepsilon_2 n\binom{|V_k|-1}{r-1}\right)/\binom{|V_k|}{r}\\ &= \rho(G[V_k]) - \varepsilon_2 n \cdot \frac{r}{|V_k|} \geq \rho(G[V_k]) - \frac{r\varepsilon_2}{\beta/2},\end{aligned} \tag{3.7}$$

which is at least $\lambda - \frac{\eta}{2}$ by (B), the induced subgraph of H on $f(V_k') \subseteq U \setminus U_{j_0}$ has edge density at least $\lambda - \eta/2$. Since n is large and $R_j = \emptyset$, we have $\lambda_{P_{j-j_0}} \geq \rho(H[f(V_k')]) - \eta/4 \geq \lambda - 3\eta/4 > \lambda - \eta_j$, contradicting our definition of η_j (that is, the minimality of P_j). Therefore, we have that $|f(V_k) \cap U_s| \geq \varepsilon_2 n$ for all $s \in [m_j]$.

For every $s \in [m_j]$, let $W_s := U_s \cap f(V_k)$. Pick some $k' \in [m_i] \setminus \{k\}$ and a vertex $u \in V_{k'}$; note that $V_{k'} \neq \emptyset$ by (A). Since $|W_s| \geq \varepsilon_2 n \geq r$ for all $s \in [m_j]$ and the E_j-link of every element of $[m_j]$ is nonempty by (2.3), there exists an edge $D \in H$ containing $f(u)$ such that $D \setminus \{f(u)\}$ is contained in $\bigcup_{s\in[m_j]} W_s$. Let X denote the profile of D and X' be the profile of $D \setminus \{f(u)\}$, both with respect to $U_1 \cup \cdots \cup U_{m_j}$. Since H is a P_I-mixing construction, all r-subsets of $V(H)$ with profile X are contained in H. Let $\mathcal{D}$ be the collection of all r-subsets $D' \ni f(u)$ such that the profile of $D' \setminus f(u)$ with respect to $W_1, \ldots, W_{m_j}$ is X'. No element of the set $f^{-1}(\mathcal{D})$ is contained in G because every member in $f^{-1}(\mathcal{D})$ has profile $\{\!\{ k^{r-1}, k' \}\!\}$ which cannot belong to E_i by Lemma 2.3. However, if we add $f^{-1}(\mathcal{D})$ into the edge set of G, the new r-graph G' is still embedded into H by the same function f. In other words, G' is a P_I-mixing subconstruction. However, the degree of u in G' satisfies

$$\begin{aligned}d_{G'}(u) \geq d_G(u) + |f^{-1}(\mathcal{D})| &\geq (\lambda - \varepsilon_0)\binom{n-1}{r-1} + \prod_{s\in[m_j]} \frac{(\varepsilon_2 n)^{X'(s)}}{X'(s)!}\\ &\geq (\lambda - \varepsilon_0)\binom{n-1}{r-1} + \frac{\varepsilon_2^{r-1} n^{r-1}}{(r-1)!} > (\lambda + \varepsilon_1)\binom{n-1}{r-1},\end{aligned}$$

which contradicts (C) above. This proves that $R_j \neq \emptyset$.

Suppose to the contrary that we cannot satisfy the first part of the claim for some $k \in R_i$; that is, for each $\ell \in R_j$ we have $|f(V_k) \cap U_\ell| < \varepsilon_2 n$. Since $|f(V_k) \cap U_\ell| < \varepsilon_2 n$ for all $\ell \in R_j$, there exists a subset $V_k' \subseteq V_k$ of size at least $|V_k| - \varepsilon_2 n|R_j|$ such that $f(V_k') \subseteq U_{[m_j]\setminus R_j} := \bigcup_{\ell\in[m_j]\setminus R_j} U_\ell$; that is, the induced subgraph $G[V_k']$ is embedded into $H[U_{[m_j]\setminus R_j}]$. Since $|V_k| \geq \beta n/2 \gg \varepsilon_2 n|R_j|$ and

$\rho(G[V_k]) \geq \lambda - \varepsilon_1$, we have

$$\rho(G[V_k']) = \frac{|G[V_k']|}{\binom{|V_k'|}{r}} \geq \frac{|G[V_k]| - \varepsilon_2 n |R_j| \binom{|V_k|-1}{r-1}}{\binom{|V_k|}{r}} = \rho(G[V_k]) - \varepsilon_2 n |R_j| \cdot \frac{r}{|V_k|}$$
$$> \lambda - \varepsilon_1 - \frac{r\,|R_j|\varepsilon_2}{\beta/2} > \lambda - \frac{\eta}{2}.$$

This means that there are arbitrarily large $(P_j - R_j)$-constructions of edge density at least $\lambda - \eta/2$, that is, $\lambda_{P_j - R_j} \geq \lambda - \eta/2 > \lambda - \eta$. This contradicts the minimality of P_j since $R_j \neq \emptyset$.

Let us restrict ourselves to those h with $h(R_i) \subseteq R_j$. Suppose that we cannot fulfill the second part of the claim. Then there is $s \in [m_j]$ such that $|f(V_k) \cap U_s| \geq \varepsilon_2 n$ for every $k \in [m_i]$. Since $E_i \neq \emptyset$, the induced subgraph $G[f^{-1}(U_s)]$ is nonempty (it has at least $\varepsilon_2 n$ vertices from each bottom part of G) and is mapped entirely into U_s. Thus, $s \in R_j$. Since $f(V(G))$ intersects at least two different bottom parts of H, we can pick some $v \in V_t$ for some $t \in [m_i]$ such that $f(v) \in U_{s'}$ and $s' \in [m_j] \setminus \{s\}$. Fix some $(r-1)$-multiset X in $L_{E_i}(t)$; note that $L_{E_i}(t) \neq \emptyset$ by (2.3) – that is, by the minimality of P_i. Take an edge $D \in G$ containing v so that $D \setminus \{v\}$ is a subset of $f^{-1}(U_s)$ and has profile X; it exists because each bottom part of G contains at least $\varepsilon_2 n \geq r$ vertices of $f^{-1}(U_s)$. The r-set $f(D)$ is an edge of H as f is an embedding. However, it has $r-1$ vertices in U_s and one vertex in $U_{s'}$. Thus, the r-multiset $\{\{ s^{(r-1)}, s' \}\}$ belongs to E_j. Since $s \in R_j$, this contradicts Lemma 2.3. The claim is proved. □

Claim 3.13.2. Each h satisfying Claim 3.13.1 is a bijection. Moreover, $j = i$ and h is an automorphism of P_i.

Proof of Claim. Fix a map h that satisfies Claim 3.13.1. First, we prove that h is injective. For every $s \in [m_j]$, let $U_s' := \bigcup_{t \in h^{-1}(s)} f(V_t) \subseteq V(H)$. Note that $U_s' = \emptyset$ for s not in the image of h. Let H' be the P_I-mixing construction on $f(V(G))$ such that $U_1' \cup \cdots \cup U_{m_j}'$ is the bottom partition of H' with base P_j (thus, $E_j((U_1', \ldots, U_{m_j}')) \subseteq H'$), and, for each $s \in R_j$, $H'[U_s']$ is the image of the P_I-mixing construction $G[f^{-1}(U_s')]$ under the bijection f.

We have just defined a new P_I-mixing construction H' so that each part V_s of G is entirely mapped by f into the $h(s)$-th part of H'; that is,

$$f(V_s) \subseteq U_{h(s)}', \quad \text{for every } s \in [m_i]. \tag{3.8}$$

This H' will be used only for proving Claim 3.13.2.

Let us show first that the same map f is an embedding of G into H'. Take any edge $D \in G$. First, suppose that $f(D)$ intersects at least two different parts U_t'. By (3.8), D has to be a bottom edge of G. Let $D' \in G$ have the same profile as D with respect to $V_1, \ldots, V_{m_i}$ and satisfy

$$D' \subseteq \bigcup_{s \in [m_i]} \Big(V_s \cap f^{-1}(U_{h(s)}) \Big), \tag{3.9}$$

which is possible because there are at least $\varepsilon_2 n$ vertices available in each part $V_s \cap f^{-1}(U_{h(s)})$. Since $f(D') \cap U_t = f(D') \cap U_t'$ for all $t \in [m_j]$, the bottom edge $f(D')$ of H has the same profile X with respect to the partitions $U_1 \cup \cdots \cup U_{m_j}$ and $U_1' \cup \cdots \cup U_{m_j}'$. Thus $X \in E_j$. Next, as each $f(V_s)$ lies entirely inside $U_{h(s)}'$, the sets $f(D)$ and $f(D')$ have the same profiles with respect to the partition $U_1' \cup \cdots \cup U_{m_j}'$. Thus $f(D)$ is an edge of $E_j((U_1', \ldots, U_{m_j}'))$, as required. It remains to consider the case that $f(D)$ lies inside some part U_t'. Let $G' := G[f^{-1}(U_t')]$. Assume that $t \in [m_j] \setminus R_j$, for otherwise, $f(D) \in f(G')$, which is a subset of H' by the definition of H'. We claim that G' has no edges in this case (obtaining a contradiction to $D \in G'$). Since $h(R_i) \subseteq R_j$, we have $h^{-1}(t) \cap R_i = \emptyset$ and D must be a bottom edge of G. As before, we can find an edge $D' \in G$ that satisfies (3.9) and has the same profile as D with respect to $V_1, \ldots, V_{m_i}$. However, f maps this D' inside a non-recursive part U_t of H, a contradiction. Thus, f is an embedding of G into H'.

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Thus, by considering H' instead of H (and without changing h), we have that $f(V_s) \subseteq U'_{h(s)}$ for all $s \in [m_i]$.

Suppose on the contrary to the claim that $|h^{-1}(s)| \geq 2$ for some $s \in [m_j]$. Let $A := h^{-1}(s)$ and $B := [m_i] \setminus A$. Since h assumes at least two different values, the set B is nonempty.

Trivially, the set U'_s is externally H'-homogeneous. We claim that $f^{-1}(U'_s) = V_A$ is externally G-homogeneous (recall that we denote $V_A := \bigcup_{\ell \in A} V_\ell$). Indeed, suppose that V_A is not externally G-homogeneous. Then there is $D \in G$ that intersects both V_A and its complement such that for some bijection σ of $V(G)$ that moves only points of V_A the r-set $D' := \sigma(D)$ is not in G. The profile of D' with respect to $V_1, \ldots, V_{m_i}$ contains at least two different elements of $[m_i]$, one in A and another in $[m_i] \setminus A$. Let $\mathcal{D}'$ consist of all r-sets that have the same profile with respect to $V_1, \ldots, V_{m_i}$ as D'. Since each bottom part V_k has at least $\beta n/2$ vertices, it holds that $|\mathcal{D}'| \geq (\beta n/2)^r/r!$. Since $D' \notin G$, no r-set in $\mathcal{D}'$ is an edge of G. With respect to $U'_1, \ldots, U'_{m_j}$, all r-sets in $f(\mathcal{D}')$ have by (3.8) the same profile as $f(D')$, which in turn is the same as the profile as that of $f(D)$ (because the bijection $f \circ \sigma \circ f^{-1}$ of $V(H')$ moves only elements inside $f(V_A) \subseteq U'_s$). Since $f(D) \in H'$, we must have $f(\mathcal{D}') \subseteq H'$. Thus, we can add $\mathcal{D}'$ to G, keeping it a P_I-mixing subconstruction. However, the new r-graph has edge density at least $\lambda - \varepsilon_0 + (\beta/2)^r$, which is a contradiction to (C) since $\beta \gg \varepsilon_1 \gg \varepsilon_0 \gg 1/n$.

Thus, V_A is externally G-homogeneous. It follows that A is externally E_i-homogeneous (since each V_t has at least $\beta n/2 \geq r$ elements).

Next, let us show that $A \cap R_i = \emptyset$. Suppose that this is false. Then fix $i_* \in A \cap R_i$. Let $\hat{G}$ be the r-graph obtained from G by replacing $G[V_A]$ with a maximum P_I-mixing construction. By Fact 3.12, the r-graph $\hat{G}$ remains a P_I-mixing construction, with A playing the role of i_*. This implies that $\rho(G[V_A]) \geq \lambda - 3\varepsilon_0/\beta^r$, for otherwise, we would have

$$\begin{aligned} |\hat{G}| = |G| - |G[V_A]| + |\hat{G}[V_A]| &\geq (\lambda - \varepsilon_0)\binom{n}{r} - \left(\lambda - \frac{3\varepsilon_0}{\beta^r}\right)\binom{|V_A|}{r} + (\lambda + o(1))\binom{|V_A|}{r} \\ &\geq (\lambda - \varepsilon_0)\binom{n}{r} + \frac{2\varepsilon_0}{\beta^r}\binom{|V_A|}{r} \\ &\geq (\lambda - \varepsilon_0)\binom{n}{r} + 2\varepsilon_0\binom{|V_A|/\beta}{r} \geq (\lambda - \varepsilon_0)\binom{n}{r} + 2\varepsilon_0\binom{n}{r}, \end{aligned}$$

a contradiction to (3.3). Here, the last inequality follows from (A) and the assumption that $|A| \geq 2$.

Recall that $B = [m_j] \setminus A$. Consider the pattern $Q := P_i - B$ obtained from P_i by removing B. Let $E' := E_i[A]$ and $R' := A \cap R_i$. Without loss of generality, we may assume that $A = [a]$ for some $a \in [m_i - 1]$. Let $\mathbf{x} := (x_1, \ldots, x_a)$, where $x_k := |V_k|/|V_A|$ for $k \in [a]$. Then it follows from $\rho(G[V_A]) \geq \lambda - 3\varepsilon_0/\beta^r$ that the obtained vector $\mathbf{x} \in \mathbb{S}_a$ satisfies that

$$\lambda_{E'}(\mathbf{x}) + \lambda \sum_{k \in R'} x_k^r = \rho(G[V_A]) + o(1) \geq \lambda - 4\varepsilon_0/\beta^r. \tag{3.10}$$

This inequality does not contradict the minimality of P_i yet, since $G[V_A]$ is not necessarily a Q-construction (for $k \in R'$, the P_I-mixing construction $G[V_k]$ can use parts indexed by B). However, if we replace $G[V_k]$ by a maximum Q-construction for every $k \in R'$, then the resulting r-graph has edge density $\lambda_{E'}(\mathbf{x}) + \lambda_Q \sum_{k \in R'} x_k^r + o(1)$. By the definition of Lagrangian, we have

$$\lambda_Q \geq \lambda_{E'}(\mathbf{x}) + \lambda_Q \sum_{k \in R'} x_k^r. \tag{3.11}$$

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By $|A| \geq 2$ and (A), we have that $x_k \leq 1 - \beta/2$ for every $k \in [a]$. So it follows from $\sum_{k=1}^{a} x_k = 1$ that, say, $\sum_{k=1}^{a} x_k^r \leq 1 - \varepsilon_2$. In particular, $\sum_{k\in R'} x_k^r \leq 1 - \varepsilon_2$. Therefore, by (3.10) and (3.11), we obtain

$$\lambda_Q\left(1 - \sum_{k\in R'} x_k^r\right) \geq \lambda_{E'}(\mathbf{x}) \geq \lambda\left(1 - \sum_{k\in R'} x_k^r\right) - 4\varepsilon_0/\beta^r,$$

which implies that $\lambda_Q \geq \lambda - \eta/2$, contradicting the minimality of P_i. Therefore, $A \cap R_i = \emptyset$.

Since $|A| \geq 2$, we can choose $t_1, t_2 \in A$ such that $t_1 \neq t_2$. Since A is externally E_i-homogeneous and, by Lemma 2.2, we have $L_{E_i}(t_1) \neq L_{E_i}(t_2)$, there exists a multiset in E_i that is completely contained inside A.

So, $G[V_A] \neq \emptyset$. Since $U_s' = f(V_A)$, we have $H'[U_s'] \neq \emptyset$, and hence, $s \in R_j$. Notice that H' is a P_I-mixing construction on n vertices with $\delta(H') \geq \delta(f(G)) \geq (\lambda - \varepsilon_0)\binom{n-1}{r-1}$. So, similarly to (B), we have $\rho(H'[U_k']) \geq \lambda - \varepsilon_1$ for all $k \in R_j$. In particular, we have $\rho(H'[U_s']) \geq \lambda - \varepsilon_1$. Therefore, $|H' \setminus H'[U_s']| \leq (\lambda + \varepsilon_1)\binom{n}{r} - (\lambda - \varepsilon_1)\binom{|U_s'|}{r}$, which implies that

$$|G \setminus G[V_A]| \leq |H' \setminus H'[U_s']| \leq (\lambda + \varepsilon_1)\binom{n}{r} - (\lambda - \varepsilon_1)\binom{|U_s'|}{r}.$$

Note that $|U_s'| = |V_A|$ since f gives a bijection between these two sets. Therefore,

$$\begin{aligned} |G[V_A]| = |G| - |G \setminus G[V_A]| &\geq (\lambda - \varepsilon_0)\binom{n}{r} - \left((\lambda + \varepsilon_1)\binom{n}{r} - (\lambda - \varepsilon_1)\binom{|V_A|}{r}\right) \\ &\geq \lambda\binom{|V_A|}{r} - 3\varepsilon_1\binom{n}{r}. \end{aligned}$$

Thus,

$$\rho(G[V_A]) = \frac{|G[V_A]|}{\binom{|V_A|}{r}} \geq \frac{\lambda\binom{|V_A|}{r} - 3\varepsilon_1\binom{n}{r}}{\binom{|V_A|}{r}} \geq \lambda - \frac{3\varepsilon_1\binom{n}{r}}{\binom{\beta n}{r}} = \lambda - \frac{3\varepsilon_1}{\beta^r + o(1)} \geq \lambda - \frac{\eta}{2}.$$

This implies that $\lambda_{P_i - B} \geq \rho(G[V_A]) \geq \lambda - \eta/2 > \lambda - \eta$, which contradicts the minimality of P_i. Therefore, h is injective.

