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ON DISCRETENESS OF SUBGROUPS OF QUATERNIONIC HYPERBOLIC ISOMETRIES

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Abstract

Let $\mathbf{H}^n_{\mathbb{H}}$ denote the n-dimensional quaternionic hyperbolic space. The linear group $\mathrm{Sp}(n, 1)$ acts on $\mathbf{H}^n_{\mathbb{H}}$ by isometries. A subgroup G of $\mathrm{Sp}(n, 1)$ is called *Zariski dense* if it neither fixes a point on $\mathbf{H}^n_{\mathbb{H}} \cup \partial\mathbf{H}^n_{\mathbb{H}}$ nor preserves a totally geodesic subspace of $\mathbf{H}^n_{\mathbb{H}}$. We prove that a Zariski dense subgroup G of $\mathrm{Sp}(n, 1)$ is discrete if for every loxodromic element $g \in G$ the two-generator subgroup $\langle f, gfg^{-1}\rangle$ is discrete, where the generator $f \in \mathrm{Sp}(n, 1)$ is a certain fixed element not necessarily from G.

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1. Introduction

The classical Jørgensen inequality [12] gives a necessary criterion to check discreteness of a two-generator subgroup of $\mathrm{SL}(2, \mathbb{C})$ that acts by Möbius transformations on the Riemann sphere. It has been generalised to the higher dimensional Möbius group that acts on n-dimensional real hyperbolic space. A well-known consequence of the generalised Jørgensen inequality is that a subgroup G of the Möbius group is discrete if and only if every two-generator subgroup is discrete (see [1, 16]). There have been several refinements of this result giving discreteness criteria in Möbius groups (see [5, 9, 17]). Generalisations of the Jørgensen inequality and related discreteness criteria have been obtained in settings such as complex hyperbolic space and normed spaces (see [7, 11, 14, 15]).

Let $\mathbb{H}$ denote the division ring of Hamilton's quaternions and $\mathbf{H}^n_{\mathbb{H}}$ the n-dimensional quaternionic hyperbolic space. Let $\mathrm{Sp}(n, 1)$ be the linear group that acts on $\mathbf{H}^n_{\mathbb{H}}$ by isometries. Following the theme sketched above, we give discreteness criteria for a subgroup of $\mathrm{Sp}(n, 1)$. The arguments restrict over the complex numbers and, as a corollary, we obtain discreteness criteria in $\mathrm{SU}(n, 1)$. To state our main result, we need the following notions.

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An element $g \in \mathrm{Sp}(n,1)$ is *elliptic* if it has a fixed point on $\mathbf{H}^n_{\mathbb{H}}$, *parabolic* if it has a unique fixed point on the boundary $\partial\mathbf{H}^n_{\mathbb{H}}$ and *loxodromic* (or hyperbolic) if it has exactly two fixed points on the boundary $\partial\mathbf{H}^n_{\mathbb{H}}$. A unipotent parabolic element, that is, a parabolic element having all eigenvalues 1, is called a *Heisenberg translation*. It is well known that an elliptic or loxodromic isometry g is conjugate to a diagonal element in $\mathrm{Sp}(n,1)$ (see [6]). If g is elliptic, then, up to conjugacy,

$$g = \mathrm{diag}(\lambda_1, \dots, \lambda_{n+1}), \tag{1.1}$$

where $|\lambda_i| = 1$ for each i and the eigenvalue λ_1 is such that the corresponding eigenvector has negative Hermitian length, while all other eigenvectors have positive Hermitian length. An elliptic element g is called *regular* if it has mutually disjoint classes of eigenvalues. A regular elliptic element has a unique fixed point on $\mathbf{H}^n_{\mathbb{H}}$. If g is loxodromic, then we may assume, up to conjugacy, that

$$g = \mathrm{diag}(\lambda_1, \bar{\lambda}_1^{-1}, \lambda_3, \dots, \lambda_{n+1}) \tag{1.2}$$

with $|\lambda_1| > 1$. Certain conjugacy invariants are associated to isometries as follows:

