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Parabolic-like mappings

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Abstract. In this paper we introduce the notion of *parabolic-like mapping*. Such an object is similar to a polynomial-like mapping, but it has a parabolic external class, i.e. an external map with a parabolic fixed point. We define the notion of parabolic-like mapping and we study the dynamical properties of parabolic-like mappings. We prove a straightening theorem for parabolic-like mappings which states that any parabolic-like mapping of degree two is hybrid conjugate to a member of the family

$$Per_1(1) = \left\{ [P_A] \;\middle|\; P_A(z) = z + \frac{1}{z} + A,\;\; A \in \mathbb{C} \right\},$$

a unique such member if the filled Julia set is connected.

1. *Introduction*

A *polynomial-like map* of degree d is a triple (f, U', U) where U' and U are open subsets of $\mathbb{C}$ isomorphic to discs, U' is compactly contained in U, and $f : U' \to U$ is a proper degree d holomorphic map (see [**DH**]). A degree d polynomial-like map is determined up to holomorphic conjugacy by its internal and external classes, that is, the (conjugacy classes of the) maps which encode the dynamics of the polynomial-like map on the filled Julia set and its complement. In particular, the external class consists of degree d real-analytic orientation preserving and strictly expanding self-coverings of the unit circle. The definition of a polynomial-like map captures the behavior of a polynomial in a neighborhood of its filled Julia set. By changing the external class of a degree d polynomial-like map to the external class of a degree d polynomial (see [**DH**]), a degree d polynomial-like map can be straightened to a polynomial of the same degree.

In this paper we introduce a new object, a *parabolic-like mapping*, similar to but different from a polynomial-like mapping. The similarity resides in the fact that a parabolic-like map is a local concept, it is characterized by a filled Julia set and an external map, and the external map of a degree d parabolic-like mapping is a degree d real-analytic orientation preserving self-covering of the unit circle. The difference resides in the fact that a parabolic-like map has a parabolic fixed point with an attracting petal outside the filled Julia set, and the external map of a parabolic-like mapping has a parabolic fixed point.

The aim of this paper is to extend the theory of polynomial-like mappings (in the dynamical plane) to parabolic-like mappings. Let us give an example which illustrates the class of maps we are considering. The map $f_1(z) = z^2 + 1/4$ has a parabolic fixed point at $z = 1/2$. Since the parabolic basin of attraction of the parabolic fixed point resides in the interior of the filled Julia set, while the repelling direction resides on the Julia set and outside of it, the external map of $f_1(z)$ is hyperbolic. The map $f_1(z)$ presents polynomial-like restrictions. On the other hand, let us interchange the roles of the filled Julia set and the closure of the basin of attraction of infinity for f_1. In other words, let us conjugate $f_1(z)$ by $\iota(z) = 1/z$ and obtain the map $f_2(z) = 4z^2/(4 + z^2)$, and let us define the closure of the basin of attraction of the superattracting fixed point $z = 0$ to be the filled Julia set for f_2. The basin of attraction of the parabolic fixed point $z = 2$ now resides outside the filled Julia set, and gives rise to an the external class with a parabolic fixed point. Appropriate restrictions of the map f_2 belong to the class of parabolic-like mappings.

As polynomial-like mappings are straightened to polynomials, we straighten degree-two parabolic-like mappings to members of a model family of maps with a parabolic external class. Our model family is the family of quadratic rational maps with a parabolic fixed point of multiplier 1, normalized by fixing the parabolic fixed point to be infinity and the critical points to be 1 and -1, this is

$$Per_1(1) = \{[P_A] \mid P_A(z) = z + 1/z + A,\ \ A \in \mathbb{C}\}.$$

All of the maps in $Per_1(1)$ have a completely invariant Fatou component Λ, namely the parabolic basin of attraction of infinity. We define the filled Julia set for these maps as

$$K_A = \widehat{\mathbb{C}} \setminus \Lambda$$

(note that for every $A \neq 0$, P_A has a unique completely invariant Fatou component Λ, hence K_A is well defined, while for the map $P_0(z) = z + 1/z$ we need to make a choice, after which the filled Julia set K_0 is well defined). The external class of this family is parabolic, and we prove in Proposition 4.2 that it is given by the class of $h_2(z) = (3z^2 + 1)/(3 + z^2)$.

In this paper we will first define parabolic-like maps and the filled Julia set of a parabolic-like map. Then we will construct and discuss the external class in this setting. Finally, we will prove that we can straighten every degree-two parabolic-like map to a member of the family $Per_1(1)$, by replacing the external map of the parabolic-like map by h_2 (see Figures 1 and 2).

2. *Preliminaries*

In this paper we are studying restrictions of maps with a parabolic fixed point of multiplier 1. By a change of coordinates we can consider the parabolic fixed point to be at $z = 0$, hence we will consider maps which locally have the form

$$f(z) = z(1 + az^n + \cdots), \qquad n \geq 1, a \neq 0.$$

The integer n is the *degeneracy/parabolic multiplicity* of the parabolic fixed point. In a neighborhood of a parabolic fixed point of parabolic multiplicity n, there are n attracting

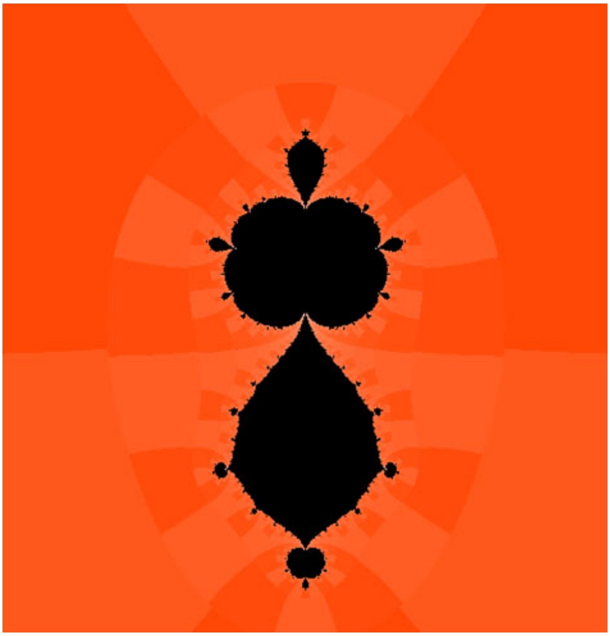

FIGURE 1. Julia set of the map $C_a(z) = z^3 + az^2 + z$, $a = i$.

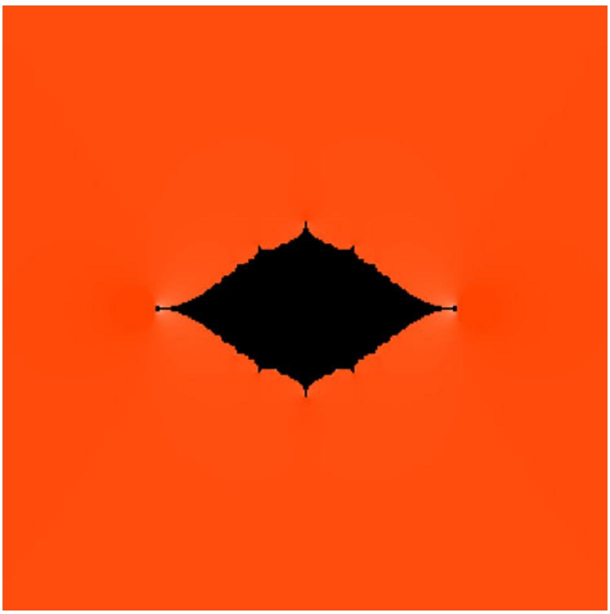

FIGURE 2. Julia set of the map $P_1(z) = z + 1/z + A$, $A = 1$.

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petals, which alternate with n repelling petals (for the definition of petal see [**Sh**] or [**M**]). We will denote the petals by Ξ. On each petal Ξ there exists a conformal map which conjugates the map f to a translation (see [**Sh**] or [**M**]). This map is called a *Fatou coordinate* for the petal Ξ, and it is unique up to composition with a translation. We will denote Fatou coordinates by ϕ. Often it is convenient to consider the quotient of a petal Ξ under the equivalence relation identifying z and $f(z)$ if both z and $f(z)$ belong to Ξ. This quotient manifold is called the *Écalle cylinder*, and it is conformally isomorphic to the infinite cylinder $\mathbb{C}/\mathbb{Z}$ (see [**Sh**] and [**M**]).

An *almost complex structure* σ on a domain $U \subset \mathbb{C}$ is a measurable field of infinitesimal ellipses $\mathcal{E}$ on the tangent bundle over U (denoted by TU). This is for almost every $u \in U$ an ellipse $\mathcal{E}_u \subset T_u U$ (defined up to scaling), with ratio of major to minor axes $K(u)$ such that the complex dilatation $\mu : U \to \mathbb{D}$, where $|\mu(u)| = (K(u) - 1)/(K(u) + 1)$ and the argument of $\mu(u)$ is twice the argument of the major axes of $\mathcal{E}_u$, is measurable. An almost complex structure is *bounded* if $\|\mu\|_\infty < 1$, and it is *standard* if $\sigma = \sigma_0$ is a field of circles. Given an ellipse field σ on U and a quasiconformal map $\phi : V \to U$, the ellipse field $\phi^*\sigma$ on V given by $\{T_v\phi^{-1}(\mathcal{E}_{\phi(v)}) \subset T_v V\}_{v \in V}$ is the *pullback of* σ *under* ϕ. The pullback of the standard structure under a quasiconformal map is a bounded almost complex structure, and the Measurable Riemann Mapping Theorem (stated below) shows that a bounded almost complex structure is the pullback of the standard structure under some quasiconformal map. For a proof of the Measurable Riemann Mapping Theorem and an exhaustive discussion about almost complex structures, quasiconformal mappings and quasisymmetric mappings, the reader is referred to [**Ah**] or, for a modern treatment, to [**Hu**].

MEASURABLE RIEMANN MAPPING THEOREM. *Let σ be a bounded almost complex structure on a domain $U \subset \mathbb{C}$. Then there exists a quasiconformal homeomorphism $\varphi : U \to \mathbb{C}$ such that*

$$\sigma = \varphi^*\sigma_0.$$

Notation. We will use the following notation:

$$\mathbb{H}_l = \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\},$$

$$\mathbb{H}_r = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}.$$

3. *Definitions and statement of the Straightening Theorem*

A parabolic-like map is an object introduced to extend the notion of polynomial-like maps to maps with a parabolic external map. The domain of a parabolic-like map is not contained in the range, and the set of points with infinite forward orbit is not contained in the intersection of the domain and the range. This calls for a partition of the set of points with infinite forward orbit into a filled Julia set compactly contained in both domain and range and exterior attracting petals.

Definition 3.1. (Parabolic-like maps) A *parabolic-like map* of degree $d \geq 2$ is a 4-tuple (f, U', U, γ) where:

- U' and U are open subsets of $\mathbb{C}$, with U', U and $U \cup U'$ isomorphic to a disc and U' not contained in U;

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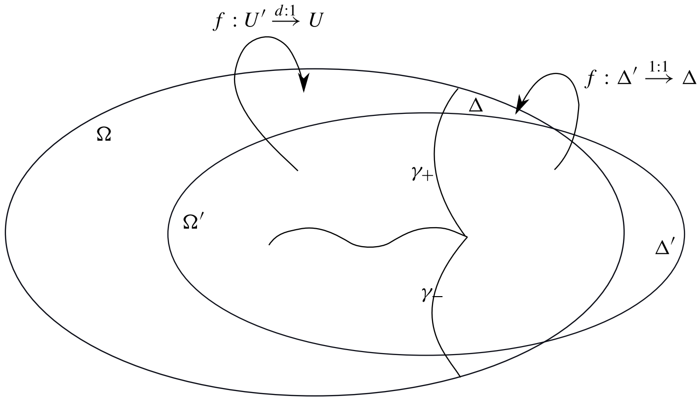

FIGURE 3. On a parabolic-like map (f, U', U, γ) the arc γ divides U' and U into Ω', Δ' and Ω, Δ, respectively. These sets are such that Ω' is compactly contained in U, $\Omega' \subset \Omega$, $f : \Delta' \to \Delta$ is an isomorphism and Δ' contains at least one attracting fixed petal of the parabolic fixed point.

- $f : U' \to U$ is a proper holomorphic map of degree $d \geq 2$ with a parabolic fixed point at $z = z_0$ of multiplier 1;
- $\gamma : [-1, 1] \to \overline{U}$ is an arc with $\gamma(0) = z_0$, forward invariant under f, C^1 on $[-1, 0]$ and on $[0, 1]$, and such that

$$f(\gamma(t)) = \gamma(dt) \quad \text{for all} -\frac{1}{d} \leq t \leq \frac{1}{d},$$

$$\gamma\left(\left[\frac{1}{d}, 1\right) \cup \left(-1, -\frac{1}{d}\right]\right) \subseteq U \setminus U', \quad \gamma(\pm 1) \in \partial U.$$

It resides in repelling petal(s) of z_0 and it divides U' and U into Ω', Δ' and Ω, Δ respectively, such that $\Omega' \subset\subset U$ (and $\Omega' \subset \Omega$), $f : \Delta' \to \Delta$ is an isomorphism (see Figure 3) and Δ' contains at least one attracting fixed petal of z_0. We call the arc γ a *dividing arc*.

Notation. We can consider $\gamma = \gamma_+ \cup \gamma_-$, where $\gamma_+ : [0, 1] \to \overline{U}$, $\gamma_- : [0, -1] \to \overline{U}$, $\gamma_\pm(0) = z_0$. Where it will be convenient (e.g. in the examples) we will refer to $\gamma_\pm$ instead of γ.

