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ASYMPTOTIC DISTRIBUTIONS FOR REGRESSION-BASED SEASONAL UNIT ROOT TEST STATISTICS IN A NEAR-INTEGRATED MODEL

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In this paper we derive representations for the limiting distributions of the regression-based seasonal unit root test statistics of Hylleberg, Engle, Granger, and Yoo (1990, *Journal of Econometrics* 44, 215–238) and Beaulieu and Miron (1993, *Journal of Econometrics* 55, 305–328), inter alia, when the underlying process displays near seasonal integration. Our results generalize those presented in previous studies by allowing for an arbitrary seasonal periodicity (including the nonseasonal case), a wide range of possible assumptions on the initial conditions, a range of (seasonal) deterministic mean effects, and finite autoregressive behavior in the driving shocks. We use these representations to simulate the asymptotic local power functions of the seasonal unit root tests, demonstrating a significant dependence on serial correlation nuisance parameters in the case of the pairs of t-statistics, but not the associated F-statistic, for unit roots at the seasonal harmonic frequencies. Monte Carlo simulation results are presented that suggest that the local limiting distribution theory provides a good approximation to the finite-sample behavior of the statistics. Our results lend further weight to the advice of previous authors that inference on the unit root hypothesis at the seasonal harmonic frequencies should be based on the F-statistic, rather than on the associated pairs of t-ratios.

1. INTRODUCTION

In a recent paper Rodrigues (2001) presents representations for the limiting distributions of the quarterly seasonal unit root test statistics of Hylleberg, Engle, Granger, and Yoo (1990) when the characteristic roots of the underlying seasonal process are local to unity. In deriving his results, Rodrigues (2001) assumes that the data are generated by a mean-zero near seasonally integrated process

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with zero starting values, whose driving shocks form an independent and identically distributed (i.i.d.) sequence.

In this paper we generalize the work of Rodrigues (2001) in four separate directions. First, we allow for any given seasonal aspect, S, say, so that our results extend those of Rodrigues (2001) for $S = 4$ to, inter alia, monthly ($S = 12$), daily trading ($S = 5$), biannual ($S = 2$), bimonthly ($S = 6$), and, indeed, nonseasonal ($S = 1$) data. Second, under the near seasonally integrated model, and following the work of Canjels and Watson (1997) and Phillips and Lee (1996), we allow for a wide spectrum of initial conditions ranging from asymptotically negligible initial conditions to the so-called unconditional case where the starting values of the process are of the same stochastic order as the subsequent data points. Third, following Smith and Taylor (1998, 1999a, 1999b) and Nabeya (2001a), we allow for (seasonal) deterministic mean effects in the process, ranging from a zero mean to seasonal intercepts and seasonal trends. Fourth, we allow for finite autoregressive (AR) behavior in the driving shocks. The final generalization also allows us to expand upon the recent work of Burridge and Taylor (2001), who derive representations for the limiting null distributions of the quarterly ($S = 4$) HEGY (Hylleberg, Engle, Granger, and Yoo) tests when the shocks follow a finite AR process.

The paper is organized as follows. In Section 2 we outline the seasonal framework, defining the hypotheses of interest and the regression-based, or so-called HEGY, approach to seasonal unit root testing, where one tests the null hypothesis of unit root behavior against the stable alternative at each of the zero and seasonal frequencies. In Section 3 we provide our main result, detailing the limiting distributions of the seasonal unit root test statistics under the near-integrated seasonal model. These representations are related to previous results in the literature, where relevant. In particular, the asymptotic local power functions of the t-tests (defined subsequently) for unit roots at the zero and Nyquist (S even) frequencies are shown to coincide with that of the conventional augmented Dickey–Fuller (ADF) unit root test. A key result that we demonstrate is that the limiting distributions of the t-statistics for testing a unit root at the zero and Nyquist frequencies are invariant to the serial correlation nuisance parameters in the shocks, as are the F-statistics for testing the null hypothesis of a complex pair of unit roots at each of the seasonal harmonic frequencies, but that the pairs of t-statistics for complex unit roots are not invariant to these nuisance parameters. Our results for the harmonic seasonal frequencies build upon the work of Burridge and Taylor (2001), who demonstrate this result under the seasonal unit root null for the particular case of $S = 4$. In Section 4 we simulate the asymptotic local power functions of the seasonal unit root tests, highlighting the relative performance of the t- and F-statistics in both serially correlated and serially uncorrelated cases and quantifying the degree of power loss seen when (seasonal) intercepts and (seasonal) time trends are included in the test regression. Finite sample simulations are also provided that suggest that the local limiting distribution theory provides a good approximation to the small sample behavior of the statistics. Section 5 concludes. A mathematical Appendix contains the proof of our main result.

2. THE SEASONAL UNIT ROOT FRAMEWORK

2.1. The Seasonal Model

Following Hylleberg et al. (1990), Beaulieu and Miron (1993), Smith and Taylor (1998, 1999a), and Nabeya (2001a), inter alia, consider the scalar process $\{x_{Sn+s}\}$, observed with seasonal periodicity S, written as the sum of a purely deterministic component, μ_{Sn+s}, and a purely stochastic process, v_{Sn+s}, namely,

$$x_{Sn+s} = \mu_{Sn+s} + v_{Sn+s}, \qquad s = 1 - S, \dots, 0, \qquad n = 1, \dots, N, \tag{2.1}$$

$$\alpha(L) v_{Sn+s} = u_{Sn+s}, \qquad s = 1 - S, \dots, 0, \qquad n = 2, \dots, N, \tag{2.2}$$

$$\mu_{Sn+s} = \gamma_s^* + \beta_s^*(Sn + s), \tag{2.3}$$

where $\alpha(L) \equiv 1 - \sum_{j=1}^{S} \alpha_j^* L^j$ in (2.2) is an Sth-order autoregressive $(AR(S))$ polynomial in the conventional lag (backshift) operator, $L^k x_{Sn+s} \equiv x_{Sn+s-k}$, $k = 0,1,\dots$. The driving shocks $\{u_{Sn+s}\}$ of (2.2) are assumed to follow an AR(p), $0 \le p < \infty$ process, namely, $\phi(L) u_{Sn+s} = \varepsilon_{Sn+s}$, the roots of $\phi(z) \equiv 1 - \sum_{j=1}^{p} \phi_j z^j = 0$ all lying outside the unit circle, $|z| = 1$, with $\varepsilon_{Sn+s} \sim \text{i.i.d.}(0, \sigma^2)$, with finite fourth moments. Exact assumptions on the initial conditions v_{S+s}, $s = 1 - S, \dots, 0$, are delayed until later. In what follows we use the notation $T \equiv SN$ to denote the total sample size.

The specification (2.1)–(2.3) allows for the presence of deterministic mean effects in $\{x_{Sn+s}\}$ through μ_{Sn+s}. For the purposes of this paper, we follow Smith and Taylor (1999a) and consider the following six cases of interest.

Case 1. No intercept, no trend: $\gamma_s^* = 0$, $\beta_s^* = 0$, $s = 1 - S, \dots, 0$.

Case 2. Constant intercept, no trend: $\gamma_s^* = \gamma$, $\beta_s^* = 0$, $s = 1 - S, \dots, 0$.

Case 3. Seasonal intercepts, no trend: $\beta_s^* = 0$, $s = 1 - S, \dots, 0$.

Case 4. Constant intercept, constant trend: $\gamma_s^* = \gamma$, $\beta_s^* = \beta$, $s = 1 - S, \dots, 0$.

Case 5. Seasonal intercepts, constant trend: $\beta_s^* = \beta$, $s = 1 - S, \dots, 0$.

Case 6. Seasonal intercepts, seasonal trends: as in (2.3) with γ_s^* and β_s^* unrestricted.

2.2. The Seasonal Unit Root Hypotheses

In this paper we are concerned with the behavior of tests for seasonal unit roots in the $AR(S)$ polynomial, $\alpha(L)$, against near seasonally integrated alternatives; that is, the null hypothesis of interest is

$$\mathrm{H}_0 : \alpha(L) = 1 - L^S \equiv \Delta_S, \tag{2.4}$$

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whereas, following Tanaka (1996, pp. 355–356), Rodrigues (2001), and Nabeya (2000, 2001b), inter alia, the near seasonally integrated alternative we consider is of the form

$$\mathsf{H}_c : \alpha(L) = \left[1 - \left(1 + \frac{c}{N}\right) L^S\right], \tag{2.5}$$

where c is a fixed nonpositive constant. Notice that H_c reduces to H_0 for $c = 0$.

Under H_0 of (2.4) the data generating process (DGP) (2.1)–(2.3) of $\{x_{Sn+s}\}$ is that of a seasonally integrated process (with or without drifts according to the form of μ_{Sn+s}), admitting unit roots at both the zero frequency, $\omega_0 \equiv 0$, and at each of the seasonal spectral frequencies, $\omega_k \equiv 2\pi k/S$, $k = 1,\ldots,[S/2]$, $[\cdot]$ denoting the integer part of its argument. Under H_c of (2.5) the process $\{x_{Sn+s}\}$ is locally stable at each of the zero and seasonal frequencies if $c < 0$. Although we constrain c to be nonpositive, all of the analysis that follows also holds for positive c, in which case the process is locally explosive.

Denoting $i \equiv \sqrt{-1}$, we may factorize the polynomial $\alpha(L)$ under H_c of (2.5) as

$$\alpha(L) = \prod_{k=0}^{[S/2]} \omega_k^c(L), \tag{2.6}$$

where the lag polynomial

$$\omega_0^c(L) \equiv \left[1 - \left(1 + \frac{c}{N}\right)^{1/S} L\right] \tag{2.7}$$

corresponds to the zero frequency $\omega_0 \equiv 0$ and the lag polynomial $\omega_k^c(L)$ corresponds to the harmonic seasonal frequencies $(\omega_k, 2\pi - \omega_k)$ and is defined by

$$\omega_k^c(L) \equiv \left[1 - 2\left(1 + \frac{c}{N}\right)^{1/S} \cos(\omega_k) L + \left(1 + \frac{c}{N}\right)^{2/S} L^2\right], \tag{2.8}$$

$k = 1,\ldots,S^*$, where $S^* \equiv (S/2) - 1$ (if S is even) or $[S/2]$ (if S is odd), together with

$$\omega_{S/2}^c(L) \equiv \left[1 + \left(1 + \frac{c}{N}\right)^{1/S} L\right] \tag{2.9}$$

corresponding to the Nyquist frequency $\omega_{S/2} \equiv \pi$, when S is even. Moreover, as demonstrated in Rodrigues (2001) for the case of $S = 4$, taking a Taylor series expansion about one on each of the factors of (2.6) allows $\alpha(L)$ of (2.2) to be written as

$$\alpha(L) = \left[1 - \left(1 + \frac{c}{T} + O(T^{-2})\right)L\right]\left[1 + \left(1 + \frac{c}{T} + O(T^{-2})\right)L\right]$$
$$\times \prod_{k=1}^{S^*}\left[1 - 2\left(1 + \frac{c}{T} + O(T^{-2})\right)\cos(\omega_k)L\right.$$
$$\left. + \left(1 + \frac{2c}{T} + O(T^{-2})\right)L^2\right]. \qquad \textbf{(2.10)}$$

Consequently, H_c of (2.5) is correspondingly partitioned as $\mathsf{H}_c \equiv \bigcap_{k=0}^{[S/2]} \mathsf{H}_{c,k}$, where the hypothesis $\mathsf{H}_{c,0}$ corresponds to a local to unit root at the zero frequency $\omega_0 = 0$, whereas $\mathsf{H}_{c,S/2}$ yields a local to unit root at the Nyquist frequency $\omega_{S/2} = \pi$, where S is even. A pair of complex conjugate local to unit roots at the harmonic seasonal frequencies $(\omega_k, 2\pi - \omega_k)$ is obtained under $\mathsf{H}_{c,k}$, $k = 1,\ldots,S^*$ (see also Gregoir, 2001). Notice therefore that the particular local alternative, H_c of (2.5), that we have considered imposes a common noncentrality parameter, c, on each of the zero and seasonal frequencies. However, as we see subsequently, this involves no loss of generality.