- for g elliptic, define
$$\delta(g) = \max\{\, |\lambda_1 - 1| + |\lambda_i - 1| : i = 2, \dots, n+1\}; \tag{1.3}$$
- for g loxodromic, following [2], define
$$\delta_{cp}(g) = \max\{|\lambda_i - 1| : i = 3, \dots, n+1\} \quad \text{and} \quad M_g = 2\delta_{cp}(g) + |\lambda_1 - 1| + |\bar{\lambda}_1^{-1} - 1|.$$

Let $T_{s,\zeta}$ be a Heisenberg translation in $\mathrm{Sp}(n,1)$. We may assume up to conjugacy that

$$T_{s,\zeta} = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & \zeta^* \\ \zeta & 0 & I \end{pmatrix},$$

where $\mathrm{Re}(s) = \frac{1}{2}|\zeta|^2$ (see [6, page 70]).

A subgroup G of $\mathrm{Sp}(n,1)$ is called *Zariski dense* if it neither fixes a point on $\mathbf{H}^n_{\mathbb{H}} \cup \partial\mathbf{H}^n_{\mathbb{H}}$ nor preserves a totally geodesic subspace of $\mathbf{H}^n_{\mathbb{H}}$. With this notation, we prove the following theorem.

THEOREM 1.1. *Let G be a Zariski dense subgroup of* $\mathrm{Sp}(n,1)$.

(1) *Let $g \in \mathrm{Sp}(n,1)$ be a regular elliptic element such that $\delta(g) < 1$. If $\langle g, hgh^{-1}\rangle$ is discrete for every loxodromic element $h \in G$, then G is discrete.*
(2) *Let $g \in \mathrm{Sp}(n,1)$ be a loxodromic element such that $M_g < 1$. If $\langle g, hgh^{-1}\rangle$ is discrete for every loxodromic element $h \in G$, then G is discrete.*
(3) *Let $g \in \mathrm{Sp}(n,1)$ be a Heisenberg translation such that $|\zeta| < \frac{1}{2}$. If $\langle g, hgh^{-1}\rangle$ is discrete for every loxodromic element h in G, then G is discrete.*

Restricting everything over the complex numbers, the above theorem also holds for $\mathrm{SU}(n,1)$.

Corollary 1.2. *Let G be a Zariski dense subgroup in* $\mathrm{Sp}(n,1)$ *or* $\mathrm{SU}(n,1)$.

(1) *Let $g \in \mathrm{Sp}(n,1)$ or $\mathrm{SU}(n,1)$ respectively be a regular elliptic element such that $\delta(g) < 1$. If $\langle g, hgh^{-1}\rangle$ is discrete for every regular elliptic element $h \in G$, then G is discrete.*
(2) *Let $g \in \mathrm{Sp}(n,1)$ or $\mathrm{SU}(n,1)$ respectively be a loxodromic element such that $M_g < 1$. If $\langle g, hgh^{-1}\rangle$ is discrete for every regular elliptic $h \in G$, then G is discrete.*
(3) *Let $g \in \mathrm{Sp}(n,1)$ or $\mathrm{SU}(n,1)$ respectively be a Heisenberg translation such that $|\zeta| < \frac{1}{2}$. If $\langle g, hgh^{-1}\rangle$ is discrete for every regular elliptic h in G, then G is discrete.*

These results show that the discreteness of a Zariski dense subgroup G of $\mathrm{Sp}(n,1)$ or $\mathrm{SU}(n,1)$ is determined by the two-generator subgroups $\langle g, hgh^{-1}\rangle$, where $h \in G$. The generator g is fixed and need not be an element from G and it is enough to take h to be loxodromic or regular elliptic. After fixing such a 'test map' g, conjugates of g by generic elements of G determine the discreteness. For isometries of the real hyperbolic space, similar discreteness criteria using a test map and its conjugates have been obtained in [18], [9, Theorem 1.2] and [8]. Theorem 1.1 and Corollary 1.2 generalise these results to $\mathrm{Sp}(n,1)$ and $\mathrm{SU}(n,1)$.