3.1. *Examples.* (1) Consider the function $h_2(z) = (3z^2 + 1)/(3 + z^2)$. This map has critical points at $z = 0$ and at ∞, and a parabolic fixed point at $z = 1$ of multiplier 1 and parabolic multiplicity 2. The attracting directions of the parabolic fixed point are along the real axis, while the repelling ones are perpendicular to the real axis. The repelling petals Ξ_+ and Ξ_- intersect the unit circle and can be taken to be reflection symmetric around the unit circle, since h_2 is autoconjugate by the reflection $T(z) = 1/\bar{z}$. Let $\phi_\pm : \Xi_\pm \to \mathbb{H}_l$ be Fatou coordinates. The image of the unit circle in the Fatou coordinate planes are horizontal lines, which we can suppose coincide with $\mathbb{R}_-$, possibly changing the normalizations of $\phi_\pm$. Choose $\epsilon > 0$ and define $U' = \{z \mid |z| < 1 + \epsilon\}$, and $U = h_2(U')$. Let $z_\pm$ be intersection points of $\Xi_\pm$, respectively, and ∂U. Thus, $\phi_+(z_+) = m_+$ with

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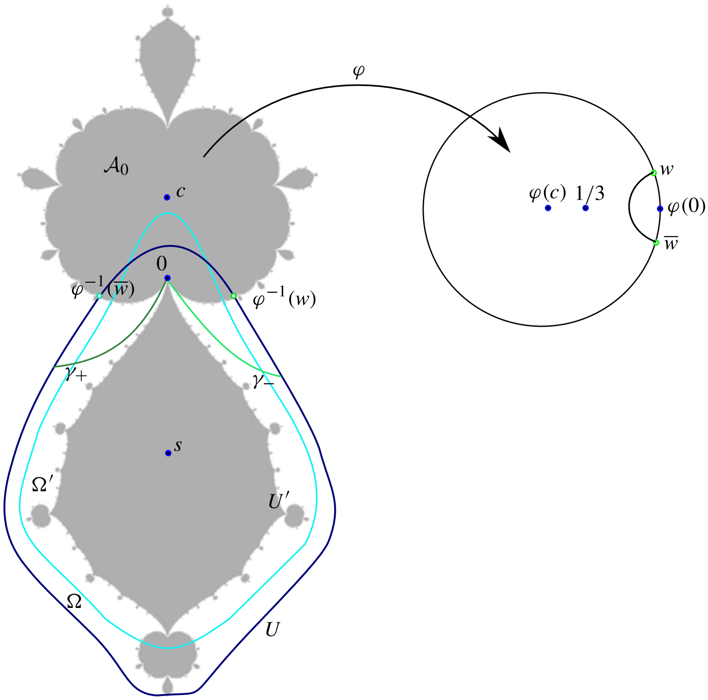

FIGURE 4. Construction of a degree-two parabolic-like map from the map $(C_a(z) = z + az^2 + z^3)$, for $a = i$. The superattracting fixed point $z = (-a - \sqrt{a^2 - 3})/3$ is denoted by s, and the critical point $z = (-a + \sqrt{a^2 - 3})/3$ in the basin of attraction of the parabolic fixed point is denoted by c.

$\mathrm{Im}(m_+) < 0$, and $\phi_-(z_-) = m_-$ with $\mathrm{Im}(m_-) > 0$. Define the dividing arcs as

$$\gamma_+ : [0, 1] \to \overline{U}, \quad \gamma_- : [0, -1] \to \overline{U},$$
$$t \to \phi_+^{-1}(\log_d(t) + m_+), \quad t \to \phi_-^{-1}(\log_d(-t) + m_-).$$

Then (h_2, U', U, γ) is a parabolic-like map of degree two.

(2) Let $(C_a(z) = z + az^2 + z^3)$, for $a = i$. This map has a superattracting fixed point s at $z = (-a - \sqrt{a^2 - 3})/3$, a critical point c at $z = (-a + \sqrt{a^2 - 3})/3$ and a parabolic fixed point at $z = 0$ with multiplier and parabolic multiplicity 1. Call $\mathcal{A}_0$ the immediate basin of attraction of the parabolic fixed point. Then the critical point c belongs to $\mathcal{A}_0$. Let $\varphi : \mathcal{A}_0 \to \mathbb{D}$ be the Riemann map normalized by setting $\varphi(c) = 0$ and $\varphi(z) \overset{z\to 0}{\longrightarrow} 1$, and let $\psi : \mathbb{D} \to \mathcal{A}_0$ be its inverse. By the Carathéodory theorem the map ψ extends continuously to $\mathbb{S}^1$. Note that $\varphi \circ f \circ \psi = h_2$. Let w be an h_2 periodic point in the first quadrant, such that the hyperbolic geodesic $\widetilde{\gamma} \in \mathbb{D}$ connecting w and $\overline{w}$ separates the critical value $z = 1/3$ from the parabolic fixed point $z = 1$. Let U be the Jordan domain bounded by $\widehat{\gamma} = \psi(\widetilde{\gamma})$, union the arcs up to potential level one of the external rays landing at $\psi(w)$ and $\psi(\overline{w})$, together with the arc of the level-one equipotential connecting these two rays around s (see Figure 4). Let U' be the connected component of $f^{-1}(U)$ containing 0 and the dividing arcs $\gamma_\pm$ be the fixed external rays landing at the parabolic fixed point 0 and parametrized by potential. Then (f, U', U, γ) is a parabolic-like map of degree two (see Figure 4).

(3) Let $f(z) = z^2 + c$, for $c = (-1 + 3\sqrt{3}i)/8$ (this is the map known as the 'fat rabbit'). Its third iterate f^3 has a parabolic fixed point at $z = (-1 + \sqrt{3}i)/4$ of multiplier 1 and parabolic multiplicity 3. Let $\mathcal{A}_0$ be the component of the immediate basin of attraction of the parabolic fixed point containing $z = 0$. Number the connected components of the immediate attracting basin in the dynamical order (which here is the counterclockwise direction around a). Let $\varphi : \mathcal{A}_0 \to \mathbb{D}$ be the Riemann map, normalized by $\varphi(0) = 0$ and $\varphi(z) \overset{z \to a}{\longrightarrow} 1$, and let $\psi : \mathbb{D} \to \mathcal{A}_0$ be its inverse. The map ψ extends continuously to $\mathbb{S}^1$, and $\varphi \circ f^3 \circ \psi = h_2$. As above let w be a h_2 periodic point in the first quadrant such that the hyperbolic geodesic $\widetilde{\gamma}$ connecting w and $\overline{w}$ separates the critical value $z = 1/3$ from the parabolic fixed point $z = 1$. Define $\widehat{\gamma} = \psi(\widetilde{\gamma})$ and $\widehat{\gamma'} = f^{-1}(\widehat{\gamma}) \cap \overline{\mathcal{A}_2}$. Let U be the Jordan domain bounded by $\widehat{\gamma}$ union the arcs up to potential level one of the external rays landing at $\psi(w)$ and $\psi(\overline{w})$ union $\widehat{\gamma'}$ union the arcs up to potential level one of the external rays landing at $f^{-1}(\psi(w)) \cap \overline{\mathcal{A}_2}$ and $f^{-1}(\psi(\overline{w})) \cap \overline{\mathcal{A}_2}$, together with the two arcs of the level-one equipotential connecting this four rays around the parabolic fixed point. Let U' be the connected component of $f^{-3}(U)$ containing $(-1 + \sqrt{3}i)/4$ and the dividing arcs γ_+ and γ_- be the external rays for angles $1/7$ and $2/7$ respectively parametrized by potential. Then (f^3, U', U, γ) is a parabolic-like map of degree two (see Figure 5).

More generally, define $\lambda_{p/q} = \exp(2\pi i p/q)$ with p and q co-prime, $c_{p/q} = \lambda_{p/q}/2 - \lambda_{p/q}^2/4$ and consider $f_q = z^2 + c_{p/q}$. The map f_q has a parabolic fixed point of multiplier $\lambda_{p/q}$ at $z = \lambda_{p/q}/2$, therefore f^q has a parabolic fixed point of multiplier 1 and parabolic multiplicity q.

Repeating the construction done above one can see that the map f^q restricts to a degree-two parabolic-like map.

Definition 3.2. Let (f, U', U, γ) be a parabolic-like map. We define the *filled Julia set* K_f of f as the set of points in U' that never leave $(\Omega' \cup \gamma_\pm(0))$ under iteration:

$$K_f := \{z \in U' \mid \forall n \geq 0,\ f^n(z) \in \Omega' \cup \gamma_\pm(0)\}.$$

Remark 3.1. An equivalent definition for the filled Julia set of f is

$$K_f = \bigcap_{n \geq 0} f^{-n}(U \setminus \Delta).$$

The filled Julia set is a compact subset of $U \cap U'$ and it is full (since it is the intersection of topological discs).

As for polynomials, we define the Julia set of f as the boundary of the filled Julia set:

$$J_f := \partial K_f.$$

3.2. *Motivations for the definition.* A parabolic-like map can be seen as the union of two different dynamical parts: a polynomial-like part (on Ω') and a parabolic one (on Δ'), which are connected by the dividing arc γ.

The parabolic fixed point belongs to the interior of the domain of a parabolic-like map in order to insure that the filled Julia set is compactly contained in the intersection of the domain and the range. The dividing arc separates the exterior attracting petals from the filled Julia set of the parabolic-like mapping, and for this reason the dividing arc is part of

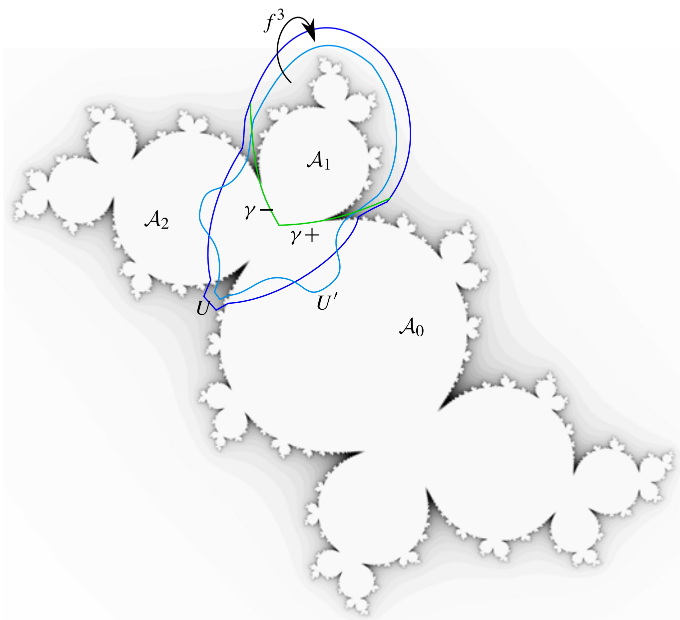

FIGURE 5. The third iterate of the map $f = z^2 + c$, for $c = (-1 + 3\sqrt{3}i)/8$, restricts to a degree-two parabolic-like map.

the definition of parabolic-like mapping (note that we could have constructed the dividing arc *a posteriori* by Fatou coordinates). The definition of parabolic-like map also guarantees the existence of an annulus, $U \setminus \Omega'$, essential in defining the external class and to perform the surgery which will give the Straightening Theorem.

There are many prospective definitions of a parabolic-like map. The one introduced here is flexible enough to capture many interesting examples, and rigid enough to allow for a viable theory.

3.3. *Conjugacies and statement of the main result.* We say that (f, U_1', U_1, γ_1) is a *parabolic-like restriction* of (f, U_2', U_2, γ_2) if $U_1' \subseteq U_2'$ and (f, U_i', U_i, γ_i), $i = 1, 2$ are parabolic-like maps with the same degree and filled Julia set.

Definition 3.3. (Conjugacy for parabolic-like mappings) We say that the parabolic-like mappings (f, U', U, γ_f) and (g, V', V, γ_g) are *topologically conjugate* if there exist parabolic-like restrictions (f, A', A, γ_f) and (g, B', B, γ_g), and a homeomorphism $\varphi : A \to B$ such that $\varphi(\gamma_{\pm f}) = \gamma_{\pm g}$ and

$$\varphi(f(z)) = g(\varphi(z)) \quad \text{on } \Omega'_{A_f} \cup \gamma_f.$$

If moreover φ is quasiconformal (and $\bar{\partial}\varphi = 0$ almost everywhere on K_f), we say that f and g are *quasiconformally* (*hybrid*) conjugate.

A topological conjugacy between parabolic-like maps is a homeomorphism defined on a neighborhood of the filled Julia set, which conjugates dynamics just on $\Omega' \cup \gamma$. This definition allows flexibility regarding the parabolic multiplicity of the parabolic fixed point.

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In this paper we will prove the following theorem.

STRAIGHTENING THEOREM. *We have the following results.*

(1) *Every degree-two parabolic-like mapping (f, U', U, γ_f) is hybrid equivalent to a member of the family $Per_1(1)$.*

(2) *Moreover, if K_f is connected, this member is unique.*

Part 1 follows from Proposition 6.2 together with Theorem 6.3, while part 2 follows from Proposition 6.5.

3.4. *Equivalence of parabolic-like mappings and Isotopy.* Two parabolic-like maps are *equivalent*, and we do not distinguish between them, if they have a common parabolic-like restriction. Given a parabolic-like map (f, U_1', U_1, γ_1), the arc $\gamma_2 : [-1, 1] \to \overline{U}$ with $\gamma_2(0) = \gamma_1(0)$ is *isotopic* to γ_1 if there exists a domain $U_2' \subseteq U_1'$ for which (f, U_i', U_i, γ_i), $i = 1, 2$ have a common parabolic-like restriction.