Since each multiset in E_i corresponds to a nonempty set of edges of G by (A) and f is an embedding, we have $h(E_i) \subseteq E_j$.

Suppose to the contrary that h is not surjective; that is, $m_j \geq m_i + 1$. Let $\mathbf{x} = (x_1, \ldots, x_{m_i}) \in \mathcal{X}_i$ be a P_i-optimal vector. Let $\mathbf{y} := (x_{h^{-1}(1)}, \ldots, x_{h^{-1}(m_j)})$, where $x_\emptyset$ means 0. Thus, $\mathbf{y}$ is defined by rearranging the entries of $\mathbf{x}$ according to h and padding them with $m_j - m_i$ zeros. By $h(R_i) \subseteq R_j$ and $h(E_i) \subseteq E_j$, we have

$$\lambda_{E_j}(\mathbf{y}) + \lambda \sum_{k\in R_j} y_k^r \geq \lambda_{E_i}(\mathbf{x}) + \lambda \sum_{k\in R_i} x_k^r = \lambda.$$

However, by Lemma 2.4(b), we have $\lambda_{E_j}(\mathbf{y}) + \lambda \sum_{k\in R_j} y_k^r \leq \lambda$. Therefore, $\lambda_{E_j}(\mathbf{y}) + \lambda \sum_{k\in R_j} y_k^r = \lambda$, which implies that $\mathbf{y} \in \mathcal{X}_j$ is a P_j-optimal vector. However, $\mathbf{y}$ has at least one entry 0, which contradicts Lemma 2.4(c). Therefore, h is surjective, and hence is bijective.

None of the inclusions $h(E_i) \subseteq E_j$ and $h(R_i) \subseteq R_j$ can be strict as otherwise, since every P_j-optimal vector has all coordinates positive by the minimality of P_j, we would have $\lambda_{P_j} > \lambda_{P_i} = \lambda$, a contradiction. Thus, h gives an isomorphism between P_i and P_j, and we conclude that $j = i$. This finishes the proof of the claim. □

By relabelling the parts of H, we can assume for notational convenience that h is the identity mapping. Now we are ready to prove the lemma – namely, that $f(V_s) \subseteq U_s$ for every $s \in [m_i]$.

Suppose on the contrary that $f(v) \in U_t$ for some $v \in V_s$ and $t \in [m_i] \setminus \{s\}$. It follows that $L_{E_i}(s) \subseteq L_{E_i}(t)$. By Lemma 2.2, this inclusion is strict and $s \in R_i$. Pick some X from $L_{E_i}(t) \setminus L_{E_i}(s) \neq \emptyset$. For every $D \in H$ containing $f(v)$ such that $D \setminus \{f(v)\}$ has the profile X with respect to both $U_1 \cup \cdots \cup U_{m_i}$ and $f(V_1) \cup \cdots \cup f(V_{m_i})$, we add $f^{-1}(D)$ into G. Denote by $\tilde{G}$ the new r-graph. Observe that f is also an embedding of $\tilde{G}$ into H. Thus, $\tilde{G}$ is a P_I-mixing subconstruction. Notice that since $|f(V_j) \cap U_j| \geq \varepsilon_2 n$, there are at least $\prod_{j \in [m_i]} (\varepsilon_2 n)^{X(j)}/X(j)! \geq \varepsilon_2^{r-1} n^{r-1}/(r-1)!$ such edges D. Since all edges in $f^{-1}(D)$ contain v, we have $d_{\tilde{G}}(v) \geq d_G(v) + \varepsilon_2^{r-1} n^{r-1}/(r-1)! > (\lambda + \varepsilon_1)\binom{n-1}{r-1}$ contradicting (C) above. This shows that G is rigid. □

Recall that I' consists of the elements $P_i \in I$ whose Lagragian λ_{P_i} attains the maximum value $\lambda = \lambda_{P_I}$. Let us denote

$$P_{I'} := \{P_i : i \in I'\}.$$

Later, in the proof of Lemma 3.16, we will need, for a given feasible tree $\mathbf{T}$ satisfying some technical conditions, the existence of a rigid P_I-mixing construction F whose tree is $\mathbf{T}$ and every part of F is sufficiently large, specifically

$$|V_{\mathbf{i}}| \geq (r-1)\max\{r, \max\{m_k : k \in I\}\}, \quad \text{for every legal (with respect to } F\text{) sequence } \mathbf{i}. \tag{3.12}$$

The following two lemmas provide such F (in fact, each obtained F will be a $P_{I'}$-mixing construction). The proofs are slightly different depending on whether $\mathbf{T}$ is extendable or not. (Recall that a feasible tree $\mathbf{T}$ is extendable if it is a subtree of some strictly larger feasible tree.)

Recall the definition of clone from the paragraph above Lemma 3.5. Note that if we add a clone v' of a vertex v of a P_I-mixing construction G, we generally obtain a subconstruction rather than a P_I-mixing construction. For example, if V_j is the bottom part containing v while some edge D of the base multiset E_i contains j with multiplicity more than 1, then the blowup of D in a P_I-mixing construction would additionally include some edges containing both v and v'. So let us define the operation of **doubling** v in G where we take a new vertex v' (called the **double** of v), put it in the partition structure of G so that v' has the same branch as v, and add all edges through v' as stipulated by the new partition structure. Of course, the degree of the new vertex v' is at least the degree of v in the old r-graph G.

Lemma 3.14. *For every non-extendable feasible tree* $\mathbf{T}$ *with all indices in* I'*, there exists a rigid* P_I*-mixing construction* F *such that* $\mathbf{T}_F = \mathbf{T}$ *and (3.12) holds.*

Proof. Take any tree $\mathbf{T}$ as in the lemma. Let $\varepsilon_0 > 0$ and n_0 be the constants given by Lemma 3.13. Let n be a sufficiently large integer – in particular, so that $n \geq n_0$ and we can apply Lemma 3.9 with ℓ equal to the height of $\mathbf{T}$. Let F be a maximum n-vertex $P_{I'}$-mixing construction under the requirement that its tree $\mathbf{T}_F$ is a subtree of $\mathbf{T}$. By taking (near) optimal parts ratio for the bottom partition and for every recursive part in $\mathbf{T}$, it is easy to see that $\rho(F) \geq \lambda - \varepsilon_0/2$.

We claim that

$$\delta(F) \geq (\lambda - \varepsilon_0)\binom{n-1}{r-1}. \tag{3.13}$$

Indeed, suppose that a vertex u violates (3.13). Remove u from F and double a vertex v with degree at least $(\lambda - \varepsilon_0/2)\binom{n-1}{r-1}$ in F. The new r-graph F' has strictly larger number of edges:

$$|F'| - |F| \geq (\lambda - \varepsilon_0/2)\binom{n-1}{r-1} - (\lambda - \varepsilon_0)\binom{n-1}{r-1} - \binom{n-2}{r-2} > 0, \tag{3.14}$$

However, the tree of F' is still a subtree of $\mathbf{T}$ (even if the part that contained u became edgeless), contradicting the maximality of F.

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It follows from Lemma 3.13 that F is rigid. Also, (3.12) holds by (3.13) and Lemma 3.9(c). In particular, we have that $\mathbf{T}_F = \mathbf{T}$, finishing the proof. □

Call a feasible tree $\mathbf{T}$ of height ℓ **maximal** if every leaf of height less than ℓ is non-recursive (or, equivalently, $\mathbf{T}$ cannot be extended to a larger feasible tree of the same height).

Lemma 3.15. *There exists constant $\ell_0 \in \mathbb{N}$ such that, for every feasible extendable tree $\mathbf{T}$ with all indices in I' which is maximal of height ℓ_0, there exists a rigid P_I-mixing construction F such that $\mathbf{T}_F = \mathbf{T}$ and (3.12) holds.*

Proof. Let ε_0 and n_0 be given by Lemma 3.13. Fix a sufficiently large ℓ_0. Let $\mathbf{T}$ be a tree as in the lemma. Choose $n \in \mathbb{N}$ to be sufficiently large such that, in particular, $n \geq n_0$,

$$(\beta/2)^{\ell_0} n \geq (r-1)\max\{r, \max\{m_k : k \in I\}\},$$

and we can apply Lemma 3.9 with ℓ equal to ℓ_0. Let G be a maximum $P_{I'}$-mixing construction provided that $\mathbf{T}_G^{\text{level} \leq \ell_0}$, the tree $\mathbf{T}_G$ restricted to levels up to ℓ_0, is a subtree of $\mathbf{T}$. Let $\mathbf{V}$ denote the partition structure of G.

Claim 3.15.1. We have $\delta(G) \geq (\lambda - \varepsilon_0/2)\binom{n-1}{r-1}$.

Proof of Claim. If we take (near) optimal parts ratio for all partitions up to level ℓ_0 and and put a maximum $P_{I'}$-mixing construction into each part corresponding to a recursive leaf of $\mathbf{T}$, then the obtained r-graph G' has edge density $\lambda + o(1)$ as $n \to \infty$. Since n is sufficiently large and $|G| \geq |G'|$ by the definition of G, we can assume that $\rho(G) \geq \lambda - \varepsilon_0/4$. Now, there cannot be a vertex $v \in V(G)$ with $d_G(v) < (\lambda - \varepsilon_0/2)\binom{n-1}{r-1}$, for otherwise, we would remove v from G and double a vertex $u \in V(G)$ with maximum degree (which is at least $\rho(G)\binom{n}{r}r/n \geq (\lambda - \varepsilon_0/4)\binom{n-1}{r-1}$), getting a contradiction as in (3.14). □

Let F be obtained from G by removing edges in $G[\mathbf{V}_{\mathbf{i}}]$ for all legal sequences $\mathbf{i}$ in G of length at least $\ell_0 + 1$.

Claim 3.15.2. We have $\delta(F) \geq (\lambda - \varepsilon_0)\binom{n-1}{r-1}$.

Proof of Claim. By Lemma 3.9(c), we have $|V_{\mathbf{i}}| \leq \left(1 - \frac{\lambda}{2r}\right)^{\ell_0} n \ll \varepsilon_0 n$ for all legal sequences $\mathbf{i}$ in G of length at least ℓ_0. So it holds that $d_F(u) = d_G(u) \geq (\lambda - \varepsilon_0)\binom{n-1}{r-1}$ if the length of $\mathrm{br}_{\mathbf{V}'}(u)$ is at most ℓ_0, and $d_F(u) \geq d_G(u) - \left(1 - \frac{\lambda}{2r}\right)^{\ell_0 \cdot (r-1)} n^{r-1} \geq (\lambda - \varepsilon_0)\binom{n-1}{r-1}$ if the length of $\mathrm{br}_{\mathbf{V}'}(u)$ is at least $\ell_0 + 1$. □

Since $\mathbf{T}$ is maximal of height ℓ_0, the tree of F is a subtree of $\mathbf{T}$. Our choice of n also makes sure that, for every legal $\mathbf{i}$ in $\mathbf{T}$ of length at most ℓ_0, we have $|V_{\mathbf{i}}| \geq (\beta/2)^{\ell_0} n \geq (r-1)\max\{r, \max\{m_k : k \in I\}\}$. In particular, we have $\mathbf{T}_F = \mathbf{T}$. Since F is a P_I-mixing construction with $\delta(F) \geq (\lambda - \varepsilon_0)\binom{n-1}{r-1}$, it follows from Lemma 3.13 that F is rigid, finishing the proof of Lemma 3.15. □

3.4. Key lemmas

The following lemma, which is proved via stability-type arguments, is the key for the proof of Theorem 1.2. Its conclusion, informally speaking, implies there is a way to replace a part of a 'reasonably good' $\mathcal{F}_{M_0}$-free r-graph G by a blowup G' of some E_i on $V(G)$ so that the new r-graph is still $\mathcal{F}_{M_0}$-free and satisfies $|G'| \geq |G| + \Omega(|G \triangle G'|)$. This is closely related to the so-called **local** (or **perfect**) stability, see, for example, [29, 32].

Lemma 3.16. *There are $c_0 > 0$ and $M_0 \in \mathbb{N}$ such that the following holds. Suppose that G is a $\mathcal{F}_{M_0}$-free r-graph on $n \geq r$ vertices that is $c_0\binom{n}{r}$-close to some P_I-mixing construction and satisfies*

$$\delta(G) \geq (\lambda - c_0)\binom{n-1}{r-1}. \tag{3.15}$$

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Then there are $i \in I$ and a partition $V(G) = V_1 \cup \cdots \cup V_{m_i}$ such that $|V_j| \in [\beta n/2, (1-\lambda/2r)n]$ for all $j \in [m_i]$ and

$$10\,|B| \le |A|, \tag{3.16}$$

where

$$A := E_i((V_1, \ldots, V_{m_i})) \setminus G, \tag{3.17}$$

$$B := G \setminus \left(E_i((V_1, \ldots, V_{m_i})) \cup \bigcup_{j \in R_i} G[V_j] \right). \tag{3.18}$$

Proof. Clearly, it is enough to establish the existence of M_0 such that the conclusion of the lemma holds for every sufficiently large n. (Indeed, it clearly holds for $n \le M_0$ by Lemma 3.2, so we can simply increase M_0 at the end to take care of finitely many exceptions; alternatively, one can decrease c_0.)

Let ℓ_0 be the constant returned by Lemma 3.15. Then let M_0 be sufficiently large. Next, choose some constants c_i in this order $c_4 \gg c_3 \gg c_2 \gg c_1 \gg c_0 > 0$, each being sufficiently small depending on the previous ones. Let n tend to infinity.

Let G be a $\mathcal{F}_{M_0}$-free r-graph on $[n]$ that satisfies (3.15) and is $c_0\binom{n}{r}$-close to some P_I-mixing construction H. We can assume that the vertices of H are already labelled so that $|G \triangle H| \le c_0\binom{n}{r}$. Let $\mathbf{V}$ be the partition structure of H. In particular, the bottom partition of H is $V_1 \cup \cdots \cup V_{m_i}$ for some $i \in I$.

One of the technical difficulties that we are going to face is that some part V_j with $j \in R_i$ may in principle contain almost every vertex of $V(G)$ (so every other part V_k has $o(n)$ vertices). This means that the 'real' approximation to G starts only at some higher level inside V_j. However, Lemma 3.9 gives us a way to rule out such cases: we have to ensure that the minimal degree of H is close to $\lambda\binom{n-1}{r-1}$. So, as our first step, we are going to modify the P_I-mixing construction H (perhaps at the expense of increasing $|G \triangle H|$ slightly) so that its minimal degree is large. When we change H here (or later), we update the partition structure $\mathbf{V}$ of H appropriately. (Note that the partition tree $\mathbf{T}_H$ may change too, when some part $V_{\mathbf{i}}$ stops spanning any edges.)

So, let $Z := \{x \in [n] : d_H(x) < (\lambda - c_1)\binom{n-1}{r-1}\}$. Due to (3.15), every vertex of Z contributes at least $(c_1 - c_0)\binom{n-1}{r-1} \ge c_1\binom{n-1}{r-1}/2 \ge c_1\binom{n-1}{r-1}/r$ to $|G \triangle H|$. We conclude that

$$|Z| \le \frac{|G \triangle H|}{c_1\binom{n-1}{r-1}/r} \le \frac{c_0\binom{n}{r}}{c_1\binom{n-1}{r-1}/r} = \frac{c_0 n}{c_1}.$$

Fix an arbitrary $y \in [n] \setminus Z$. Let us change H by making all vertices in Z into doubles of y. Clearly, we have now

$$\begin{aligned}\delta(H) &\ge (\lambda - c_1)\binom{n-1}{r-1} - |Z|\binom{n-2}{r-2} \ge (\lambda - c_1)\binom{n-1}{r-1} - \frac{c_0 n}{c_1}\cdot\frac{r-1}{n-1}\binom{n-1}{r-1} \\ &\ge (\lambda - 2c_1)\binom{n-1}{r-1}\end{aligned} \tag{3.19}$$

while $|G \triangle H| \le c_0\binom{n}{r} + |Z|\binom{n-1}{r-1} \le c_0\binom{n}{r} + \frac{c_0 n}{c_1}\cdot\frac{r}{n}\binom{n}{r} \le c_1\binom{n}{r}$.

By Lemma 3.9 we can conclude that, in the new P_I-mixing construction H satisfying (3.19), part ratios up to height ℓ_0 are close to optimal ones and $|V_{\mathbf{i}}| \ge 2c_4 n$ for each legal sequence $\mathbf{i}$ of length at most ℓ_0.