We note some preliminary notions in Section 2 and prove the main result in Section 4. To prove the results, we use some generalised Jørgensen inequalities in $\mathrm{Sp}(n,1)$. We use the Jørgensen inequality of Cao and Parker [2] to deal with subgroups having a loxodromic generator. For subgroups having a unipotent parabolic generator, we use a quaternionic version of Shimizu's lemma following Hersonsky and Paulin [10]. To deal with subgroups having a regular elliptic generator, we use a variation of the inequality of Cao and Tan [4]. For this case, we have introduced the new conjugacy invariant $\delta(g)$ given above. The invariant $\delta(g)$ is different from the conjugacy invariant $\delta_{ct}(g)$ used by Cao and Tan and may be considered as a restriction of the Cao–Parker invariant $\delta_{cp}(g)$ to subgroups having at least one elliptic generator. This new invariant gives quantitatively better bounds in a larger domain. We refer to Section 3 for more details.

2. Preliminaries

2.1. Quaternionic hyperbolic space. We begin with some background material on quaternionic hyperbolic geometry. Much of this can be found in [6, 13].

Let $\mathbb{H}^{n,1}$ be the right vector space over $\mathbb{H}$ of quaternionic dimension $n+1$ (so that its real dimension is $4n+4$) equipped with the quaternionic Hermitian form

$$\langle z, w\rangle = -(\bar{z}_0 w_1 + \bar{z}_1 w_0) + \sum_{i=2}^{n} \bar{z}_i w_i$$

for $z = (z_0, \dots, z_n)$, $w = (w_0, \dots, w_n)$. Thus, the quaternionic Hermitian form is defined

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by the matrix

$$J_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix}.$$

Equivalently, when convenient, one may also use the Hermitian form given by the matrix

$$J_1 = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}.$$

Following [6, Section 2], let

$$V_0 = \{\mathbf{z} \in \mathbb{H}^{n,1} - \{0\} : \langle \mathbf{z}, \mathbf{z} \rangle = 0\}, \quad V_- = \{\mathbf{z} \in \mathbb{H}^{n,1} : \langle \mathbf{z}, \mathbf{z} \rangle < 0\}.$$

Clearly, V_0 and V_- are invariant under $\mathrm{Sp}(n, 1)$. We define an equivalence relation $\sim$ on $\mathbb{H}^{n,1}$ by $\mathbf{z} \sim \mathbf{w}$ if and only if there exists a nonzero quaternion λ so that $\mathbf{w} = \mathbf{z}\lambda$. Let $[\mathbf{z}]$ denote the equivalence class of $\mathbf{z}$. Let $\mathbb{P} : \mathbb{H}^{n,1} - \{0\} \longrightarrow \mathbb{HP}^n$ be the *right projection* map given by $\mathbb{P} : \mathbf{z} \longmapsto z$, where $z = [\mathbf{z}]$. The n-dimensional quaternionic hyperbolic space is defined to be $\mathbf{H}^n_{\mathbb{H}} = \mathbb{P}(V_-)$ with boundary $\partial\mathbf{H}^n_{\mathbb{H}} = \mathbb{P}(V_0)$.

In the model using J_2, there are two distinct points 0 and ∞ on $\partial\mathbf{H}^n_{\mathbb{H}}$. For $z_1 \neq 0$, the projection map $\mathbb{P}$ is given by

$$\mathbb{P}(z_1, z_2, \dots, z_{n+1}) = (z_2 z_1^{-1}, \dots, z_{n+1} z_1^{-1})$$

and accordingly we choose boundary points

$$\mathbb{P}(0, 1, \dots, 0, 0)^t = 0,$$
$$\mathbb{P}(1, 0, \dots, 0, 0)^t = \infty.$$

In the model using J_1, we mark $\mathbb{P}(1, 0, \dots, 0, 0)^t$ as the origin $0 = (0, 0, \dots, 0)^t$ of the quaternionic hyperbolic ball. The Bergmann metric on $\mathbf{H}^n_{\mathbb{H}}$ is given by the distance formula

$$\cosh^2 \frac{\rho(z, w)}{2} = \frac{\langle \mathbf{z}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{z} \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle \langle \mathbf{w}, \mathbf{w} \rangle} \quad \text{where } z, w \in \mathbf{H}^n_{\mathbb{H}}, \mathbf{z} \in \mathbb{P}^{-1}(z), \mathbf{w} \in \mathbb{P}^{-1}(w).$$

The above formula is independent of the choice of $\mathbf{z}$ and $\mathbf{w}$.