LEMMA 3.1. *Let (f, U', U, γ) be a parabolic-like map, and let $\gamma_s : [-1, 1] \to \overline{U}$ be an arc forward invariant under f, with $\gamma_s(0) = \gamma(0)$ and C^1 on $[-1, 0]$ and $[0, -1]$. Then γ_s and γ are isotopic if and only if their projections to Écalle cylinders are isotopic and the isotopies are disjoint from the projections of the filled Julia set and the critical points.*

Proof. Let us prove that, if the projections of γ and γ_s to Écalle cylinders are isotopic and the isotopies are disjoint from the projections of the filled Julia set and the critical points, then γ_s and γ are isotopic. The converse is trivial.

Let Ξ_+ and Ξ_- be repelling petals where γ_+ and γ_- respectively reside (note that Ξ_+ and Ξ_- may coincide). Then the quotient manifolds Ξ_+/f, Ξ_-/f are conformally isomorphic to the bi-infinite cylinder. Call β the isomorphism between Ξ_+/f and $\mathbb{C}/\mathbb{Z}$, and δ the isomorphism between Ξ_-/f and $\mathbb{C}/\mathbb{Z}$. Let

$$H_+ : [0, 1] \times \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$$
$$(s, t) \to H_+(s, t),$$
$$H_- : [0, 1] \times \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$$
$$(s, t) \to H_-(s, t),$$

be isotopies, disjoint from the projections of the filled Julia set and the critical points, such that for every fixed $s \in [0, 1]$, both $H_\pm(s, t) : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ are at least C^1. Set $\gamma_{s+}[\tau, d\tau] = \beta^{-1}(H_+(s, \cdot))$ (where $0 < \tau \le 1/d$) and $\gamma_{s-}[d\hat{\tau}, \hat{\tau}] = \delta^{-1}(H_-(s, \cdot))$ (where $-1/d \le \hat{\tau} < 0$). Define γ_s by extending γ_{s+} and γ_{s-} by the dynamics of f to forward invariant curves in Ξ_+ and Ξ_- respectively (see Figure 6), i.e.

(1) $\gamma_{s+}(d^n t) = f^n(\gamma_{s+}(t))$, $\gamma_{s+}(t/d^n) = f(\gamma_{s+}(t))^{-n}$ for all $\tau \le t \le d\tau$;

(2) $\gamma_{s-}(d^n t) = f^n(\gamma_{s-}(t))$, $\gamma_{s-}(t/d^n) = f(\gamma_{s-}(t))^{-n}$ for all $d\hat{\tau} \le t \le \hat{\tau}$;

(3) $\gamma_s(\pm 1) \in \partial U$ and $\gamma_s(0) = \gamma(0)$;

where $f(\gamma_s)^{-n}$ is the branch which gives continuity. Then γ_s divides U and U' in Ω_s, Δ_s and Ω_s', Δ_s' respectively, and by construction Ω_s' contains K_f and all of the critical points of (f, U', U, γ). Hence, (f, U', U, γ_s) is a parabolic-like restriction of (f, U', U, γ), and thus the arcs γ and γ_s are isotopic. □

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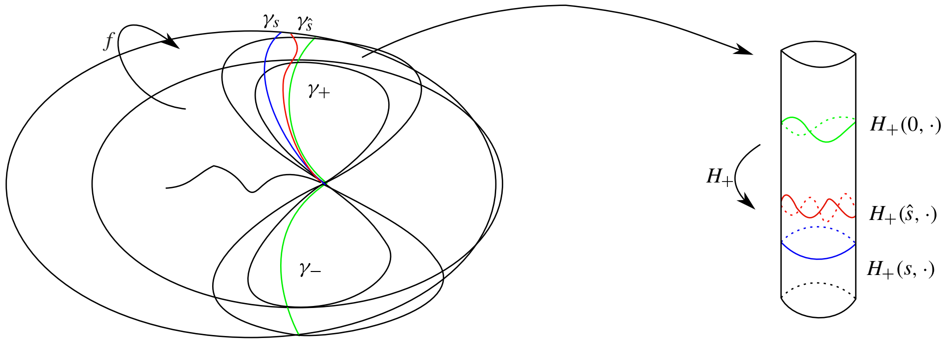

FIGURE 6. Construction of dividing arcs isotopic to γ.

Note that, by construction, if (f, U', U, γ) is a parabolic-like map and γ_s is isotopic to γ, then the arc γ_{+s} resides in the same petal as γ_+ and the arc γ_{-s} resides in the same petal as γ_-.

4. *The external class of a parabolic-like map*

In analogy with the polynomial-like setting, we want to associate to any degree d parabolic-like map (f, U', U, γ), a degree d real-analytic map $h_f : \mathbb{S}^1 \to \mathbb{S}^1$ with a parabolic fixed point, unique up to conjugacy by a real-analytic diffeomorphism. We will call h_f an *external map* of f, and we will call $[h_f]$ (its conjugacy class under real-analytic diffeomorphisms) the *external class* of f.

4.1. *Construction of an external map of a parabolic-like map f with connected Julia set.*

The construction of an external map of a parabolic-like map with connected Julia set follows the construction of an external map in [**DH**], up to the differences given by the geometry of our setting. Let (f, U', U, γ) be a parabolic-like map of degree d with connected filled Julia set K_f. Then K_f contains all of the critical points of f and, hence, $f : U' \setminus K_f \to U \setminus K_f$ is a holomorphic degree d covering map. Let

$$\alpha : \widehat{\mathbb{C}} \setminus K_f \longrightarrow \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \tag{1}$$

be the Riemann map, normalized by $\alpha(\infty) = \infty$ and $\alpha(\gamma(t)) \to 1$ as $t \to 0$. Write $W' = \alpha(U' \setminus K_f)$ and $W = \alpha(U \setminus K_f)$ (see Figure 7) and define the map:

$$h^+ := \alpha \circ f \circ \alpha^{-1} : \ W' \to W.$$

Then the map h^+ is a holomorphic degree d covering. Let $\tau(z) = 1/\bar{z}$ denote the reflection with respect to the unit circle, and define $W_- = \tau(W)$, $W'_- = \tau(W')$, $\widetilde{W} = W \cup \mathbb{S}^1 \cup W_-$ and $\widetilde{W'} = W' \cup \mathbb{S}^1 \cup W'_-$. Applying the strong reflection principle with respect to $\mathbb{S}^1$ we can extend analytically the map $h^+ : W' \to W$ to $h : \widetilde{W'} \to \widetilde{W}$. Let h_f be the restriction of h to the unit circle, then the map $h_f : \mathbb{S}^1 \to \mathbb{S}^1$ is an *external map* of f. An external map of a parabolic-like map is defined up to a real-analytic diffeomorphism.

4.2. *The general case.*

Let (f, U', U, γ) be a parabolic-like map of degree d. To deal with the case where the filled Julia set is not connected, we will lean on the similar

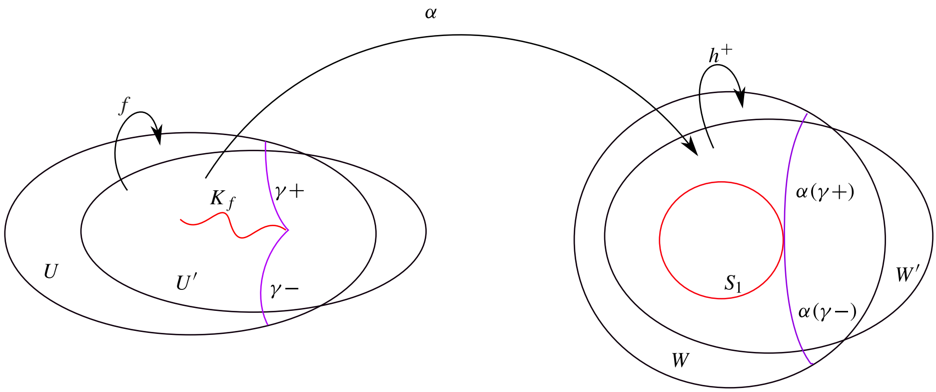

FIGURE 7. Construction of an external map in the case K_f connected. We set $W' = \alpha(U' \setminus K_f)$, $W = \alpha(U \setminus K_f)$ and $h^+ : W' \to W$.

construction in the polynomial-like case. We construct annular Riemann surfaces T and T' that will play the role of $U' \setminus K_f$ and $U \setminus K_f$, respectively, and an analytic map $F : T \to T'$ that will play the role of f.

Let $V \approx \mathbb{D}$ be a full relatively compact connected subset of U containing $\overline{\Omega}'$, the critical values of f and such that $(f, f^{-1}(V), V, \gamma)$ (after rescaling γ, where rescaling an arc means precomposing it with a scaling) is a parabolic-like restriction of (f, U', U, γ). Call $L = f^{-1}(\overline{V}) \cap \overline{\Omega}'$ and $M = f^{-1}(\overline{V}) \cap \Delta'$. Define $X_0' = (U \cup U') \setminus L$, $U_0 = U \setminus \overline{V}$, $A_0 = U \cap U' \setminus L$, $X_0 = U \setminus L$, $A_0' = U' \setminus L$ and $A_0'' = U' \setminus f^{-1}(\overline{V})$. Note that X_0 is an annular domain.

Let $\rho_0 : X_1 \to X_0$ be a degree d covering map for some Riemann surface X_1, and define $V_1 = \rho_0^{-1}(V \setminus L)$. Define $X_1'' = X_1 \setminus \overline{V_1}$. The map $f : A_0'' \to U_0$ is proper holomorphic of degree d, and $\rho_0 : X_1'' \to U_0$ is a proper holomorphic map of degree d. Therefore, we can choose $\pi_0 : A_0'' \to X_1''$, a lift of $f : A_0'' \to U_0$ to $\rho_0 : X_1'' \to U_0$, and π_0 is an isomorphism. The subset Δ has d preimages under the map ρ_0. Let us call Δ_1 the preimage of Δ under ρ_0 such that $\Delta_1 \cap \pi_0(A_0'' \cap \Delta') \neq \emptyset$. Since $f : \Delta' \to \Delta$ is an isomorphism, we can extend the map π_0 to Δ'. Let us call $B_1' = X_1'' \cup \Delta_1$. Since $\pi_0(\Delta' \setminus A_0'') \cap X_1'' = \emptyset$, the extension $\pi_0 : A_0' \to B_1'$ is an isomorphism (see Figure 8). Let us call $B_1 = \pi_0(A_0)$. Define $A_1' = \rho_0^{-1}(A_0)$ and $f_1 = \pi_0 \circ \rho_0 : A_1' \to B_1$. The map f_1 is proper, holomorphic and of degree d (see Figure 9). Indeed $\rho_0 : A_1' \to A_0$ is a degree d covering by definition and $\pi_0 : A_0 \to B_1$ is an isomorphism because it is a restriction of an isomorphism. Define $X_1' = X_1 \setminus \pi_0(A_0' \setminus A_0)$, then $B_1 \subset X_1'$.

Let $\rho_1 : X_2 \to X_1'$ be a degree d covering map for some Riemann surface X_2, and call $B_2' = \rho_1^{-1}(B_1)$. Define $\pi_1 : A_1' \to B_2'$ as a lift of f_1 to ρ_1. Then π_1 is an isomorphism, since $f_1 : A_1' \to B_1$ is a degree d covering and $\rho_1 : B_2' \to B_1$ is a degree d covering as well. Define $A_1 = A_1' \cap X_1'$ and $B_2 = \pi_1(A_1)$. Define $A_2' = \rho_1^{-1}(A_1)$ and $f_2 = \pi_1 \circ \rho_1 : A_2' \to B_2$. The map f_2 is proper, holomorphic and of degree d, indeed $\rho_1 : A_2' \to A_1$ is a degree d covering and $\pi_1 : A_1 \to B_2$ is an isomorphism. Define $X_2' = X_2 \setminus \pi_1(A_1' \setminus A_1)$, then $B_2 \subset X_2'$.

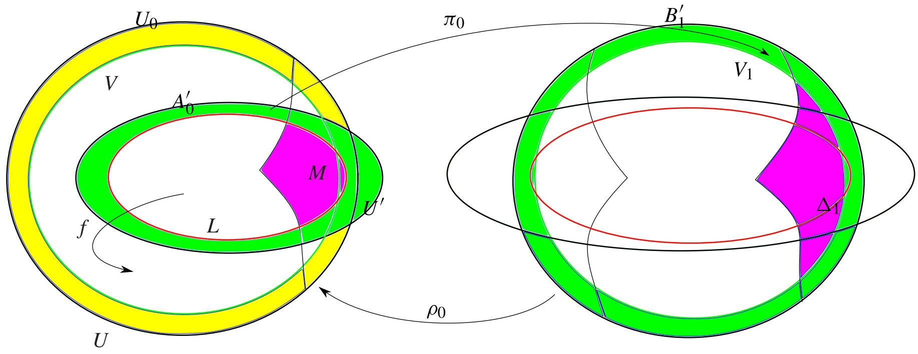

FIGURE 8. Left: the colored outer annulus is $U_0 = U \setminus \overline{V}$, the colored inner annulus is $A'_0 = U' \setminus L$. Right: the colored annulus is $B'_1 = X''_1 \cup \Delta_1$. The map $\pi_0 : A'_0 \to B'_1$ is an isomorphism.