In order to satisfy the lemma, we may need to modify the current partition $V_1, \ldots, V_{m_i}$ of $V(G)$ further. It will be convenient now to keep track of the sets A and B defined by (3.17) and (3.18), respectively, updating them when the partition changes. Recall that the set A consists of edges that are in $E_i((V_1, \ldots, V_{m_i}))$ but not in G. Let us call these edges **absent**. Call an r-multiset D on $[m_i]$ **bad** if

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$D \notin E_i$ and $D \neq \{\{ j^{(r)} \}\}$ for some $j \in R_i$. Call an edge of G **bad** if its profile with respect to $V_1, \ldots, V_{m_i}$ is bad. Thus, B is precisely the set of bad edges, and our aim is to prove that there are at least 10 times more absent edges than bad edges.

Our next modification is needed to ensure later that (3.25) holds. Roughly speaking, we want a property that the number of bad edges cannot be decreased much if we move one vertex between parts. Unfortunately, we cannot just take a partition structure $\mathbf{V}$ that minimises $|B|$ because then we do not know how to guarantee that (3.19) holds (another property important in our proof). Nonetheless, we can simultaneously satisfy both properties, although with weaker bounds.

Namely, we modify H as follows (updating A, B, $\mathbf{V}$, etc, as we proceed). If there is a vertex $x \in [n]$ such that by moving it to another part V_j we decrease $|B|$ by at least $c_2\binom{n-1}{r-1}$, then we pick $y \in V_j$ of maximum H-degree and make x a double of y. Clearly, we perform this operation at most $c_1\binom{n}{r}/c_2\binom{n-1}{r-1} = c_1 n/(c_2 r)$ times because we initially had $|B| \leq |G \triangle H| \leq c_1\binom{n}{r}$. Thus, we have at all steps of this process (which affects at most $c_1 n/(c_2 r)$ vertices of H) that, trivially,

$$|V_{\mathbf{j}}| \geq 2c_4 n - \frac{c_1 n}{c_2 r} \;\geq\; c_4 n, \qquad \text{for all legal } \mathbf{j} \text{ with } |\mathbf{j}| \leq \ell_0, \tag{3.20}$$

$$|G \triangle H| \leq c_1\binom{n}{r} + \frac{c_1 n}{c_2 r}\binom{n-1}{r-1} \;\leq\; c_2\binom{n}{r}. \tag{3.21}$$

It follows that at every step each part V_j had a vertex of degree at least $(\lambda - c_2/2)\binom{n-1}{r-1}$, for otherwise, by (3.15) and (3.20), the edit distance between H and G at that moment would be at least

$$\frac{1}{r} \cdot |V_j| \cdot \left((\lambda - c_0)\binom{n-1}{r-1} - \left(\lambda - \frac{c_2}{2}\right)\binom{n-1}{r-1}\right) \geq \frac{1}{r} \cdot c_4 n \cdot \frac{c_2}{3}\binom{n-1}{r-1} = \frac{c_2 c_4}{3}\binom{n}{r},$$

contradicting the first inequality in (3.21). This implies that every time we double a vertex it has a high degree. Thus, we have by (3.19) that, additionally to (3.20) and (3.21), the following holds at the end of this process:

$$\delta(H) \geq \left(\lambda - \max\{3c_1, c_2/2\}\right)\binom{n-1}{r-1} - \frac{c_1 n}{c_2 r}\binom{n-2}{r-2} \;\geq\; (\lambda - c_2)\binom{n-1}{r-1}. \tag{3.22}$$

So by Lemma 3.9(c), $|V_j| \geq \beta n/2$ and $|V_j| \leq (1 - \lambda/2r)n$ (note that we may choose $c_2 > 0$ small enough and n large enough in the beginning so that Lemma 3.9(c) applies).

Suppose that $B \neq \emptyset$, for otherwise, the lemma holds trivially.

Let H' be obtained from H by removing edges contained in $H[\mathbf{V_j}]$ for all legal $\mathbf{j}$ of length at least $\ell_0 + 1$. Let $\mathbf{T} := \mathbf{T}_{H'}$. It follows from the definition that $\mathbf{T} = \mathbf{T}_H^{\text{level} \leq \ell_0}$; that is, $\mathbf{T}$ is obtained from $\mathbf{T}_H$ by restricting it to levels up to ℓ_0. By Lemma 3.9(a), all indices in $\mathbf{T}$ belong to I'. If $\mathbf{T}$ is non-extendable (and thus $\mathbf{T} = \mathbf{T}_H$), then we let F be the rigid construction given by Lemma 3.14 whose tree $\mathbf{T}_F$ is the same as $\mathbf{T}$. If $\mathbf{T}$ is extendable then, due to Lemma 3.9(d) and $\ell_0 \ll 1/c_1 \ll n$, every recursive part $V_{\mathbf{i}}$ with $|\mathbf{i}| < \ell_0$ spans at least one edge in H', and thus, the tree $\mathbf{T}$ is maximal of height ℓ_0. In this case, we let F be the rigid construction given by Lemma 3.15 whose tree is $\mathbf{T}$. In either case, let $\mathbf{W} = (W_{\mathbf{i}})$ be the partition structure of F. Since the number of possible trees $\mathbf{T}$ is bounded by a function of ℓ_0 and we have $\ell_0 \ll M_0$, we can assume that

$$M_0 \geq v(F) + r. \tag{3.23}$$

Let us show that the maximal degree of B is small – namely, that

$$\Delta(B) < c_3\binom{n-1}{r-1}. \tag{3.24}$$

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Suppose on the contrary that $d_B(x) \geq c_3\binom{n-1}{r-1}$ for some $x \in [n]$. For $j \in [m_i]$, let the $(r-1)$-graph $B_{x,j}$ consist of those $D \in L_G(x)$ such that if we add j to the profile of D, then the obtained r-multiset is bad. In other words, if we move x to V_j, then $B_{x,j}$ will be the link of x with respect to the updated bad r-graph B. By the definition of H, we have

$$|B_{x,j}| \geq (c_3 - c_2)\binom{n-1}{r-1} \quad \text{for every } j \in [m_i]. \tag{3.25}$$

For $\mathbf{D} = (D_1, \ldots, D_{m_i}) \in \prod_{j=1}^{m_i} B_{x,j}$, let $F_{\mathbf{D}}$ be the r-graph that is constructed as follows. Recall that F is the rigid P_I-mixing construction given by Lemma 3.14 or 3.15, and $\mathbf{W}$ is its partition structure. By relabelling vertices of F, we can assume that $x \notin V(F)$ while $D := \bigcup_{j=1}^{m_i} D_j$ is a subset of $V(F)$ so that for every $y \in D$, we have $\mathrm{br}_F(y) = \mathrm{br}_{H'}(y)$; that is, y has the same branches in both F and H'. This is possible because these P_I-mixing constructions have the same trees of height at most ℓ_0 while each part of F of height at most ℓ_0 has at least $m_i(r-1) \geq |D|$ vertices. Finally, add x as a new vertex and the sets $D_j \cup \{x\}$ for $j \in [m_i]$ as edges, obtaining the r-graph $F_{\mathbf{D}}$.

Claim 3.16.1. For every $\mathbf{D} \in \prod_{j=1}^{m_i} B_{x,j}$, we have $F_{\mathbf{D}} \in \mathcal{F}_{M_0}$.

Proof of Claim. Recall that M_0 was chosen to be sufficiently large depending on ℓ_0. When we applied Lemma 3.14 or 3.15, the input tree $\mathbf{T}$ had height ℓ_0. By (3.23), we have $v(F_{\mathbf{D}}) = v(F) + 1 \leq M_0$.

So, suppose on the contrary that we have an embedding f of $F_{\mathbf{D}}$ into some P_I-mixing construction F' with the partition structure $\mathbf{U}$. By the rigidity of F, we can assume that the base of F' is P_i and that $f(W_j) \subseteq U_j$ for every $j \in [m_i]$. Let $j \in [m_i]$ satisfy $f(x) \in U_j$. Then the edge $D_j \cup \{x\} \in F_{\mathbf{D}}$ is mapped into a non-edge because $f(D_j \cup \{x\})$ has bad profile with respect $U_1, \ldots, U_{m_i}$ by the choice of $D_j \in B_{x,j}$, a contradiction. □

For every vector $\mathbf{D} = (D_1, \ldots, D_{m_i}) \in \prod_{j=1}^{m_j} B_{x,j}$ and every map $f\colon V(F_{\mathbf{D}}) \to V(G)$ such that f is the identity on $\{x\} \cup (\bigcup_{j=1}^{m_i} D_j)$ and f preserves branches of height up to ℓ_0 on all other vertices, the image $f(F_{\mathbf{D}})$ has to contain some $X \in \overline{G}$ by Claim 3.16.1. (Recall that G is $\mathcal{F}_{M_0}$-free.) Also,

$$f\big(F_{\mathbf{D}} \setminus \{D_1 \cup \{x\}, \ldots, D_{m_i} \cup \{x\}\}\big) \subseteq H';$$

that is, the underlying copy of F on which $F_{\mathbf{D}}$ was built is embedded by f into H'. However, each of the edges $D_1 \cup \{x\}, \ldots, D_{m_i} \cup \{x\}$ of $F_{\mathbf{D}}$ that contain x is mapped to an edge of G (to itself). Thus, $X \in H' \setminus G$ and $X \not\ni x$. Any such X can appear, very roughly, for at most $\binom{w}{r-1}^{m_i}(w+1)!\,n^{w-r}$ choices of $(\mathbf{D}, f)$, where $w := v(F) = v(F_{\mathbf{D}}) - 1$. However, the total number of choices of $(\mathbf{D}, f)$ is at least $\prod_{j=1}^{m_i} |B_{x,j}| \times (c_4 n/2)^{w-(r-1)m_i} \geq \left((c_3 - c_2)\binom{n-1}{r-1}\right)^{m_i} \times (c_4 n/2)^{w-(r-1)m_i}$ (since every part of H' has at least $c_4 n$ vertices by (3.20)). We conclude that

$$|H \setminus G| \geq |H' \setminus G| \geq \frac{\left((c_3 - c_2)\binom{n-1}{r-1}\right)^{m_i} \times (c_4 n/2)^{w-(r-1)m_i}}{\binom{w}{r-1}^{m_i}(w+1)!\,n^{w-r}} > c_2\binom{n}{r}.$$

However, this contradicts (3.21). Thus, (3.24) is proved.

Take any bad edge $D \in B$. We are going to show (in Claim 3.16.3 below) that D must intersect $\Omega(c_3 n^{r-1})$ absent edges. We need some preparation first.

For each $j \in R_i$ and $y \in D \cap V_j$, pick some $D_y \in G[V_j]$ such that $D_y \cap D = \{y\}$; it exists by Part (d) of Lemma 3.9, which gives that

$$d_{G[V_j]}(y) \geq c_4\binom{n-1}{r-1} \quad \text{for all } j \in R_i \text{ and } y \in V_j. \tag{3.26}$$

Let $\mathbf{D} := (D, \{D_y : y \in D \cap V_{R_i}\})$. (Recall that we denote $V_{R_i} = \bigcup_{j \in R_i} V_j$.) We define the r-graph $F^{\mathbf{D}}$ using the rigid r-graph F as follows. By relabelling $V(F)$, we can assume that $X \subseteq V(F)$, where

$$X := \bigcup_{y \in D \cap V_{R_i}} D_y \setminus \{y\}, \tag{3.27}$$

so that for every $x \in X$, its branches in F and H' coincide. Again, there is enough space inside F to accommodate all $|X| \leq r(r-1)$ vertices of X. Assume also that D is disjoint from $V(F)$. The vertex set of $F^{\mathbf{D}}$ is $V(F) \cup D$. The edge set of $F^{\mathbf{D}}$ is defined as follows. Starting with the edge-set of F, add D and each D_y with $y \in D \cap V_{R_i}$. Finally, for every $y \in D \cap V_j$ with $j \in [m_i] \setminus R_i$ pick some $z \in W_j$ and add $\{Z \cup \{y\} \mid Z \in F_z\}$ to the edge set, obtaining the r-graph $F^{\mathbf{D}}$. The last step can be viewed as enlarging the part W_j by $D \cap V_j$ and adding those edges that are stipulated by the pattern P_i and intersect D in at most one vertex.

Claim 3.16.2. For every $\mathbf{D}$ as above, we have $F^{\mathbf{D}} \in \mathcal{F}_{M_0}$.

Proof of Claim. By (3.23), we have that $v(F^{\mathbf{D}}) = v(F) + r \leq M_0$. Suppose on the contrary that we have an embedding f of $F^{\mathbf{D}}$ into some P_I-mixing construction F' with the partition structure $\mathbf{U}$. We can assume by the rigidity of F, that the base of F' is P_i and $f(W_j) \subseteq U_j$ for each i.

Let us show that for any $y \in D$, we have $f(y) \in U_j$, where the index $j \in [m_i]$ satisfies $y \in W_j$. First, suppose that $j \in R_i$. The $(r-1)$-set $f(D_y \setminus \{y\})$ lies entirely inside U_j. We cannot have $f(y) \in U_k$ with $k \neq j$ because otherwise the profile of the edge $f(D_y)$ is $\{\{ j^{(r-1)}, k \}\}$, contradicting Lemma 2.3. Thus, $f(y) \in U_j$, as claimed. Next, suppose that $j \in [m_i] \setminus R_i$. Pick some $z \in W_j$. By the rigidity of F, if we fix the restriction of f to $V(F) \setminus \{z\}$, then U_j is the only part where z can be mapped to. By definition, y and z have the same link $(r-1)$-graphs in $F^{\mathbf{D}}$ when restricted to $V(F) \setminus \{y, z\}$. Hence, $f(y)$ necessarily belongs to U_j, as claimed.

Thus, the edge $f(D)$ has the same profile as $D \in B$, a contradiction. □

Claim 3.16.3. For every $D \in B$ there are at least $10rc_3\binom{n-1}{r-1}$ absent edges $Y \in A$ with $|D \cap Y| = 1$.

Proof of Claim. Given $D \in B$, choose the sets D_y for $y \in D \cap V_{R_i}$ as before Claim 3.16.2. The condition $D_y \cap D = \{y\}$ rules out at most $r\binom{n-2}{r-2}$ edges for this y. Thus, by (3.26) there are, for example, at least $(c_4/2)\binom{n-1}{r-1}$ choices of each D_y. Form the r-graph $F^{\mathbf{D}}$ as above and consider potential injective embeddings f of $F^{\mathbf{D}}$ into G that are the identity on $D \cup X$ and map every other vertex of F into a vertex of H' with the same branch, where X is defined by (3.27). For every vertex $x \notin D \cup X$, we have at least $c_4 n/2$ choices for $f(x)$ by (3.20). By Claim 3.16.2, G does not contain $F^{\mathbf{D}}$ as a subgraph so its image under f contains some $Y \in \overline{G}$. Since f maps D and each D_y to an edge of G (to itself) and

$$f\Big(F^{\mathbf{D}} \setminus (\{D\} \cup \{D_y : y \in D \cap V_{R_i}\})\Big) \subseteq H',$$

we have that $Y \in H'$. The number of choices of $(\mathbf{D}, f)$ is at least

$$\left((c_4/2)\binom{n-1}{r-1}\right)^{|D \cap V_{R_i}|} \times (c_4 n/2)^{w-(r-1)|D \cap V_{R_i}|} \geq \left(\frac{c_4 n}{4r}\right)^w,$$

where $w := v(F)$. Assume that for at least half of the time the obtained set Y intersects D for otherwise we get a contradiction to (3.21):

$$|H' \setminus G| \geq \frac{1}{2} \times \frac{(c_4 n/4r)^w}{\binom{w}{r-1}^r (w+r)!\, n^{w-r}} > c_2\binom{n}{r}.$$

By the definitions of $F^{\mathbf{D}}$ and f, we have that $|Y \cap D| = 1$ and $Y \in A$. Each such $Y \in A$ is counted for at most $\binom{w}{r-1}^r (w+r)!\, n^{w-r+1}$ choices of f and $F^{\mathbf{D}}$. Thus, the number of such Y is at least $\frac{1}{2}(c_4 n/4r)^w / (\binom{w}{r-1}^r (w+r)!\, n^{w-r+1})$, implying the claim. □

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Let us count the number of pairs (Y, D) where $Y \in A$, $D \in B$, and $|Y \cap D| = 1$. On one hand, each bad edge $D \in B$ creates at least $10rc_3\binom{n-1}{r-1}$ such pairs by Claim 3.16.3. On the other hand, we trivially have at most $r|A| \cdot \Delta(B)$ such pairs. Therefore, $|B| \cdot 10rc_3\binom{n-1}{r-1} \le r|A|\Delta(B)$, which, by (3.24), implies that $|A| \ge 10\,|B|$, as desired. This proves Lemma 3.16. □

Let us state a special case of a result of Rödl and Schacht [33, Theorem 6] that we will need.

Lemma 3.17 (Strong Removal Lemma [33]). *For every r-graph family $\mathcal{F}$ and $\varepsilon > 0$, there are $\delta > 0$, M_1, and n_0 such that the following holds. Let G be an r-graph on $n \ge n_0$ vertices such that for every $F \in \mathcal{F}$ with $v(F) \le M_1$, the number of F-subgraphs in G is at most $\delta n^{v(F)}$. Then G can be made $\mathcal{F}$-free by removing at most $\varepsilon\binom{n}{r}$ edges.*

Lemma 3.18. *For every $c_0 > 0$, there is M_1 such that every maximum $\mathcal{F}_{M_1}$-free G with $n \ge M_1$ vertices is $c_0\binom{n}{r}$-close to a P_I-mixing construction.*

Proof. Lemma 3.17 for $c_0/2$ gives M_1 such that any $\mathcal{F}_{M_1}$-free r-graph G on $n \ge M_1$ vertices can be made into an $\mathcal{F}_\infty$-free r-graph G' by removing at most $c_0\binom{n}{r}/2$ edges. By Lemma 3.2, G' embeds into some P_I-mixing construction H with $v(H) = v(G')$. Assume that $V(H) = V(G')$ and the identity map is an embedding of G' into H.