Now consider the noncompact linear Lie group

$$\mathrm{Sp}(n, 1) = \{A \in \mathrm{GL}(n + 1, \mathbb{H}) : A^* J_i A = J_i\}.$$

An element $g \in \mathrm{Sp}(n, 1)$ acts on $\overline{\mathbf{H}_{\mathbb{H}}}^n = \mathbf{H}^n_{\mathbb{H}} \cup \partial\mathbf{H}^n_{\mathbb{H}}$ as $g(z) = \mathbb{P}g\mathbb{P}^{-1}(z)$. Thus, the isometry group of $\mathbf{H}^n_{\mathbb{H}}$ is given by $\mathrm{PSp}(n, 1) = \mathrm{Sp}(n, 1)/\{I, -I\}$.

2.2. The inequality of Cao and Parker. The quaternionic cross ratio of four distinct points z_1, z_2, z_3, z_4 on $\partial\mathbf{H}^n_{\mathbb{H}}$ is

$$[z_1, z_2, z_3, z_4] = \langle \mathbf{z}_3, \mathbf{z}_1 \rangle \langle \mathbf{z}_3, \mathbf{z}_2 \rangle^{-1} \langle \mathbf{z}_4, \mathbf{z}_2 \rangle \langle \mathbf{z}_4, \mathbf{z}_1 \rangle^{-1},$$

where $\mathbf{z}_i$ denotes the lift to $\mathbb{H}^{n+1}$ of a point z_i on $\partial\mathbf{H}^n_{\mathbb{H}}$. We note the following lemma concerning cross ratios.

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LEMMA 2.1 [2]. *Let* $0, \infty \in \partial \mathbf{H}^n_{\mathbb{H}}$ *stand for the respective points* $(0, 1, \ldots, 0)^t$ *and* $(1, 0, \ldots, 0)^t \in \mathbb{H}^{n,1}$ *under the projection map* $\mathbb{P}$ *and let* $h \in \mathrm{PSp}(n, 1)$ *be given by (2.2). Then*

$$|[h(\infty), 0, \infty, h(0)]| = |bc|, \quad |[h(\infty), \infty, 0, h(0)]| = |ad|, \quad |[\infty, 0, h(\infty), h(0)]| = \frac{|bc|}{|ad|}.$$

The theorem of Cao and Parker may be stated as follows.

THEOREM 2.2 (Cao and Parker [2]). *Let* g *and* h *be elements of* $\mathrm{Sp}(n, 1)$ *such that* g *is a loxodromic element with fixed points* $u, v \in \partial \mathbf{H}^n_{\mathbb{H}}$ *and* $M_g < 1$. *If* $\langle g, h \rangle$ *is nonelementary and discrete, then*

$$|[h(u), u, v, h(v)]|^{1/2} |[h(u), v, u, h(v)]|^{1/2} \geq \frac{1 - M_g}{M_g^2}.$$

2.3. Shimizu's lemma in Sp(*n*, 1). We use the Hermitian form J_2 in this section. Up to conjugacy, we assume that a Heisenberg translation fixes the boundary point 0, that is, it is of the form

$$T_{s,\zeta} = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & \zeta^* \\ \zeta & 0 & I \end{pmatrix}, \tag{2.1}$$

where $\mathrm{Re}(s) = \frac{1}{2}|\zeta|^2$.

Let A be an element in $\mathrm{Sp}(n, 1)$. Then one can choose A to be of the form

$$A = \begin{pmatrix} a & b & \gamma^* \\ c & d & \delta^* \\ \alpha & \beta & U \end{pmatrix}, \tag{2.2}$$

where a, b, c, d are scalars, $\gamma, \delta, \alpha, \beta$ are column matrices and U is an element in $M(n-1, \mathbb{H})$. It is easy to compute

$$A^{-1} = \begin{pmatrix} \bar{d} & \bar{b} & -\beta^* \\ \bar{c} & \bar{a} & -\alpha^* \\ -\delta & -\gamma & U^* \end{pmatrix}.$$

The next theorem follows by mimicking the arguments of Hersonsky and Paulin in [10, Appendix]. Hersonsky and Paulin proved it over the complex numbers. Over the quaternions, only a slight variation is needed and it is straightforward.