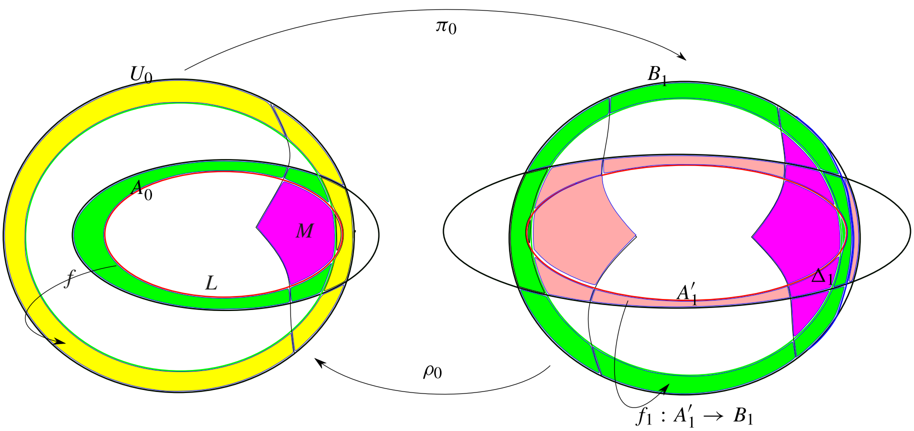

FIGURE 9. The map $f_1 = \pi_0 \circ \rho_0 : A'_1 \to B_1$ is proper holomorphic of degree d.

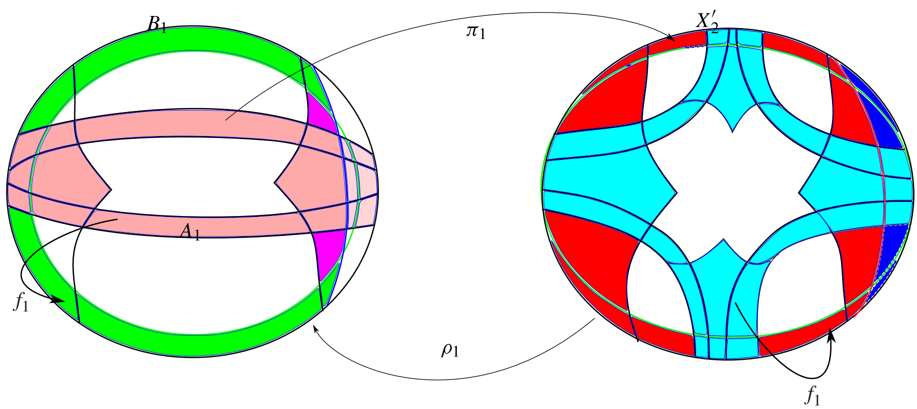

FIGURE 10. The map $\pi_1 : A'_1 \to B'_2$ is a lift of f_1 to ρ_1, and it is an isomorphism.

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Define recursively $\rho_{n-1} : X_n \to X'_{n-1}$ for $n > 1$ as a holomorphic degree d covering for some Riemann surface X_n and call $B'_n = \rho_{n-1}^{-1}(B_{n-1})$. Define recursively $\pi_{n-1} : A'_{n-1} \to B'_n \subset X_n$ as a lift of f_{n-1} to ρ_{n-1}. Then π_{n-1} is an isomorphism. Define $A_{n-1} = A'_{n-1} \cap X'_{n-1}$ and $B_n = \pi_{n-1}(A_{n-1})$. Define $A'_n = \rho_{n-1}^{-1}(A_{n-1})$ and $f_n = \pi_{n-1} \circ \rho_{n-1} : A'_n \to B_n$. Then all of the f_n are proper holomorphic maps of degree d, indeed $\rho_{n-1} : A'_n \to A_{n-1}$ are degree d coverings and $\pi_{n-1} : A_{n-1} \to B_n$ are isomorphisms. Define $X'_n = X_n \setminus \pi_{n-1}(A'_{n-1} \setminus A_{n-1})$, then $B_n \subset X'_n$.

We define $X' = \coprod_{n \geq 0} X'_n$ and $X = \coprod_{n \geq 1} X_n$ (disjoint union). Let T' be the quotient of X' by the equivalence relation identifying $x \in A'_n$ with $x' = \pi_n(x) \in X_{n+1}$, and T be the quotient of X by the same equivalence relation. Then T' is an annulus, since it is constructed by identifying at each level an inner annulus $A_i \subset X'_i$ with an outer annulus $B_{i+1} \subset X'_{i+1}$ in the next level. Similarly T is an annulus, since it is constructed by identifying at each level an inner annulus $A'_i \subset X_i$ with an outer annulus $B'_{i+1} \subset X_{i+1}$ in the next level. Since for all $i > 1$, $X'_i \subset X_i$, $T \cup T' = T \cup X'_0 / \sim$, which is an annulus because X'_0 is an annulus, T is an annulus, and π_0 identifies an inner annulus of X'_0 (which is A'_0) with an outer annulus of X_1 (which is B'_1). The covering maps ρ_n induce a degree d holomorphic covering map $F : T \to T'$. Indeed, F is well defined, since at each level $f_n = \pi_{n-1} \circ \rho_{n-1}$ by definition and π_n is a lift of f_n to ρ_n. Therefore, $\rho_n \circ \pi_n = f_n = \pi_{n-1} \circ \rho_{n-1}$ and the following diagram commutes:

$$\begin{array}{ccc} A'_n & \xrightarrow{\pi_n} & B'_{n+1} \\ \big\downarrow{\scriptstyle \rho_{n-1}} & & \big\downarrow{\scriptstyle \rho_n} \\ A_n & \xrightarrow{\pi_{n-1}} & B_n \end{array} \tag{2}$$

Finally, the map F is proper of degree d since by definition $F_{|X_n} = \rho_{n-1} : X_n \to X'_{n-1}$ is a proper map (and $F_{|X_1} = \rho_0 : X_1 \to X'_0$ is proper onto its range, which is X_0).

Now, let us construct an external map for f. Let $m > 0$ be the modulus of the annulus $T \cup T'$. Let $A \subseteq \mathbb{C}$ be any annulus with inner boundary $\mathbb{S}^1$ and modulus m. Then there exists an isomorphism

$$\alpha : T \cup T' \longrightarrow A \tag{3}$$

with $|\alpha(z)| \to 1$ when $z \to L$ and $\alpha(z) \to 1$ when $z \to z_0$ within $\Delta/\sim$ (where $\Delta/\sim = \{z \mid \exists\, n : \pi_0^{-1} \circ \cdots \circ \pi_{n-1}^{-1} \circ \pi_n^{-1}(z) \in \Delta \cup \Delta'\}$). Then we just have to repeat the construction done for the case K_f connected, with T and T' playing the role of $U' \setminus K_f$ and $U \setminus K_f$, respectively, and F playing the role of f.

4.3. *External equivalence.*

Definition 4.1. Two parabolic-like maps (f, U', U, γ_f) and (g, V', V, γ_g) are *externally equivalent* if their external maps are conjugate by a real-analytic diffeomorphism, i.e. if their external maps belong to the same external class.

Let (f, U', U, γ_f) and (g, V', V, γ_g) be two parabolic-like mappings with connected Julia sets. By the construction of an external map we gave (see §4.1), it is easy to see that (f, U', U, γ_f) and (g, V', V, γ_g) are externally equivalent if and only if there exist parabolic-like restrictions (f, A', A, γ_f) and (g, B', B, γ_g), and a biholomorphic map

$$\psi : (A \cup A') \setminus K_f \to (B \cup B') \setminus K_g$$

such that $\psi(\gamma_{\pm f}) = \gamma_{\pm g}$ and $\psi \circ f = g \circ \psi$ on $A' \setminus K_f$. We call ψ an *external equivalence* between f and g.

The following lemma shows that the situation is analogous also in the case where the Julia sets are not connected.

LEMMA 4.1. *Let $(f_i, U_i', U_i, \gamma_i)$, $i = 1, 2$, be two parabolic-like mappings with disconnected Julia sets. Let $W_i \approx \mathbb{D}$ be a full relatively compact connected subset of U_i containing $\overline{\Omega_i'}$ and the critical values of f_i, and such that $(f_i,\ f_i^{-1}(W_i),\ W_i,\ \gamma_i)$ is a parabolic-like restriction of $(f_i,\ U_i',\ U_i,\ \gamma_i)$. Define $L_i := f_i^{-1}(\overline{W}_i) \cap \overline{\Omega}_i'$. Suppose*

$$\overline{\varphi} : (U_1 \cup U_1') \setminus L_1 \to (U_2 \cup U_2') \setminus L_2$$

is a biholomorphic map such that $\overline{\varphi} \circ f_1 = f_2 \circ \overline{\varphi}$ on $U_1' \setminus L_1$. Then $(f_1, U_1', U_1, \gamma_1)$ and $(f_2, U_2', U_2, \gamma_2)$ are externally equivalent, and we say that $\overline{\varphi}$ is an external equivalence between them.

Proof. Let $(X_{n,i}, \rho_{(n-1),i}, \pi_{(n-1),i}, f_{n,i})_{n\geq 1,\ i=1,2}$ be as in the construction of an external map for a parabolic-like map with disconnected Julia set. Let us set $\varphi_0 = \overline{\varphi}$ and define recursively $\varphi_n = \rho^{-1}_{(n-1),2} \circ \varphi_{n-1} \circ \rho_{(n-1),1} : X_{n,1} \to X_{n,2}$. Then the following diagram commutes:

$$\begin{array}{ccc} X'_{n,1} \subset X_{n,1} & \xrightarrow{\varphi_n} & X_{n,2} \supset X'_{n,2} \\ \big\downarrow \rho_{(n-1),1} & & \big\downarrow \rho_{(n-1),2} \\ X'_{(n-1),1} & \xrightarrow{\varphi_{n-1}} & X'_{(n-1),2} \end{array} \tag{4}$$

(for $n = 0$, $\rho_{0,i} : X_{1,i} \to X_{0,i} \subset X'_{0,i}$). Then every $\varphi_n : X_{n,1} \to X_{n,2}$ thus defined is an isomorphism and a conjugacy between $f_{n,1}$ and $f_{n,2}$, and the following diagram commutes:

$$\begin{array}{ccc} X_{n,1} \supset A'_{n,1} & \xrightarrow{f_{n,1}} & B_{n,1} \subset X'_{n,1} \\ \big\downarrow \varphi_n & & \big\downarrow \varphi_n \\ X_{n,2} \supset A'_{n,2} & \xrightarrow{f_{n,2}} & B_{n,2} \subset X'_{n,2} \end{array} \tag{5}$$

Thus, the family of isomorphisms φ_n induces an isomorphism $\Phi : T_1 \cup T_1' \to T_2 \cup T_2'$ compatible with dynamics (where T_i and T_i', $i = 1, 2$, are as in the construction of an external map for a parabolic-like map with disconnected Julia set), and so the external maps of f_1 and f_2 are real-analytically conjugate. □

4.3.1. *External map for the members of the family $Per_1(1)$.* The filled Julia set K_P of a polynomial $P : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is defined as the complement of the basin of attraction of infinity, which is a completely invariant Fatou component. For a degree d rational map $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with a completely invariant Fatou component Λ we may define the filled Julia set as

$$K_R = \widehat{\mathbb{C}} \setminus \Lambda.$$

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Note that a degree d map can have up to two completely invariant Fatou components $\Lambda_1,\ \Lambda_2$ (since a degree d map defined on the Riemann sphere has $2d-2$ critical points, and a completely invariant Fatou component contains at least $d-1$ critical points). In the case R has precisely one completely invariant Fatou component Λ, the filled Julia set $K_R = \widehat{\mathbb{C}} \setminus \Lambda$ is well defined. In the case R has two such components $\Lambda_1,\ \Lambda_2$, there are two possibilities for the filled Julia set, hence we need to make a choice. After choosing a completely invariant component Λ_*, the filled Julia set $K_R = \widehat{\mathbb{C}} \setminus \Lambda_*$ is well defined.

Every member of the family $Per_1(1)$ has a parabolic fixed point at ∞ with multiplier 1, and the basin of attraction of the parabolic fixed point is a completely invariant Fatou component. For all the members of the family $Per_1(1)$ with $A \neq 0$ the parabolic multiplicity of the parabolic fixed point is 1, hence all of these maps have precisely one completely invariant Fatou component Λ. Thus, for all of the members of the family $Per_1(1)$ with $A \neq 0$ the filled Julia set $K_{P_A} = \widehat{\mathbb{C}} \setminus \Lambda$ is well defined. On the other hand, since for the map $P_0(z) = z + 1/z$ the parabolic multiplicity of ∞ is 2, this map has two completely invariant Fatou components, namely $\mathbb{H}_r$ and $\mathbb{H}_l$. Since $P_0(z) = z + 1/z$ is conformally conjugate to the map $h_2(z) = (3z^2+1)/(3+z^2)$ under the map $\varphi(z) = (z+1)/(z-1)$, for consistency with Example 1 in § 3.1 we consider $K_{P_0} = \overline{\mathbb{H}}_l = \varphi(\overline{\mathbb{D}})$.

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree d. The map f has a *parabolic-like restriction* if there exist open connected sets U and U' and a dividing arc γ such that (f, U', U, γ) is a parabolic-like map of some degree $d' \leq d$. A parabolic-like restriction of a member P_A of the family $Per_1(1)$ has degree two, hence the filled Julia set K_{P_A} defined as above coincides with the filled Julia set of the parabolic-like restriction of P_A. Therefore, we consider as external class of P_A the external class of its parabolic-like restriction.

PROPOSITION 4.2. *For every* $A \in \mathbb{C}$ *the* external class of P_A *is given by the class of* $h_2(z) = (z^2 + (1/3))/(1 + (z^2/3))$.