Since H is $\mathcal{F}_{M_1}$-free, the maximality of G implies that $|G| \ge |H|$. Thus, $|H \setminus G'| \le c_0\binom{n}{r}/2$, and we can transform G' into H by changing at most $c_0\binom{n}{r}/2$ further edges. □

3.5. Proof of Theorem 1.2: Putting All Together

We are ready to prove Part (a) of Theorem 1.2. Let all assumptions of Section 3.1 apply.

Proof of Theorem 1.2(a). Let Lemma 3.16 return c_0 and M_0. Then let Lemma 3.18 on input c_0 return some M_1. Finally, take sufficiently large M depending on the previous constants.

Let us argue that this M works in Theorem 1.2(a). As every graph in ΣP_I is $\mathcal{F}_M$-free, it is enough to show that every maximum $\mathcal{F}_M$-free r-graph is a P_I-mixing construction. We use induction on the number of vertices n. Let G be any maximum $\mathcal{F}_M$-free r-graph on $[n]$. Suppose that $n > M$, for otherwise, we are done by Lemma 3.2. Thus, Lemma 3.18 applies and shows that G is $c_0\binom{n}{r}$-close to some P_I-mixing construction. Lemma 3.7 shows additionally that the minimum degree of G is at least $(\lambda - c_0)\binom{n-1}{r-1}$. Thus, Lemma 3.16 applies and returns a partition $[n] = V_1 \cup \cdots \cup V_{m_i}$ for some $i \in I$ such that (3.16) holds; that is, $|A| \ge 10\,|B|$, where the sets A of absent and B of bad edges are defined by (3.17) and (3.18), respectively. Now, if we take the union of $E_i((V_1, \ldots, V_{m_i}))$ with $\bigcup_{j \in R_i} G[V_j]$, then the obtained r-graph is still $\mathcal{F}_M$-free by Lemma 3.6 and has exactly $|A| - |B| + |G|$ edges. The maximality of G implies that $|B| \ge |A|$. By (3.16), this is possible only if $A = B = \emptyset$. Thus, G coincides with the blowup $E_i((V_1, \ldots, V_{m_i}))$, apart edges inside the recursive parts V_j, $j \in R_i$.

Let $j \in R_i$ be arbitrary. By Lemma 3.6, if we replace $G[V_j]$ by any $\mathcal{F}_M$-free r-graph, then the new r-graph on V is still $\mathcal{F}_M$-free. By the maximality of G, we conclude that $G[V_j]$ is a maximum $\mathcal{F}_M$-free r-graph. By the induction hypothesis (note that $|V_j| \le n-1$), the induced subgraph $G[V_j]$ is a P_I-mixing construction. It follows that G is a P_I-mixing construction itself, which implies Theorem 1.2(a). □

In order to prove Part (b) of Theorem 1.2, we prove the following partial result first, where stability is proved for the 'bottom' edges only.

Lemma 3.19. *There exists $M_2 \in \mathbb{N}$ so that for every $\varepsilon > 0$, there exist $\delta_0 > 0$ and n_0 such that the following holds for all $n \ge n_0$. Suppose that G is a $\mathcal{F}_{M_2}$-free r-graph on n vertices with $|G| \ge (1-\delta_0)\mathrm{ex}(n, \mathcal{F}_{M_2})$. Then there exist $i \in I$ and a partition $V(G) = V_1 \cup \cdots \cup V_{m_i}$ with $|V_j| \in [\beta n/4, (1-\lambda/4r)n]$ for all $j \in [m_i]$ such that $G' := G \setminus (\bigcup_{j\in R_i} G[V_j])$ satisfies $|G' \triangle E_i((V_1, \ldots, V_{m_i}))| \le \varepsilon n^r$.*

Proof. Let c_0 and M_0 be the constants returned by Lemma 3.16. Then let M_2 be sufficiently large, in particular so that it satisfies Lemma 3.18 for $c_0/4$. Let us show that this M_2 works in the lemma.

Given any $\varepsilon > 0$, choose sufficiently small positive constants $\delta_1 \gg \delta_0$. Let G be an $\mathcal{F}_{M_2}$-free r-graph on $n \to \infty$ vertices with $|G| \geq (1-\delta_0)\mathrm{ex}(n, \mathcal{F}_{M_2})$. By our choice of M_2, the r-graph G can be embedded into some n-vertex P_I-mixing construction H by removing at most $c_0\binom{n}{r}/4$ edges. Since $|G| \geq (1-\delta_0)\mathrm{ex}(n, \mathcal{F}_{M_2}) \geq |H| - \delta_0\binom{n}{r}$, we have $|G \triangle H| \leq 2 \cdot c_0\binom{n}{r}/4 + \delta_0\binom{n}{r} \leq 3c_0\binom{n}{r}/4$.

Define

$$Z := \left\{ v \in V(G) \colon d_G(v) \leq (\lambda - r\delta_1)\binom{n-1}{r-1} \right\}.$$

Claim 3.19.1. We have that $|Z| < \delta_1 n$.

Proof of Claim. Suppose to the contrary that $|Z| \geq \delta_1 n$. Let G' be obtained from G by removing some $\delta_1 n$ vertices of Z. We have that

$$\begin{aligned} |G'| - \lambda\binom{n-\delta_1 n}{r} &\geq |G| - \delta_1 n(\lambda - r\delta_1)\binom{n-1}{r-1} - \lambda(1-\delta_1)^r\binom{n}{r} + o(n^r) \\ &\geq \left((\lambda - \delta_0) - r\delta_1(\lambda - r\delta_1) - \lambda\left(1 - r\delta_1 + \binom{r}{2}\delta_1^2\right)\right)\binom{n}{r} + o(n^r) \\ &\geq \left(-\delta_0 + r^2\delta_1^2 - \lambda\binom{r}{2}\delta_1^2\right)\binom{n}{r} + o(n^r) > \Omega(n^r). \end{aligned}$$

Thus, the $\mathcal{F}_{M_2}$-free r-graph G' contradicts the consequence of Theorem 1.2(a) that $\pi(\mathcal{F}_{M_2}) = \lambda$. □

Let $n_1 := n - |Z|$ and $G_1 := G - Z$. So G_1 is an r-graph on $n_1 \geq (1-\delta_1)n$ vertices with

$$\delta(G_1) \geq (\lambda - r\delta_1)\binom{n-1}{r-1} - \delta_1 n\binom{n-2}{r-2} \geq (\lambda - 2r\delta_1)\binom{n-1}{r-1}, \quad \text{and}$$

$$|G_1| \geq (1-\delta_0)\mathrm{ex}(n, \mathcal{F}_{M_2}) - \delta_1 n\binom{n-1}{r-1} \geq \mathrm{ex}(n, \mathcal{F}_{M_2}) - 2r\delta_1\binom{n}{r}.$$

Let H_1 be the induced subgraph of H on $V(G) \setminus Z$ and note that H_1 is also a P_I-mixing construction. Since $|G_1 \triangle H_1| \leq |G \triangle H| \leq 3c_0\binom{n}{r}/4 \leq c_0\binom{n_1}{r}$, by Lemma 3.16, there are $i \in I'$ and a partition $V(G) \setminus Z = V_1' \cup \cdots \cup V_{m_i}'$ such that $|V_j'| \in [\beta n_1/2, (1-\lambda/2r)n_1]$ for all $j \in [m_i]$ and it holds that $|A_1| \geq 10\,|B_1|$, where $A_1 := E_i((V_1', \ldots, V_{m_i}')) \setminus G$ and $B_1 := G_1 \setminus \left(E_i((V_1', \ldots, V_{m_i}')) \cup \bigcup_{j \in R_i} G_1[V_j]\right)$. If we take the union of $E_i((V_1', \ldots, V_{m_i}'))$ with $\bigcup_{j \in R_i} G_1[V_j']$, then the obtained r-graph is still $\mathcal{F}_{M_2}$-free (by Lemma 3.6) and has exactly $|G_1| + |A_1| - |B_1| \geq |G_1| + \frac{9}{10}|A_1|$ edges. Therefore, $|G_1| + \frac{9}{10}|A_1| \leq \mathrm{ex}(n_1, \mathcal{F}_{M_2})$, which implies that

$$|A_1| + |B_1| \leq \frac{11}{10} \cdot \frac{10}{9}\left(\mathrm{ex}(n_1, \mathcal{F}_{M_2}) - |G_1|\right) \leq 4r\delta_1\binom{n}{r}.$$

Now, extend the partition $V_1', \ldots, V_{m_i}'$ arbitrarily to $V(G)$, for example, by defining

$$V_j := \begin{cases} V_1' \cup Z, & j = 1, \\ V_j', & j \neq 1. \end{cases}$$

Then simple calculations show that $|V_j| \in [\beta n/4, (1-\lambda/4r)n_1]$ for all $j \in [m_i]$, and the r-graph $G' := G \setminus (\bigcup_{j \in R_i} G[V_j])$ satisfies

$$|G' \triangle E_i((V_1, \ldots, V_{m_i}))| \leq |Z|\binom{n-1}{r-1} + |A_1| + |B_1| \leq \varepsilon n^r.$$

This completes the proof of Lemma 3.19. □

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Now we are ready to prove Part (b) of Theorem 1.2.

Proof of Theorem 1.2(b). Let M_0 and c_0 be returned by Lemma 3.16. Let M_1 be returned by Lemma 3.18 for c_0, and let M_2 be returned by Lemma 3.19. Let us show that $M := \max\{M_0, M_1, M_2\}$ works in Theorem 1.2(b).

Take any $\varepsilon > 0$. Let $\ell \in \mathbb{N}$ be a sufficiently large integer such that, in particular, $\left(1-\frac{\lambda}{4r}\right)^{\ell} \ll \varepsilon$. Next, choose sufficiently small positive constants $\delta_\ell \gg \cdots \gg \delta_1 \gg \delta$. Let n be sufficiently large. Let G be an $\mathcal{F}_M$-free r-graph on n vertices with $|G| \geq (1-\delta)\mathrm{ex}(n,\mathcal{F}_M)$. By Lemma 3.19, there exist $i \in I$ and a partition $V(G) = V_1 \cup \cdots \cup V_{m_i}$ with $|V_j| \in [\beta n/4, (1-\lambda/4r)n]$ such that $G' := G \setminus \left(\bigcup_{j\in R_i} G[V_j]\right)$ satisfies $|G' \triangle E_i((V_1,\ldots,V_{m_i}))| \leq \delta_1 n^r$.

Let $\hat{G} := E_i((V_1,\ldots,V_{m_i})) \cup \left(\bigcup_{j\in R_i} G[V_j]\right)$. Then Lemma 3.6 implies that $\hat{G}$ is still $\mathcal{F}_M$-free, and our argument above shows that $|\hat{G}| \geq |G| - \delta_1 n^r \geq \mathrm{ex}(n,\mathcal{F}_M) - 2\delta_1 n^r$.

Now take any $j \in R_i$. Note that we have $|G[V_j]| \geq \mathrm{ex}(|V_j|,\mathcal{F}_M) - 2\delta_1 n^r$, since otherwise, by replacing $G[V_j]$ in $\hat{G}$ by a maximum $\mathcal{F}_M$-free r-graph on V_j, we would get an r-graph which is $\mathcal{F}_M$-free (by Lemma 3.6) and has more than $\mathrm{ex}(n,\mathcal{F}_M)$ edges, a contradiction. Since $|V_j| \geq \beta n/4$ is sufficiently large and $\delta_1 \ll \delta_2$, there exist, by Lemma 3.19 again, an index $i' \in I$ and a partition $V_j = V'_{j,1} \cup \cdots \cup V'_{j,m_{i'}}$ such that $|V'_{j,k}| \in [(\beta/4)^2 n, (1-\lambda/4r)^2 n]$ for all $k \in [m_{i'}]$ and $G'_j := G[V_j] \setminus \left(\bigcup_{k\in R_{i'}} G[V_{j,k}]\right)$ satisfies $|G'_j \triangle E_{i'}((V_{j,1},\ldots,V_{j,m_{i'}}))| \leq \delta_2 |V_j|^r$. Summing over all $j \in R_i$, we get

$$\sum_{j\in R_i} |G'_j \triangle E_{i'}((V_{j,1},\ldots,V_{j,m_{i'}}))| \leq \delta_2 \sum_{j\in R_i} |V_j|^r \leq \delta_2 \left(\sum_{j\in R_i} |V_j|\right)^r \leq \delta_2 n^r.$$

Repeating this argument until the ℓ-th level, we see that G can be made into a P_I-mixing construction by removing and adding at most

$$\sum_{i=1}^{\ell} \delta_i n^r + \sum_{\mathbf{i}} \binom{|V_{\mathbf{i}}|}{r} \leq \sum_{i=1}^{\ell} \delta_i n^r + \frac{n}{\left(1-\frac{\lambda}{4r}\right)^{\ell} n} \binom{\left(1-\frac{\lambda}{4r}\right)^{\ell} n}{r}$$
$$\leq \left(\sum_{i=1}^{\ell} \delta_i + \left(1-\frac{\lambda}{4r}\right)^{\ell}\right) n^r \leq \varepsilon n^r$$

edges. Here, the second summation is over all legal (with respect to G) vectors of length ℓ. This completes the proof of Theorem 1.2(b). □

4. Finite r-graph families with rich extremal Turán constructions

In this section, we prove Theorem 1.3. We need some preliminaries first.

Recall that a **Steiner triple system** on a set V is a 3-graph D with vertex set V such that every pair of distinct elements of V is covered by exactly one edge of D. Let $\mathcal{STS}_t$ be the set of all Steiner triple systems on $[t]$. It is known that such a design D exists (and thus $\mathcal{STS}_t \neq \emptyset$) if and only if the residue of $t \geq 3$ modulo 6 is 1 or 3, a result that was proved by Kirkman [21] already in 1847.

We will need the following result, which is a special case of [27, Proposition 2.2].

Lemma 4.1 [27]. *Let $t \geq 55$ and $D \in \mathcal{STS}_t$ be arbitrary. Then, for every $(x_1,\ldots,x_t) \in \mathbb{S}_t$, it holds that*

$$\lambda_{\overline{D}}(x_1,\ldots,x_t) \leq \lambda_{\overline{D}}\left(\frac{1}{t},\ldots,\frac{1}{t}\right) - \frac{2}{3}\sum_{i=1}^{t}\left(x_i - \frac{1}{t}\right)^2.$$

(Recall that $\overline{D} = \binom{[t]}{3} \setminus D$ denotes the complement of D.)

The above lemma implies, in particular, that the uniform weight is the unique $\overline{D}$-optimal vector. Also, note that $|\overline{D}| = \binom{t}{3} - \frac{1}{3}\binom{t}{2} = \frac{t(t-1)(t-3)}{6}$, and thus,

$$\lambda_{\overline{D}} = \lambda_{\overline{D}}\left(\frac{1}{t}, \ldots, \frac{1}{t}\right) = \frac{3!\,|\overline{D}|}{t^3} = \frac{(t-1)(t-3)}{t^2}.$$

Here, $\lambda_{\overline{D}} := \lambda_{(t,\overline{D},\emptyset)}$ denotes the maximum of the Lagrange polynomial $\lambda_{\overline{D}}(\mathbf{x})$ of the 3-graph $\overline{D}$ over $\mathbf{x} \in \mathbb{S}_t$.

For $D \in \mathcal{STS}_t$, let $P_D := (t, \overline{D}, [t])$ be the pattern where every vertex of the complementary 3-graph $\overline{D}$ is made recursive. If we take the uniform blowups of $\overline{D}$ at all levels when making a P_D-construction, then the obtained limiting edge density λ_1 satisfies the relation $\lambda_1 = \lambda_{\overline{D}} + \lambda_1 t(1/t)^3$ which gives that

$$\lambda_{P_D} \geq \lambda_1 = \frac{\lambda_{\overline{D}}}{1 - 1/t^2} = \frac{(t-1)(t-3)}{t^2 - 1} = \frac{t-3}{t+1}.$$

Lemma 4.2. *Let t be sufficiently large and let $D \in STS_t$. Let $f(\mathbf{x}) := \lambda_{\overline{D}}(\mathbf{x}) + \lambda_1 \sum_{i=1}^t x_i^3$. Then $f(\mathbf{x}) \leq \lambda_1$ for every $\mathbf{x} \in \mathbb{S}_t^*$ with equality if and only if $\mathbf{x}$ is the uniform vector $(\frac{1}{t}, \ldots, \frac{1}{t})$.*

It follows that $\lambda_{P_D} = \frac{t-3}{t+1}$ and the set of P_D-optimal vectors consists only of the uniform vector $(\frac{1}{t}, \ldots, \frac{1}{t}) \in \mathbb{S}_t^*$.

Proof. Take any $\mathbf{x} \in \mathbb{S}_t^*$. In order to prove that $f(\mathbf{x}) \leq \lambda_1$, we split the argument into two cases.