2.3. Regression-Based Seasonal Unit Root Tests

Following Hylleberg et al. (1990) and Smith and Taylor (1999a), inter alia, the regression-based approach to testing for seasonal unit roots in $\alpha(L)$ consists of two stages. First, one obtains the ordinary least squares (OLS) demeaned series $x_{Sn+s}^{\kappa} = x_{Sn+s} - \hat{\mu}_{Sn+s}^{\kappa}$, where $\hat{\mu}_{Sn+s}^{\kappa}$ is the fitted value from the OLS regression of x_{Sn+s} on the intercept and trend variables relevant to each of Cases 1–6, $\kappa \in \{1,\ldots,6\}$ indicating the case of interest. Notice that for Case 1, $\hat{\mu}_{Sn+s}^{1}$ will be zero, and hence $x_{Sn+s}^{1} = x_{Sn+s}$, by definition. In what follows we assume that μ_{Sn+s} is not estimated under an overly restrictive case, such that the resulting unit root tests will be exact invariant to the parameters characterizing the mean function μ_{Sn+s} (see Burridge and Taylor, 2004).

Following Smith and Taylor (1999a, equation (4.17), p. 9), we then linearize $\alpha(L)$ in (2.2) around the seasonal unit roots $\exp(\pm\, i2\pi k/S)$, $k = 0,\ldots,[S/2]$, to obtain the auxiliary regression equation

$$\Delta_S x_{Sn+s}^{\kappa} = \pi_0 x_{0,Sn+s-1}^{\kappa} + \pi_{S/2} x_{S/2,Sn+s-1}^{\kappa}$$
$$+ \sum_{k=1}^{S^*} (\pi_{\alpha,k} x_{k,Sn+s-1}^{\alpha,\kappa} + \pi_{\beta,k} x_{k,Sn+s-1}^{\beta,\kappa})$$
$$+ \sum_{j=1}^{p^*} \phi_j^* \Delta_S x_{Sn+s-j}^{\kappa} + \varepsilon_{Sn+s}^{\kappa}, \qquad \textbf{(2.11)}$$

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which may be estimated along $Sn+s = p^*+S+1,\ldots,T$, $p^* \ge p$, omitting the term $\pi_{S/2}x^{\kappa}_{S/2,Sn+s-1}$ if S is odd, and where corresponding to the zero and seasonal frequencies $\omega_k = 2\pi k/S$, $k = 0,\ldots,[S/2]$,

$$x^{\kappa}_{0,Sn+s} \equiv \sum_{j=0}^{S-1} x^{\kappa}_{Sn+s-j}, \qquad x^{\kappa}_{S/2,Sn+s} \equiv \sum_{j=0}^{S-1}\cos[(j+1)\pi]x^{\kappa}_{Sn+s-j},$$

$$x^{\alpha,\kappa}_{k,Sn+s} \equiv \sum_{j=0}^{S-1}\cos[(j+1)\omega_k]x^{\kappa}_{Sn+s-j}, \tag{2.12}$$

$$x^{\beta,\kappa}_{k,Sn+s} \equiv -\sum_{j=0}^{S-1}\sin[(j+1)\omega_k]x^{\kappa}_{Sn+s-j},$$

$k = 1,\ldots,S^*$, together with $\Delta_S x^{\kappa}_{Sn+s} \equiv x^{\kappa}_{Sn+s} - x^{\kappa}_{S(n-1)+s}$. For the case of quarterly data, $S = 4$, the relevant transformations are $x^{\kappa}_{0,Sn+s} \equiv (1+L+L^2+L^3)x^{\kappa}_{Sn+s}$, $x^{\kappa}_{2,Sn+s} \equiv -(1-L+L^2-L^3)x^{\kappa}_{Sn+s}$, $x^{\alpha,\kappa}_{1,Sn+s} \equiv -L(1-L^2)x^{\kappa}_{Sn+s}$, and $x^{\beta,\kappa}_{1,Sn+s} \equiv -(1-L^2)x^{\kappa}_{Sn+s}$.

It is the elements of $\boldsymbol{\pi} \equiv (\pi_0, \pi_{S/2}, \pi_{1,\alpha}, \ldots, \pi_{S^*,\beta})'$, omitting $\pi_{S/2}$ where S is odd, from (2.11) that are of focal interest. From the characterization theorem of Smith and Taylor (1999a, p. 7), the following expressions for the elements of $\boldsymbol{\pi}$, omitting that for $\pi_{S/2}$ where S is odd, obtain. This result is proved in the accompanying working paper, Rodrigues and Taylor (2003).

PROPOSITION 2.1. *The elements of the parameter vector* $\boldsymbol{\pi}$ *from the test regression (2.11), when the* DGP *is (2.1)–(2.3) under* H_c *of (2.5), are given by*

$$\pi_0 = \phi(1)\frac{c}{T} + O(T^{-2}), \qquad \pi_{S/2} = \phi(-1)\frac{c}{T} + O(T^{-2}), \tag{2.13}$$

$$\pi_{k,\alpha} = \mathrm{Re}\{\phi[\exp(-i\omega_k)]\}\left(\frac{2c}{T}\right) - \mathrm{Im}\{\phi[\exp(-i\omega_k)]\}\delta^*_k + O(T^{-2}), \tag{2.14}$$

$$\pi_{k,\beta} = \mathrm{Re}\{\phi[\exp(-i\omega_k)]\}\delta^*_k + \mathrm{Im}\{\phi[\exp(-i\omega_k)]\}\left(\frac{2c}{T}\right) + O(T^{-2}), \tag{2.15}$$

where $\delta^*_k \equiv [1/\sin(\omega_k)][((1+2c/T)^{1/2}-1)^2\cos(\omega_k)]$, $k = 1,\ldots,S^*$.

Remark 2.1. It follows immediately from (2.13) that $T\pi_0 = \phi(1)c + o(1)$ and $T\pi_{S/2} = \phi(-1)c + o(1)$ (S even). Similarly, from (2.14) and (2.15), $T\pi_{k,\alpha} = 2c\,\mathrm{Re}\{\phi(\exp(-i\omega_k))\} + o(1)$, and $T\pi_{k,\beta} = 2c\,\mathrm{Im}\{\phi[\exp(-i\omega_k)]\} + o(1)$, $k = 1,\ldots,S^*$, where we have used the result that $\lim_{N\to\infty}[T/\sin(\omega_k)]\{[(1+(2c/T)+O(T^{-2}))^{1/2}-1]^2\cos(\omega_k)\} = 0$, $k = 1,\ldots,S^*$.

Remark 2.2. Suppose that, instead of using a single noncentrality parameter c in (2.6), we used a set of frequency-specific noncentrality parameters (which may or may not be equal), that is, $\alpha(L) = \prod_{j=0}^{[S/2]}\omega_j^{c_j}(L)$, where the $\omega_j^{c_j}(L)$ are as defined in (2.7)–(2.9) but replacing c by c_j, $j = 0,\ldots,[S/2]$. Then the results

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in Proposition 2.1 remain valid on replacing c in (2.13)–(2.15) by the appropriate noncentrality parameter c_j, $j = 0,\dots,[S/2]$. This result is proved in Rodrigues and Taylor (2003).

Noting that for $c = 0$ the expansion in (2.10) is exact,[1] it is clear from Proposition 2.1 that under H_0 of (2.4), $\boldsymbol{\pi} = \mathbf{0}$, regardless of the lag parameters characterizing $\phi(z)$. Under H_c of (2.5) with $c < 0$ and T finite, it can be shown from Proposition 2.1 that π_0, $\pi_{S/2}$ (S even) and $\pi_{k,\alpha}$, $k = 1,\dots,S^*$, are negative, regardless of $\phi(z)$, whereas the $\pi_{k,\beta}$, $k = 1,\dots,S^*$, can either be zero or nonzero, depending on the form of $\phi(z)$. Consequently, to test H_0 of (2.4) against the alternative of stationarity at one or more of the zero and seasonal frequencies, Hylleberg et al. (1990) and Smith and Taylor (1999a), inter alia, have suggested using standard regression t- and F-statistics from (2.11).

Specifically, tests for the presence or otherwise of a unit root at the zero and Nyquist (S even) frequencies are conventional lower tailed regression t-tests, denoted t_0 and $t_{S/2}$, for the exclusion of $x^{\kappa}_{0,Sn+s-1}$ and $x^{\kappa}_{S/2,Sn+s-1}$, respectively, from (2.11). Similarly, the hypothesis of a pair of complex unit roots at the kth harmonic seasonal frequency may be tested by the lower tailed t_k^{α} and two-tailed t_k^{β} regression t-tests from (2.11) for the exclusion of $x^{\alpha,\kappa}_{k,Sn+s-1}$ and $x^{\beta,\kappa}_{k,Sn+s-1}$, respectively, or by the regression F-test, denoted F_k, for the exclusion of both $x^{\alpha,\kappa}_{k,Sn+s-1}$ and $x^{\beta,\kappa}_{k,Sn+s-1}$ from (2.11), $k = 1,\dots,S^*$. Ghysels, Lee, and Noh (1994), Taylor (1998), and Smith and Taylor (1998, 1999a) also consider the joint frequency regression F-tests from (2.11), $F_{1\dots[S/2]}$, for the exclusion of $x^{\kappa}_{S/2,Sn+s-1}$ (S even) and $\{x^{\alpha,\kappa}_{k,Sn+s-1}, x^{\beta,\kappa}_{k,Sn+s-1}\}_{k=1}^{S^*}$, and $F_{0\dots[S/2]}$, for the exclusion of $x^{\kappa}_{0,Sn+s-1}$, $x^{\kappa}_{S/2,Sn+s-1}$ (S even) and $\{x^{\alpha,\kappa}_{k,Sn+s-1}, x^{\beta,\kappa}_{k,Sn+s-1}\}_{k=1}^{S^*}$. The former tests the null hypothesis of unit roots at all of the seasonal frequencies, whereas the latter tests the overall null, H_0.

Various percentiles from approximations to the finite sample null distributions of the preceding t- and F-statistics for certain choices of the seasonal aspect S, obtained by Monte Carlo simulation, assuming that $\{u_{Sn+s}\} \sim IN(0,1)$, appear in the literature; see, inter alia, Hylleberg et al. (1990, Tables 1a and 1b, pp. 226–277), Smith and Taylor (1998, Tables 1a and 1b, p. 276), Beaulieu and Miron (1993, Table A.1, pp. 325–326) and Ghysels et al. (1994, Tables C.1 and C.2, pp. 440–441). Asymptotic critical values are also provided in Beaulieu and Miron (1993, Table A.1, pp. 325–326) and Taylor (1998, Tables I and II, pp. 356–357).

3. ASYMPTOTIC REPRESENTATIONS

In this section we derive representations for the limiting distributions of the OLS seasonal unit root statistics computed from the test regression (2.11) for each of Cases 1–6 of (2.3), under the near seasonally integrated alternative, H_c of (2.5).

In what follows, what we assume about the initial conditions v_{S+s}, $s = 1 - S,\dots,0$, is important. One possibility, following Elliott, Rothenberg, and

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Stock (1996), is to assume that the starting values satisfy $T^{-1/2}v_{S+s} \to^p 0$, $s = 1 - S,\dots,0$. One particular example of this is the so-called conditional case where $v_{S+s} = u_{S+s}$, $s = 1 - S,\dots,0$, so that the initial observation in each of the S seasons, x_{S+s}, has variance equal to that of the shocks. In contrast, Pantula, Gonzalez-Farias, and Fuller (1994) argue that "there are a modest number of situations" (p. 459) where one might reasonably assume that the conditional case applies. For macroeconomic data in particular, the conditional assumption seems untenable. Pantula et al. (1994) suggest instead the unconditional case, where the initial conditions obey the same data generating process as the rest of the process. Within our near seasonally integrated model we can contain both the conditional and unconditional cases within a more general framework by making the following assumption, which is the seasonal generalization of Assumption 3 of Canjels and Watson (1997, p. 185; see also Phillips and Lee, 1996).