THEOREM 2.3. *Suppose that* $T_{s,\zeta}$ *is a Heisenberg translation in* $\mathrm{Sp}(n, 1)$ *and* A *is an element in* $\mathrm{Sp}(n, 1)$ *of the form (2.2). Suppose that* A *does not fix* 0. *Set*

$$t = \sup\{|b|, |\beta|, |\gamma|, |U - I|\}, \qquad M = |s| + 2|\zeta|.$$

If $Mt + 2|\zeta| < 1$, *then the group generated by* A *and* $T_{s,\zeta}$ *is either nondiscrete or fixes* 0.

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This is the simplest quaternionic version of Shimizu's lemma for two-generator subgroups of $\mathrm{Sp}(n, 1)$ with a unipotent parabolic generator. Stronger versions of Shimizu's lemma in $\mathrm{Sp}(n, 1)$ have been obtained by Kim and Parker [13, Theorem 4.8] and Cao and Parker [3]. The version in Theorem 2.3 is easier to apply for our purpose.

2.4. Useful results. A subgroup G of $\mathrm{Sp}(n, 1)$ is called elementary if it has a finite orbit in $\mathbf{H}^n_{\mathbb{H}} \cup \partial\mathbf{H}^n_{\mathbb{H}}$. If all its orbits are infinite, then G is nonelementary. In particular, G is nonelementary if it contains two nonelliptic elements of infinite order with distinct fixed points.

Theorem 2.4 [6]. *Let G be a Zariski dense subgroup of $\mathrm{Sp}(n, 1)$. Then G is either discrete or dense in $\mathrm{Sp}(n, 1)$.*

3. The inequality of Cao and Tan revisited

Theorem 3.1. *Let g and h be elements of $\mathrm{Sp}(n, 1)$. Suppose that g is a regular elliptic element with fixed point q and $\delta(g)$ is as in (1.3). If*

$$\cosh \frac{\rho(q, h(q))}{2}\, \delta(g) < 1, \tag{3.1}$$

then the group $\langle g, h\rangle$ generated by g and h is either elementary or not discrete.

The proof of Theorem 3.1 is a variation of the proof of [4, Theorem 1.1]. The initial computations are very similar, except that at a crucial stage we replace the Cao–Tan invariant by $\delta(g)$ and observe that the proof still works. We sketch the proof for completeness. We follow similar notation to [4] and use the ball model, that is, the Hermitian form J_1.

Proof. Using conjugation, we may assume that g is of the form (1.2) having fixed point $q = (0, \dots, 0)^t \in \mathbf{H}^n_{\mathbb{H}}$ and

$$h = (a_{i,j})_{i,j=1,\dots,n+1} = \begin{pmatrix} a_{1,1} & \beta \\ \alpha & A \end{pmatrix}.$$

For $L = \mathrm{diag}(\lambda_2, \dots, \lambda_{n+1})$, write

$$g = \begin{pmatrix} \lambda_1 & 0 \\ 0 & L \end{pmatrix}.$$

Then

$$\cosh \frac{\rho(q, h(q))}{2} = |a_{1,1}|, \qquad \delta(g) = \max\{|\lambda_1 - 1| + |\lambda_i - 1| : i = 2, \dots, n + 1\}.$$

The inequality (3.1) becomes

$$|a_{1,1}|\delta(g) < 1.$$

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Let $h_0 = h$ and $h_{k+1} = h_k g h_k^{-1}$ and write

$$h_k = (a_{i,j}^{(k)})_{i,j=1,\dots,n+1} = \begin{pmatrix} a_{1,1}^{(k)} & \beta^{(k)} \\ \alpha^{(k)} & A^{(k)} \end{pmatrix}.$$

If $\beta^{(k)} = 0$ for some k, it follows as in the proof of [4, Theorem 1.1] that $\langle g, h\rangle$ is elementary. So, assume that $\beta^{(k)} \neq 0$ and that the group $\langle g, h\rangle$ is nonelementary. By similar computations to those in the proof of [4, Theorem 1.1],

$$|a_{1,1}^{(k+1)}|^2 \leq |a_{1,1}^{(k)}|^4 + |\beta^{(k)}|^4 - \sum_{i=2}^{n+1} |a_{1,1}^{(k)}|^2 |a_{1,i}^{(k)}|^2 (2 - |u_1 - u_i|^2), \tag{3.2}$$

where

$$u_i = \overline{a_{1,i}^{(k)}}^{-1} \lambda_i \overline{a_{1,i}^{(k)}}, \quad i = 2, \dots, n+1.$$