First, suppose that $\max\{x_i : i \in [t]\} \leq 1/2$. Here, we have by Lemma 4.1 that

$$f(\mathbf{x}) \leq \lambda_{\overline{D}}\left(\frac{1}{t}, \ldots, \frac{1}{t}\right) - \frac{2}{3}\sum_{i=1}^t \left(x_i - \frac{1}{t}\right)^2 + \lambda_1 \sum_{i=1}^t x_i^3 = \lambda_{\overline{D}} + \sum_{i=1}^t g(x_i),$$

where $g(x) := -\frac{2}{3}(x - \frac{1}{t})^2 + \lambda_1 x^3$ for $x \in [0, \frac{1}{2}]$.

The second derivative of the cubic polynomial g has zero at $x_0 := \frac{2(t+1)}{9(t-3)}$. We have that $x_0 > 1/t$. While this can be checked to hold for every t, it is trivial here since t was assumed to be sufficiently large. Thus function g is concave on $[0, \frac{1}{t}]$. Unfortunately, it is not concave on the whole interval $[0, 1/2]$, so we consider a different function g^* which equals g on $[0, \frac{1}{t}]$ and whose graph on $[\frac{1}{t}, \frac{1}{2}]$ is the line tangent to the graph of g at $1/t$; that is, we set

$$g^*(x) := g\left(\frac{1}{t}\right) + g'\left(\frac{1}{t}\right)\left(x - \frac{1}{t}\right), \quad \text{for } x \in [1/t, 1/2].$$

By above, g^* is a concave function on $[0, 1/2]$. Also, we have that $g \leq g^*$ on this interval. Indeed, since the second derivative g'' changes sign from negative to positive at $x_0 > 1/t$, it is enough to check only that $g(1/2) \leq g^*(1/2)$ and routine calculations give that

$$g^*(1/2) - g(1/2) = \frac{(t-2)^2(t^2 + t + 36)}{24t^3(t+1)} > 0,$$

as desired. Thus, since g^* is a concave majorant of g, we have that

$$\frac{1}{t}\sum_{i=1}^t g(x_i) \leq \frac{1}{t}\sum_{i=1}^t g^*(x_i) \leq g^*\left(\frac{x_1 + \cdots + x_t}{t}\right) = g^*(1/t) = g(1/t).$$

This gives that $f(\mathbf{x}) \leq \lambda_{\overline{D}} + tg(1/t) = \lambda_1$. Moreover, if we have equality, then $x_1 = \cdots = x_t = 1/t$ (because g^* is strictly concave on $[0, 1/t]$).

Thus, it remains to consider the case when some x_i is strictly larger than $1/2$. Without loss of generality, assume that $x_1 > 1/2$. Here, we can bound

$$f(\mathbf{x}) \le h(x_1) := 3!\, x_1(1-x_1)^2 \frac{t-3}{2(t-1)} + \frac{3!\binom{t-1}{3}}{(t-1)^3}(1-x_1)^3 + \lambda_1\Big(x_1^3 + (1-x_1)^3\Big),$$

where the stated three terms come from the following arguments. The first term accounts for the triples containing x_1 in the Lagrange polynomial $\lambda_{\overline{D}}(\mathbf{x})$. The link graph $L_D(1)$ is just a perfect matching M on $\{2,\ldots,t\}$ (because D is a Steiner triple system) and receives total weight $1-x_1$. As it is well-known (see, for example, [28]), the Largrangian of a graph is maximised by putting the uniform weight on a maximum clique which, for the complement $L_{\overline{D}}(1)$ of a perfect matching, has size $s := (t-1)/2$. Thus, $(1-x_1)^{-2}\sum_{ij\in L_D(1)} x_i x_j \le \binom{s}{2}/s^2 = \frac{s-1}{2s} = \frac{t-3}{2(t-1)}$, giving the first term. The second term just upper bounds the Lagrangian of $\overline{D} - 1 = \overline{D}[\{2,\ldots,t\}]$ by the Largrangian of the complete 3-graph on $t-1$ vertices, scaling the result by the cube of the total weight $1-x_1$. The third term uses the fact that the sum of cubes of nonnegative entries with sum $1-x_1$ is maximised when we put all weight on a single element.

The coefficient at x^3 in the cubic polynomial $h(x)$ is $\frac{2t^2-10t+9}{(t-1)^2} > 0$. Also, the derivative of h has two roots, which are $\pm 1/\sqrt{2} + o(1)$ as $t \to \infty$. Thus (since t is large), the function h, when restricted to the interval on $[1/2, 1]$, first decreases and then increases. So, in order to show that $f(\mathbf{x}) \le \lambda_1$, it is enough to check the inequality $h(x) \le \lambda_1$ only at the points $x = 1/2$ and $x = 1$. There, the values of $h(x) - \lambda_1$ are, respectively, $-\frac{2t^3-20t^2+47t-27}{8(t-1)^2(t+1)} < 0$ and 0. Thus, $f(\mathbf{x}_1) \le \lambda_1$ also in Case 2. Furthermore, the equality can only be attained if $x_1 = 1$ (and $x_2 = \cdots = x_t = 0$); however, we exclude standard basis vectors from $\mathbb{S}_t^*$. This proves the first part of the lemma.

Using the proved inequality $f \le \lambda_1$, one can show by a simple induction on the number of levels that every P-construction on $n \to \infty$ vertices has edge density at most $\lambda_1 + o(1)$. Thus, $\lambda_{P_D} = \lambda_1$. Also, the set of P_D-extremal vectors, which by definition consists of $\mathbf{x} \in \mathbb{S}_t^*$ with $f(\mathbf{x}) = \lambda_1$, is exactly as claimed. □

In the rest of the this section, whenever we have any $I \subseteq \mathcal{STS}_t$, a set of Steiner triple systems on $[t]$, we denote $P_I := \{P_D : D \in I\}$. (Recall that $P_D = (t, \overline{D}, [t])$.) Also, let us call a partition $V = V_1 \cup \cdots \cup V_t$ **balanced** if for all $i, j \in [t]$ we have $\big||V_i| - |V_j|\big| \le 1$.

Lemma 4.3. *Let t be sufficiently large and take any nonempty $I \subseteq \mathcal{STS}_t$. Then there is $n_0 = n_0(I)$ such that every maximum P_I-mixing construction G on $n \ge n_0$ vertices has its bottom partition balanced.*

Proof. View n as tending to ∞ and take any maximum P_I-mixing construction G on $[n]$. Let G have the base P_D and the bottom partition $[n] = V_1 \cup \cdots \cup V_t$. Let $v_i := |V_i|$ and $e_i := |G[V_i]|$ for $i \in [t]$. By Lemma 4.2, we have $v_i = (1/t + o(1))n$ for each $i \in [t]$.

Suppose on the contrary that some two part sizes differ by more than 1, say, $v_1 \ge v_2 + 2$. Let u (resp. w) be a vertex of minimum degree in $G[V_1]$ (resp. maximum degree in $G[V_2]$). Let G' be obtained from G by removing u and adding a double w' of w. Thus, G' is also a P_I-mixing construction, with the bottom parts $V_1' := V_1 \setminus \{u\}$, $V_2' := V_2 \cup \{w'\}$, and $V_i' := V_i$ for $3 \le i \le t$. By the maximality of G, we have $|G| \ge |G'|$.

Let us estimate $|G'| - |G|$. The contribution from the edges that intersect both the first part and the second part is

$$((v_1-1)(v_2+1) - v_1 v_2) \sum_{i:\ \{1,2,i\}\in\overline{D}} |V_i| = (v_1 - v_2 - 1)\left(\frac{t-3}{t} + o(1)\right)n,$$

where the equality uses Lemma 4.2 and the fact that there are exactly $t-3$ triples containing the pair $\{1,2\}$. By the maximality of G, each part V_i spans a maximum P_I-mixing construction; thus,

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$e_i = \Lambda_{P_I}(v_i)$. Since $v_1 \geq v_2$, we have by Lemma 3.3 that $\Lambda_{P_I}(v_1)/\binom{v_1}{3} \leq \Lambda_{P_I}(v_2)/\binom{v_2}{3}$. Thus, the degree of u in $G[V_1]$ is at most

$$\frac{3e_1}{v_1} = \frac{3\Lambda_{P_I}(v_1)}{v_1} \leq \frac{3\Lambda_{P_I}(v_2)\binom{v_1}{3}}{v_1\binom{v_2}{3}} = \frac{3e_2}{v_2}\cdot\frac{(v_1-1)(v_1-2)}{(v_2-1)(v_2-2)}.$$

The degree of w' in G' is at least the degree of w in G which in turn is at least $3e_2/v_2$. Thus, the contribution to $|G'|-|G|$ of edges inside the first or second part is at least

$$\frac{3e_2}{v_2}-\frac{3e_1}{v_1} \geq \frac{3e_2}{v_2}\left(1-\frac{(v_1-1)(v_1-2)}{(v_2-1)(v_2-2)}\right) = -\frac{3e_2}{v_2}\cdot\frac{(v_1-v_2)(v_1+v_2-3)}{(v_2-1)(v_2-2)}.$$

This is, by $e_2 = (\lambda_{P_D}+o(1))\binom{v_2}{3} = \left(\frac{t-3}{t+1}+o(1)\right)\binom{n/t}{3}$,

$$(-1+o(1))\,\frac{3\,\frac{t-3}{t+1}\binom{n/t}{3}}{n/t}\cdot\frac{(v_1-v_2)\,2n/t}{(n/t)^2} = -\frac{(v_1-v_2)(t-3)n}{t(t+1)}+o((v_1-v_2)n).$$

By putting both contributions together and using $v_1-v_2-1 \geq (v_1-v_2)/2$, we obtain that

$$\begin{aligned} 0 \geq |G'|-|G| \;&\geq\; (v_1-v_2-1)\,\frac{t-3}{t}n-\frac{(v_1-v_2)(t-3)n}{t(t+1)}+o((v_1-v_2)n)\\ &\geq (v_1-v_2)\,\frac{(t-3)n}{t}\left(\frac{1}{2}-\frac{1}{t+1}\right)+o((v_1-v_2)n) > 0, \end{aligned}$$

which is the desired contradiction. □

For every $D \in \mathcal{STS}_t$, we construct a parameter $F(D)$ of much lower complexity than D that nonetheless contains enough information to compute the sizes of maximum balanced $\overline{D}$-blowups of all large orders. More precisely, we do the following for every $q \in \{0,\ldots,t-1\}$. Let $\ell \to \infty$. For every q-set $Q \subseteq [t]$, consider a $\overline{D}$-blowup G on $[t\ell+q]$ with partition $V_1 \cup \cdots \cup V_t$, where

$$|V_i| = \begin{cases} \ell+1, & \text{if } i \in Q,\\ \ell, & \text{if } i \in [t]\setminus Q.\end{cases}$$

Thus, the q larger parts are exactly those specified by Q. Clearly, the size of G is

$$p_{D,Q}(\ell) := (\ell+1)^3 t_{D,Q,3}+\ell(\ell+1)^2 t_{D,Q,2}+\ell^2(\ell+1)t_{D,Q,1}+\ell^3\left(|\overline{D}|-t_{D,Q,3}-t_{D,Q,2}-t_{D,Q,1}\right),$$

where, for $i \in [3]$, we let $t_{D,Q,i}$ denote the number of triples in $\overline{D}$ that have exactly i vertices in Q. This is a polynomial function of ℓ. If we take another q-set Q' then the polynomial $p_{D,Q}(\ell)-p_{D,Q'}(\ell)$ does not change sign for large ℓ (possibly being the zero polynomial). Since there are finitely many choices of Q (namely, $\binom{t}{q}$ choices), there are $Q_{D,q} \in \binom{[t]}{q}$ and ℓ_0 such that

$$p_{D,Q_{D,q}}(\ell) \geq p_{D,Q'}(\ell), \quad \text{for all } Q' \in \binom{[t]}{q} \text{ and } \ell \geq \ell_0. \tag{4.1}$$

Fix one such $Q_{D,q}$ for each $q \in \{0,\ldots,t-1\}$ and define

$$F(D) := ((t_{D,Q_{D,q},i})_{i=1}^3)_{q=0}^{t-1}. \tag{4.2}$$

The number of possible values of F is upper bounded by, say, t^{9t} because each individual $t_{D,Q,i}$ assumes at most $\binom{t}{3}+1 \leq t^3$ possible values.

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Lemma 4.4. *Let t be sufficiently large. Let $I \subseteq \mathcal{STS}_t$ be any subset on which the above function F is constant. Then there is n_0 such that for all $n \geq n_0$ and all $D \in I$ there is a maximum P_I-mixing construction G on $[n]$ with the base P_D.*

Proof. Choose n_0 sufficiently large, in particular so that, for every $n \geq n_0$, (4.1) holds for every $D \in I$ with respect to $\ell := \lfloor n/t \rfloor$ and $q := n - t\ell$.

Take any $n \geq n_0$ and $D \in I$. We have to exhibit a maximum P_I-mixing construction G with base P_D. Let $\ell := \lfloor n/t \rfloor$, $q := n - t\ell$, and $Q := Q_{D,q}$. Take a balanced partition $V_1 \cup \cdots \cup V_t$ of $[n]$ with $|V_j| = \ell + 1$ exactly for $j \in Q$. Let G be obtained from $\overline{D}((V_1, \ldots, V_t))$ by adding for every $j \in [t]$ a maximum P_I-mixing construction on V_j.

Let us show that the P_I-mixing construction G is maximum. Note that the size of the graph G is

$$|G| = p_{D,Q}(\ell) + q\,\Lambda_{P_I}(\ell + 1) + (t - q)\,\Lambda_{P_I}(\ell). \tag{4.3}$$

Let G' be any maximum P_I-mixing construction on $[n]$. Let G' have the base D' and the bottom partition $V_1' \cup \cdots \cup V_t'$. This partition must be balanced by Lemma 4.3, since n is large. Let $Q' \in \binom{[t]}{q}$ consist of the indices of parts of size $\ell + 1$. Clearly, $|Q'| = q$. By the maximality of G', every part V_s', $s \in [t]$, induces a maximum P_I-mixing construction. Thus, the obvious analogue of (4.3) holds for G' as well. Let $Q'' := Q_{D',q}$. Since $\ell \geq \lfloor n_0/t \rfloor$ is large, we have $p_{D',Q'}(\ell) \leq p_{D',Q''}(\ell)$. Since $F(D) = F(D')$, the polynomials $p_{D',Q''}$ and $p_{D,Q}$ are the same. Putting all together, we obtain

$$|G'| - |G| = p_{D',Q'}(\ell) - p_{D,Q}(\ell) \leq p_{D',Q''}(\ell) - p_{D,Q}(\ell) = 0.$$

Thus, $|G|$ is indeed a maximum P_I-mixing construction. □

Let us remark that Lemma 4.4 need not hold for small n when it is in general possible that some of the patterns P_D for $D \in I$ cannot be the base in a maximum P_I-mixing construction.

Proof of Theorem 1.3. Keevash [20, Theorem 2.2] proved that if $t \to \infty$ is 1 or 3 modulo 6, then the number of Steiner triples systems on $[t]$ is $(t/\mathrm{e}^2 + o(1))^{t^2/6}$. Note that the function F assumes at most t^{9t} values while each isomorphism class of $\mathcal{STS}_t$ has at most $t!$ elements. Thus, we can fix a sufficiently large t and a subset $I \subseteq \mathcal{STS}_t$ consisting of non-isomorphic 3-graphs such that F is constant on I while

$$|I| \geq \frac{(t/\mathrm{e}^2 + o(1))^{t^2/6}}{t!\,t^{9t}} > t!.$$

Let $\mathcal{F}$ be the finite family of 3-graphs returned by Theorem 1.2; thus, maximum $\mathcal{F}$-free 3-graphs are exactly maximum P_I-mixing constructions. Let us show that this family $\mathcal{F}$ satisfies both parts of Theorem 1.3. Let n_0 be sufficiently large.

Given any $n \in \mathbb{N}$, let $V := [n]$ and consider the family $\mathcal{P}$ of 3-graphs G that can be recursively constructed as follows. If the current vertex set V has less than n_0 vertices, put any maximum P_I-mixing construction on V and stop. So suppose that $|V| \geq n_0$. Pick any $D \in I$. Let G' be a maximum P_I-mixing construction on V with the base P_D, which exists by Lemma 4.4. Let $V = V_1 \cup \cdots \cup V_t$ be the bottom partition of G'. Let G be obtained by taking all bottom edges of G' and adding for each $i \in [t]$ a 3-graph on V_i that can be recursively constructed by the above procedure.

Let us observe some easy properties of any obtained $G \in \mathcal{P}$. By definition, G is a P_I-mixing construction. In fact, it is a maximum one, which can be shown by induction on the number of vertices: the initial P_I-mixing construction G' is maximum, and when we 'erase' edges in $G'[V_i]$, we add back the same number of edges by induction.

Let the **size-m truncated tree** $\mathbf{T}_G^{\mathrm{size}\geq m}$ of P_I-mixing construction G be the subtree of $\mathbf{T}_G$ where we keep only those nodes that corresponds to parts in $\mathbf{V}_G$ of size at least m. Of course, if a node is not included into $\mathbf{T}_G^{\mathrm{size}\geq m}$, then all its descendants are not, so $\mathbf{T}_G^{\mathrm{size}\geq m}$ is a subtree of $\mathbf{T}_G$.