Assumption 3.1. The initial conditions satisfy $v_{S+s} = \sum_{k=0}^{m+[N\lambda]}(1 + c/N)^k u_{S(1-k)+s}$, $s = 1 - S,\dots,0$, where $\lambda \geq 0$ and $m \in \{0,1,\dots,m^*\}$, m^* finite. As noted in Canjels and Watson (1997), the unconditional case obtains taking the limit as $\lambda N \to \infty$. The conditional case obtains for $\lambda = m = 0$.

In the results given in Theorem 3.1, which follows, we will use the superscript ξ to indicate which of Cases 1–6 of μ_{Sn+s} of (2.3) hold. For the zero frequency ω_0 tests: Case 1: $\xi = 0$; Cases 2 and 3: $\xi = 1$; Cases 4–6: $\xi = 2$. For the seasonal frequency ω_k tests, $k = 1,\dots,[S/2]$: Cases 1, 2, and 4: $\xi = 0$; Cases 3 and 5: $\xi = 1$; Case 6: $\xi = 2$. For example, substituting $\xi = 0$ into the expression given in (3.1), which follows, for t_0 gives the limiting representation for the t_0 statistic under Case 1 (no deterministics) of μ_{Sn+s}, whereas $\xi = 1$ gives the limiting representation under Case 2 (nonseasonal intercept, no trend) and Case 3 (seasonal intercepts, no trend). Similarly, substituting $\xi = 2$ into (3.2) and (3.3) gives the limiting representations for the t_k^α and t_k^β statistics, respectively, under Case 6 (seasonal intercepts and seasonal trends) of μ_{Sn+s}.

We now state our main theorem. Some remarks follow.

THEOREM 3.1. *Let the process $\{x_{Sn+s}\}$ be generated by (2.1)–(2.3) and let the (unique) polynomial $\psi(z)$ be defined such that $\psi(z)\phi(z) \equiv 1$. Then, under H_c of (2.5) and Assumption 3.1, and denoting weak convergence of the associated probability measures by $\Rightarrow$,*

$$t_i \Rightarrow c\left\{\int_0^1 [\bar{J}_{i,c}^{\xi}(r,\lambda)]^2\,dr\right\}^{1/2} + \frac{\int_0^1 \bar{J}_{i,c}^{\xi}(r,\lambda)\,d\bar{J}_{i,0}^{0}(r,0)}{\left\{\int_0^1 [\bar{J}_{i,c}^{\xi}(r,\lambda)]^2\,dr\right\}^{1/2}}$$

$$\equiv \Im_{i,c}^{\xi}(\lambda), \tag{3.1}$$

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$$
\begin{aligned}
t_k^{\alpha} \Rightarrow\ & cb_k^* \left\{ (a_k^2 + b_k^2) \int_0^1 \{[\bar{J}_{\alpha,k,c}^{\xi}(r,\lambda)]^2 + [\bar{J}_{\beta,k,c}^{\xi}(r,\lambda)]^2\}\, dr \right\}^{1/2} \\
& + \left[\left\{ a_k \left\{ \int_0^1 [\bar{J}_{\beta,k,c}^{\xi}(r,\lambda)\, d\bar{J}_{\alpha,k,0}^{0}(r,0) - \bar{J}_{\alpha,k,c}^{\xi}(r,\lambda)\, d\bar{J}_{\beta,k,0}^{0}(r,0)] \right\} \right. \right. \\
& \qquad \left. + b_k \left\{ \int_0^1 [\bar{J}_{\alpha,k,c}^{\xi}(r,\lambda)\, d\bar{J}_{\alpha,k,0}^{0}(r,0) + \bar{J}_{\beta,k,c}^{\xi}(r,\lambda)\, d\bar{J}_{\beta,k,0}^{0}(r,0)] \right\} \right\} \\
& \qquad \left. \times \left\{ (a_k^2 + b_k^2) \int_0^1 \{[\bar{J}_{\alpha,k,c}^{\xi}(r,\lambda)]^2 + [\bar{J}_{\beta,k,c}^{\xi}(r,\lambda)]^2\}\, dr \right\}^{-1/2} \right] \\
\equiv\ & \Im_{\alpha,k,c}^{\xi}(\lambda, a_k, b_k, b_k^*),
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
t_k^{\beta} \Rightarrow\ & ca_k^* \left\{ (a_k^2 + b_k^2) \int_0^1 \{[\bar{J}_{\alpha,k,c}^{\xi}(r,\lambda)]^2 + [\bar{J}_{\beta,k,c}^{\xi}(r,\lambda)]^2\}\, dr \right\}^{1/2} \\
& + \left[\left\{ b_k \int_0^1 [\bar{J}_{\alpha,k,c}^{\xi}(r,\lambda)\, d\bar{J}_{\beta,k,0}^{0}(r,0) - \bar{J}_{\beta,k,c}^{\xi}(r,\lambda)\, d\bar{J}_{\alpha,k,0}^{0}(r,0)] \right. \right. \\
& \qquad \left. + a_k \int_0^1 [\bar{J}_{\alpha,k,c}^{\xi}(r,\lambda)\, d\bar{J}_{\alpha,k,0}^{0}(r,0) + \bar{J}_{\beta,k,c}^{\xi}(r,\lambda)\, d\bar{J}_{\beta,k,0}^{0}(r,0)] \right\} \\
& \qquad \left. \times \left\{ (a_k^2 + b_k^2) \int_0^1 \{[\bar{J}_{\alpha,k,c}^{\xi}(r,\lambda)]^2 + [\bar{J}_{\beta,k,c}^{\xi}(r,\lambda)]^2\}\, dr \right\}^{-1/2} \right] \\
\equiv\ & \Im_{\beta,k,c}^{\xi}(\lambda, a_k, b_k, a_k^*),
\end{aligned} \tag{3.3}
$$

$$
F_k \Rightarrow \frac{1}{2} [(\Im_{\alpha,k,c}^{\xi}(\lambda,0,1,1))^2 + (\Im_{\beta,k,c}^{\xi}(\lambda,0,1,0))^2] \equiv \Im_{k,c}^{\xi}(\lambda), \tag{3.4}
$$

$$
F_{1\ldots[S/2]} \Rightarrow \frac{1}{S-1} \left\{ (\Im_{S/2,c}^{\xi}(\lambda))^2 + 2 \sum_{k=1}^{S} \Im_{k,c}^{\xi}(\lambda) \right\}, \tag{3.5}
$$

$$
F_{0\ldots[S/2]} \Rightarrow \frac{1}{S} \left\{ \sum_{i=0,S/2} (\Im_{i,c}^{\xi}(\lambda))^2 + 2 \sum_{k=1}^{S} \Im_{k,c}^{\xi}(\lambda) \right\}, \tag{3.6}
$$

omitting $\Im_{S/2,c}^{\xi}(\lambda)$ in (3.5) and (3.6) where S is odd. In (3.1), $i = 0, S/2$ if S is even and $i = 0$ if S is odd, whereas $k = 1,\ldots S^*$ in (3.2)–(3.4). The nomenclature ξ is as defined following Assumption 3.1, $a_k \equiv \mathrm{Im}\{\psi[\exp(i\omega_k)]\}$, $a_k^* \equiv \mathrm{Im}\{\phi[\exp(-i\omega_k)]\}$, $b_k \equiv \mathrm{Re}\{\psi[\exp(i\omega_k)]\}$, $b_k^* \equiv \mathrm{Re}\{\phi[\exp(-i\omega_k)]\}$. Finally, the independent limiting processes $\bar{J}_{i,c}^{\xi}(r,\lambda)$, $i = 0, S/2$, and $\bar{J}_{\alpha,k,c}^{\xi}(r,\lambda)$ and $\bar{J}_{\beta,k,c}^{\xi}(r,\lambda)$, $k = 1,\ldots,S^*$, are as defined in Definition A.1 of the Appendix.

Remark 3.1. Setting $c = 0$ and $S = 4$, the representations provided in Theorem 3.1 reduce to the limiting null representations provided in Burridge and Taylor (2001). Theorem 3.1 therefore generalizes the results of Burridge and Taylor to an arbitrary seasonal aspect, S, and to the near seasonally integrated case, $c \neq 0$. For $c = 0$ and $\phi(z) = 1$, the representations in Theorem 3.1 also

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reduce to those given in Smith and Taylor (1999a) and Nabeya (2001a). Moreover, notice that the representations in (3.1)–(3.6) for $c = 0$ do not depend on the initial conditions v_{S+s}, $s = 1 - S, \dots, 0$, provided $\xi > 0$. Indeed, for each of Cases 3, 5, and 6 of μ_{Sn+s} of (2.3), the statistics also yield exact similar tests (for detailed discussion on this point, see Smith and Taylor 1998, 1999a).

Remark 3.2. The representations given in Theorem 3.1 delineate the asymptotic local power functions of the seasonal unit root tests from (2.11) in all cases indexed by a common noncentrality parameter c. Numerical tabulations of certain of these functions are given in Section 4, which allow us to investigate the relative power properties of the tests and also to quantify the degree of power loss incurred when (seasonal) intercepts and (seasonal) time trends are included in the test regression. Notice also, for example, from the results in Theorem 3.1 that the asymptotic local power functions of the seasonal frequency $t_{S/2}$ (S even), t_k^{α}, t_k^{β} and F_k, $k = 1, \dots, S^*$, and $F_{1\dots[S/2]}$ tests are not affected by including a nonseasonal intercept or time trend in (2.11) but are affected by the inclusion of seasonal intercepts and seasonal trends. Similarly, the asymptotic local power function of the t_0 test is not affected by the inclusion of seasonal intercepts (trends) vis-à-vis the case where a nonseasonal intercept (trend) is included in (2.11). The asymptotic local power function of the $F_{0\dots[S/2]}$ test is, however, clearly affected by both seasonal and nonseasonal intercepts and time trends.

Remark 3.3. In practice, we would probably want to permit the noncentrality parameter to vary across the seasonal frequencies $\omega_k \equiv 2\pi k/S$, $k = 0, \dots, [S/2]$, as in Remark 2.2. Just as argued in Rodrigues (2001, p. 80), the asymptotic orthogonality result stated in Remark A.1 of the Appendix ensures that the representations given in Theorem 3.1 remain appropriate in such cases on substituting c for $c_k \le 0$, $k = 0, \dots, [S/2]$, in (3.1) and (3.2), respectively, throughout and appropriately redefining Assumption 3.1.

Remark 3.4. From (3.1)–(3.6), it is seen that the limiting distributions of the t_0, $t_{S/2}$ (S even), F_k, $k = 1, \dots, S^*$, $F_{1\dots[S/2]}$, and $F_{0\dots[S/2]}$ statistics do not depend on the serial correlation nuisance parameters $\{\phi_i\}_{i=1}^{p}$ under H_c of (2.5), whereas those of the harmonic frequency t-statistics t_k^{α} and t_k^{β}, $k = 1, \dots S^*$, do, in general. We investigate this dependence for the case of an AR(1) error process in Section 4. An obvious exception occurs when $\phi(z)$ is expressible as $\phi(z^S)$, that is, $\phi(z)$ is a purely seasonal polynomial. In this case the limiting distributions of the t_k^{α} and t_k^{β} statistics in (3.2) and (3.3) are invariant to the parameters characterizing $\phi(z)$, $k = 1, \dots S^*$. Other exceptions can occur at specific frequencies. For example, if S is an integer multiple of four, then the limiting distributions of the $t_{S/4}^{\alpha}$ and $t_{S/4}^{\beta}$ statistics, corresponding to the harmonic frequency pair $(\pi/2, 3\pi/2)$, will be invariant to the parameters of $\phi(z)$, provided $\phi(z)$ is expressible as $\phi(z^2)$. Notice that in the serially uncorrelated case, $\phi(z) = 1$, $a_k = 0$, $k = 1, \dots, S^*$, and $b_k = 1$, $k = 0, \dots, [S/2]$. For the special case of $S = 4$, $c = 0$, and $\lambda = 0$, these results reproduce those proved in Burridge and Taylor (2001).