Noting that $|a_{1,1}^{(k)}|^2 - |\beta^{(k)}|^2 = 1$, by (3.2),

$$\begin{aligned} |a_{1,1}^{(k+1)}|^2 - 1 &\leq |a_{1,1}^{(k)}|^2 \sum_{i=2}^{n+1} |a_{1,i}^{(k)}|^2 |u_1 - u_i|^2 \\ &\leq |a_{1,1}^{(k)}|^2 \sum_{i=2}^{n+1} |a_{1,i}^{(k)}|^2 (|u_1 - 1|^2 + |u_i - 1|^2) \\ &\leq |a_{1,1}^{(k)}|^2 \sum_{i=2}^{n+1} |a_{1,i}^{(k)}|^2 \, (|u_1 - 1| + |u_i - 1|)^2. \end{aligned}$$

Therefore,

$$|a_{1,1}^{(k+1)}|^2 - 1 \leq (|a_{1,1}^{(k)}|^2 - 1)\, |a_{1,1}^{(k)}|^2 \delta^2(g).$$

Then, by induction,

$$|a_{1,1}^{(k+1)}| < |a_{1,1}^{(k)}|$$

and

$$|a_{1,1}^{(k+1)}|^2 - 1 < (|a_{1,1}|^2 - 1)(|a_{1,1}|^2 \delta^2(g))^{k+1}.$$

Since $|a_{1,1}|\delta(g) < 1$, it follows that $|a_{1,1}^{(k)}| \to 1$ and, as in the last part of the proof of [4, Theorem 1.1],

$$\beta^{(k)} \to 0, \quad \alpha^{(k)} \mapsto 0, \quad A^{(k)}(A^{(k)})^* \to I_n.$$

By passing to a subsequence, we may assume that

$$A^{(k_t)} \to A_\infty, \;\; a_{1,1}^{(k_t)} \to a_\infty.$$

Thus, h_{k+1} converges to

$$h_\infty = \begin{pmatrix} a_\infty & 0 \\ 0 & A_\infty \end{pmatrix} \in \mathrm{Sp}(n, 1),$$

which implies that $\langle g, h\rangle$ is not discrete. This completes the proof. □

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Using the embedding of $\mathrm{SL}(2,\mathbb{C})$ in $\mathrm{Sp}(1,1)$ and applying similar arguments to those in the proof of [4, Theorem 1.2] gives the following corollary. It may be thought of as a generalised version of the classical Jørgensen inequality in $\mathrm{SL}(2,\mathbb{C})$ for two-generator subgroups with an elliptic generator.

Corollary 3.2. *Let g and h be elements in $\mathrm{SL}(2,\mathbb{C})$, say*

$$g = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \ (\textit{with } \theta \in [0,\pi]), \quad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $\|h\|^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2$. If $\langle g,h\rangle$ is nonelementary and discrete, then

$$4\sin^2\frac{\theta}{2}(\|h\|^2 + 2) \geq 1. \tag{3.3}$$

Proof. Let $\hat{g}$ be the image of g in $\mathrm{Sp}(1,1)$. By calculations similar to those in [4, Section 4],

$$\delta(\hat{g}) = 4\sin\frac{\theta}{2}, \quad \cosh^2\left(\frac{\rho(0,\hat{h}(0))}{2}\right) = \|h\|^2.$$

This gives the proof. □

3.1. Comparison of the conjugacy invariants. Let g be elliptic. Up to conjugacy in $\mathrm{Sp}(n,1)$,

$$g = \mathrm{diag}(\lambda_1, \ldots, \lambda_{n+1}),$$

where $|\lambda_i| = 1$ for all i. Instead of $\delta(g)$, Cao and Tan used the conjugacy invariant

$$\delta_{ct}(g) = \max\{|\lambda_i - \lambda_1|^2 : i = 2, \ldots, n+1\}.$$

Let $\lambda_j = e^{i\theta_j}$ with $\theta_j \in [0,\pi]$ for $j = 1, \ldots, n$. Note that

$$|e^{i\theta} - 1| + |e^{i\phi} - 1| = 2\left(\left|\sin\frac{\theta}{2}\right| + \left|\sin\frac{\phi}{2}\right|\right).$$