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In the rest of Section 4, we would need to work with **unordered** trees where the t children of a non-leaf vertex are not ordered and the first part $\mathbf{i}$ of each node $(\mathbf{i}, x)$, that can be used to order children, is erased (but we keep the second part x). For these objects we will use the (non-bold) symbol T instead of $\mathbf{T}$. In particular, the **size-$\mathbf{n_0}$ truncated tree** $T_G^{\text{size}\ge n_0}$ is the unordered subtree of the tree of the P_I-mixing construction G, where we keep only those nodes that correspond to parts of size at least n_0. Suppose now that $G \in \Sigma P_I$ is maximum. Then every branch of $T_G^{\text{size}\ge n_0}$ has length at least, say, $\log_{t+1}(n/n_0)$, because when we subdivide any set V with at least n_0 vertices, its partition is balanced by Lemma 4.3, and thus, the smallest part size is at least $\lfloor |V|/t \rfloor \ge |V|/(t+1)$. Thus, any resulting tree has at least $s := t^{\log_{t+1}(n/n_0)-1}$ non-leaf vertices. Since the children of every non-leaf can be labelled by any element of I (as long as we use the same element for all children), we have at least $|I|^s$ choices here. The number of ways that an isomorphic copy of any feasible I-labelled rooted tree T can be generated as above, rather roughly, is most $(t!)^s$. Thus, there are at least $(|I|/t!)^s$ non-isomorphic (unordered I-labelled rooted) trees obtainable this way. This is exponential in n (since n_0 is fixed); thus, the first part of Theorem 1.3 follows from the following claim.

Claim 4.4.1. If $G, G' \in \mathcal{P}$ are isomorphic 3-graphs, then their n_0-truncated unordered trees $T_G^{\text{size}\ge n_0}$ and $T_{G'}^{\text{size}\ge n_0}$ are isomorphic.

Proof of Claim. We use induction on n, the number of vertices in G. If $n < n_0$, then the truncated trees of G and G' are empty, and thus, the conclusion vacuously holds. Suppose $n \ge n_0$ and that the identity map on $[n]$ gives an isomorphism between some $G, G' \in \mathcal{P}$.

By Lemmas 3.7 and 3.13, the maximum P_I-mixing construction G is rigid. (In fact, the proof of Lemma 3.13 simplifies greatly in this case and every P_I-mixing construction G with all bottom parts nonempty is rigid since we can identify bottom edges of G as precisely those that do not contain the symmetric difference of some two distinct edges of G.)

Thus, G and G' have the same base P_D, and the isomorphism between their bottom edge-sets, $\overline{D}((V_1, \ldots, V_t)) \subseteq G$ and $\overline{D}((V_1', \ldots, V_t')) \subseteq G'$, comes from an automorphism h of D, that is, $V_{h(i)}' = V_i$ for every $i \in [t]$. Now, using the automorphism h relabel the bottom parts of G' so that $V_i' = V_i$ for $i \in [t]$ and apply induction to each pair $G[V_i], G'[V_i] \in \mathcal{P}$ of isomorphic (in fact, equal) 3-graphs. □

We now turn to the second part of the theorem. Take any s. Let ℓ satisfy $(|I|/t!)^{\ell-1} \ge s$. Fix sufficiently small $\varepsilon > 0$ and let $n \to \infty$. It is enough to find s maximum P_I-mixing constructions $\mathcal{G}_1, \ldots, \mathcal{G}_s$ on $[n]$ so that every two are at edit distance at least $\varepsilon^3 n^3$, as this will demonstrate that $\xi(\mathcal{F}) \ge s$. Indeed, suppose on the contrary that $\xi(\mathcal{F}) \le s-1$. Then there exist $s-1$ 3-graphs $\mathcal{H}_1, \ldots, \mathcal{H}_{s-1}$ on $[n]$ such that, for each $i \in [s]$, there is some $j \in [s-1]$ where the edit distance between $\mathcal{G}_i$ and $\mathcal{H}_j$ is at most $\varepsilon^3 n^3/3$. By the Pigeonhole Principle, there must be two 3-graphs, say $\mathcal{G}_{i_1}$ and $\mathcal{G}_{i_2}$, that are both within edit distance of at most $\varepsilon^3 n^3/3$ to the same $\mathcal{H}_j$. However, by the triangle inequality, this implies that the edit distance between $\mathcal{G}_{i_1}$ and $\mathcal{G}_{i_2}$ is at most $2\varepsilon^3 n^3/3 < \varepsilon^3 n^3$, a contradiction.

Call two I-labelled unordered trees T and T' **isomorphic up to level m** if their first m levels span isomorphic trees. For example, two trees are isomorphic up to level 1 if and only if the children of the root are labelled by the same element of I in the both trees. (Recall that the root always get label $\emptyset$, while all children of a node always get the same label.) For convenience, let us agree that every two trees are isomorphic up to level 0.

By our choice of ℓ, there are s trees that are pairwise non-isomorphic up to level ℓ. Since n is sufficiently large, we can assume by Lemma 4.4 that for each of these trees T, there is a maximum P_I-mixing construction G on $[n]$ whose unordered tree is isomorphic to T up to level ℓ; that is, $T_G^{\text{level}\le\ell} \cong T$. Thus, it is enough to show that any two of the obtained 3-graphs, say G and G' with unordered trees T and T', respectively, where T' and T' are non-isomorphic up to level ℓ, are at the edit distance at least $\varepsilon^3 n^3$. Suppose that this is false and the identity map exhibits this; that is, $|G \triangle G'| < \varepsilon^3 n^3$. Let $\mathbf{V}$ and $\mathbf{V}'$ be the partition structures of G and G', respectively. Also, for a tree T and a sequence $(i_1, \ldots, i_m)$, let $T[i_1, \ldots, i_m]$ be the subtree formed by the vertex with the first coordinate $(i_1, \ldots, i_m)$ and all its descendants in T.

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Claim 4.4.2. For every $m \le \ell$, there are sequences $(i_1, \dots, i_m), (i'_1, \dots, i'_m) \in [t]^m$ such that $T[i_1, \dots, i_m]$ and $T'[i'_1, \dots, i'_m]$ are non-isomorphic up to level $\ell - m$ and

$$\left| V_{i_1,\dots,i_m} \triangle V'_{i'_1,\dots,i'_m} \right| \le 2(t-1)m \cdot \varepsilon n.$$

Proof of Claim. We use induction on m with the base case $m = 0$ being satisfied by the empty sequences. Let $m \ge 1$ and suppose that we already constructed some sequences $(i_1, \dots, i_{m-1}), (i'_1, \dots, i'_{m-1}) \in [t]^{m-1}$ that satisfy the claim for $m - 1$. Let $U := V_{i_1,\dots,i_{m-1}}$ and $U' := V'_{i'_1,\dots,i'_{m-1}}$. For $j \in [t]$, let $U_j := V_{i_1,\dots,i_{m-1},j}$ and $U'_j := V'_{i'_1,\dots,i'_{m-1},j}$. In other words, $U = U_1 \cup \dots \cup U_t$ and $U' = U'_1 \cup \dots \cup U'_t$ are the bottom partitions of the P_I-mixing constructions $G[U]$ and $G'[U']$, respectively. These constructions are maximum; thus, by Lemma 4.3, their bottom partitions are balanced and, by simple induction on m, each part has size $n/t^m + O(1)$.

For each $i \in [t]$, there cannot be distinct $j, k \in [t]$ with each of $A := U_i \cap U'_j$ and $B := U_i \cap U'_k$ having at least εn elements. Indeed, otherwise the co-degree of each pair $(a, b) \in A \times B$ is at most $|U_i| - 2 = n/t^m + O(1)$ in $G[U]$ as all such edges lie inside U_i, and at least $(t-3)n/t^m + O(1)$ in $G'[U']$ (since the bottom pattern of $G'[U']$ has $t - 3$ triples containing the pair $\{i, j\}$). This is impossible, since then

$$|G \triangle G'| \ge |G' \setminus G| \ge |A| \cdot |B| \cdot \left(\frac{(t-4)n}{t^m} + O(1) \right) \ge \varepsilon^3 n^3. \tag{4.4}$$

Thus, there is $h : [t] \to [t]$ with $|U_i \cap U'_j| < \varepsilon n$ for each $j \in [t]$ different from $h(i)$. Since $\varepsilon \ll 1/t^\ell \le 1/t^m$, we have

$$|U_i \cap U'_{h(i)}| \ge |U_i| - (t-1)\varepsilon n - |U' \setminus U| \ge \frac{n}{t^m} - 2(t-1)m \cdot \varepsilon n + O(1), \tag{4.5}$$

which is strictly larger than half of $|U'_{h(i)}|$. Thus, h is an injection and, by the equal finite cardinality of its domain and its range, a bijection.

The map h has to be an isomorphism between the base patterns D and D' of $G[U]$ and $G'[U']$, respectively. Indeed, if h does not preserve the adjacency for at least one triple, then (4.5) implies that, say, $|G \triangle G'| \ge (n/(2t^m))^3 > \varepsilon^3 n^3$, a contradiction.

Since the trees $T[i_1, \dots, i_{m-1}]$ and $T'[i'_1, \dots, i'_{m-1}]$ (which are the trees of $G[U]$ and $G'[U']$) are non-isomorphic up to level $\ell - m + 1$, there must be i such that, letting $i_m := i$ and $i'_m := h(i)$, the trees $T[i_1, \dots, i_m]$ and $T'[i'_1, \dots, i'_m]$ are non-isomorphic up to level $\ell - m$. Also, by (4.5) (and its version when the roles of G and G' are exchanged), we have that

$$\left| V_{i_1,\dots,i_m} \triangle V'_{i'_1,\dots,i'_m} \right| \le |U \triangle U'| + 2(t-1) \cdot \varepsilon n \le 2(t-1)m \cdot \varepsilon n.$$

Thus, the obtained sequences $(i_1, \dots, i_m), (i'_1, \dots, i'_m) \in [t]^m$ satisfy the claim. □

Finally, a desired contradiction follows by taking $m := \ell$ in the above claim (as every two trees are isomorphic up to level 0), finishing the proof of the second part of Theorem 1.3. □

5. Feasible region

In this section, we prove Theorem 1.5. We need some auxiliary results first.

Lemma 5.1. *Suppose that $P = (m, E, R)$ is a minimal pattern. Then every pair $\{i, j\} \in \binom{[m]}{2}$ is contained in some multiset in E.*

Proof. Suppose to the contrary that there exists a pair in $[m]$ that is not contained in any multiset in E. By relabelling the vertices in P, we may assume that this pair is $\{1, 2\}$. Let $\lambda := \lambda_P$. For $\mathbf{z} \in \mathbb{S}_m$, let

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$f(\mathbf{z}) := \lambda_E(\mathbf{z}) + \lambda \sum_{t\in R} z_t^r$. Note that no term in $f(\mathbf{z})$ contains the product $z_1 z_2$. So we can rewrite $f(\mathbf{z})$ as

$$f(\mathbf{z}) = \sum_{i=1}^{r} \left(\alpha_i z_1^i + \beta_i z_2^i\right) + \gamma,$$

where $\alpha_i, \beta_i, \gamma \ge 0$ all depend only on $z_3, \ldots, z_m$.

Let $\mathbf{x} \in \mathcal{X}$ be an optimal vector for P. By symmetry, we may assume that $\sum_{i=1}^{r} \alpha_i x_1^{i-1} \ge \sum_{i=1}^{r} \beta_i x_2^{i-1}$. Let $\mathbf{y} := (x_1 + x_2, 0, x_3, \ldots, x_m) \in \mathbb{S}_m$. Notice that

$$\begin{aligned} f(\mathbf{y}) - f(\mathbf{x}) &= \sum_{i=1}^{r} \alpha_i (x_1 + x_2)^i - \sum_{i=1}^{r} \left(\alpha_i x_1^i + \beta_i x_2^i\right) \\ &= (x_1 + x_2) \sum_{i=1}^{r} \alpha_i (x_1 + x_2)^{i-1} - \left(x_1 \sum_{i=1}^{r} \alpha_i x_1^{i-1} + x_2 \sum_{i=1}^{r} \beta_i x_2^{i-1}\right) \\ &\ge (x_1 + x_2) \sum_{i=1}^{r} \alpha_i (x_1 + x_2)^{i-1} - \left(x_1 \sum_{i=1}^{r} \alpha_i x_1^{i-1} + x_2 \sum_{i=1}^{r} \alpha_i x_1^{i-1}\right) \\ &= (x_1 + x_2) \sum_{i=1}^{r} \left(\alpha_i (x_1 + x_2)^{i-1} - \alpha_i x_1^{i-1}\right) \ge 0. \end{aligned}$$

Since $\mathbf{x}$ is an optimal vector for P, we must have $f(\mathbf{y}) - f(\mathbf{x}) = 0$, implying that $\mathbf{y}$ is an optimal vector for P or that $\mathbf{y} = \mathbf{e}_1$ is the first standard basis vector. In either case, one can easily derive that $\lambda_P = \lambda_{P-2}$, contradicting the minimality of P. □

For every $i \in I$ and pattern $P_i = (m_i, E_i, R_i)$ we let $S_i \subseteq [m_i] \setminus R_i$ be the collection of $j \in [m_i] \setminus R_i$ such that for every blowup $E_i((V_1, \ldots, V_{m_i}))$ of E_i the shadow of $E_i((V_1, \ldots, V_{m_i}))$ has no edge inside V_j.

Lemma 5.2. *For every finite collection P_I of minimal 3-graph patterns with the same Lagrangian $\lambda \in (0, 1)$, there is an integer M such that for every $\varepsilon > 0$, there exist $\delta > 0$ and $N_0 > 0$ such that the following holds for all $n \ge N_0$. If G is an $\mathcal{F}_M$-free 3-graph on $[n]$ with at least $(\lambda - \delta)\binom{n}{3}$ edges, then there exists a P_I-mixing construction H on $[n]$ with base P_i and bottom partition $V(G) = V_1 \cup \cdots \cup V_{m_i}$ for some $i \in I$ such that $\delta(H) \ge (\lambda - \varepsilon)\binom{n-1}{2}$, $|H \triangle G| \le \varepsilon\binom{n}{3}$, $(|V_1|/n, \ldots, |V_{m_i}|/n)$ is ε-close to a P_i-optimal vector, and $\sum_{j\in S_i} |(\partial G)[V_j]| \le \varepsilon\binom{n}{2}$.*

Proof. Many steps of this proof are similar to the analogous parts of the proof of Lemma 3.16. So we may be brief when the appropriate adaptation is straightforward,

Let ℓ_0 be the constant returned by Lemma 3.15 and then let M be sufficiently large. Given $\varepsilon > 0$, choose sufficiently small positive constants in this order: $\delta_4 \gg \delta_3 \gg \delta_2 \gg \delta_1 \gg \delta$. Let $n \to \infty$ and let G be any $\mathcal{F}_M$-free r-graph on $V := [n]$ with at least $(\lambda - \delta)\binom{n}{3}$ edges.

Due to Theorem 1.2(b), we may assume that $|G \triangle H| \le \delta_1\binom{n}{3}$ for some P_I-mixing construction H. Let H have the partition structure $\mathbf{V}$, the base P_i and the bottom partition $V_1, \ldots, V_{m_i}$. By modifying H as in the argument leading to (3.19), we can further assume that

$$\delta(H) \ge (\lambda - \delta_2)\binom{n-1}{2} \quad \text{and} \quad |G \triangle H| \le \delta_2\binom{n}{3}. \tag{5.1}$$

By Lemma 3.9(b), the bottom part ratios are within ε from a P_i-optimal vector. Thus, this H satisfies the first three properties stated in the lemma. The rest of the proof is dedicated to proving the remaining property that the total shadow of G inside the parts V_j with $j \in S_i$ is 'small'.

Let

$$Z_1 := \left\{v \in V : d_G(v) \le (\lambda - \delta_3)\binom{n-1}{2}\right\} \quad \text{and} \quad G_1 := G - Z_1.$$

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By (5.1), we have the following inequalities.

Claim 5.2.1. It holds that

$$|Z_1| \le \frac{3|G \bigtriangleup H|}{(\delta_3-\delta_2)\binom{n-1}{2}} \le \delta_3 n,$$
$$|G_1| \ge |G| - |Z_1|\binom{n-1}{2} \ge (\lambda-\delta)\binom{n}{3} - \delta_3 n\binom{n-1}{2} \ge \lambda\binom{n}{3} - \delta_3 n^3,$$
$$\delta(G_1) \ge (\lambda-\delta_3)\binom{n-1}{2} - |Z_1|n \ge (\lambda - 4\delta_3)\binom{n}{2}.$$

□

Let $V' := V \setminus Z_1$, and let H_1 be the induced subgraph of H on V'. Clearly, we have $|G_1 \bigtriangleup H_1| \le |G \bigtriangleup H| \le \delta_2\binom{n}{3}$.

Define

$$Z_2 := \left\{ v \in V' \colon |L_{G_1}(v) \cap L_{H_1}(v)| \le (\lambda-\delta_4)\binom{n}{2}\right\} \quad \text{and} \quad G_2 := G_1 - Z_2.$$

Similarly to Claim 5.2.1, we have the following.

Claim 5.2.2. We have $|Z_2| \le \delta_4 n$, $|G_2| \ge \lambda\binom{n}{3} - \delta_4 n^3$ and $\delta(G_2) \ge (\lambda - 4\delta_4)\binom{n}{2}$.