Remark 3.5. It can be seen from (3.1) of Theorem 3.1 that for a given value of ξ, t_0 and $t_{S/2}$ have identical limiting representations, and hence asymptotic local power functions, under H_c of (2.5). These limiting distributions are also seen to be independent, by virtue of the independence of $\bar{J}^{\xi}_{0,c}(r,\lambda)$ and $\bar{J}^{\xi}_{S/2,c}(r,\lambda)$, $\xi \in \{0,1,2\}$. For $c = 0$ and $\xi = 1,2$, these coincide with the corresponding limiting null representations for the augmented Dickey–Fuller (ADF) t-statistic provided in Theorem 10.1.3, p. 561, and Theorem 10.1.6, pp. 567–568, of Fuller (1996), respectively. For $\xi = 0$ the representations obtained for $\lambda = 0$ coincide with the representation given in Corollary 10.1.1.5 of Fuller (1996, p. 554), noting that Fuller imposes zero starting values on his analysis. For $c < 0$ and $\xi = 2$ they replicate the representation given in Canjels and Watson (1997, Footnote 13, p. 192) for the asymptotic power function of the conventional ADF test. For $\lambda = 0$ the representation in (3.1) replicates those given for the ADF test in, inter alia, Chan and Wei (1987), Nabeya and Tanaka (1990), Perron (1989), Phillips (1987), and Elliott et al. (1996), with graphs of the associated asymptotic power function for $\xi = 0,1,2$ provided in Figures 1, 2, and 3, respectively, of Elliott et al. (1996, pp. 822–824).

Remark 3.6. Representations (3.2)–(3.5) of Theorem 3.1 demonstrate that, under H_c of (2.5), the t^{α}_k and t^{α}_l, $k \neq l$, and t^{β}_k and t^{β}_l, $k \neq l$, $k,l = 1,\dots,S^*$, statistics are asymptotically mutually independent and are also asymptotically independent of t_0 and $t_{S/2}$; the F-statistics F_k, $k = 1,\dots,S^*$, possess mutually independent and identical limiting representations and are asymptotically independent of the t_0 and $t_{S/2}$ statistics; and the F-statistic $F_{1\dots[S/2]}$ is asymptotically independent of the zero frequency t_0 statistic.

Remark 3.7. The limiting representation in (3.2) for $c = \lambda = 0$ can be shown to coincide with that of the limiting null distribution of the t-statistic of Dickey, Hasza, and Fuller (1984) for the case of a biannual ($S = 2$) seasonal process. Moreover, for $c \leq 0$, the stated representation in (3.2) for the t^{α}_k statistic when $\lambda = 0$, $a_k = 0$ and $b_k = 1$, $k = 1,\dots,S^*$, reduces to that provided by Chan (1989, Theorem 1, p. 282) and Perron (1992, Theorem 2, p. 127) for $\xi = 0$ and Tanaka (1996, pp. 355–362) and Nabeya (2000, 2001b) for $\xi = 0,1,2$ for the limiting distribution of the Dickey et al. (1984) statistic for $S = 2$.

4. NUMERICAL RESULTS

In Figures 1a–c we graph the asymptotic local power functions of the t^{α}_k, t^{β}_k, and F_k tests for unit roots at the seasonal harmonic frequencies ω_k, $k \in \{1,\dots,S^*\}$, for $\phi(z) = 1$. Figures 2a and b graph the corresponding results for the $t^{\alpha}_{S/4}$ and $t^{\beta}_{S/4}$ tests, respectively, for unit roots at the harmonic frequency pair $(\pi/2, 3\pi/2)$ in cases where S is an integer multiple of four, when

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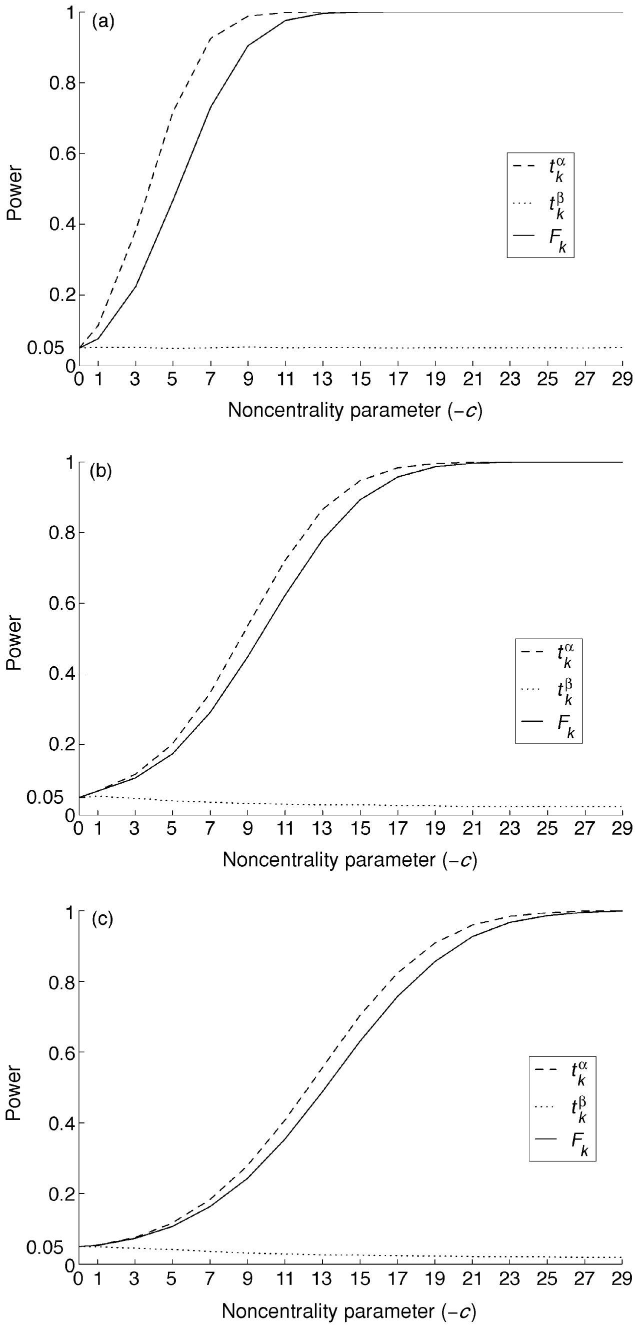

FIGURE 1. Asymptotic local power functions of harmonic unit root tests: (a) $\phi(L) = 1$, $\xi = 0$; (b) $\phi(L) = 1$, $\xi = 1$; (c) $\phi(L) = 1$, $\xi = 2$.

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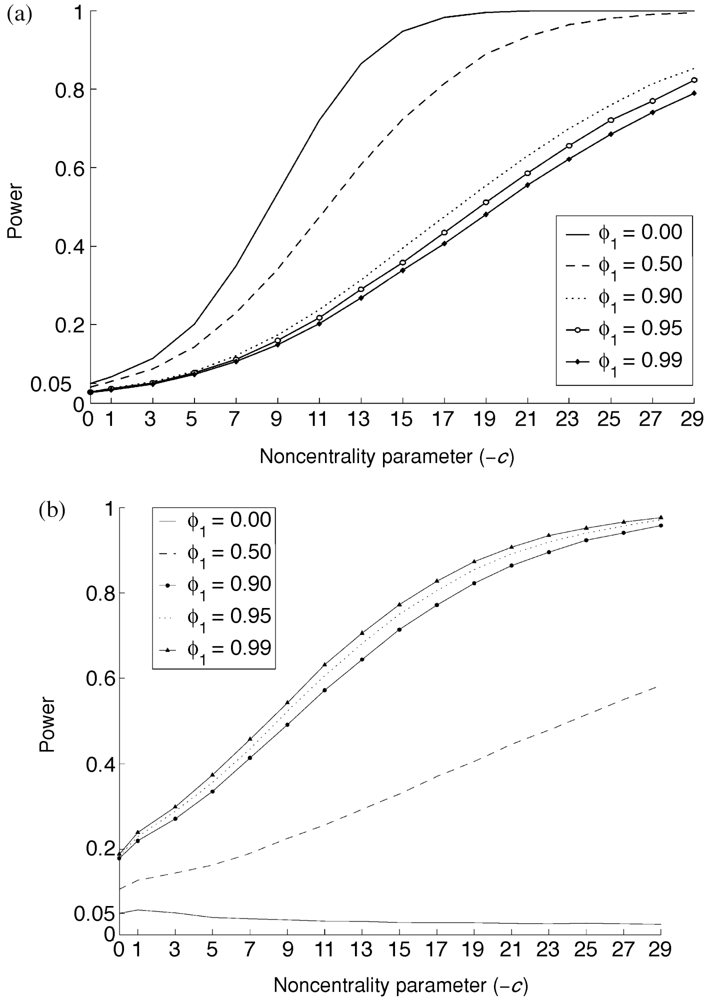

FIGURE 2. Asymptotic local power functions of (a) $t^{\alpha}_{S/4}$: $\phi(L) = 1 - \phi_1 L$, $\xi = 1$; and (b) $t^{\beta}_{S/4}$: $\phi(L) = 1 - \phi_1 L$, $\xi = 1$.

$\phi(z) = 1 - \phi_1 L$, a first-order autoregression. We report results for $\phi_1 = \{0.5, 0.9, 0.95, 0.99\}$.[2]

In the case of Figures 1a–c these functions are reported for $\xi = \{0,1,2\}$, whereas for Figures 2a and b only results for $\xi = 1$ are reported, these displaying the same qualitative features with respect to ϕ_1 as were seen for $\xi = 0$ and $\xi = 2$.[3] All reported results were for tests conducted at the asymptotic 0.05 significance level, but in unreported results we considered other significance levels. These results were qualitatively no different from those reported. We tabulate the limiting powers of these tests under the conditional environment, the initial conditions generated by Assumption 3.1 with $\lambda = 0$. Corresponding simulations for other values of λ, including the unconditional case, were also computed but, comfortingly given that λ is unknown in practice, differed very little from the results reported here; full details may be obtained on request. The reported results were obtained by direct simulation of the appropriate limiting functionals from Theorem 3.1; that is, they were calculated using discrete approximations to the relevant stochastic integrals appearing in Theorem 3.1, as in, for example, Beaulieu and Miron (1993, p. 317). All simulations were based on 60,000 replications and a sample size of 4,000 and used the RNDN function of Gauss 3.1. As noted in Remark 3.5, graphs of the asymptotic local power functions of the t_0 and $t_{S/2}$ (S even) tests can be found in Elliott et al. (1996).

Consider first Figures 1a–c. A number of interesting features are apparent from these asymptotic local power functions. First, the power loss from using the F_k rather than t_k^α statistics in testing for unit roots at the seasonal harmonic frequencies is not too large. The biggest differences are seen in the case of $\xi = 0$. However, this case is arguably of no practical interest because the resulting unit root tests will neither be similar with respect to the starting values of the process nor be invariant to the seasonal intercepts, γ_s^*, $s = 1 - S,\ldots,0$, of (2.3) (for details, see Smith and Taylor, 1998, 1999a; Burridge and Taylor, 2004). For $\xi = 1$ the difference between the power functions of the two tests is relatively small and is further reduced for $\xi = 2$. Figures 1a–c also highlight the fact that the t_k^β tests do not display power in the case of $\phi(z) = 1$; indeed, it is straightforward to show from (3.3) that $t_k^\beta \to^p 0$ as $c \to -\infty$ in this case.

Figures 1a–c also show the power loss in the t_k^α and F_k tests when deterministic components are included in (2.11). As has also been observed by Elliott et al. (1996, p. 823) for the asymptotic local power functions of the ADF test, the power loss incurred when moving from $\xi = 0$ to $\xi = 1$ is somewhat larger than that when moving from $\xi = 1$ to $\xi = 2$ for both the t_k^α and F_k tests.