Proof. Since the maps $P_0(z) = z + 1/z$ and $h_2(z) = (3z^2+1)/(3+z^2)$ are conformally conjugate, in order to prove that h_2 is an external map of P_A, it is sufficient to prove that P_0 is externally equivalent to P_A, for $A \in \mathbb{C}$. Let Ξ^0 be an attracting petal of P_0 containing the critical value $z = 2$, and let $\phi_0 : \Xi^0 \to \mathbb{H}_r$ be the incoming Fatou coordinates of P_0 normalized by $\phi_0(2) = 1$. Let Ξ^A be an attracting petal of the parabolic fixed point ∞ of P_A and let $\phi_A : \Xi^A \to \mathbb{H}_r$ be the incoming Fatou coordinates of P_A with $\phi_A(2+A) = 1$. We call the critical point in the boundary of the maximal domain which is sent univalently to the right half-plane by ϕ_A *the first critical point attracted to the parabolic fixed point* ∞. Replacing A by $-A$ if necessary, we can assume that $z = 1$ is the first critical point attracted by the parabolic fixed point ∞.

Let us construct an external equivalence between P_0 and P_A first in the case K_{P_A} is connected. The map $\eta := \phi_A^{-1} \circ \phi_0 : \Xi^0 \to \Xi^A$ is a conformal conjugacy between P_0 and P_A on Ξ^0. Defining $\Xi^0_{-n},\ n > 0$ as the connected component of $P_0^{-n}(\Xi^0)$ containing Ξ^0, and Ξ^A_{-n}, $n > 0$ as the connected component of $P_A^{-n}(\Xi^A)$ containing Ξ^A, we can lift the map η to $\eta_n : \Xi^0_{-n} \to \Xi^A_{-n}$. Since K_{P_A} is connected, by iterated lifting of η we obtain a conformal conjugacy $\overline{\eta} : \widehat{\mathbb{C}} \setminus K_{P_0} \to \widehat{\mathbb{C}} \setminus K_{P_A}$ between P_0 and P_A.

In the case K_{P_A} is not connected the map η is a conformal conjugacy between P_0 and P_A on the region delimited by the Fatou equipotential passing through $z = 1$. We are now going to construct parabolic-like restrictions $(P_0, U_0', U_0, \gamma_0)$ and $(P_A, U_A', U_A, \gamma_A)$ of the maps P_0 and P_A, respectively, and extend the map η to an external equivalence between them. The critical point $z = 1$ is the first attracted by infinity for both the maps P_0 and P_A, so it cannot belong to the domains U_0', U_A' of their parabolic-like restrictions but it may belong to the codomains U_0, U_A, while the critical point $z = -1$ belongs to Ω_0' and Ω_A'. Let us denote by $\widehat{\phi_A}$ and $\widehat{\phi_0}$ the Fatou coordinates of P_A and P_0, respectively (normalized by $\widehat{\phi_A}(2 + A) = 1$ and $\widehat{\phi_0}(2) = 1$), extended to the whole basin of attraction of ∞ by iterated lifting. The maps $\widehat{\phi_A}$ and $\widehat{\phi_0}$ have univalent inverse branches

$$\psi_A : \mathbb{C} \setminus \{z = x + iy \mid x < 0 \wedge y \in [0, \mathrm{Im}(\widehat{\phi_A}(-2 + A))]\} \to \widehat{\Xi}_A$$

and $\psi_0 : \mathbb{C} \setminus \mathbb{R}_- \to \widehat{\Xi}_0$, respectively, and the map

$$\eta = \psi_A \circ \widehat{\phi_0} : \psi_0^{-1}(\mathbb{C} \setminus \{z = x + iy \mid x < 0 \wedge y \in [0, \mathrm{Im}(\widehat{\phi_A}(-2 + A))]\}) \to \widehat{\Xi}_A$$

is a biholomorphic extension of η conjugating dynamics. Choose $r > \max\{1 + \mathrm{Im}(\widehat{\phi_A}(A - 2)),\ 2\}$ and z_0, $r < z_0 < r + 1$ such that $A - 2 \notin \phi_A^{-1}(\overline{\mathbb{D}(z_0, r)})$. Then for $r < r' < z_0$ with r' sufficiently close to r we have $A - 2 \notin \phi_A^{-1}(\mathbb{D}(z_0, r'))$. Let $\widetilde{\gamma}_+$, $\widetilde{\gamma}_-$ be horizontal lines, symmetric with respect to the real axis, starting at $-\infty$ and landing at $\partial\mathbb{D}(z_0, r)$, such that the point $\widehat{\phi_A}(A - 2)$ is contained in the strip between them (see Figure 11) and they do not leave the disc $T^{-1}(\mathbb{D}(z_0, r))$ (where $T^{-1}(\mathbb{D}(z_0, r))$ is the disc of radius r and center $z_1 = z_0 - 1$) after having entered into it. Define $U_0 = (\phi_0^{-1}(\mathbb{D}(z_0, r))^c$, $U_0' = P_0^{-1}(U_0)$, $\gamma_{+_0} = \psi_0(\widetilde{\gamma}_+)$, and $\gamma_{-_0} = \psi_0(\widetilde{\gamma}_-)$. In the same way define $U_A = (\phi_A^{-1}(\mathbb{D}(z_0, r))^c$, $U_A' = P_A^{-1}(U_A)$, $\gamma_{+_A} = \psi_A(\widetilde{\gamma}_+)$, and $\gamma_{-_A} = \psi_A(\widetilde{\gamma}_-)$. Then the parabolic-like restriction of P_0 we consider is $(P_0, U_0', U_0, \gamma_{+_0}, \gamma_{-_0})$, and the parabolic-like restriction of P_A we consider is $(P_A, U_A', U_A, \gamma_{+_A}, \gamma_{-_A})$. Note that, by construction, the map η is a conformal conjugacy between P_0 and P_A on Δ_0'. In order to obtain an external equivalence we need η to be defined on a fundamental annulus. Define $D_0 = \phi_0^{-1}(\mathbb{D}(z_0,\ r'))$, $D_0' = P_0^{-1}(D_0)$, $D_A = \phi_A^{-1}(\mathbb{D}(z_0,\ r'))$, and $D_A' = P_A^{-1}(D_A)$ (see Figure 12). Since D_0 and D_A belong to the regions delimited by the Fatou equipotential passing through $z = 1$, the restriction $\eta : D_0 \to D_A$ is a holomorphic conjugacy between P_0 and P_A. Since $-2 \notin D_0$ and $-2 + A \notin D_A$, the restrictions $P_0 : D_0' \setminus \{1\} \to D_0 \setminus \{2\}$ and $P_A : D_A' \setminus \{1\} \to D_A \setminus \{2 + A\}$ are degree two coverings. Hence, we can lift the map η to $\eta : D_0' \setminus \{1\} \to D_A' \setminus \{1\}$. Finally, we obtain a biholomorphic map $\eta : D_0' \cup \Delta_0' \to D_A' \cup \Delta_A'$ which conjugates dynamics.

Define $V_0 = (\overline{D_0})^c$ and $V_A = (\overline{D_A})^c$, and consequently $L = \overline{\Omega_0' \setminus D_0'}$ and $M = \overline{\Omega_A' \setminus D_A'}$. The sets V_0 and V_A are compactly contained in U_0 and U_A, respectively, containing $\overline{\Omega_0'}$ (which contains the critical value -2) and $\overline{\Omega_A'}$ and the critical value $-2 + A$, respectively, and such that $P_0 : (\overline{D}_0')^c \to (\overline{D}_0)^c$ and $P_A : (\overline{D}_A')^c \to (\overline{D}_A)^c$ are parabolic-like restrictions of $(P_0, U_0, U_0', \gamma_0)$ and $(P_A, U_A, U_A', \gamma_A)$, respectively, and the map $\eta : (U_0 \cup U_0') \setminus L \to (U_A \cup U_A') \setminus M$ is a biholomorphic conjugacy between P_0 and P_A. Therefore, the result follows by Lemma 4.1. □

4.4. *Properties of external maps.* Let (f, U', U, γ) be a parabolic-like map of degree d, and let h_f be a representative of its external class. The map $h_f : \mathbb{S}^1 \to \mathbb{S}^1$ is by construction real-analytic, symmetric with respect to the unit circle, and it has a parabolic fixed point

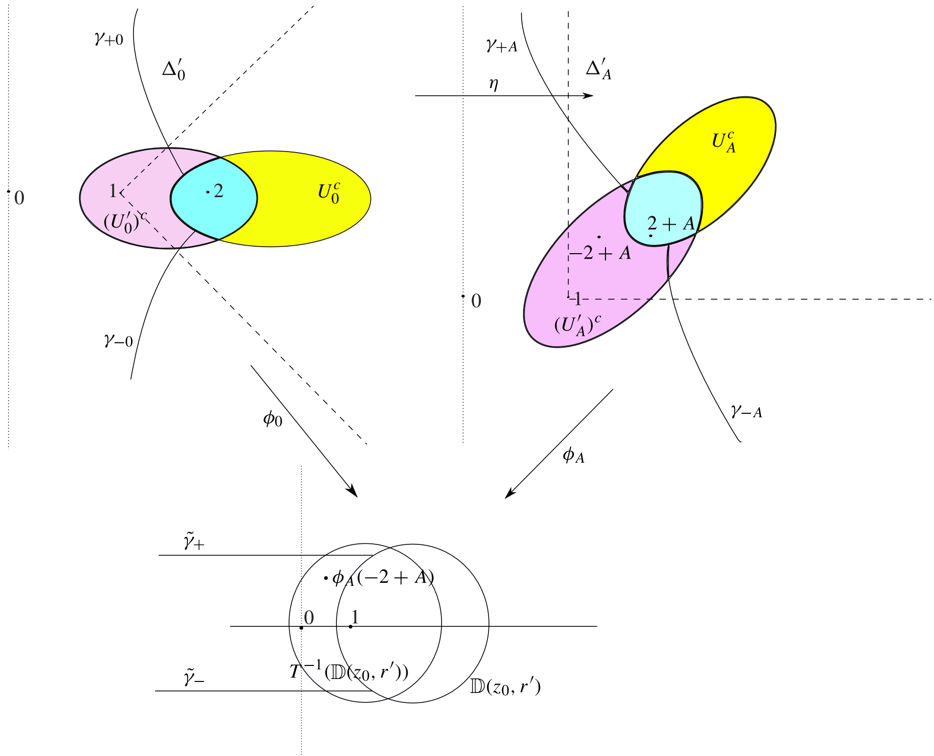

FIGURE 11. The construction of parabolic-like restrictions of P_0 and P_A. In the picture we are assuming the critical value $z = -2 + A$ in $\Omega_A \setminus \Omega'_A$. In this case the critical value $z = -2 + A$ belongs to the attracting petal Ξ_A.

z_1 of multiplier 1 and even parabolic multiplicity $2n$ (where n is the number of petals of z_0 outside K_f). Let α be an isomorphism which defines h_f. Hence, h_f inherits via α dividing arcs $\gamma_{h_f+} := \alpha(\gamma_+ \setminus \{z_0\}) \cup \{z_1\}$ and $\gamma_{h_f-} := \alpha(\gamma_- \setminus \{z_0\}) \cup \{z_1\}$, which divide $W'_f \setminus \mathbb{D}$ and $W_f \setminus \mathbb{D}$ into Ω'_W, Δ'_W and Ω_W, Δ_W, respectively, such that $h_f : \Delta'_W \to \Delta_W$ is an isomorphism and Δ'_W contains at least one attracting fixed petal of z_1. Note that Ω'_W is not compactly contained in W (since they share the inner boundary), and that γ_{h_f+} and γ_{h_f-} form a positive angle (since there is at least one attracting fixed petal of z_1 in Δ_W; we prove in [**L**] that this angle is π). Moreover, we prove in [**L**, Theorem 2.3.3] that there exists $\hat{h} \in [h_f]$ such that for all $z \in \mathbb{S}^1$, $|\hat{h}(z)| \geq 1$, and the equality holds only at the parabolic fixed point.

5. *Parabolic external maps*

So far we have considered external maps only in relation to parabolic-like maps (and members of the family $Per_1(1)$). We now want to separate these two concepts, and then consider external maps as maps of the unit circle to itself with some specific properties, without referring to a particular parabolic-like map. In order to do so we need to give an abstract definition of external map, which endows it with all of the properties it would have, if it would have been constructed from a parabolic-like map.