Let $V'' := V' \setminus Z_2$ and $V''_{\mathbf{i}} := V_{\mathbf{i}} \cap V''$ for every legal sequence $\mathbf{i}$. Let H_2 be the induced subgraph of H on V''. Similarly to above, we have $|G_2 \bigtriangleup H_2| \le |G \bigtriangleup H| \le \delta_2 n^3$ and

$$\delta(H_2) \ge \delta(H) - (|Z_1| + |Z_2|)\binom{n-1}{2} \ge (\lambda - 3\delta_4)\binom{n}{2}. \tag{5.2}$$

In addition, it follows from the definition of Z_2 that for every $v \in V''$,

$$\left|L_{G_2}(v) \cap L_{H_2}(v)\right| \ge (\lambda-\delta_4)\binom{n}{2} - |Z_2|(n-2) \ge (\lambda - 3\delta_4)\binom{n}{2}. \tag{5.3}$$

Take any $j \in S_i$. Our next aim is to show that $\partial G \cap \binom{V''_j}{2} = \emptyset$. Suppose to the contrary that there exists $D = \{u_1, u_2, u_3\}$ in G with $u_1, u_2 \in V''_j$.

Let H_3 be obtained from H_2 by removing edges contained in $H_2[V''_{\mathbf{i}}]$ for all legal sequences $\mathbf{i}$ of length at least ℓ_0. Let $\mathbf{T} := \mathbf{T}_{H_3}$. By the min-degree property of H and Lemma 3.9(c), each part of $\mathbf{V}_H$ at level at most ℓ_0 has at least $(\beta/2)^{\ell_0} n$ vertices. This is much larger than $|Z_1 \cup Z_2| \le (\delta_3 + \delta_2)n$, the number of the removed vertices when building H_3 from H. Thus, in particular, $\mathbf{T}_{H_3} = \mathbf{T}_H^{\text{level} \le \ell_0}$. Note that if the tree $\mathbf{T}$ is extendible, then it is maximal up to level ℓ_0. Indeed, each involved recursive part $V_{\mathbf{i}}$ of H has at least $(\beta/2)^{\ell_0} n$ vertices and thus must span some edges, for otherwise, the edge density of the P_I-mixing construction H, which is $\rho(H) \ge \rho(G) - \delta_2 \ge \lambda - \delta - \delta_2$, can be increased by

$$(\lambda - o(1))\binom{|V_{\mathbf{i}}|}{3}\Big/\binom{n}{3} \ge \lambda(\beta/2)^{3\ell_0} - o(1) \gg \delta + \delta_2,$$

thus jumping above the maximum density $\lambda + o(1)$, a contradiction. Let F be the rigid P_I-mixing construction with $\mathbf{T}_F = \mathbf{T}$, returned by Lemma 3.15 or 3.14. Since $M \gg \ell_0$, we can assume that F has at most $M-1$ vertices.

By relabelling $V(F)$, we can assume that $u_1, u_2 \in V(F)$ and these vertices have the same branch in F and H (which is just the single-element sequence (j)) while $u_3 \notin V(F)$. Let F^D be obtained from F by adding the edge D (and the new vertex u_3). Let $\mathbf{W}$ be the partition structure of F.

Claim 5.2.3. $F^D \in \mathcal{F}_M$

Proof of Claim. Suppose to the contrary that there exists an embedding f of F^D into some P_I-mixing construction Q with base P_t and bottom partition $U_1, \ldots, U_{m_t}$. We can assume that $f(V(F^D))$ intersects at least two parts U_s. First, suppose that already the image of $V(F) \subseteq V(F^D)$ under f intersects at least two bottom parts of Q. By the rigidity of F, we have that $t = i$, and by relabelling the parts U_s, we can assume that $W_s \subseteq U_s$ for every $s \in [m_i]$. Thus, $f(\{u_1, u_2\}) \subseteq U_j$. By $j \in S_i$, no edge of Q can cover the pair $\{u_1, u_2\}$, a contradiction to f mapping D to an edge of Q. Thus, suppose that $f(V(F)) \subseteq U_s$ for some $s \in [m_t]$. Since $|F| > 0$, we have $s \in R_t$. By $V(F^D) = V(F) \cup \{u_3\}$, it holds that $f(u_3) \in U_k$ for some $k \neq s$. However, then the profile of the edge $f(D)$ in Q is $\{\!\{ s^{(2)}, k \}\!\}$, contradicting by Lemma 2.3 our assumption that $\lambda_{P_t} = \lambda < 1$. □

Now, we apply the familiar argument where we consider all maps $f : V(F^D) \to V(H_3)$ such that f is the identity on D and, for every vertex u of F^D different from u_3, it holds that $\mathrm{br}_F(u) = \mathrm{br}_{H_3}(f(u))$, that is, the branch of $f(u)$ in H_3 is the same as the branch of u in F. We have by Lemma 3.9(c) that for every legal sequence $\mathbf{i}$ of length at most ℓ_0, we have $|V''_{\mathbf{i}}| \geq (\beta/2)^{\ell_0}|V''|$. Thus, the number of choices of f is at least, say, $((\beta/2)^{\ell_0} n/3)^{v(F)-2}$. By Claim 5.2.3, each choice of f reveals a missing edge $Y \in H_3 \setminus G_2$. It is impossible that Y is disjoint from $\{u_1, u_2\}$ for at least half of the choices, for otherwise, the size of $H_3 \setminus G_2$ is too large, since every such Y is overcounted at most $n^{v(F)-5}$ times. Thus, at least half of the time, we have $Y \cap \{u_1, u_2\} \neq \emptyset$. By $j \in S_i$, each such Y intersects $\{u_1, u_2\}$ in exactly one vertex. However, note that, for each $j \in \{1, 2\}$, we have by (5.3) and Lemma 3.11 that

$$\begin{aligned} |L_{H_3}(u_j) \setminus L_{G_2}(u_j)| &\leq |L_{H_2}(u_j) \setminus L_{G_2}(u_j)| \\ &= |L_{H_2}(u_j)| - |L_{G_2}(u_j) \cap L_{H_2}(u_j)| \\ &\leq \Delta(H) - |L_{G_2}(u_j) \cap L_{H_2}(u_j)| \leq (\lambda + \delta_3)\binom{n}{2} - (\lambda - 3\delta_4)\binom{n}{2} \leq 4\delta_4 n^2. \end{aligned}$$

Thus, the total number of such choices of f can be upper bounded by $2 \times 4\delta_4 n^2 = 8\delta_4 n^2$, the number of choices of Y, times the trivial upper bound $n^{v(F)-4}$ on the number of extensions to the remaining vertices of $V(F)$. This contradicts that δ_4 was sufficiently small depending on β and ℓ_0.

We have shown that, for every $j \in S_i$, the set V''_j spans no edges in ∂G; that is, every edge of ∂G in V_j must contain at least one vertex from $Z_1 \cup Z_2$. Thus, $\sum_{j \in S_i} |(\partial G)[V_j]| \leq |Z_1 \cup Z_2| n \leq \varepsilon\binom{n-1}{2}$. This proves Lemma 5.2. □

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. Again, we can assume that the assumptions and terminology of Section 3.1 apply to P_I.

Let M be a sufficiently large integer, in particular, such that M satisfies Lemma 5.2 and the family $\mathcal{F}_M$ satisfies Theorem 1.2. Let us show that $\mathcal{F} := \mathcal{F}_M$ satisfies Theorem 1.5. Theorem 1.2 implies that $\pi(\mathcal{F}_M) = \pi(\mathcal{F}_\infty) = \lambda$, from which it easily follow that $M(\mathcal{F}_\infty) \subseteq M(\mathcal{F}_M)$. Let us show the converse implication.

Fix any x in $M(\mathcal{F}_M)$ and $\varepsilon > 0$. We have to approximate the point $(x, \lambda) \in \Omega(\mathcal{F}_M)$ by an element of $\Omega(\mathcal{F}_\infty)$ within ε in, say, the supremum norm. Let $(G_n)_{n=1}^{\infty}$ be a sequence of $\mathcal{F}_M$-free 3-graphs that realizes (x, λ). Let $v_n := v(G_n)$ for $n \geq 1$. By passing to a subsequence if necessary, we can assume that the sequence $(v_n)_{n \in \mathbb{N}}$ is strictly increasing. Since $\lim_{n\to\infty} \rho(G_n) = \pi(\mathcal{F}_M)$, Lemma 5.2 applies for all large n and returns a P_I-mixing construction H_n on $V(G_n)$. In particular, it holds that $\delta(H_n) \geq (\lambda - o(1))\binom{v_n - 1}{2}$ and $|G_n \triangle H_n| = o(v_n^3)$ as $n \to \infty$. Since H_n is a P_I-mixing construction and $\lim_{n\to\infty} \rho(H_n) = \lim_{n\to\infty} \rho(G_n) = \pi(\mathcal{F}_M)$, we have $\lim_{n\to\infty} \rho(\partial H_n) \in M(\mathcal{F}_\infty)$.

Let ℓ be a sufficiently large integer. Define $\delta_{\ell+1} := (1 - \lambda/(2r))^{\ell+1}/2$ and then let $\delta_\ell \gg \cdots \gg \delta_1$ be sufficiently small positive constants, chosen in this order. Let $n \to \infty$. Let $\mathbf{V}$ be the partition structure of H_n. By choosing n sufficiently large, we can assume that $\delta(H_n) \geq (\lambda - \delta_1)\binom{v_n - 1}{2}$, $|G_n \triangle H_n| \leq \delta_1\binom{v_n}{3}$, and $|G_n| \geq (\lambda - \delta_1)\binom{v_n}{3}$.

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We call edges in $\partial G_n \setminus \partial H_n$ **bad shadows**. By Lemma 5.1, every edge $e \in \partial G_n \setminus \partial H_n$ is contained entirely inside some bottom part of H_n. Let $\mathrm{br}_{\mathbf{V}}(e)$ denote the (unique, nonempty) maximal sequence $\mathbf{i}$ such that $e \subseteq V_{\mathbf{i}}$.

Every bad shadow with branch of length at least $\ell+1$ lies inside some part $V_{\mathbf{i}}$ with $|\mathbf{i}| \geq \ell+1$, whose size is at most $(1-\lambda/(2r))^{\ell+1} v_n$ by Lemma 3.9(c). By the Hand-Shaking Lemma, the number of such pairs is at most $\frac{1}{2}(1-\lambda/(2r))^{\ell+1} v_n \cdot v_n = \delta_{\ell+1} v_n^2$.

Recall that each part of height at most ℓ in H_n has size at least $(\beta/2)^\ell v_n$ by Lemma 3.9(c). In particular, by Lemma 5.1, the collection of all bad shadows whose branch has length 1 is exactly the set $\bigcup_{j \in S_{b(H_n)}} \partial G_n[V_j]$, and by Lemma 5.2, we have that

$$\sum_{j \in S_{b(H_n)}} |\partial G_n[V_j]| \leq \delta_2 v_n^2.$$

Now, repeat the following for every recursive index $j_1 \in R_{b(H_n)}$. Since $\delta(H_n) \geq (\lambda - \delta_1)\binom{v_n - 1}{2}$, we have by Lemma 3.9 that $|V_{j_1}| \geq \beta v_n/2$ and $\delta(H_n[V_{j_1}]) \geq (\lambda - \delta_2)\binom{|V_{j_1}|-1}{2}$. Since $|G_n[V_{j_1}] \triangle H_n[V_{j_1}]| \leq |G_n \triangle H_n| \leq \delta_1 \binom{v_n}{3}$, we have

$$|G_n[V_{j_1}]| \geq |H_n[V_{j_1}]| - \delta_1 \binom{v_n}{3} \geq (\lambda - 2\delta_2)\binom{|V_{j_1}|}{3}.$$

So applying Lemma 5.2 to $H_n[V_{j_1}]$, we obtain that, for example,

$$\sum_{j \in S_{b(H_n[V_{j_1}])}} |\partial G_n[V_{j_1,j}]| \leq \frac{1}{m}\,\delta_3 \binom{v_n}{2},$$

where $m := \max\{m_k : k \in I\}$. Therefore,

$$\sum_{j_1 \in R_{b(H_n)}} \sum_{j \in S_{b(H_n[V_{j_1}])}} |\partial G_n[V_{j_1,j}]| \leq m \frac{\delta_3 \binom{v_n}{2}}{m} = \delta_3 \binom{v_n}{2};$$

that is, the number of bad shadows whose branch has length 2 is at most $\delta_3 \binom{v_n}{2}$. Repeating this argument, we can show that the the number of bad shadows whose length of branch is $h \leq \ell$ is at most $\delta_{h+1}\binom{v_n}{2}$. Therefore, the total number of bad shadows is bounded by

$$\delta_2 \binom{v_n}{2} + \delta_3 \binom{v_n}{2} + \cdots + \delta_\ell \binom{v_n}{2} + \delta_{\ell+1}\binom{v_n}{2} \leq 2\delta_{\ell+1}\binom{v_n}{2}.$$

Therefore, $|\partial G_n \setminus \partial H_n| / \binom{v_n}{2} < \varepsilon/2$. However, every pair $xy \in \partial H_n$ at level at most ℓ is covered by at least $(\beta/2)^\ell v_n$ triples in H_n. This observation and our previous estimate of the total number of pairs at levels at least $\ell+1$ give that

$$|\partial H_n \setminus \partial G_n| \leq \delta_{\ell+1} v_n^2 + \frac{3|G_n \triangle H_n|}{(\beta/2)^\ell v_n} < \frac{\varepsilon}{2}\binom{v_n}{2}.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $x \in M(\mathcal{F}_M)$. This proves Theorem 1.5. □

6. Proof of Corollary 1.6

In this section, we prove Corollary 1.6 by applying Theorem 1.5 to two specific patterns.

Consider the following two patterns $P_1 = (5, K_5^3, \{1\})$ and $P_2 = (7, B_{5,3}, \{1\})$, where the 3-graph $B_{5,3}$ was defined in Section 2.4.

Lemma 6.1. *The following statements hold.*

(a) *We have* $\lambda_{P_1} = \lambda_{P_2} = \lambda$, *where* $\lambda := 3\left(\sqrt{7}-2\right)/4 \approx 0.484313$.

(b) *For* $\mathbf{x} \in \mathbb{S}_5^*$, *we have* $\lambda_{K_5^3}(\mathbf{x}) + \lambda x_1^3 = \lambda$ *if and only if*

$$x_1 = \frac{\sqrt{7}-2}{3}, \quad \textit{and} \quad x_2 = \cdots = x_5 = \frac{5-\sqrt{7}}{12}. \tag{6.1}$$

(c) *For* $\mathbf{y} \in \mathbb{S}_7^*$, *we have* $\lambda_{B_{5,3}}(\mathbf{y}) + \lambda y_1^3 = \lambda$ *if and only if*

$$y_1 = \frac{\sqrt{7}-2}{3}, \quad y_2 = y_3 = \frac{5-\sqrt{7}}{12}, \quad \textit{and} \quad y_4 = \cdots = y_7 = \frac{5-\sqrt{7}}{24}. \tag{6.2}$$

Remark. Parts (b) and (c) imply that P_1 and P_2 are minimal.

Proof. For $\mathbf{x} \in \mathbb{S}_5^*$ and $\mathbf{y} \in \mathbb{S}_7^*$, let

$$g_1(\mathbf{x}) := \frac{\lambda_{K_5^3}(\mathbf{x})}{1-x_1^3} \quad \text{and} \quad g_2(\mathbf{y}) := \frac{\lambda_{B_{5,3}}(\mathbf{y})}{1-y_1^3}.$$

It follows from the AM-GM Inequality that

$$\begin{aligned} g_1(\mathbf{x}) = \frac{\lambda_{K_5^3}(\mathbf{x})}{1-x_1^3} &= \frac{6\left(x_1 \sum_{ij\in\binom{[2,5]}{2}} x_i x_j + \sum_{ijk\in\binom{[2,5]}{3}} x_i x_j x_k\right)}{1-x_1^3} \\ &\le \frac{6\left(x_1 \binom{4}{2}\left(\frac{1-x_1}{4}\right)^2 + \binom{4}{3}\left(\frac{1-x_1}{4}\right)^3\right)}{1-x_1^3} \\ &= \frac{3(1-x_1)(1+5x_1)}{8(1+x_1+x_1^2)} \le \frac{3(\sqrt{7}-2)}{4}, \end{aligned}$$

where the equality holds if and only if (6.1) holds.

Similarly, for $g_2(\mathbf{y})$, we have

$$\begin{aligned} \lambda_{B_{5,3}}(\mathbf{y}) = & \, 6y_1(y_2y_3 + (y_2+y_3)(y_4+y_5+y_6+y_7) + (y_4+y_5)(y_6+y_7)) \\ & + 6(y_2y_3(y_4+y_5+y_6+y_7) + y_2(y_4+y_6)(y_5+y_7) + y_3(y_4+y_7)(y_5+y_6)). \end{aligned}$$

Notice that

$$\begin{aligned} & y_2y_3 + (y_2+y_3)(y_4+y_5+y_6+y_7) + (y_4+y_5)(y_6+y_7) \\ & = y_2y_3 + y_2(y_4+y_5) + y_2(y_6+y_7) + y_3(y_4+y_5) + y_3(y_6+y_7) + (y_4+y_5)(y_6+y_7) \\ & = \sigma_2(y_2, y_3, y_4+y_5, y_6+y_7) \le \binom{4}{2}\left(\frac{y_2+\cdots+y_7}{4}\right)^2 = \frac{3}{8}(1-y_1)^2, \end{aligned}$$

where

$$\sigma_k(x_1,\ldots,x_s) := \sum_{X\in\binom{[s]}{k}} \prod_{i\in X} x_i$$

is the **i-th symmetric polynomial**, and the last inequality follows from the Maclaurin Inequality (see, for example, [4, Theorem 11.2]) that $\sigma_k(x_1,\ldots,x_s) \le \binom{s}{k}(x_1+\cdots+x_s)^k/s^k$ for any nonnegative x_i's and $k \in \mathbb{N}$.