Turning to Figures 2a and b one observes a strong dependence of the asymptotic local power functions of the harmonic frequency t-tests on the AR parameter, ϕ_1. From Figure 2a it is seen that for $c = 0$, the $t_{S/4}^\alpha$ test is undersized, the more so as ϕ_1 increases toward unity. For a given value of c, this undersizing is translated into large reductions in power relative to $\phi_1 = 0$. Indeed, comparing Figures 2a and 1b it is seen that the $F_{S/4}$ test is considerably more powerful than the $t_{S/4}^\alpha$ test when $\phi \neq 0$. A similar but reversed pattern is seen in the $t_{S/4}^\beta$ test, where considerable oversizing is seen for $c = 0$, with test size approaching 20% in many cases,[4] with associated power gains when $c < 0$. Notice that under this design $\pi_{S/4}^\beta$ is nonzero when $c < 0$ and $\phi_1 \neq 0$, so that the $t_{S/4}^\beta$ test will diverge as $c \to -\infty$, whereas it is zero when $\phi_1 = 0$.

We now use Monte Carlo simulation methods to investigate the finite sample local power properties of the t_0, t_1^α, t_1^β, F_1, and t_2 tests from (2.11) for the case of quarterly data, $S = 4$, when the true DGP for $\{x_{4n+s}\}$ is the near seasonally integrated AR model:

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$$\left(1-\left[1+\frac{c}{N}\right]L^4\right)x_{4n+s} = u_{4n+s}, \qquad s=-3,\dots,0, \qquad n=2,\dots,N, \tag{4.1}$$

$$(1-\phi_1 L)u_{4n+s} = \epsilon_{4n+s} \sim IN(0,1),$$
$$s=-3,\dots,0, \qquad n=1,\dots,N, \tag{4.2}$$

with $u_{4k+s} = 0$, $s = -3,\dots,0$, $k \leq 0$, for the conditional case, $x_{4+s} = u_{4+s}$, $s = -3,\dots,0$.[5] We report the effects of varying the noncentrality parameter c among $c = \{0,-1,-5,-9,-13,-17,-21\}$ and the first-order autoregressive parameter ϕ_1 among $\phi_1 = \{0.0,0.5,0.9\}$. Results for $\phi_1 = 0$ with $p^* = 0$ in (2.11) are reported in Table 1, whereas those for $\phi_1 = 0.5$ and 0.9 with $p^* = 1$ in (2.11) are reported in Table 2.

We focus on the sample sizes $N = 25$, 50, 100 and on Case 3 of (2.11), where $\xi = 1$ for all reported tests. All tests were run at the nominal 0.05 level using finite sample critical values. These were computed using data generated according to (4.1) and (4.2) with $c = \phi_1 = 0$ and were based on 60,000 Monte Carlo

TABLE 1. Empirical power of quarterly unit root tests (nominal 0.05 level)

c	N	t_0	t_k^α	t_k^β	t_2	F_1
−1	25	0.060	0.068	0.053	0.061	0.069
−1	50	0.063	0.069	0.052	0.058	0.067
−1	100	0.059	0.068	0.055	0.061	0.065
−5	25	0.136	0.225	0.037	0.134	0.195
−5	50	0.128	0.214	0.039	0.126	0.183
−5	100	0.123	0.207	0.040	0.124	0.175
−9	25	0.338	0.622	0.029	0.334	0.543
−9	50	0.297	0.582	0.030	0.292	0.493
−9	100	0.279	0.562	0.033	0.277	0.467
−13	25	0.667	0.944	0.025	0.665	0.902
−13	50	0.570	0.911	0.027	0.567	0.848
−13	100	0.527	0.889	0.028	0.523	0.810
−17	25	0.919	0.998	0.024	0.915	0.994
−17	50	0.831	0.994	0.025	0.832	0.982
−17	100	0.772	0.989	0.027	0.777	0.970
−21	25	0.992	1.00	0.022	0.992	1.00
−21	50	0.965	1.00	0.025	0.964	0.999
−21	100	0.931	1.00	0.025	0.931	0.998

Note: DGP (4.1) with $\phi_1 = 0$. Auxiliary regression (2.11), $p^* = 0$: Case 3.

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TABLE 2. Empirical size and power of quarterly unit root tests (nominal 0.05 level)

		$\phi_1 = 0.5$					$\phi_1 = 0.9$				
c	N	t_0	t_k^α	t_k^β	t_2	F_1	t_0	t_k^α	t_k^β	t_2	F_1
0	25	0.049	0.042	0.105	0.049	0.051	0.061	0.028	0.172	0.051	0.049
0	50	0.052	0.040	0.101	0.050	0.049	0.056	0.029	0.167	0.049	0.048
0	100	0.051	0.041	0.107	0.048	0.050	0.053	0.029	0.177	0.049	0.051
−1	25	0.056	0.058	0.123	0.059	0.068	0.062	0.040	0.207	0.059	0.065
−1	50	0.059	0.054	0.120	0.058	0.066	0.062	0.038	0.207	0.056	0.065
−1	100	0.058	0.054	0.126	0.058	0.067	0.061	0.039	0.216	0.058	0.066
−5	25	0.101	0.164	0.158	0.125	0.185	0.079	0.095	0.327	0.126	0.180
−5	50	0.110	0.151	0.155	0.117	0.176	0.095	0.086	0.320	0.117	0.172
−5	100	0.113	0.145	0.166	0.117	0.177	0.102	0.083	0.337	0.118	0.177
−9	25	0.223	0.430	0.221	0.308	0.517	0.108	0.236	0.513	0.320	0.520
−9	50	0.245	0.381	0.217	0.277	0.471	0.157	0.203	0.492	0.275	0.475
−9	100	0.251	0.361	0.224	0.263	0.461	0.194	0.188	0.503	0.269	0.464
−13	25	0.420	0.751	0.311	0.618	0.876	0.139	0.462	0.709	0.646	0.885
−13	50	0.453	0.677	0.296	0.537	0.825	0.227	0.378	0.672	0.550	0.831
−13	100	0.460	0.637	0.302	0.502	0.803	0.311	0.342	0.664	0.505	0.805
−17	25	0.636	0.935	0.415	0.885	0.990	0.162	0.706	0.855	0.908	0.993
−17	50	0.676	0.885	0.378	0.802	0.977	0.298	0.581	0.808	0.813	0.979
−17	100	0.690	0.850	0.380	0.753	0.968	0.436	0.523	0.795	0.761	0.970
−21	25	0.813	0.989	0.524	0.983	1.00	0.185	0.881	0.939	0.990	1.00
−21	50	0.847	0.972	0.474	0.950	0.999	0.364	0.762	0.902	0.960	0.999
−21	100	0.862	0.957	0.462	0.919	0.998	0.555	0.691	0.886	0.925	0.998

Note: DGP (4.1) with $\phi_1 = 0.5, 0.9$. Auxiliary regression (2.11), $p^* = 1$: Case 3.

replications of the statistics from (2.11), for each of the sample sizes considered.[6] The remaining cases of (2.11) were also considered, as were tests run at other nominal levels, and the corresponding tests for other values of S, but in each case yielded qualitatively similar conclusions to those reported and are hence omitted. The corresponding results for $N \to \infty$ are given in Figures 1b, 2a, and 2b for the t_1^α, t_1^β, and F_1 tests and in Elliott et al. (1996) for t_0 and t_2.

As N is increased the reported quantities appear to be converging rapidly toward the corresponding asymptotic power levels as might be expected. As an example, for $c = -7$ the asymptotic local power of the t_1^α and F_1 tests is 0.202 and 0.174, respectively, both of which are very close indeed to the local power of these tests for $N = 100$ given in Table 1. Mirroring the asymptotic local power results in Figure 1b we also see from Table 1 that for $\phi_1 = 0$ (the only case where the t_1^α statistic is correctly sized) only relatively small losses in power are incurred from using the F_1 rather than the t_1^α test.

A general feature of the results reported in Table 1 for $\phi_1 = 0$ is that for all of the tests the finite sample powers for a given value of c approach the limiting value for that value of c from above, as has also been noted by Tanaka

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(1996, p. 351) for the ADF test. In Table 2 this general pattern is reversed in the case of the t_0 test, which displays a relatively strong finite sample dependence on ϕ_1. This behavior is to be expected given that positive values of ϕ_1 will imply spectral mass at frequency zero. Indeed, in the case of $\phi_1 = 0.9$ there are cases where this yields the dominant root at frequency zero. This occurs for $(c, N) = (-9,25)$, $(-13,25)$, $(-17,25)$, $(-21,25)$, and $(-21,50)$. The results for the t_2 and F_1 tests in Table 2 are very similar to those reported in Table 1, showing very little finite sample dependence on ϕ_1. The results for the t_1^α and t_1^β tests show a clear dependence on ϕ_1 with the finite sample results approaching the limiting values for those values of ϕ_1 and c.

In practice it is therefore quite clear from these results that proper inference cannot be based on the t_k^α and t_k^β tests, their size properties under autocorrelated errors being unknown, even asymptotically. The fact that the asymptotic local power functions of the F_k tests are invariant to the parameters of $\phi(z)$ and that the F_k tests display only small power losses relative to the t_k^α test when $\phi(z) = 1$, and are otherwise more powerful, makes the case in favor of the use of the F_k tests overwhelming.

5. CONCLUSIONS

In this paper we have derived representations for the limiting distributions of regression-based seasonal unit root test statistics when the characteristic roots of the underlying seasonal process are local to unity. We derived our results for the case of a process observed for any given seasonal periodicity, S, under very general assumptions on the initial values of the process, for test regressions that included either no deterministic variables or variables that ranged from a nonseasonal intercept to seasonal intercepts and seasonal trends, and for driving shocks that followed a stationary AR(p) process.

Our results have built upon and generalized, in various directions, earlier representations provided in Rodrigues (2001), Burridge and Taylor (2001), Smith and Taylor (1999a), and Nabeya (2001a). In our key result we have demonstrated that the limiting distributions of the t_0 and $t_{S/2}$ statistics for testing for a unit root at the zero and Nyquist (S even) frequencies, respectively, under near seasonal integration coincide with that of the conventional ADF statistic and are invariant to the serial correlation nuisance parameters in the shocks. This invariance property has also been shown to hold for the F_k statistics for testing the null hypothesis of a complex pair of unit roots at the seasonal harmonic frequencies, $k = 1, \ldots, S^*$, but not for the corresponding t_k^α and t_k^β statistics, whose asymptotic distributions under near seasonal integration are not, in general, invariant to these nuisance parameters. Our findings confirmed the recommendations of earlier authors against the use of the t_k^α and t_k^β statistics for practical data analysis in favor of the F_k statistics; see, inter alia, Burridge and Taylor (2001) and Smith and Taylor (1999a).

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NOTES

1. That is, $(1 - L^S) \equiv (1 - L)(1 + L)^\delta \prod_{k=1}^{S^*}(1 - 2\cos\omega_k L + L^2)$, where $\delta = 1(0)$ if S is even (odd).

2. We need only consider positive values of ϕ_1 because the asymptotic local power functions in this case are easily shown to be invariant to the sign of ϕ_1. Moreover, there is no need to report results for $F_{S/4}$ because its asymptotic local power function is invariant to ϕ_1, as demonstrated in Theorem 3.1.

3. The unreported results may be obtained from the authors on request.

4. For $\xi = 2$, empirical test size rises to around 40% in these cases.

5. We also ran our experiments under the unconditional case. The results in this case were little different from those reported.

6. For this reason the results for $c = 0$ are omitted from Table 1.

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APPENDIX

Proof of Theorem 3.1. In what follows we simplify the exposition by setting $\gamma_s^* = \beta_s^* = 0$, $s = 1-S,\dots,0$, in (2.3). Before proving our main theorem, we need to set up some notation and establish some preparatory lemmas.