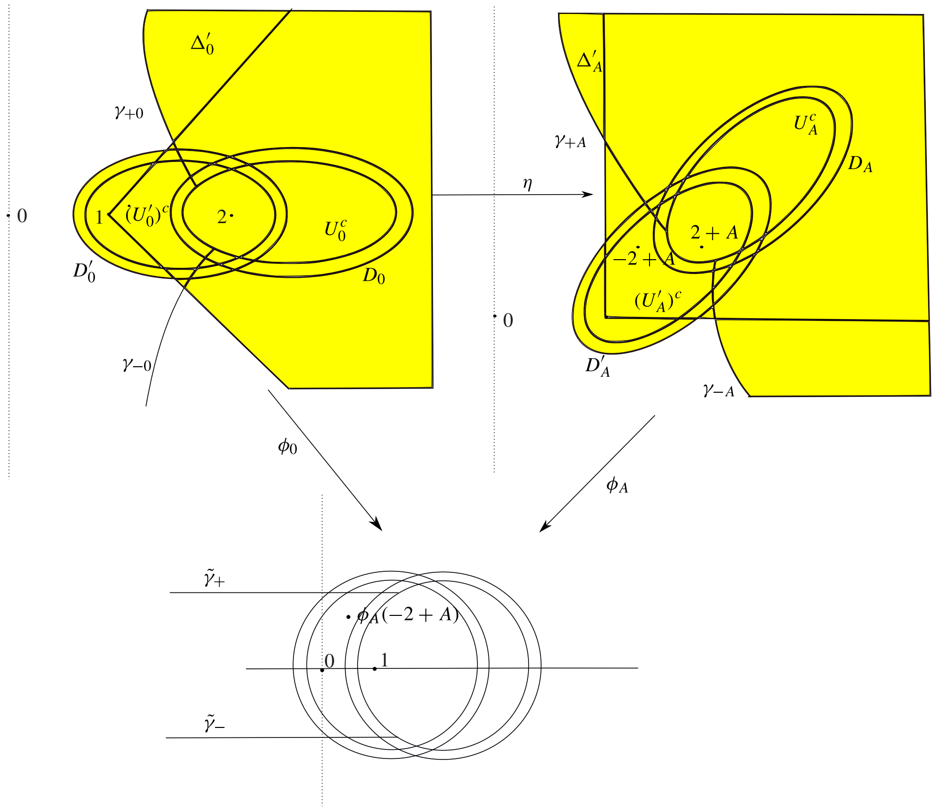

FIGURE 12. The construction of the external equivalence η between the parabolic-like restriction of P_0 and the parabolic-like restriction of P_A. For $r < r'$, $\mathbb{D}(z_0, r) \subset\subset \mathbb{D}(z_0, r')$ and $T^{-1}(\mathbb{D}(z_0, r)) \subset\subset T^{-1}(\mathbb{D}(z_0, r'))$. In the picture we are assuming the critical value $z = -2 + A$ in $\Omega_A \setminus \Omega'_A$.

Definition 5.1. (Parabolic external map) Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be a degree d orientation-preserving real-analytic and metrically expanding (i.e. $|h'(z)| \geq 1$) map. We say that h is a *parabolic external map*, if there exists a unique $z = z_*$ such that $h(z_*) = z_*$ and $h'(z_*) = 1$, and $|h'(z)| > 1$ for all $z \neq z_*$.

The multiplicity of z_* as parabolic fixed point of h is even (since the map h is symmetric with respect to the unit circle). As h is metrically expanding, the repelling petals of z_* intersect the unit circle. Let $h : W' \to W$ be an extension which is a degree d covering (where $W = \{z : e^{-\epsilon} < |z| < e^{\epsilon}\}$ for an $\epsilon > 0$, and $W' = h^{-1}(W)$). We define a *dividing arc for h* to be an arc $\widetilde{\gamma} : [-1, 1] \to \overline{W} \setminus \mathbb{D}$, forward invariant under h, C^1 on $[-1, 0]$ and $[0, -1]$, residing in the union of the repelling petals which intersect the unit circle and such that

$$h(\widetilde{\gamma}(t)) = \widetilde{\gamma}(dt) \quad \text{for all } -\frac{1}{d} \leq t \leq \frac{1}{d},$$

$$\widetilde{\gamma}\left(\left[\frac{1}{d}, 1\right) \cup \left(-1, -\frac{1}{d}\right]\right) \subseteq W \setminus W', \quad \widetilde{\gamma}(\pm 1) \in \partial W.$$

Remark 5.1. A dividing arc for a parabolic external map can be constructed by taking preimages of horizontal lines by repelling Fatou coordinates defined on (disjoint) repelling petals intersecting the unit circle.

The dividing arc divides $W' \setminus \mathbb{D}$ and $W \setminus \mathbb{D}$ into Ω'_W, Δ'_W and Ω_W, Δ_W, respectively. The restriction $h : \Delta'_W \to \Delta_W$ is an isomorphism and Δ'_W contains at least an attracting fixed petal of z_*. We prove in [**L**, Lemma 2.3.9] that there exists a range $\widetilde{W}$ for an extension $h : \widetilde{W}' \to \widetilde{W}$ degree d covering such that $\Omega_{\widetilde{W}} \setminus \Omega'_{\widetilde{W}}$ is a topological quadrilateral. Therefore, external maps constructed from parabolic-like mappings and parabolic external maps are equivalent concepts.

Remarks 5.2. We make the following remarks.

- For clarity of exposition we consider in this paper parabolic-like maps with external map having exactly one parabolic fixed point (Cf. Definition 3.1). This concept naturally generalizes to maps with external maps having several parabolic fixed points. A general parabolic-like map has as many pairs of dividing arcs $\gamma_\pm$ (which divide U and U' in $\Omega, \Delta_1, \Delta_2, \ldots, \Delta_n$ and $\Omega', \Delta'_1, \Delta'_2, \ldots, \Delta'_n$, respectively) as the number of parabolic fixed points.
- Moreover, this concept generalizes in a similar way to maps with external maps having several parabolic periodic orbits. An external map for such an object is an orientation-preserving real-analytic and metrically expanding map $h : \mathbb{S}^1 \to \mathbb{S}^1$ with $h'(z_*) = 1$ for every z_* belonging to a parabolic cycle and $|h'(z)| > 1$ for all of the other points of the unit circle.

Definition 5.3. A *degree d covering extension* $h : W' \to W$ of a parabolic external map $h : \mathbb{S}^1 \to \mathbb{S}^1$ is an extension to some neighborhood $W = \{z \mid e^{-\epsilon} < |z| < e^{\epsilon}\}$ for an $\epsilon > 0$, and $W' = h^{-1}(W)$ such that the map $h : W' \to W$ is a degree d covering and there exists a dividing arc $\widetilde{\gamma}$ which divides $W' \setminus \mathbb{D}$, $W \setminus \mathbb{D}$ into Ω'_W, Δ'_W and Ω_W, Δ_W, respectively, such that $\Omega_W \setminus \Omega'_W$ is a topological quadrilateral.

The concept of *parabolic-like restriction* naturally applies to parabolic external maps ($h : \hat{W}' \to \hat{W}$ is a parabolic-like restriction of $h : W' \to W$ if they are both degree d covering extension of the same parabolic external map and $\hat{W} \subseteq W$). Let γ be a dividing arc for some parabolic external map. We say that $\gamma_s : [-1, 1] \to \mathbb{C} \setminus \mathbb{D}$ is *isotopic* to γ if their projections to Écalle cylinders are isotopic and the isotopies are disjoint from the projections of the unit circle. From the definitions of dividing arc and isotopy of arcs for parabolic external maps is easy to see that dividing arcs for the same external map are isotopic.

PROPOSITION 5.4. *Let $h_i : \mathbb{S}^1 \to \mathbb{S}^1$, $i = 1, 2$ be parabolic external maps of the same degree d, $h_i : W'_i \to W_i$ degree d covering extensions, and γ_i dividing arcs. Then the following statements hold.*

(1) *Let $\gamma_s : [-1, 1] \to \mathbb{C} \setminus \mathbb{D}$ be a forward invariant arc for h_1, isotopic to γ_1, and C^1 on $[-1, 0]$ and $[0, 1]$. Then (possibly after rescaling) γ_s is a dividing arc for h_1.*

(2) *Assume γ_{i+} and γ_{i-} are constructed by taking preimages of the same periodic curves by repelling Fatou coordinates ϕ_{i+} and ϕ_{i-} (defined on disjoint repelling petals*

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intersecting the unit circle) of h_i. Then the map $\phi_2^{-1} \circ \phi_1 : \gamma_1 \to \gamma_2$ defined as

$$\phi_2^{-1} \circ \phi_1(z) = \begin{cases} \phi_{2+}^{-1} \circ \phi_{1+} & \text{on } \gamma_{1+}, \\ \phi_{2-}^{-1} \circ \phi_{1-} & \text{on } \gamma_{1-}, \end{cases}$$

is a quasisymmetric conjugacy between $h_{1|\gamma_1}$ and $h_{2|\gamma_2}$.

Proof. Property (1) comes from the definition of isotopy in the parabolic external maps setting and a similar argument as in the proof of Lemma 3.1. Let us prove property (2). In the following, $i = 1, 2$. Let Ξ_{i+} and Ξ_{i-} be repelling petals where γ_{i+} and γ_{i-} respectively reside (so Ξ_{i+} and Ξ_{i-} intersect the unit circle), and let $\phi_{i+} : \Xi_{i+} \to \mathbb{H}_l$ and $\phi_{i-} : \Xi_{i-} \to \mathbb{H}_l$ be repelling Fatou coordinates. Then there exist γ_+ and γ_-, 1-periodic curves in $\mathbb{H}_l$ bounded from above and below, such that $\gamma_{i+} = \phi_{i+}^{-1}(\gamma_+)$ and $\gamma_{i-} = \phi_{i-}^{-1}(\gamma_-)$. The map $\phi_2^{-1} \circ \phi_1 : \gamma_1 \to \gamma_2$ is clearly a conjugacy between $h_{1|\gamma_1}$ and $h_{2|\gamma_2}$. Let us prove that this map is quasisymmetric. To fix the notation let us assume the multiplicity of z_i as parabolic fixed point of h_i is $2n_i$. By an iterative local change of coordinates applied to eliminate lower-order terms one by one, we obtain conformal diffeomorphisms g_i which conjugate h_i to the map $z \to z(1 + z^{2n_i} + cz^{4n_i} + O(z^{6n_i}))$ on $\Xi_{i\pm}$. Since the forward invariant arcs $\gamma_{i\pm}$ reside in the repelling petals $\Xi_{i\pm}$, it suffices to consider $h_i(z) = z(1 + z^{2n_i} + cz^{4n_i} + O(z^{6n_i}))$. The map $I_i(z) = -1/(2n_i z^{2n_i})$ conjugates h_i to $h_i^*(z) = z + 1 + \hat{c}_i(1/z) + O(1/z^2)$. Shishikura proved in [**Sh**] that Fatou coordinates which conjugate the map h_i^* to $T(z) = z + 1$ on $I_i(\Xi_{i\pm})$ take the form $\Phi_{i\pm}(z) = z - \hat{c}_i \log(z) + c_{i\pm} + o(1)$. Therefore, $\phi_{i\pm} = \Phi_{i\pm} \circ I_i$, and we can write

$$\gamma_{i+} = (\Phi_{i+} \circ I_i)^{-1}(\gamma_+)_{\{i=1,2\}},$$
$$\gamma_{i-} = (\Phi_{i-} \circ I_i)^{-1}(\gamma_-)_{\{i=1,2\}},$$

$$\begin{array}{ccc} \gamma_i & \xrightarrow{h_i} & \gamma_i \\ \downarrow I_i & & \downarrow I_i \\ \mathbb{H}_l & \xrightarrow{h_i^*} & \mathbb{H}_l \\ \downarrow \Phi_i & & \downarrow \Phi_i \\ \mathbb{H}_l & \xrightarrow{T} & \mathbb{H}_l \end{array} \tag{6}$$

Call $\gamma_{i+}^* = I_i(\gamma_{i+})$, $\gamma_{i-}^* = -I_i(\gamma_{i-})$ and $\gamma_i^* = \gamma_{i+}^* \cup \infty \cup \gamma_{i-}^*$. The map $\widehat{I_i} : \gamma_i \to \gamma_i^*$:

$$\widehat{I_i}(z) = \begin{cases} I_i(z) & \text{on } \gamma_{i+}, \\ -I_i(z) & \text{on } \gamma_{i-}, \end{cases}$$

is quasisymmetric on a neighborhood of 0. Define $\widehat{\gamma} = \gamma_+ \cup \infty \cup -\gamma_-$, and the map $\widehat{\Phi}_i : \gamma_i^* \to \widehat{\gamma}$ as follows:

$$\widehat{\Phi}_i(z) = \begin{cases} \Phi_{i+}(z) & \text{on } \gamma_{i+}^*, \\ -\Phi_{i-}(-z) & \text{on } \gamma_{i-}^*. \end{cases}$$

The map $\widehat{\Phi}_i$ is the restriction to $\gamma_i^* \setminus \infty$ of a conformal map. Again by Shishikura [**Sh**] the maps Φ_{i+}, Φ_{i-} have derivatives $\Phi_{i\pm}' = 1 + o(1)$, hence the map $\widehat{\Phi}_i : \gamma_i^* \to \widehat{\gamma}$ is a

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diffeomorphism (one may take $1/x$ as a chart). The map $\widehat{\Phi}_i \circ \widehat{I}_i : \gamma_i \to \widehat{\gamma}$ conjugates the map h_i to the map $T_+(z) = z + 1$ on γ_{i+}, and to the map $T_-(z) = z - 1$ on γ_{i-}. Hence, $\phi_2^{-1} \circ \phi_1 = (\widehat{\Phi}_2 \circ \widehat{I}_2)^{-1} \circ (\widehat{\Phi}_1 \circ \widehat{I}_1) : \gamma_1 \to \gamma_2$. The map $\widehat{\Phi}_2^{-1}$ is a diffeomorphism because it has the same analytic expression as $\widehat{\Phi}_2$, and therefore the map $\widehat{\Phi}_2^{-1} \circ \widehat{\Phi}_1$ is a diffeomorphism. Since the map $\widehat{I}_i$ is quasisymmetric on a neighborhood of 0, the inverses are quasisymmetric on a neighborhood of ∞. Hence the composition $\phi_2^{-1} \circ \phi_1 = \widehat{I}_2^{-1} \circ \widehat{\Phi}_2^{-1} \circ \widehat{\Phi}_1 \circ \widehat{I}_1 : \gamma_1 \to \gamma_2$ is quasisymmetric. □

