In all cases, the new solution has a smaller hypotenuse than the original triangle. Since there cannot be an infinite series of decreasing hypotenuses, there can be no triangle with the required properties.

Concluding remarks

A positive rational solution of any of the equations

$$x^4 + y^4 = z^2 \qquad x^4 - y^4 = z^2$$

$$x^4 + 4y^4 = z^2 \qquad x^4 - 4y^4 = z^2$$

would give rise to a right-angled triangle contradicting the Theorem. All of these equations are therefore insoluble.

It is worth noting that it is not obvious how to use a supposed solution of $x^4 + y^4 = z^4$ to find another solution of the same equation. However, by strengthening the conjecture as shown above it was possible to carry out the proof relatively simply. This illustrates an important feature of Fermat's method of proof; that the choice of conjecture can be crucial to the success of the method. Sometimes it can be much easier to prove a stronger result than the one which is needed.

Reference

1. Sir Thomas L. Heath, *Diophantus of Alexandria*, Dover, 1964, p. 293.

STAN DOLAN

126A Harpenden Road, St. Albans AL3 6BZ

95.25 Goldbach variations

The Goldbach conjecture is one of the most famous unsolved problems in number theory, indeed in all of mathematics. It states that each even number $2n > 2$ is expressible as the sum of two primes

$$2n = p + q, \text{ where } p \text{ and } q \text{ are prime.}$$

The conjecture has been verified for all $2n$ less than 10^{18}, but a general proof is lacking. Looking at the conjecture from another angle, let us pick an even number, say 10 000 and consider $10\,000 - p$, for each prime p less than 10 000. We would probably be surprised if the resulting number was never prime, and this viewpoint gives the conjecture a certain amount of plausibility.

Theorem 1: There are infinitely many natural numbers n for which $2n$ is the sum of two prime numbers.

Proof: A famous result of Euclid states that there are infinitely many prime numbers p. So for $n = p$, $2n = p + p$, the sum of two primes!

Unfortunately, infinitely many does not mean all and any prize this proof deserves is pretty small—perhaps zero! But, as someone has said, inside every hard problem there are several easier problems struggling to get

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out. Recall that a natural number is prime if it has exactly two distinct divisors, namely itself and one; a natural number is composite if it has more than two distinct divisors. Clearly 1 is neither prime nor composite. The following are some pretty variants on the Goldbach conjecture.

Theorem 2: Every natural number greater than 11 is the sum of two composite numbers.

Proof: A quick calculation shows that $8 = 4 + 4$ and $10 = 6 + 4$ are the only numbers less than 11 which are expressible as the sum of two composites and that 11 is not so expressible. We need therefore to consider only numbers greater than 11. The 'divide and conquer' approach suggests that we consider even and odd natural numbers separately.

(i) Consider $2n$ for $n > 5$. Now, $2n = (2n - 4) + 4 = 2(n - 2) + 4$, which is the sum of two composites.

(ii) Consider $2n + 1$, for $n > 5$. Now, $2n + 1 = 2n - 8 + 9 = 2(n - 4) + 9$, which is again the sum of two composites.

Of course, in general, this decomposition can be achieved in several ways. For example, $12 = 4 + 8 = 6 + 6$ and $19 = 4 + 15 = 9 + 10$, so this suggests the following

Investigation 1: In how many different ways can a given n be expressed as the sum of composites $a + b$, where a is less than or equal to b?

The reader is invited to draw up a chart to see if any pattern can be discerned.

How about mixing primes and composites? After a little experimentation, one is led to the following result.

Theorem 3: Every natural number greater than 5 is the sum of a composite number and a prime number.

Proof: A quick check shows that any number less than or equal to 5 cannot be so expressed, so we need consider only numbers greater than 5. Again, we consider the even and odd cases separately.

(i) For $n > 2$, $2n = 2n - 2 + 2 = 2(n - 1) + 2$, which is the sum of a composite and the prime 2.

(ii) For $n > 2$, $2n + 1 = 2n - 2 + 3 = 2(n - 1) + 3$, which is the sum of a composite and the prime 3.

Again this decomposition can be achieved in several ways, for example, $11 = 2 + 9 = 3 + 8 = 4 + 7 = 5 + 6$ and $12 = 2 + 10 = 3 + 9$. This prompts

Investigation 2: In how many different ways can a given n be expressed as the sum of a prime number and a composite number?

Again the reader is invited to draw up a chart or use a computer to see if any conclusions can be drawn or any further conjectures can be made. And always remember, mathematics is an experimental science!

DES MACHALE
Department of Mathematics, University College, Cork, Ireland
e-mail: *d.machale@ucc.ie*

95.26 Nice cubics

Introduction

A discussion with some other mathematics teachers raised the question whether it is possible to find a 'nice' cubic, that is one which has three rational zeros and two rational stationary points. The idea was to use it as an exercise for students to find the zeros and stationary points without having to substitute surds back into the cubic.

I was not aware of a solution without a repeated zero and my initial reaction was that it was not possible with three distinct zeros, but then a colleague managed to find one. So I decided to try and see if it was possible to find all solutions.

To simplify the problem, by a suitable transformation we can assume that one of the zeros is zero, the others are coprime integers and the cubic is monic (i.e. with a leading coefficient of 1). So the cubic can be assumed to be $y = x(x - a)(x - b) = x^3 - (a + b)x^2 + abx$ where a and b are coprime integers.

On differentiating and setting to zero, the stationary points are given by

$$3x^2 - 2(a + b)x + ab = 0, \tag{1}$$

which has roots $\frac{1}{3}\left(a + b \pm \sqrt{(a + b)^2 - 3ab}\right)$, which are rational if, and only if, $(a + b)^2 - 3ab = a^2 - ab + b^2$ is the square of a rational number (hence the square of an integer as a and b are integers).

So we need to find integer solutions of

$$a^2 - ab + b^2 = d^2. \tag{2}$$

Finding solutions

Such equations (solving polynomials in integers) are known as Diophantine equations (after Diophantus whose work Fermat annotated with his famous marginal note). A famous Diophantine equation is the Pythagoras equation and one method of finding Pythagorean triples is to factorise over the Gaussian integers, which are numbers of the form $a + bi$ where a and b are integers. To solve (2) we need to factorise in the ring of Eisenstein integers, $\mathbb{Z}[\omega]$, which are numbers of the form $p + q\omega$ where p and q are integers and ω is a complex cube root of one. Such structures are rings of algebraic integers. An algebraic integer is a zero of a monic polynomial with integer coefficients. Loosely speaking, a ring is an algebraic structure with an arithmetic similar to $\mathbb{Z}$.