Under H_c of (2.5) and Assumption 3.1 it follows from (2.1)–(2.3) that

$$x_{Sn+s} = \sum_{i=2}^{n} \exp\left\{(n-i)\frac{c}{N}\right\} u_{Si+s} + \sum_{j=0}^{[N\lambda]+m} \exp\left\{(n+j-1)\frac{c}{N}\right\} u_{s-S(j-1)},$$

$$s = 1-S,\dots,0. \qquad \textbf{(A.1)}$$

Defining the annualized processes

$$\mathbf{u}_n = [u_{S(n-1)+1}, u_{S(n-1)+2},\dots,u_{Sn}]', \qquad \boldsymbol{\varepsilon}_n = [\varepsilon_{S(n-1)+1}, \varepsilon_{S(n-1)+2},\dots,\varepsilon_{Sn}]'$$

we may write $\mathbf{u}_n = \boldsymbol{\Psi}^*(L)\boldsymbol{\varepsilon}_n \equiv \sum_{k=0}^{\infty}\boldsymbol{\Psi}_k^*\boldsymbol{\varepsilon}_{n-k}$, where the sequence of $S\times S$ matrices $\{\boldsymbol{\Psi}_k^*\}_{k=0}^{\infty}$ are as defined in the Appendix of Burridge and Taylor (2001). Notice that the sequence $\{k\boldsymbol{\Psi}_k^*\}$ is absolutely summable by virtue of the stationarity of $\{u_{Sn+s}\}$ (cf. Burridge and Taylor, 2001). Moreover, $E(\boldsymbol{\varepsilon}_n\boldsymbol{\varepsilon}_n') = \sigma^2\mathbf{I}_S$ and $E(\mathbf{u}_n\mathbf{u}_{n-q}') = \sigma^2\sum_{k=0}^{\infty}\boldsymbol{\Psi}_{k+q}^*\boldsymbol{\Psi}_k^{*\prime}$, $q = 0,1,\dots$.

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LEMMA A.1. *Let* $\{x_{Sn+s}\}$ *be generated by (2.1)–(2.3) under* H_c *of (2.5) and Assumption 3.1. Defining the annualized process* $\mathbf{X}_n^\kappa \equiv [x_{S(n-1)+1}^\kappa, x_{S(n-1)+2}^\kappa, \ldots, x_{Sn}^\kappa]'$*, then as* $N \to \infty$*,*

$$[\boldsymbol{\Psi}^*(1)]^{-1}(\sigma^2 N)^{-1/2}\mathbf{X}_{[rN]}^\kappa \Rightarrow \mathbf{J}_c^\kappa(r) + \delta_c(\exp\{rc\}\hat{\mathbf{J}}_c^\kappa(\lambda)) \equiv \mathbf{J}_c^\kappa(r,\lambda),$$
$$r \in [0,1], \qquad \lambda \geq 0, \qquad \textbf{(A.2)}$$

where $[\boldsymbol{\Psi}^*(1)]^{-1}$ *is the inverse of the matrix* $\boldsymbol{\Psi}^*(1)$ *and* δ_c *is a scalar indicator variable such that* $\delta_c = 1$ *if* $c \neq 0$ *or if* $c = 0$ *and* $\kappa = 1,2,4$*, and* $\delta_c = 0$ *otherwise. In (A.2),* $\mathbf{J}_c^\kappa(r) \equiv (J_{1-S,c}^\kappa(r), \ldots, J_{0,c}^\kappa(r))'$ *and* $\hat{\mathbf{J}}_c^\kappa(\lambda) \equiv (\hat{J}_{1-S,c}^\kappa(\lambda), \ldots, \hat{J}_{0,c}^\kappa(\lambda))'$ *are mutually independent S-vectors each with independent elements such that* $\mathbf{J}_c^\kappa(r,\lambda) \equiv (J_{1-S,c}^\kappa(r,\lambda), \ldots, J_{0,c}^\kappa(r,\lambda))'$*, where* $J_{s,c}^1(r,\lambda) = J_{s,c}^1(r) + \delta_c(\exp\{rc\}\hat{J}_{s,c}^1(\lambda))$*, with* $J_{s,c}^1(r)$ *and* $\hat{J}_{s,c}^1(\lambda)$ *independent Ornstein–Uhlenbeck* (OU) *processes,* $s = 1 - S, \ldots, 0$*, and*

$$J_{s,c}^2(r,\lambda) \equiv J_{s,c}^1(r,\lambda) - \int_0^1 \left[\frac{1}{S}\sum_{s=1-S}^{0} J_{s,c}^1(r,\lambda)\right] dr, \qquad \textbf{(A.3)}$$

$$J_{s,c}^3(r,\lambda) \equiv J_{s,c}^1(r,\lambda) - \int_0^1 J_{s,c}^1(r,\lambda)\, dr, \qquad \textbf{(A.4)}$$

$$J_{s,c}^4(r,\lambda) \equiv J_{s,c}^2(r,\lambda) - 12\left(r - \frac{1}{2}\right)\int_0^1 \left(r - \frac{1}{2}\right)\left[\frac{1}{S}\sum_{s=1-S}^{0} J_{s,c}^2(r,\lambda)\right] dr, \qquad \textbf{(A.5)}$$

$$J_{s,c}^5(r,\lambda) \equiv J_{s,c}^3(r,\lambda) - 12\left(r - \frac{1}{2}\right)\int_0^1 \left(r - \frac{1}{2}\right)\left[\frac{1}{S}\sum_{s=1-S}^{0} J_{s,c}^3(r,\lambda)\right] dr, \qquad \textbf{(A.6)}$$

$$J_{s,c}^6(r,\lambda) \equiv J_{s,c}^3(r,\lambda) - 12\left(r - \frac{1}{2}\right)\int_0^1 \left(r - \frac{1}{2}\right) J_{s,c}^3(r,\lambda)\, dr. \qquad \textbf{(A.7)}$$

Proof. The proof follows from a straightforward extension of results proved in Rodrigues (2001), using the multivariate invariance principle of Phillips (1988, p. 1026), coupled with (A.1) of Canjels and Watson (1997, p. 197) and applications of the continuous mapping theorem (CMT). ■

LEMMA A.2. *Under the conditions of Lemma A.1 and as* $N \to \infty$*,*

$$N^{-2}\sum_{n=2}^{N} \mathbf{X}_{n-1}^\kappa(\mathbf{X}_{n-1}^\kappa)' \Rightarrow \sigma^2\boldsymbol{\Psi}^*(1)\left\{\int_0^1 \mathbf{J}_c^\kappa(r,\lambda)[\mathbf{J}_c^\kappa(r,\lambda)]'\, dr\right\}\boldsymbol{\Psi}^*(1)' \qquad \textbf{(A.8)}$$

and

$$N^{-1}\sum_{n=2}^{N} \mathbf{X}_{n-1}^\kappa \boldsymbol{\varepsilon}_n' \Rightarrow \sigma^2\boldsymbol{\Psi}^*(\mathbf{1})\int_0^1 \mathbf{J}_c^\kappa(r,\lambda)\, d\mathbf{J}_0^1(r,0)'. \qquad \textbf{(A.9)}$$

Proof. The proof follows from Lemma A.1 and applications of the CMT. ■

We next define the mutually orthogonal $S \times 1$ selection vectors,

$$c_0 = [1,1,\ldots,1]', \qquad c_{S/2} = [\cos(\pi),\cos(2\pi),\ldots,\cos(S\pi)]', \tag{A.10}$$

$$c_{\alpha,k} = [\cos(\omega_k),\cos(2\omega_k),\ldots,\cos(S\omega_k)]', \tag{A.11}$$

$$c_{\beta,k} = -[\sin(\omega_k),\sin(2\omega_k),\ldots,\sin(S\omega_k)]', \tag{A.12}$$

omitting $c_{S/2}$ if S is odd. In the analysis that follows we will require the following definition, which makes use of the preceding selection vectors.

DEFINITION A.1. *For each of Cases 1–6 and $r \in [0,1]$, we define $\bar{J}^{\xi}_{0,c}(r,\lambda)$, $\bar{J}^{\xi}_{S/2,c}(r,\lambda)$ (S even), $\bar{J}^{\xi}_{\alpha,k,c}(r,\lambda)$, and $\bar{J}^{\xi}_{\beta,k,c}(r,\lambda)$, $k = 1,\ldots,S^*$, for $\xi = 0,1,2$, as follows:*

$$\sqrt{S}\bar{J}^{0}_{0,c}(r,\lambda) \equiv c_0'\mathbf{J}^{1}_{c}(r,\lambda),$$

$$\sqrt{S}\bar{J}^{1}_{0,c}(r,\lambda) \equiv c_0'\mathbf{J}^{\kappa}_{c}(r,\lambda), \qquad \kappa = 2,3,$$

$$\sqrt{S}\bar{J}^{2}_{0,c}(r,\lambda) \equiv c_0'\mathbf{J}^{\kappa}_{c}(r,\lambda), \qquad \kappa = 4,5,6,$$

$$\sqrt{\frac{S}{2}}\bar{J}^{0}_{\alpha,k,c}(r,\lambda) \equiv c_{\alpha,k}'\mathbf{J}^{\kappa}_{c}(r,\lambda), \qquad \sqrt{\frac{S}{2}}\bar{J}^{0}_{\beta,k,c}(r,\lambda) \equiv c_{\beta,k}'\mathbf{J}^{\kappa}_{c}(r,\lambda), \qquad \kappa = 1,2,4,$$

$$\sqrt{\frac{S}{2}}\bar{J}^{1}_{\alpha,k,c}(r,\lambda) \equiv c_{\alpha,k}'\mathbf{J}^{\kappa}_{c}(r,\lambda), \qquad \sqrt{\frac{S}{2}}\bar{J}^{1}_{\beta,k,c}(r,\lambda) \equiv c_{\beta,k}'\mathbf{J}^{\kappa}_{c}(r,\lambda), \qquad \kappa = 3,5,$$

$$\sqrt{\frac{S}{2}}\bar{J}^{2}_{\alpha,k,c}(r,\lambda) \equiv c_{\alpha,k}'\mathbf{J}^{6}_{c}(r,\lambda), \qquad \sqrt{\frac{S}{2}}\bar{J}^{2}_{\beta,k,c}(r,\lambda) \equiv c_{\beta,k}'\mathbf{J}^{6}_{c}(r,\lambda),$$

$k = 1,\ldots,S^*$, and, for S even,

$$\sqrt{S}\bar{J}^{0}_{S/2,c}(r,\lambda) \equiv c_{S/2}'\mathbf{J}^{\kappa}_{c}(r,\lambda), \qquad \kappa = 1,2,4,$$

$$\sqrt{S}\bar{J}^{1}_{S/2,c}(r,\lambda) \equiv c_{S/2}'\mathbf{J}^{\kappa}_{c}(r,\lambda), \qquad \kappa = 3,5,$$

$$\sqrt{S}\bar{J}^{2}_{S/2,c}(r,\lambda) \equiv c_{S/2}'\mathbf{J}^{6}_{c}(r,\lambda).$$

These are straightforwardly seen to be mutually independent because they are mutually orthogonal transformations of the OU-based functionals from (A.2). Moreover, it is straightforwardly seen that for $\lambda = 0$ these are independent standard, demeaned, and demeaned and detrended OU processes for $\xi = 0$, $\xi = 1$, and $\xi = 2$, respectively.

Noting that the selection vectors defined in (A-10)–(A.12) are precisely those used to define the transformed level variables in (2.12), it is straightforward but tedious to show that the following relations hold, omitting (A.14) where S is odd:

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$$\frac{1}{\sqrt{T}} x^{\kappa}_{0,S[rN]+s} = \frac{1}{\sqrt{T}} c_0' \mathbf{X}^{\kappa}_{[rN]} + o_p(1), \tag{A.13}$$

$$\frac{1}{\sqrt{T}} x^{\kappa}_{S/2,S[rN]+s} = \frac{1}{\sqrt{T}} (-1)^i c_{S/2}' \mathbf{X}^{\kappa}_{[rN]} + o_p(1), \qquad i = s \bmod 2, \tag{A.14}$$

$$\frac{1}{\sqrt{T}} x^{\alpha,\kappa}_{k,S[rN]+s} = \frac{1}{\sqrt{T}} e_s'(c^*_{\alpha,k} c'_{\alpha,k} \mathbf{X}^{\kappa}_{[rN]} + c^*_{\beta,k} c'_{\beta,k} \mathbf{X}^{\kappa}_{[rN]}) + o_p(1), \tag{A.15}$$

$$\frac{1}{\sqrt{T}} x^{\beta,\kappa}_{k,S[rN]+s} = \frac{1}{\sqrt{T}} e_s'(c^*_{\beta,k} c'_{\alpha,k} \mathbf{X}^{\kappa}_{[rN]} - c^*_{\alpha,k} c'_{\beta,k} \mathbf{X}^{\kappa}_{[rN]}) + o_p(1), \tag{A.16}$$

$s = 1 - S, \ldots, 0$, and where $\{e_s\}_{s=1-S}^{0}$ are a collection of S-dimensional selection vectors whose $(S+s)$th element is unity and all other elements are equal to zero, $c^*_{\alpha,k} = [\cos(\omega_k), \cos(0), \ldots, \cos((2-S)\omega_k)]'$, and $c^*_{\beta,k} = [\sin(\omega_k), \sin(0), \ldots, \sin((2-S)\omega_k)]'$.