From (1.3),

$$\begin{aligned}
\delta(g) &= 2\max\left\{\left|\sin\frac{\theta_1}{2}\right| + \left|\sin\frac{\theta_{j+1}}{2}\right| : j = 1, \ldots, n\right\} \\
&= \max\left\{2\left(\sin\frac{\theta_1}{2} + \sin\frac{\theta_{j+1}}{2}\right) : j = 1, \ldots, n\right\} \\
&= \max\left\{4\sin\frac{\theta_1 + \theta_{j+1}}{4}\cos\frac{\theta_1 - \theta_{j+1}}{4} : j = 1, \ldots, n\right\}.
\end{aligned}$$

On the other hand, the Cao–Tan invariant in [4] is given by

$$\delta_{ct}(g) = \max\left\{4\sin^2\frac{\theta_1 \pm \theta_{j+1}}{2} : j = 1, \ldots, n\right\}.$$

By [4, Corollary 1.2], under the hypotheses of Corollary 3.2,

$$4\sin^2\theta(\|h\|^2 + 2) \geq 1. \tag{3.4}$$

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By comparing the sine terms on the left-hand sides of the inequalities (3.3) and (3.4),

$$\sin^2(\theta/2) \le \sin^2\theta \quad \text{for } \theta \in [0, 2\pi/3],$$

showing that the inequality (3.3) is stronger than the inequality (3.4) of Cao and Tan. But $\sin^2(\theta/2) > \sin^2\theta$ for $\theta \in (2\pi/3, \pi]$, so the inequality of Cao and Tan is better in this subinterval.

4. Proof of Theorem 1.1

Proof. Given g, let F_g denote the subgroup of $\mathrm{Sp}(n,1)$ that stabilises the set of fixed points of g. The subgroup F_g is closed in $\mathrm{Sp}(n,1)$.

Suppose, if possible, that G is not discrete. Then G is dense in $\mathrm{Sp}(n,1)$, by Theorem 2.4. Since the set of loxodromic elements $\mathcal{L}$ forms an open subset of $\mathrm{Sp}(n,1)$, it follows that $\mathcal{L} \setminus F_g$ is also an open subset in $\mathrm{Sp}(n,1)$.

Part (1). Suppose first that g is regular elliptic. We shall use the ball model. Up to conjugacy, we may assume that $q = 0$ is a fixed point of g and it is of the form (1.1). Since G is dense in $\mathrm{Sp}(n,1)$, there is a sequence of loxodromic elements $\{h_m\}$ in $\mathcal{L} \cap G$ such that $h_m \to I$. For each m, the element $h_m g h_m^{-1}$ is also regular elliptic with fixed point $h_m(q)$. Let

$$h_m g h_m^{-1} = (a_{i,j}^{(m)}) = \begin{pmatrix} a_{1,1}^{(m)} & \beta^{(m)} \\ \alpha^{(m)} & A^{(m)} \end{pmatrix}. \tag{4.1}$$

Then $h_m g h_m^{-1} \to g$. In particular, $a_{1,1}^{m} \to \lambda_1$, where $|\lambda_1| = 1$. Since $q = 0$ is a fixed point of g, the left-hand side of (3.1) becomes $|a_{1,1}^{(m)}|\delta(g)$. The group $\langle g, h_m g h_m^{-1}\rangle$ is clearly discrete.

If possible, suppose that $\langle g, h_m g h_m^{-1}\rangle$ is elementary. Then $h_m(0) \neq 0$ since loxodromic elements have no fixed point on $\mathbf{H}^n_{\mathbb{H}}$. Thus, g and $h_m g h_m^{-1}$ do not have a common fixed point. Then $\langle g, h_m g h_m^{-1}\rangle$ must keep two boundary points p_1, p_2 invariant and hence will keep invariant the quaternionic line l passing through p_1 and p_2. Then $g|_l$ acts as a regular elliptic element of $\mathrm{Isom}(l) \approx \mathrm{Sp}(1,1)$. Hence, q must belong to l, otherwise g would have at least two fixed points, contradicting regularity of g. Now, note that $g^2|_l$ is also an elliptic element that fixes p_1, p_2 and q. With respect to a chosen basis $\mathbf{p}_1$ and $\mathbf{p}_2$, $g^2|_l$ must be of the form $g^2|_l = \mathrm{diag}(\lambda, \lambda)$, where $|\lambda| = 1$. This implies that g has an eigenvalue class represented by $\lambda^{1/2}$ of multiplicity at least two, contradicting the regularity of g.