6. *The Straightening Theorem*

Definition 6.1. Let (f, U', U, γ_f) and (g, V', V, γ_g) be two parabolic-like mappings. We say that f and g are *holomorphically equivalent* if there exist parabolic-like restrictions (f, A', A, γ_f) and (g, B', B, γ_g), and a biholomorphic map $\varphi : (A \cup A') \to (B \cup B')$ such that $\varphi(\gamma_{\pm f}) = \gamma_{\pm g}$ and

$$\varphi(f(z)) = g(\varphi(z)) \quad \text{on } A'.$$

PROPOSITION 6.2. *A degree-two parabolic-like map is holomorphically conjugate to a member of the family $Per_1(1)$ if and only if its external class is given by the class of h_2.*

Proof. By Proposition 4.2, the external class of every member of the family $Per_1(1)$ is given by the class of h_2, hence a parabolic-like map holomorphically conjugate to a member of the family $Per_1(1)$ has external map in the class of h_2. Let us prove that a degree-two parabolic-like map $g : V' \to V$ with external map h_2 is holomorphically conjugate to a member of the family $Per_1(1)$. Let $\overline{\psi}$ be an external equivalence between the maps g and h_2. Let S be the Riemann surface obtained by gluing $V \cup V'$ and $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, by the equivalence relation identifying z to $\overline{\psi}(z)$, i.e.

$$S = (V \cup V') \coprod (\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}) / z \sim \overline{\psi}(z).$$

By the Uniformization Theorem, S is isomorphic to the Riemann sphere. Consider the map

$$\widetilde{g}(z) = \begin{cases} g & \text{on } V', \\ h_2 & \text{on } \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}. \end{cases}$$

Since the map h_2 is an external map of g, the map $\tilde{g}$ is holomorphic. Let $\widehat{\varphi} : S \to \widehat{\mathbb{C}}$ be an isomorphism that sends the parabolic fixed point of $\widetilde{g}$ to infinity, the critical point of $\widetilde{g}$ to $z = -1$, and the preimage of the parabolic fixed point of $\widetilde{g}$ to $z = 0$. Define $P_2 = \widehat{\varphi} \circ \tilde{g} \circ \widehat{\varphi}^{-1} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. The map P_2 is a degree two holomorphic map defined on the Riemann sphere, so it is a quadratic rational function. By construction it has a parabolic fixed point of multiplier 1 at $z = \infty$ with preimage $z = 0$, and it has a critical point at $z = -1$. Hence, P_2 belongs to the family $Per_1(1)$. □

THEOREM 6.3. *Let (f, U', U, γ_f) be a parabolic-like mapping of some degree $d > 1$, and $h : \mathbb{S}^1 \to \mathbb{S}^1$ be a parabolic external map of the same degree d. Then there exists a parabolic-like mapping (g, V', V, γ_g) which is hybrid equivalent to f and whose external class is $[h]$.*

Throughout this proof we assume, in order to simplify the notation, U and U' with C^1 boundaries (if U and U' do not have C^1 boundaries we consider a parabolic-like restriction of (f, U', U, γ_f) with C^1 boundaries).

Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be a parabolic external map of degree $d > 1$, z_* be its parabolic fixed point and $h : W' \longrightarrow W$ be a degree d covering extension. Define $B = W \cup \mathbb{D}$ and $B' = W' \cup \mathbb{D}$. We are going to construct now a dividing arc $\widetilde{\gamma} : [-1, 1] \to \overline{B} \setminus \mathbb{D}$ for h, such that on $\widetilde{\gamma}$ the dynamics of h is conjugate to the dynamics of f.

Let h_f be an external map of f, z_1 its parabolic fixed point, $h_f : W'_f \to W_f$ a degree d covering extension and α an external equivalence between f and h_f. The dividing arcs $\gamma_{h_f\pm}$ are tangent to $\mathbb{S}^1$ at the parabolic fixed point z_1, and they divide W_f and W'_f in Δ_W, Ω_W and Δ'_W, Ω'_W, respectively (see § 4.4).

Let $\Xi_{h_f\pm}$ be repelling petals for the parabolic fixed point z_1 which intersect the unit circle and $\phi_\pm : \Xi_{h_f\pm} \to \mathbb{H}_l$ be Fatou coordinates. On the other hand, let $\Xi_{h\pm}$ be repelling petals for the parabolic fixed point z_* of h which intersect the unit circle and $\widetilde{\phi}_\pm : \Xi_{h\pm} \to \mathbb{H}_l$ be Fatou coordinates. Define

$$\widetilde{\gamma}_+ = \widetilde{\phi}_+^{-1}(\phi_{h_f+}(\gamma_{h_f+}))$$

and

$$\widetilde{\gamma}_- = \widetilde{\phi}_-^{-1}(\phi_{h_f-}(\gamma_{h_f-})).$$

The arc $\widetilde{\gamma} = \widetilde{\gamma}_+ \cup \widetilde{\gamma}_-$ is (possibly after rescaling) a dividing arc for h. It divides the set B into Ω_B and Δ_B (with $\mathbb{D} \subset \Omega_B$) and the set B' into Ω'_B and Δ'_B (with $\mathbb{D} \subset \Omega'_B$). Define the map $\widetilde{\phi}^{-1} \circ \phi_{h_f} : \gamma_{h_f} \to \widetilde{\gamma}$ as follows:

$$\widetilde{\phi}^{-1} \circ \phi_{h_f}(z) = \begin{cases} \widetilde{\phi}_+^{-1} \circ \phi_{h_f+} & \text{on } \gamma_{h_f+}, \\ \widetilde{\phi}_-^{-1} \circ \phi_{h_f-} & \text{on } \gamma_{h_f-}. \end{cases}$$

By Proposition 5.4(2) the map $\widetilde{\phi}^{-1} \circ \phi_{h_f}$ is a quasisymmetric conjugacy between $h_{f|\gamma_{h_f}}$ and $h_{|\widetilde{\gamma}}$. Let z_0 be the parabolic fixed point of f, and define the map $\psi : \gamma_f \to \widetilde{\gamma}$ as follows:

$$\psi(z) = \begin{cases} \widetilde{\phi}_+^{-1} \circ \phi_{h_f+} \circ \alpha & \text{on } \gamma_{f+} \setminus \{z_0\}, \\ \widetilde{\phi}_-^{-1} \circ \phi_{h_f-} \circ \alpha & \text{on } \gamma_{f-} \setminus \{z_0\}, \\ z_* & \text{on } z_0. \end{cases}$$

The map $\psi : \gamma_f \to \widetilde{\gamma}$ is an orientation-preserving homeomorphism, real-analytic on $\gamma_f \setminus \{z_0\}$, which conjugates the dynamics of f and h. Let $\psi_0 : \partial U \to \partial B$ be an orientation-preserving C^1-diffeomorphism coinciding with ψ on $\gamma_f \cap \partial U$ (it exists because both U and B have smooth boundaries).

CLAIM 6.1. *There exists a quasiconformal map $\Phi_\Delta : \Delta \to \Delta_B$ which extends to ψ on γ_f, and to ψ_0 on $\partial U \cap \partial\Delta$.*

Proof. It is sufficient to construct a quasiconformal map $\Phi_{\Delta_W} : \Delta_W \to \Delta_B$ which extends to $\widetilde{\phi}^{-1} \circ \phi_{h_f}$ on γ_{h_f} and to $\psi_0 \circ \alpha^{-1}$ on $\alpha(\partial U \cap \partial\Delta)$. Then we will set $\Phi_\Delta = \Phi_{\Delta_W} \circ \alpha$.

The set $\partial\Delta_W$ is a quasicircle, since it is a piecewise C^1 closed curve with non-zero interior angles. Indeed, γ_{h_f+} and γ_{h_f-} form a positive angle since they are separated

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by at least one attracting petal, and we can assume the angles between γ_{h_f} and ∂W_f to be positive (we may take parabolic-like restrictions). The same argument shows that $\partial\Delta_B$ is a quasicircle. Let $\Phi_f : \Delta_W \to \mathbb{D}$ and $\Phi_h : \Delta_B \to \mathbb{D}$ be Riemann maps, and let $\Psi_f : \mathbb{D} \to \Delta_W$ and $\Psi_h : \mathbb{D} \to \Delta_B$ be their inverse maps. By the Carathéodory theorem the maps Ψ_f and Ψ_h extend continuously to the boundaries, and since $\partial\Delta_W$ and $\partial\Delta_B$ are quasicircles, the restrictions $\Psi_f : \mathbb{S}^1 \to \partial\Delta_W$ and $\Psi_h : \mathbb{S}^1 \to \partial\Delta_B$ are quasisymmetric. Define the map $\widetilde{\Phi}_0 : \mathbb{S}^1 \to \mathbb{S}^1$ as follows:

$$\widetilde{\Phi_0}(z) = \begin{cases} \Psi_h^{-1} \circ \widetilde{\phi}^{-1} \circ \phi_{h_f} \circ \Psi_f & \text{on } \Psi_f^{-1}(\gamma_{h_f}), \\ \Psi_h^{-1} \circ \psi_0 \circ \alpha^{-1} \circ \Psi_f & \text{on } \Psi_f^{-1}(\partial\Delta_W \cup \partial W_f). \end{cases}$$

The map $\widetilde{\Phi}_0 : \mathbb{S}^1 \to \mathbb{S}^1$ is quasisymmetric, because the extensions of Ψ_h and Ψ_f to the unit circle are quasisymmetric, α is conformal, the map ψ_0 is a C^1-diffeomorphism and by Proposition 5.4(2) the map $\widetilde{\phi}^{-1} \circ \phi_{h_f} : \gamma_{h_f} \to \widetilde{\gamma}$ is quasisymmetric. Hence, it extends by the Douady–Earle extension (see [**DE**]) to a quasiconformal map $\widetilde{\phi}_0 : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ which is a real-analytic diffeomorphism on $\mathbb{D}$. Thus, $\Phi_{\Delta_W} := \Psi_h \circ \widetilde{\phi}_0 \circ \Phi_f$ is a quasiconformal map between $\overline{\Delta_W}$ and $\overline{\Delta_B}$, which is a real-analytic diffeomorphism on Δ_W, and which coincides with $\widetilde{\phi}^{-1} \circ \phi_{h_f}$ on γ_{h_f} and with $\psi_0 \circ \alpha^{-1}$ on $\alpha(\partial U \cap \partial\Delta)$. □

Let us define $\widetilde{\Delta}_B = h(\Delta_B \cap \Delta'_B)$, $\widetilde{B} = \Omega_B \cup \widetilde{\gamma} \cup \widetilde{\Delta}_B$, $\widetilde{B}' = h^{-1}(\widetilde{B})$, $\widetilde{\Omega}'_B = \Omega'_B \cap \widetilde{B}'$, $\widetilde{\Delta}'_B = \Delta'_B \cap \widetilde{B}'$. On the other hand define $\widetilde{\Delta} = \Phi_\Delta^{-1}(\widetilde{\Delta}_B)$, $\widetilde{\Delta}' = \Phi_\Delta^{-1}(\widetilde{\Delta}'_B)$, $\widetilde{U} = (\Omega \cup \gamma_f \cup \widetilde{\Delta}) \subset U$. Consider

$$\widetilde{f}(z) = \begin{cases} \Phi_\Delta^{-1} \circ h \circ \Phi_\Delta & \text{on } \widetilde{\Delta}', \\ f & \text{on } \Omega' \cup \gamma_f. \end{cases}$$

Define $\widetilde{U'} = \widetilde{f}^{-1}(\widetilde{U})$, and $\widetilde{\Omega}' = \widetilde{U'} \cap \Omega'$. The map $\widetilde{f} : \widetilde{U'} \to \widetilde{U}$ is a degree d proper and quasiregular map which coincides with f on $(\widetilde{\Omega}' \cup \gamma_f) \subset (\Omega' \cup \gamma_f)$. Define $\widehat{U'} = f^{-1}(\widetilde{U})$, $\widehat{\Delta}' = \Delta' \cap \widehat{U'}$ and $\widehat{\Omega}' = \Omega' \cap \widehat{U'}$. Then $(f, \widehat{U'}, \widetilde{U}, \gamma_f)$ is a parabolic-like restriction of (f, U', U, γ_f), and $\widehat{\Omega}' = \widetilde{\Omega}'$. Set $Q_f = \Omega \setminus \overline{\widetilde{\Omega}'}$, and $Q_h = \Omega_B \setminus \overline{\widetilde{\Omega}'}_B$. Let $\overline{\psi}_0 : \partial\widetilde{U} \to \partial\widetilde{B}$ be an orientation-preserving C^1-diffeomorphism coinciding with ψ_0 on $\partial\Omega$, and let $\psi_1 : \partial\widetilde{U'} \to \partial\widetilde{B}'$ be a lift of $\overline{\psi}_0 \circ \widetilde{f}$ to h.

CLAIM 6.2. *There exists a homeomorphism $\widetilde{\psi} : \overline{U} \setminus \widetilde{\Omega} \to \overline{B} \setminus \widetilde{\Omega}_B$ quasiconformal on $\overline{U} \setminus (\widetilde{\Omega} \cup \{z_0\})$ such that the almost complex structure σ defined as*

$$\sigma(z) = \begin{cases} \sigma_0 & \text{on } K_f, \\ \sigma_1 = \widetilde{\psi}^*(\sigma_0) & \text{on } \overline{U} \setminus \overline{\widetilde{\Omega}}, \\ (\widetilde{f}^n)^* \sigma_1 & \text{on } \widetilde{f}^{-n}(\overline{\widetilde{U}} \setminus \overline{\widetilde{\Omega}}), \end{cases}$$

is bounded and $\widetilde{f}$-invariant.