In Lemma A.3, which follows, we will make use of the following identities: $c_i' \mathbf{\Psi}^*(1) \equiv b_i c_i'$, where $i = 0, S/2$ if S is even and $i = 0$ otherwise, $c'_{\alpha,k} \mathbf{\Psi}^*(1) \equiv a_k c'_{\beta,k} + b_k c'_{\alpha,k}$, and $c'_{\beta,k} \mathbf{\Psi}^*(1) \equiv -a_k c'_{\alpha,k} + b_k c'_{\beta,k}$, $k = 1, \ldots, S^*$, where $b_0 \equiv \psi(1)$, $b_{S/2} \equiv \psi(-1)$ (S even), $a_k \equiv \text{Im}[\psi(\exp(i\omega_k))]$, and $b_k \equiv \text{Re}[\psi(\exp(i\omega_k))]$, with $\psi(z)$ as defined in Theorem 3.1.

LEMMA A.3. *Under the conditions of Lemma A.1 and as $N \to \infty$, and letting $\ell \equiv S + p^* + 1$,*

(i) $$\frac{1}{T^2} \sum_{Sn+s=\ell}^{T} (x^{\kappa}_{i,Sn+s-1})^2 \Rightarrow \sigma^2 b_i^2 \int_0^1 [\bar{J}^{\xi}_{i,c}(r,\lambda)]^2\, dr,\ i = 0, S/2\ (S \text{ even}).$$

(ii) $$\frac{1}{T^2} \sum_{Sn+s=\ell}^{T} (x^{\alpha,\kappa}_{k,Sn+s-1})^2 \overset{a}{\equiv} \frac{1}{T^2} \sum_{Sn+s=\ell}^{T} (x^{\beta,\kappa}_{k,Sn+s-1})^2$$

$$\Rightarrow \frac{\sigma^2}{4} (a_k^2 + b_k^2) \int_0^1 \{[\bar{J}^{\xi}_{\alpha,k,c}(r,\lambda)]^2 + [\bar{J}^{\xi}_{\beta,k,c}(r,\lambda)]^2\}\, dr,$$

$k = 1, \ldots, S^*$, and where $\overset{a}{\equiv}$ denotes asymptotic equivalence.

(iii) $$\frac{1}{T} \sum_{Sn+s=\ell}^{T} x^{\kappa}_{i,Sn+s-1} \varepsilon_{Sn+s} \Rightarrow \sigma^2 b_i \int_0^1 \bar{J}^{\xi}_{i,c}(r,\lambda)\, d\bar{J}^{0}_{i,0}(r,0),\ i = 0, S/2\ (S \text{ even}).$$

(iv) $$\frac{1}{T} \sum_{Sn+s=\ell}^{T} x^{\alpha,\kappa}_{k,Sn+s-1} \varepsilon_{Sn+s}$$

$$\Rightarrow \frac{\sigma^2}{2} \left\{ b_k \left[\int_0^1 \bar{J}^{\xi}_{\alpha,k,c}(r,\lambda)\, d\bar{J}^{0}_{\alpha,k,0}(r,0) + \int_0^1 \bar{J}^{\xi}_{\beta,k,c}(r,\lambda)\, d\bar{J}^{0}_{\beta,k,0}(r,0) \right] \right.$$

$$\left. + a_k \left[\int_0^1 \bar{J}^{\xi}_{\beta,k,c}(r,\lambda)\, d\bar{J}^{0}_{\alpha,k,0}(r,0) - \int_0^1 \bar{J}^{\xi}_{\alpha,k,c}(r,\lambda)\, d\bar{J}^{0}_{\beta,k,0}(r,0) \right] \right\}.$$

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(v) $\frac{1}{T}\sum_{Sn+s=\ell}^{T} x_{k,Sn+s-1}^{\beta,\kappa}\varepsilon_{Sn+s}$

$$\Rightarrow \frac{\sigma^2}{2}\left\{b_k\left[\int_0^1 \bar{J}_{\alpha,k,c}^{\xi}(r,\lambda)\,d\bar{J}_{\beta,k,0}^{0}(r,0) - \int_0^1 \bar{J}_{\beta,k,c}^{\xi}(r,\lambda)\,d\bar{J}_{\alpha,k,0}^{0}(r,0)\right]\right.$$

$$\left. + a_k\left[\int_0^1 \bar{J}_{\alpha,k,c}^{\xi}(r,\lambda)\,d\bar{J}_{\alpha,k,0}^{0}(r,0) + \int_0^1 \bar{J}_{\beta,k,c}^{\xi}(r,\lambda)\,d\bar{J}_{\beta,k,0}^{0}(r,0)\right]\right\},$$

$k = 1,\ldots,S^*$.

Proof. The proofs of parts (i) and (iii) follow straightforwardly from (A.13), (A.14), (A.8), (A.9), and the CMT (for full details, see Rodrigues and Taylor, 2003). ∎

(ii) From (A.15) and (A.16) it is straightforward to show that

$$\frac{1}{T^2}\sum_{Sn+s=\ell}^{T}(x_{k,Sn+s-1}^{\alpha,\kappa})^2 = \frac{1}{T^2}\sum_{Sn+s=\ell}^{T}(x_{k,Sn+s-1}^{\beta,\kappa})^2 + o_p(1),$$

so we need only establish results for $T^{-2}\sum(x_{k,Sn+s-1}^{\alpha,\kappa})^2$ in what follows. From (A.15),

$$\frac{1}{T^2}\sum_{Sn+s=\ell}^{T}(x_{k,Sn+s-1}^{\alpha,\kappa})^2 \equiv \frac{S}{2T^2}\left(c_{\alpha,k}'\left[\sum_{n=2}^{N}\mathbf{X}_{n-1}^{\kappa}(\mathbf{X}_{n-1}^{\kappa})'\right]c_{\alpha,k}\right.$$

$$\left. + c_{\beta,k}'\left[\sum_{n=2}^{N}\mathbf{X}_{n-1}^{\kappa}(\mathbf{X}_{n-1}^{\kappa})'\right]c_{\beta,k}\right) + o_p(1).$$

It then follows from (A.8), the orthogonality of $c_{\alpha,k}$ and $c_{\beta,k}$, and the CMT that

$$\frac{1}{T^2}\sum_{Sn+s=\ell}^{T}(x_{k,Sn+s-1}^{\alpha,\kappa})^2$$

$$\Rightarrow \frac{\sigma^2}{2S}\left\{(a_k c_{\beta,k}' + b_k c_{\alpha,k}')\int_0^1 \mathbf{J}_c^{\kappa}(r,\lambda)[\mathbf{J}_c^{\kappa}(r,\lambda)]'\,dr(a_k c_{\beta,k}' + b_k c_{\alpha,k}')'\right.$$

$$\left. + (-a_k c_{\alpha,k}' + b_k c_{\beta,k}')\int_0^1 \mathbf{J}_c^{\kappa}(r,\lambda)[\mathbf{J}_c^{\kappa}(r,\lambda)]'\,dr(-a_k c_{\alpha,k}' + b_k c_{\beta,k}')'\right\}$$

$$\equiv \frac{\sigma^2}{2S}(a_k^2 + b_k^2)\left\{c_{\alpha,k}'\left\{\int_0^1 \mathbf{J}_c^{\kappa}(r,\lambda)[\mathbf{J}_c^{\kappa}(r,\lambda)]'\,dr\right\}c_{\alpha,k}\right.$$

$$\left. + c_{\beta,k}'\left\{\int_0^1 \mathbf{J}_c^{\kappa}(r,\lambda)[\mathbf{J}_c^{\kappa}(r,\lambda)]'\,dr\right\}c_{\beta,k}\right\}$$

$$\equiv \frac{\sigma^2}{4}(a_k^2 + b_k^2)\int_0^1 \{[\bar{J}_{\alpha,k,c}^{\xi}(r,\lambda)]^2 + [\bar{J}_{\beta,k,c}^{\xi}(r,\lambda)]^2\}\,dr.$$

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(iv) From (A.15) we obtain that

$$\frac{1}{T}\sum_{Sn+s=\ell}^{T} x_{k,Sn+s-1}^{\alpha,\kappa}\varepsilon_{Sn+s} = \frac{1}{T}\left(c_{\alpha,k}'\left[\sum_{n=2}^{N}\mathbf{X}_{n-1}^{\kappa}\boldsymbol{\varepsilon}_n'\right]c_{\alpha,k} + c_{\beta,k}'\left[\sum_{n=2}^{N}\mathbf{X}_{n-1}^{\kappa}\boldsymbol{\varepsilon}_n'\right]c_{\beta,k}\right) + o_p(1).$$

It then follows from (A.9) and applications of the CMT that

$$\begin{aligned}
&\frac{1}{T}\sum_{Sn+s=\ell}^{T} x_{k,Sn+s-1}^{\alpha,\kappa}\varepsilon_{Sn+s}\\
&\Rightarrow \frac{\sigma^2}{S}\left\{(a_k c_{\beta,k}' + b_k c_{\alpha,k}')\int_0^1 \mathbf{J}_c^{\kappa}(r,\lambda)\, d\mathbf{J}_0^1(r,0)' c_{\alpha,k}\right.\\
&\qquad \left. + (-a_k c_{\alpha,k}' + b_k c_{\beta,k}')\int_0^1 \mathbf{J}_c^{\kappa}(r,\lambda)\, d\mathbf{J}_0^1(r,0)' c_{\beta,k}\right\}\\
&\equiv \frac{\sigma^2}{S}\left\{b_k\left[c_{\alpha,k}'\left[\int_0^1 \mathbf{J}_c^{\kappa}(r,\lambda)\, d\mathbf{J}_0^1(r,0)'\right]c_{\alpha,k} + c_{\beta,k}'\left[\int_0^1 \mathbf{J}_c^{\kappa}(r,\lambda)\, d\mathbf{J}_0^1(r,0)'\right]c_{\beta,k}\right]\right.\\
&\qquad + a_k\left[c_{\beta,k}'\left[\int_0^1 \mathbf{J}_c^{\kappa}(r,\lambda)\, d\mathbf{J}_0^1(r,0)'\right]c_{\alpha,k}\right.\\
&\qquad\qquad \left.\left. - c_{\alpha,k}'\left[\int_0^1 \mathbf{J}_c^{\kappa}(r,\lambda)\, d\mathbf{J}_0^1(r,0)'\right]c_{\beta,k}\right]\right\},
\end{aligned}$$

from which the stated result follows immediately.