So, the group $\langle g, h_m g h_m^{-1}\rangle$ must be nonelementary. By Theorem 3.1,

$$|a_{1,1}^{(m)}|\,\delta(g) \ge 1.$$

But $|a_{1,1}^{(m)}| \to 1$ and $\delta(g) < 1$, which is a contradiction. This proves Part (1) of Theorem 1.1.

Part (2). Suppose that g is loxodromic. We shall use the Siegel domain model. Up to conjugacy, let 0 and ∞ be the fixed points of g, so that g is of the form (1.2). Since G is

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dense in $\mathrm{Sp}(n,1)$, there exists a sequence $\{h_n\}$ of loxodromic elements in $(\mathcal{L} \setminus F_g) \cap G$ such that $h_n \to g$. Let

$$h_n g h_n^{-1} = \begin{pmatrix} a_n & b_n & \gamma_n^* \\ c_n & d_n & \delta_n^* \\ \alpha_n & \beta_n & U_n \end{pmatrix}. \tag{4.2}$$

Since $h_n \in \mathcal{L} \setminus F_g$, it follows that g and h_n cannot have a common fixed point and neither can have a two-point invariant subset. So, $\langle g, h_n g h_n^{-1} \rangle$ is nonelementary for each n. By Theorem 2.2,

$$|a_n d_n|^{1/2} |b_n c_n|^{1/2} \geq \frac{1 - M_f}{M_f^2}.$$

But $b_n c_n \to 0$ as $n \to \infty$ and hence

$$\frac{1 - M_f}{M_f^2} \leq 0,$$

which is a contradiction. This proves Part (2) of Theorem 1.1.

Part (3). Let g be a Heisenberg translation and again use the Siegel domain model. Up to conjugacy, let 0 be the fixed point of g, so that g has the form (2.1). As $g \in \bar{G}$, there exists a sequence of loxodromic elements $\{h_n\} \in (\mathcal{L} \setminus F_g) \cap G$ such that

$$h_n \to g.$$

Let $h_n g h_n^{-1}$ be of the form (4.2). Since $h_n g h_n^{-1} \to g$, it follows that $t_n \to 0$.

Since g and $h_n g h_n^{-1}$ have no fixed points in common, $\langle g, h_n g h_n^{-1} \rangle$ is discrete and nonelementary; hence, by Theorem 2.3,

$$M t_n + 2|\zeta| > 1.$$

But $t_n \to 0$ as $n \to \infty$. Thus, for large n, $|\zeta| \geq \frac{1}{2}$, contrary to the hypothesis in Part (3) of Theorem 1.1.

This completes the proof. □

4.1. Proof of Corollary 1.2. Note that the set of regular elliptic elements in $\mathrm{Sp}(n,1)$ forms an open subset $\mathcal{E}$.

Part (1). Let g be regular elliptic and use the ball model. Up to conjugacy, we may assume that g is of the form (1.1) and thus $g(0) = 0$. Since G is dense in $\mathrm{Sp}(n,1)$, there exists a sequence of regular elliptic elements $\{h_m\}$ in $(\mathcal{E} \setminus F_g) \cap G$ such that $h_m \to I$. For each m, the element $h_m g h_m^{-1}$ is also regular elliptic with fixed point $h_m(0)$. Suppose that $h_m g h_m^{-1}$ is of the form (4.1). The group $\langle g, h_m g h_m^{-1} \rangle$ is clearly discrete. We claim that it is also nonelementary. For, otherwise, g and $h_m g h_m^{-1}$ must have a common fixed point different from 0 and $h_m(0)$, which will contradict the regularity of the isometries. Now, by Theorem 3.1, $|a_{1,1}^{(m)}|\,\delta(g) \geq 1$. Since $|a_{1,1}^{(m)}| \to 1$ and $\delta(g) < 1$, this is a contradiction. This proves Part (1).

Using similar arguments to those in the proof of Theorem 1.1, Parts (2) and (3) also follow.

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