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Also, it follows from the AM-GM and Maclaurin Inequalities that

$$\begin{aligned}
&y_2y_3(y_4+y_5+y_6+y_7)+y_2(y_4+y_6)(y_5+y_7)+y_3(y_4+y_7)(y_5+y_6)\\
&\le y_2y_3(y_4+y_5+y_6+y_7)+y_2\Big(\frac{y_4+y_6+y_5+y_7}{2}\Big)^2+y_3\Big(\frac{y_4+y_7+y_5+y_6}{2}\Big)^2\\
&=\sigma_3(y_2,y_3,(y_4+y_5+y_6+y_7)/2,(y_4+y_5+y_6+y_7)/2)\le\binom{4}{3}\Big(\frac{y_2+\cdots+y_7}{4}\Big)^3=\frac{1}{16}(1-y_1)^3.
\end{aligned}$$

Therefore, $\lambda_{B_{5,3}}(\mathbf{y})\le 6\Big(\frac{3}{8}y_1(1-y_1)^2+\frac{1}{16}(1-y_1)^3\Big)$. Similar to $g_1(\mathbf{x})$, we obtain

$$g_2(\mathbf{y})=\frac{\lambda_{B_{5,3}}(\mathbf{y})}{1-y_1^3}\le\frac{6\Big(\frac{3}{8}y_1(1-y_1)^2+\frac{1}{16}(1-y_1)^3\Big)}{1-y_1^3}\le\frac{3(\sqrt{7}-2)}{4},$$

and, as it is easy to check, equality holds if and only if (6.2) holds. This gives all claims of the lemma. □

For the next lemma, we need the following definitions. Given a collection $\{P_i=(m_i,E_i,R_i)\colon i\in I\}$ of r-graph patterns, we define a family $\Sigma^k P_I$ recursively for every integer $k\ge 0$ in the following way. (Note that it is different from the family $\Sigma_k P_I$ that appeared in the proof of Lemma 3.4.)

Definition 6.2 (The k-th P_I-mixing construction). Let

$$\Sigma^0 P_I := \bigcup_{i\in I}\{H\colon H \text{ is a } P_i\text{-construction}\}.$$

For every integer $k\ge 1$, an r-graph H is a **k-th $\mathbf{P_I}$-mixing construction** if there exist $i\in I$ and a partition $V(H)=V_1\cup\cdots\cup V_{m_i}$ such that H can be obtained from the blowup $E_i((V_1,\ldots,V_{m_i}))$ by adding, for each $j\in R_i$, an arbitrary $(k-1)$-th P_I-mixing construction on V_j. Let $\Sigma^k P_I$ denote the collection of all k-th P_I-mixing constructions.

Informally speaking, in a k-th P_I-mixing construction, we have to fix $i\in I$ and are required to use only pattern P_i on all levels larger than k. It easy to see that $\Sigma^k P_I\subseteq\Sigma^{k'}P_I$ for all $k'\ge k$, and if $\lambda_{P_i}=\lambda$ for each $i\in I$, then the maximum asymptotic density attainable by $\Sigma^k P_I$ for each $k\ge 0$ is λ. Moreover,

$$\bigcup_{k\ge 0}\Sigma^k P_I=\Sigma P_I.$$

For every integer $k\ge 0$, let $M_{\Sigma^k P_I}$ (resp. $M_{\Sigma P_I}$) be the collection of points $x\in[0,1]$ such that there exists a sequence $(H_n)_{n=1}^{\infty}$ of r-graphs in $\Sigma^k P_I$ (resp. ΣP_I) with

$$\lim_{n\to\infty}v(H_n)=\infty,\quad \lim_{n\to\infty}\rho(H_n)=\lambda_{P_I},\quad\text{and}\quad\lim_{n\to\infty}\rho(\partial H_n)=x.$$

It is easy to observe that the set $M_{\Sigma P_I}$ is the closure of $\bigcup_{k\ge 0}M_{\Sigma^k P_I}$; that is,

$$M_{\Sigma P_I}=\overline{\bigcup_{k\ge 0}M_{\Sigma^k P_I}},$$

which gives a more precise version of Observation 1.4.

We will need the following theorem (which is a rather special case of, for example, [16, Theorems (1) and (3)]) for determining the Hausdorff dimension of a self-similar set.

Theorem 6.3. *Suppose that $m\ge 2$ and, for each $i\in[m]$, $\psi_i(x):=r_i(x-x_i)+x_i$ a linear map with $r_i,x_i\in\mathbb{R}$ and $|r_i|<1$. Additionally, suppose that this collection of maps $\{\psi_1,\ldots,\psi_m\}$ satisfies the* ***open set condition*** *– namely, that there exists a nonempty open set V such that*

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1. $\bigcup_{i\in[m]} \psi_i(V) \subseteq V$, *and*
2. *the sets in* $\{\psi_i(V) \colon i \in [m]\}$ *are pairwise disjoint.*

Then the Hausdorff dimension d of the (unique) bounded closed nonempty set A that satisfies $A = \bigcup_{i\in[m]} \psi_i(A)$ *is the (unique) solution of the equation*

$$\sum_{i\in[m]} |r_i|^d = 1.$$

The following lemma determines the Hausdorff dimension of the set $M_{\Sigma\{P_1,P_2\}}$.

Lemma 6.4. *We have* $M_{\Sigma^0\{P_1,P_2\}} = \left\{\frac{6-\sqrt{7}}{4}, \frac{22-3\sqrt{7}}{16}\right\}$, *and for every integer* $k \geq 0$, *we have*

$$M_{\Sigma^{k+1}\{P_1,P_2\}} = \left\{\psi_1(x) \colon x \in M_{\Sigma^k\{P_1,P_2\}}\right\} \cup \left\{\psi_2(x) \colon x \in M_{\Sigma^k\{P_1,P_2\}}\right\}, \tag{6.3}$$

where we define, for every $x \in \mathbb{R}$,

$$\psi_1(x) := \frac{11-4\sqrt{7}}{9}\left(x - \frac{6-\sqrt{7}}{4}\right) + \frac{6-\sqrt{7}}{4} \quad \textit{and}$$
$$\psi_2(x) := \frac{11-4\sqrt{7}}{9}\left(x - \frac{22-3\sqrt{7}}{16}\right) + \frac{22-3\sqrt{7}}{16}.$$

Moreover, the Hausdorff dimension of $M_{\Sigma\{P_1,P_2\}}$ *is*

$$\delta := \frac{\log 2}{\log\left(4\sqrt{7}+11\right)} \approx 0.225641.$$

Proof. Lemma 6.1 implies that $M_{\Sigma^0\{P_1,P_2\}} = \{a, b\}$, where $a := \frac{6-\sqrt{7}}{4}$ and $b := \frac{22-3\sqrt{7}}{16}$. Here, a is obtained by solving the equation

$$1 - \left(\frac{\sqrt{7}-2}{3}\right)^2 - 4\left(\frac{5-\sqrt{7}}{12}\right)^2 + \left(\frac{\sqrt{7}-2}{3}\right)^2 x = x,$$

and b is obtained by solving the equation

$$1 - \left(\frac{\sqrt{7}-2}{3}\right)^2 - 2\left(\frac{5-\sqrt{7}}{12}\right)^2 - 4\left(\frac{5-\sqrt{7}}{24}\right)^2 + \left(\frac{\sqrt{7}-2}{3}\right)^2 x = x.$$

Let $k \geq 1$. It follows from the definition of $M_{\Sigma^{k+1}\{P_1,P_2\}}$ that $\alpha \in M_{\Sigma^{k+1}\{P_1,P_2\}}$ if and only if there exists $\beta \in M_{\Sigma^k\{P_1,P_2\}}$ such that

$$\alpha = 1 - \left(\frac{\sqrt{7}-2}{3}\right)^2 - 4\left(\frac{5-\sqrt{7}}{12}\right)^2 + \left(\frac{\sqrt{7}-2}{3}\right)^2 \beta = \frac{13\sqrt{7}-20}{18} + \frac{11-4\sqrt{7}}{9}\beta = \psi_1(\beta),$$

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or

$$\alpha = 1 - \left(\frac{\sqrt{7}-2}{3}\right)^2 - 2\left(\frac{5-\sqrt{7}}{12}\right)^2 - 4\left(\frac{5-\sqrt{7}}{24}\right)^2 + \left(\frac{\sqrt{7}-2}{3}\right)^2 \beta$$
$$= \frac{47\sqrt{7}-64}{72} + \frac{11-4\sqrt{7}}{9}\beta = \psi_2(\beta).$$

This proves (6.3).

Next, we prove that the Hausdorff dimension of $M_{\Sigma\{P_1,P_2\}}$ is δ. Let $A := M_{\Sigma\{P_1,P_2\}}$ and $V := (a,b)$. Note that A is a bounded closed set that, by (6.3), satisfies $A = \psi_1(A) \cup \psi_2(A)$. Also, routine calculations show that $\psi_1(V) = (a,c)$ and $\psi_2(V) = (d,b)$, where $c := \frac{166-17\sqrt{7}}{144}$ is less than $d := \frac{124-23\sqrt{7}}{72}$. Thus, the open set V and the maps ψ_1 and ψ_2 satisfy the Open Set Condition; that is,

$$\psi_1(V) \cup \psi_2(V) \subseteq V \quad \text{and} \quad \psi_1(V) \cap \psi_2(V) = \emptyset.$$

However, recall that $A \subseteq [0,1]$ is a closed set satisfying $A = \psi_1(A) \cup \psi_2(A)$. So, by Theorem 6.3, the Hausdorff dimension of A is the unique solution x to

$$\left(\frac{11-4\sqrt{7}}{9}\right)^x + \left(\frac{11-4\sqrt{7}}{9}\right)^x = 1,$$

which is δ, as desired. □

Now Corollary 1.6 is an easy consequence of Theorem 1.5 and Lemma 6.4.

7. Concluding remarks

Since our paper is quite long, we restricted applications of Theorem 1.2 to 3-graphs only. Some of these results extends to general r, while such an extension for others seems quite challenging. For example, our proof of Theorem 1.5 also works for r-graph patterns and the $(r-2)$-th shadow (when we consider pairs of vertices covered by edges). However, we do not know if the analogue of Theorem 1.5 holds for the $(r-i)$-th shadow when $i \geq 3$.

However, we tried to present a fairly general version of Theorem 1.2 (for example, allowing P_I to contain patterns with different Lagrangians) in case it may be useful for some other applications.

In some rather special cases of Theorem 1.2, it may be possible to drop the constraint that each $P_i \in P$ is minimal. For example, it is shown in [15] that for every (not necessarily minimal) pattern $(m, H, \emptyset)$ where H consists of simple r-sets, there exists a finite forbidden family whose extremal Turán constructions are exactly maximum blowups of H. However, we do not know if this is true in general since the minimality condition is crucially used in the proof of Theorem 1.2.

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Competing interest. The authors have no competing interest to declare.

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References

[1] R. Baber and J. Talbot, 'Hypergraphs do jump', *Combin. Probab. Comput.* **20**(2) (2011), 161–171.

[2] J. Balogh, F. C. Clemen and H. Luo, 'Non-degenerate hypergraphs with exponentially many extremal constructions', Preprint, 2022, arxiv:2208.00652.

[3] W. G. Brown, 'On an open problem of Paul Turán concerning 3-graphs', in *Studies in Pure Mathematics* (Birkhäuser, Basel, 1983), 91–93.

[4] Z. Cvetkovski, *Inequalities* (Springer, Heidelberg, 2012). Theorems, techniques and selected problems.
[5] P. Erdős, 'On extremal problems of graphs and generalized graphs', *Israel J. Math.* **2** (1964), 183–190.
[6] P. Erdős, 'Some recent results on extremal problems in graph theory. Results', in *Theory of Graphs (Internat. Sympos., Rome, 1966)* (Gordon and Breach, New York, 1967), 117–123 (English); 124–130 (French).
[7] P. Erdős, 'On the combinatorial problems I would most like to see solved', *Combinatorica* **1** (1981), 25–42.
[8] P. Erdős and M. Simonovits, 'A limit theorem in graph theory', *Stud. Sci. Math. Hungar.* **1** (1966), 51–57.
[9] P. Erdős and A. H. Stone, 'On the structure of linear graphs', *Bull. Amer. Math. Soc.* **52** (1946), 1087–1091.
[10] D. G. Fon-der Flaass, 'A method for constructing $(3, 4)$-graphs', *Mat. Zametki* **44** (1988), 546–550.
[11] P. Frankl and Z. Füredi, 'An exact result for 3-graphs', *Discrete Math.* **50** (1984), 323–328.
[12] P. Frankl, Y. Peng, V. Rödl and J. Talbot, 'A note on the jumping constant conjecture of Erdős' *J. Combin. Theory (B)* **97** (2007), 204–216.
[13] P. Frankl and V. Rödl, 'Hypergraphs do not jump', *Combinatorica* **4**(2–3) (1984), 149–159.
[14] A. Frohmader, 'More constructions for Turán's $(3, 4)$-conjecture', *Electron. J. Combin.* **15**(1) (2008), Research Paper 137, 23.
[15] J. Hou, H. Li, X. Liu, D. Mubayi and Y. Zhang, 'Hypergraphs with infinitely many extremal constructions', *Discrete Anal.* (2023), Paper No. 18, 34.
[16] J. E. Hutchinson, 'Fractals and self-similarity', *Indiana Univ. Math. J.* **30**(5) (1981), 713–747.
[17] G. Katona, 'A theorem of finite sets', in *Theory of Graphs (Proc. Colloq., Tihany, 1966)* (Academic Press, New York-London, 1968), 187–207.
[18] G. O. H. Katona, T. Nemetz and M. Simonovits, 'On a graph problem of Turán (In Hungarian)', *Mat. Fiz. Lapok* **15** (1964), 228–238.
[19] P. Keevash, 'Hypergraph Turán problems', in *Surveys in Combinatorics 2011* (London Math. Soc. Lecture Note Ser.) vol. 392 (Cambridge Univ. Press, Cambridge, 2011), 83–139.
[20] P. Keevash, 'Counting designs', *J. Europ. Math. Soc.* **20** (2018), 903–927.
[21] T. P. Kirkman, 'On a problem in combinations', *Cambridge and Dublin Math. J.* **2** (1847), 191–204.
[22] A. V. Kostochka, 'A class of constructions for Turán's (3,4)-problem', *Combinatorica* **2** (1982), 187–192.
[23] J. B. Kruskal, 'The number of simplices in a complex', in *Mathematical Optimization Techniques* (Univ. of California Press, Berkeley, CA, 1963), 251–278.
[24] X. Liu, 'Cancellative hypergraphs and Steiner triple systems', *J. Combin. Theory Ser. B* **167** (2024), 303–337,
[25] X. Liu and D. Mubayi, 'The feasible region of hypergraphs', *J. Comb. Theory, Ser. B* **148** (2021), 23–59.
[26] X. Liu and D. Mubayi, 'A hypergraph Turán problem with no stability', *Combinatorica* (2022), 1–30.
[27] X. Liu, D. Mubayi and C. Reiher, 'Hypergraphs with many extremal configurations*', to appear*, *Israel. J. Math.*
[28] T. S. Motzkin and E. G. Straus, 'Maxima for graphs and a new proof of a theorem of Turán', *Canadian J. Math.* **17** (1965), 533–540.
[29] S. Norin and L. Yepremyan, 'Turán number of generalized triangles', *J. Combin. Theory (A)* **146** (2017), 312–343.
[30] O. Pikhurko, 'On possible Turán densities', *Israel J. Math.* **201**(1) (2014), 415–454.
[31] O. Pikhurko, 'The maximal length of a gap between r-graph Turán densities', *Electronic J. Combin.* **22** (2015), 7pp.
[32] O. Pikhurko, J. Sliacăn and K. Tyros, 'Strong forms of stability from flag algebra calculations', *J. Combin. Theory (B)* **135** (2019), 129–178.
[33] V. Rödl and M. Schacht, 'Generalizations of the removal lemma', *Combinatorica* **29**(4) (2009), 467–501.
[34] A. Sidorenko, 'What we know and what we do not know about Turán numbers', *Graphs Combin.* **11**(2) (1995), 179–199.
[35] M. Simonovits, 'A method for solving extremal problems in graph theory, stability problems', in *Theory of Graphs (Proc. Colloq., Tihany, 1966)* (Academic Press, New York, 1968), 279–319.
[36] P. Turán, 'On an extremal problem in graph theory (in Hungarian)', *Mat. Fiz. Lapok* **48** (1941), 436–452.
[37] Z. Yan and Y. Peng, 'Non-jumping Turán densities of hypergraphs', Discrete Math. 346(1) (2023), Paper No. 113195, 11.

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