(v) From (A.16) we obtain that

$$\frac{1}{T}\sum_{Sn+s=\ell}^{T} x_{k,Sn+s-1}^{\beta,\kappa}\varepsilon_{Sn+s} = \frac{1}{T}\left(c_{\alpha,k}'\left[\sum_{n=2}^{N}\mathbf{X}_{n-1}^{\kappa}\boldsymbol{\varepsilon}_n'\right]c_{\beta,k} - c_{\beta,k}'\left[\sum_{n=2}^{N}\mathbf{X}_{n-1}^{\kappa}\boldsymbol{\varepsilon}_n'\right]c_{\alpha,k}\right) + o_p(1).$$

Again, from (A.9) and applications of the CMT it follows that

$$\begin{aligned}
&\frac{1}{T}\sum_{Sn+s=\ell}^{T} x_{k,Sn+s-1}^{\beta,\kappa}\varepsilon_{Sn+s}\\
&\Rightarrow \frac{\sigma^2}{S}\left\{(a_k c_{\beta,k}' + b_k c_{\alpha,k}')\int_0^1 \mathbf{J}_c^{\kappa}(r,\lambda)\, d\mathbf{J}_0^1(r,0)' c_{\beta,k}\right.\\
&\qquad \left. - (-a_k c_{\alpha,k}' + b_k c_{\beta,k}')\int_0^1 \mathbf{J}_c^{\kappa}(r,\lambda)\, d\mathbf{J}_0^1(r,0)' c_{\alpha,k}\right\},\\
&\equiv \frac{\sigma^2}{S}\left\{b_k\left[c_{\alpha,k}'\int_0^1 \mathbf{J}_c^{\kappa}(r,\lambda)\, d\mathbf{J}_0^1(r,0)' c_{\beta,k} - c_{\beta,k}'\int_0^1 \mathbf{J}_c^{\kappa}(r,\lambda)\, d\mathbf{J}_0^1(r,0)' c_{\alpha,k}\right]\right.\\
&\qquad \left. + a_k\left[c_{\beta,k}'\int_0^1 \mathbf{J}_c^{\kappa}(r,\lambda)\, d\mathbf{J}_0^1(r,0)' c_{\beta,k} + c_{\alpha,k}'\int_0^1 \mathbf{J}_c^{\kappa}(r,\lambda)\, d\mathbf{J}_0^1(r,0)' c_{\alpha,k}\right]\right\},
\end{aligned}$$

from which the stated result follows immediately. ■

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Now we move to the proof of our main result. Consider the appropriately scaled OLS estimator of $\boldsymbol{\pi}$ from (2.11), $\mathbf{D}_T^{-1}\hat{\boldsymbol{\pi}} = [\mathbf{D}_T\mathbf{R}(\mathbf{X}'\mathbf{X})\mathbf{R}'\mathbf{D}_T]^{-1}\mathbf{D}_T\mathbf{R}(\mathbf{X}'\mathbf{Y})$, where $\mathbf{D}_T = T^{-1}\mathbf{I}_S$, $\mathbf{R} = [\mathbf{I}_S : \mathbf{0}_{S\times p^*}]$, $\mathbf{X}$ is the $(T-p^*-S)\times(S+p^*)$ matrix,

$$\mathbf{X} = \begin{bmatrix} x^{\kappa}_{0,p^*+S} & x^{\kappa}_{S/2,p^*+S} & x^{\alpha,\kappa}_{1,p^*+S} & x^{\beta,\kappa}_{1,p^*+S} & \cdots & \cdots & x^{\alpha,\kappa}_{S^*,p^*+S} & x^{\beta,\kappa}_{S^*,p^*+S} & \Delta_S x^{\kappa}_{p^*+S} & \cdots & \Delta_S x^{\kappa}_{S+1} \\ x^{\kappa}_{0,p^*+S+1} & x^{\kappa}_{S/2,p^*+S+1} & x^{\alpha,\kappa}_{1,p^*+S+1} & x^{\beta,\kappa}_{1,p^*+S+1} & \cdots & \cdots & x^{\alpha,\kappa}_{S^*,p^*+S+1} & x^{\beta,\kappa}_{S^*,p^*+S+1} & \Delta_S x^{\kappa}_{p^*+S+1} & \cdots & \Delta_S x^{\kappa}_{S+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x^{\kappa}_{0,T-1} & x^{\kappa}_{S/2,T-1} & x^{\alpha,\kappa}_{1,T-1} & x^{\beta,\kappa}_{1,T-1} & \cdots & \cdots & x^{\alpha,\kappa}_{S^*,T-1} & x^{\beta,\kappa}_{S^*,T-1} & \Delta_S x^{\kappa}_{T-1} & \cdots & \Delta_S x^{\kappa}_{T-p^*} \end{bmatrix},$$

omitting the second column if S is odd, and $\mathbf{Y} = [\Delta_S x^{\kappa}_{p^*+S+1}, \Delta_S x^{\kappa}_{p^*+S+2}, \ldots, \Delta_S x^{\kappa}_T]'$.

Remark A.1. Using results from Jeganathan (1991) it is straightforward to show that the off-diagonal elements of $\mathbf{D}_T\mathbf{R}(\mathbf{X}'\mathbf{X})\mathbf{R}'\mathbf{D}_T$ are all of $o_p(1)$. Moreover, notice that we may consider $\mathbf{D}_T^{-1}\hat{\boldsymbol{\pi}}$ directly, because the matrix $\mathbf{D}^*_T(\mathbf{X}'\mathbf{X})\mathbf{D}^*_T$, where $\mathbf{D}^*_T$ is a diagonal $(S+p^*)\times(S+p^*)$ matrix whose first S leading diagonal elements are T^{-1} and whose remaining p^* leading diagonal elements are $T^{-1/2}$, is block diagonal between its upper $(S\times S)$ and lower $(p^*\times p^*)$ blocks.

The following proposition, whose proof is given in Rodrigues and Taylor (2003), gives a convenient form for the OLS t_0, $t_{S/2}$ (S even), t^{α}_k and t^{β}_k, $k = 1,\ldots,S^*$, statistics from (2.11), where $\hat{\sigma}^2$ is used to denote the usual OLS estimator of σ^2 from (2.11), and $\ell \equiv S + p^* + 1$.

PROPOSITION A.1. *The* OLS *t-statistics from (2.11) can be written as*

$$t_i = \frac{T}{\hat{\sigma}}\pi_i(\Lambda^{\kappa}_i)^{1/2} + \frac{d^{*\kappa}_i}{\hat{\sigma}(\Lambda^{\kappa}_i)^{1/2}} + o_p(1), \qquad i = 0, S/2\ (S\text{ even}), \tag{A.17}$$

$$t^{\alpha}_k = \frac{T}{\hat{\sigma}}\pi_{\alpha,k}(\Lambda^{\alpha,\kappa}_k)^{1/2} + \frac{d^{*\alpha,\kappa}_k}{\hat{\sigma}(\Lambda^{\alpha,\kappa}_k)^{1/2}} + o_p(1), \tag{A.18}$$

$$t^{\beta}_k = \frac{T}{\hat{\sigma}}\pi_{\beta,k}(\Lambda^{\beta,\kappa}_k)^{1/2} + \frac{d^{*\beta,\kappa}_k}{\hat{\sigma}(\Lambda^{\beta,\kappa}_k)^{1/2}} + o_p(1), \tag{A.19}$$

$k = 1,\ldots S^*$, where

$$\Lambda^{\kappa}_i = \frac{1}{T^2}\sum_{Sn+s=\ell}^{T}(x^{\kappa}_{i,Sn+s-1})^2, \tag{A.20}$$

$$\Lambda^{\alpha,\kappa}_k = \frac{1}{T^2}\sum_{Sn+s=\ell}^{T}(x^{\alpha,\kappa}_{k,Sn+s-1})^2, \qquad \Lambda^{\beta,\kappa}_k = \frac{1}{T^2}\sum_{Sn+s=\ell}^{T}(x^{\beta,\kappa}_{k,Sn+s-1})^2, \tag{A.21}$$

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and

$$d_i^{*\kappa} = \frac{1}{T} \sum_{Sn+s=\ell}^{T} x_{i,Sn+s-1}^{\kappa} \varepsilon_{Sn+s}, \tag{A.22}$$

$$d_k^{*\alpha,\kappa} = \frac{1}{T} \sum_{Sn+s=\ell}^{T} x_{k,Sn+s-1}^{\alpha,\kappa} \varepsilon_{Sn+s}, \qquad d_k^{*\beta,\kappa} = \frac{1}{T} \sum_{Sn+s=\ell}^{T} x_{k,Sn+s-1}^{\beta,\kappa} \varepsilon_{Sn+s}. \tag{A.23}$$

The results stated for the t-statistics in (3.1)–(3.3) then follow directly from Proposition 2.1, Proposition A.1, Lemma A.3, and applications of the CMT.

Turning to the F_k-statistics, $k = 1,\ldots,S^*$, observe from the asymptotic orthogonality result (see Remark A.1) that $F_k = \frac{1}{2}[(t_k^\alpha)^2 + (t_k^\beta)^2] + o_p(1)$. It therefore follows from (3.2) and (3.3) and the CMT that, as $N \to \infty$, on grouping terms, that

$$\begin{aligned}
F_k \Rightarrow \frac{1}{2}\Bigg\{ & c^2(a_k^{*2} + b_k^{*2})\left\{(a_k^2 + b_k^2)\int_0^1 \{[\bar{J}_{\alpha,k,c}^{\xi}(r,\lambda)]^2 + [\bar{J}_{\beta,k,c}^{\xi}(r,\lambda)]^2\}\,dr\right\} \\
& + cb_k^*\left\{a_k\int_0^1 [\bar{J}_{\beta,k,c}^{\xi}(r,\lambda)\,d\bar{J}_{\alpha,k,0}^{0}(r,0) - \bar{J}_{\alpha,k,c}^{\xi}(r,\lambda)\,d\bar{J}_{\beta,k,0}^{0}(r,0)]\right. \\
& \qquad \left. + b_k\int_0^1 [\bar{J}_{\alpha,k,c}^{\xi}(r,\lambda)\,d\bar{J}_{\alpha,k,0}^{0}(r,0) + \bar{J}_{\beta,k,c}^{\xi}(r,\lambda)\,d\bar{J}_{\beta,k,0}^{0}(r,0)]\right\} \\
& + \left\{(a_k^2 + b_k^2)\left\{\int_0^1 [\bar{J}_{\alpha,k,c}^{\xi}(r,\lambda)\,d\bar{J}_{\beta,k,0}^{0}(r,0) - \bar{J}_{\beta,k,c}^{\xi}(r,\lambda)\,d\bar{J}_{\alpha,k,0}^{0}(r,0)]\right\}^2\right. \\
& \qquad \left. + (a_k^2 + b_k^2)\left\{\int_0^1 [\bar{J}_{\alpha,k,c}^{\xi}(r,\lambda)\,d\bar{J}_{\alpha,k,0}^{0}(r,0) + \bar{J}_{\beta,k,c}^{\xi}(r,\lambda)\,d\bar{J}_{\beta,k,0}^{0}(r,0)]\right\}^2\right\} \\
& \times \left\{(a_k^2 + b_k^2)\int_0^1 \{[\bar{J}_{\alpha,k,c}^{\xi}(r,\lambda)]^2 + [\bar{J}_{\beta,k,c}^{\xi}(r,\lambda)]^2\}\,dr\right\}^{-1} \\
& + ca_k^*\left\{b_k\int_0^1 [\bar{J}_{\alpha,k,c}^{\xi}(r,\lambda)\,d\bar{J}_{\beta,k,0}^{0}(r,0) - \bar{J}_{\beta,k,c}^{\xi}(r,\lambda)\,d\bar{J}_{\alpha,k,0}^{0}(r,0)]\right. \\
& \qquad \left. + a_k\int_0^1 [\bar{J}_{\alpha,k,c}^{\xi}(r,\lambda)\,d\bar{J}_{\alpha,k,0}^{0}(r,0) + \bar{J}_{\beta,k,c}^{\xi}(r,\lambda)\,d\bar{J}_{\beta,k,0}^{0}(r,0)]\right\}\Bigg\},
\end{aligned} \tag{A.24}$$

where a_k^* and b_k^* are as defined in Theorem 3.1. Because $\phi(z)$ and, hence, $\psi(z)$ are power series functions in z, it is trivial to show that the following identities hold: $(a_k^{*2} + b_k^{*2})(a_k^2 + b_k^2) \equiv 1$, $b_k^* a_k - a_k^* b_k \equiv 0$, and $b_k^* b_k + a_k^* a_k \equiv 1$. Substituting these identities into (A.24), the stated result follows immediately. The stated results for the $F_{1,\ldots,[S/2]}$ and $F_{0,\ldots,[S/2]}$ statistics then follow directly using the asymptotic orthogonality result; cf. Remark A.1. ∎

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