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Proof. Let us start by constructing a quasiconformal map Ψ_Q between $\overline{Q_f}$ and $\overline{Q_h}$ which agrees with ψ on γ_f, with ψ_0 on ∂U and with ψ_1 on $\partial\widetilde{U}'$. The sets ∂Q_f and ∂Q_h are quasicircles, since they are piecewise C^1 closed curves with non-zero interior angles. Let $\varphi_f : Q_f \to \mathbb{D}$ and $\varphi_h : Q_h \to \mathbb{D}$ be Riemann maps, and let $\psi_f : \mathbb{D} \to Q_f$ and $\psi_h : \mathbb{D} \to Q_h$ be their inverses. By the Carathéodory theorem ψ_f and ψ_h extend continuously to the boundaries, and since ∂Q_f and ∂Q_h are quasicircles, $\psi_{f_{|\mathbb{S}^1}}$ and $\psi_{h_{|\mathbb{S}^1}}$ are quasisymmetric. Hence, the map $\widehat{\Phi}_0 : \mathbb{S}^1 \to \mathbb{S}^1$ defined as

$$\widehat{\Phi_0}(z) = \begin{cases} \psi_h^{-1} \circ \psi_+ \circ \psi_f & \text{on } \psi_f^{-1}(\gamma_{f+}), \\ \psi_h^{-1} \circ \psi_0 \circ \psi_f & \text{on } \psi_f^{-1}(\overline{Q_f} \cap \partial U), \\ \psi_h^{-1} \circ \psi_- \circ \psi_f & \text{on } \psi_f^{-1}(\gamma_{f-}), \\ \psi_h^{-1} \circ \psi_1 \circ \psi_f & \text{on } \psi_f^{-1}(\overline{Q_f} \cap \partial\widetilde{U}'), \end{cases}$$

is quasisymmetric, and it extends (see [**DE**]) to a quasiconformal map $\widehat{\varphi}_0 : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ which is a real-analytic diffeomorphism on $\mathbb{D}$. Finally, the map $\Psi_Q :=: \psi_h \circ \widehat{\varphi}_0 \circ \psi_f^{-1} : \overline{Q_f} \to \overline{Q_h}$ is a quasiconformal map which coincides with ψ on γ_f, with ψ_0 on ∂U and with ψ_1 on $\partial\widetilde{U}'$. Moreover, the map Ψ_Q is a real-analytic diffeomorphism on Q_f.

Define the homeomorphism $\widetilde{\psi} : \overline{U} \setminus \widetilde{\Omega} \to \overline{B} \setminus \widetilde{\Omega}_B$ quasiconformal on $\overline{U} \setminus (\widetilde{\Omega} \cup \{z_0\})$ as follows:

$$\widetilde{\psi}(z) = \begin{cases} \psi & \text{on } \gamma_f, \\ \psi_0 & \text{on } \partial U, \\ \psi_1 & \text{on } \partial\widetilde{\Omega}' \cap \partial\widetilde{U}', \\ \Psi_Q & \text{on } Q_f, \\ \Phi_\Delta & \text{on } \Delta. \end{cases}$$

Therefore, the almost complex structure

$$\sigma(z) = \begin{cases} \sigma_0 & \text{on } K_f, \\ \sigma_1 = \widetilde{\psi}^*(\sigma_0) & \text{on } \overline{U} \setminus \overline{\widetilde{\Omega}'}, \\ (\widetilde{f}^n)^*\sigma_1 & \text{on } \widetilde{f}^{-n}(\overline{\widetilde{U}} \setminus \overline{\widetilde{\Omega}}), \end{cases}$$

is bounded and $\widetilde{f}$-invariant. □

By the measurable mapping theorem, there exists a quasiconformal map $\varphi : U \to \mathbb{C}$ such that $\varphi^*\sigma_0 = \sigma$. Let

$$g := \varphi \circ \widetilde{f} \circ \varphi^{-1} : \varphi(\widetilde{U}') \to \varphi(\widetilde{U}).$$

Let us call $V' = \varphi(\widetilde{U}')$, $V = \varphi(\widetilde{U})$, $\gamma_{g+} = \varphi(\gamma_{f+})$ and $\gamma_{g-} = \varphi(\gamma_{f-})$. Then (g, V', V, γ_g) is a parabolic-like map hybrid equivalent to f. Indeed, since $\widetilde{f}_{|\widetilde{\Omega}' \cup \gamma_f} = f$, $\widehat{\Omega} = \widetilde{\Omega}$ and $(f, \widehat{U'}, \widetilde{U}, \gamma_f)$ is a parabolic-like restriction of (f, U', U, γ_f), the map φ is a quasiconformal conjugacy between f and g, and $\varphi^*\sigma_0 = \sigma_0$ on K_f by construction.

If K_f is connected, define the quasiconformal map $\widehat{\psi} : U \setminus K_f \to B \setminus \overline{\mathbb{D}}$ as follows:

$$\widehat{\psi}(z) = \begin{cases} \widetilde{\psi} & \text{on } U \setminus (\widetilde{\Omega}' \cup \{z_0\}), \\ h^{-n} \circ \widetilde{\psi} \circ \widetilde{f}^n & \text{on } \widetilde{f}^{-n}(\widetilde{U} \setminus \widetilde{\Omega}'). \end{cases}$$

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Then the quasiconformal map $\overline{\psi} = \widehat{\psi} \circ \varphi^{-1} : (V \cup V') \setminus K_g \to B \setminus \overline{\mathbb{D}}$ is an external equivalence between g and h, since by construction on $V' \setminus K_g$ $\overline{\psi} \circ g = h \circ \overline{\psi}$, and $\overline{\psi}$ is holomorphic (indeed $(\widehat{\psi} \circ \varphi^{-1})^* \sigma_0 = \sigma_0$).

If K_f is not connected, let $V_f \approx \mathbb{D}$ be a full relatively compact connected subset of $\widetilde{U}$, containing $\overline{\widetilde{\Omega}'}$, the critical values of $\widetilde{f}$ and such that $(f, f^{-1}(V_f), V_f, \gamma_f)$ is a parabolic-like restriction of $(f, \widehat{U'}, \widetilde{U}, \gamma_f)$. Call $L = \widetilde{f}^{-1}(\overline{V}_f) \cap \overline{\widetilde{\Omega}'}$. Define the map $\widehat{\psi} : U \setminus L \to B \setminus \overline{\mathbb{D}}$ as follows:

$$\widehat{\psi}(z) = \begin{cases} \widetilde{\psi} & \text{on } U \setminus (\widetilde{\Omega}' \cup \{z_0\}), \\ h^{-1} \circ \widetilde{\psi} \circ \widetilde{f} & \text{on } \widetilde{f}^{-1}(\widetilde{U} \setminus \widetilde{\Omega}') \setminus L. \end{cases}$$

Let $V_g \approx \mathbb{D}$ be a full relatively compact connected subset of V containing $\overline{\Omega}'_g$, the critical values of g and such that $(g, g^{-1}(V_g), V_g, \gamma_g)$ is a parabolic-like restriction of (g, V, V', γ_g). Call $M = g^{-1}(\overline{V}_g) \cap \overline{\Omega}'_g$. Then the map $\overline{\psi} = \widehat{\psi} \circ \varphi^{-1} : (V \cup V') \setminus M \to B \setminus \overline{\mathbb{D}}$ is an external equivalence between g and h (cf. Lemma 4.1).

6.1. *Unicity.*

PROPOSITION 6.4. *Let $f : U' \to U$ and $g : V' \longrightarrow V$ be two parabolic-like mappings of degree d with connected Julia sets. If they are hybrid and externally equivalent, then they are holomorphically equivalent.*

Proof. Let $\varphi : A \to B$ be a hybrid equivalence between f and g, and $\psi : (A_1 \cup A_1') \setminus K_f \to (B_1 \cup B_1') \setminus K_g$ an external equivalence between f and g. Let $h : \widetilde{W}' \to \widetilde{W}$ be an external map of f constructed from the Riemann map $\alpha : \widehat{\mathbb{C}} \setminus K_f \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Let A_f be a topological disc compactly contained in $(A_1 \cup A_1') \cap A$ and such that $B_\varphi = \varphi(A_f)$ is compactly contained in $(B_1 \cup B_1')$, and $B_{\psi^o} = \psi(A_f \setminus K_f)$ is compactly contained in B. Set $W_\beta = \alpha \circ \psi^{-1}(B_\varphi \setminus K_g)$. The map $\beta = \alpha \circ \psi^{-1} : B_\varphi \setminus K_g \to W_\beta$ restricts to an external equivalence between g and h.

Define $B_\psi = B_{\psi^o} \cup K_g$ and the map $\Phi : A_f \to B_\psi$ as

$$\Phi(z) = \begin{cases} \varphi & \text{on } K_f, \\ \psi & \text{on } A_f \setminus K_f. \end{cases}$$

By construction, the map $\Phi : A_f \to B_\psi$ conjugates the maps f and g conformally on $A_f \setminus K_f$ and quasiconformally with $\overline{\partial}\Phi = 0$ on K_f. We want to prove that the map Φ is holomorphic. Owing to the Rickmann lemma, it is enough to prove that the map Φ is continuous (for a proof of the Rickmann lemma we refer the reader to [**DH**, Lemma 2, p. 303]).

LEMMA 6.1. (Rickmann) *Let $U \subset \mathbb{C}$ be open, $K \subset U$ be compact, $\varphi : U \to \mathbb{C}$ and $\Phi : U \to \mathbb{C}$ be two maps which are homeomorphisms onto their images. Suppose that φ is quasiconformal, that Φ is quasiconformal on $U \setminus K$ and that $\Phi = \varphi$ on K. Then Φ is quasiconformal and $D\Phi = D\varphi$ almost everywhere on K.*

Let us show that the map Φ is continuous. Define $W_f = h(h^{-1}(\alpha(A_f \setminus K_f)) \cap \alpha(A_f \setminus K_f)) \subset \alpha(A_f \setminus K_f)$ and $W_f' = h^{-1}(W_f)$. The restriction $h : W_f' \to W_f$ is proper

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holomorphic and of degree d. The map $\chi := \beta \circ \varphi \circ \alpha^{-1} : W'_f \to W_\beta$ is a quasiconformal homeomorphism (into its image) which autoconjugates h on $\Omega'_W \cup \gamma_h \setminus \{\gamma_h(0)\}$.

Setting $\tau(z) = 1/\bar{z}$, $\widetilde{W}'_f = W'_f \cup \mathbb{S}^1 \cup \tau(W'_f)$, $\widetilde{W}_\beta = W_\beta \cup \mathbb{S}^1 \cup \tau(W_\beta)$, and applying the strong reflection principle with respect to the unit circle, we obtain a quasiconformal homeomorphism (into its image) $\widetilde{\chi} : \widetilde{W}'_f \to \widetilde{W}_\beta$, which autoconjugates h on $\widetilde{\Omega}'_W$. Thus, the restriction $\widetilde{\chi} : \mathbb{S}^1 \to \mathbb{S}^1$ is a quasisymmetric autoconjugacy of h on the unit circle. Since the preimages of the parabolic fixed point $z = 1$ are dense in $\mathbb{S}^1$, an autoconjugacy of h on the unit circle is the identity. Therefore, $\widetilde{\chi}\,|_{\mathbb{S}^1} = \mathrm{Id}$.

Since the map $\widetilde{\chi} : \widetilde{W}'_f \to \widetilde{W}_\beta$ is a quasiconformal homeomorphism which coincides with the identity on $\mathbb{S}^1$, the hyperbolic distance between a point near $\mathbb{S}^1$ and its image is uniformly bounded, i.e. there exists $M > 0$ and $r > 1$ such that

$$\text{for all } z,\ 1 < |z| < r, \quad d_{W'_f}(z, \beta \circ \varphi \circ \alpha^{-1}(z)) \leq M.$$

Since α and β are isometries, we obtain

$$d_{A_f \setminus K_f}(\beta^{-1} \circ \alpha(z), \varphi(z)) \leq M \quad \text{for } z \notin K_f,\ z \text{ in a neighborhood of } K_f.$$

Then $\beta^{-1} \circ \alpha(z)$ and $\varphi(z)$ converge to the same value as z converges to J_f, i.e. $\beta^{-1} \circ \alpha$ extends continuously to J_f by $\beta^{-1} \circ \alpha(z) = \varphi(z),\ z \in J_f$. Thus Φ is continuous, and this completes the proof. □

PROPOSITION 6.5. *If $P_A = z + 1/z + A$ and $P_{A'} = z + 1/z + A'$ are hybrid conjugate and K_A is connected, then they are holomorphically conjugate, i.e. $A^2 = (A')^2$.*

Proof. Since K_A and $K_{A'}$ are connected, the external conjugacies between P_A and $P_{A'}$, respectively, and h_2 can be extended to the discs $\widehat{\mathbb{C}} \setminus K_A$ and $\widehat{\mathbb{C}} \setminus K_{A'}$ (see Proposition 4.2), i.e. there exist holomorphic conjugacies $\alpha : \widehat{\mathbb{C}} \setminus K_A \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and $\beta : \widehat{\mathbb{C}} \setminus K_A \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ between P_A and $P_{A'}$, respectively, and h_2. Therefore, $\beta^{-1} \circ \alpha : \widehat{\mathbb{C}} \setminus K_{A'} \to \widehat{\mathbb{C}} \setminus K_{A'}$ is a holomorphic conjugacy between P_A and $P_{A'}$.

Let (f, U', U, γ_f) and (g, V', V, γ_g) be parabolic-like restrictions of P_A and $P_{A'}$, respectively, and let $\varphi : A \to B$ be a hybrid equivalence between them. Define the map $\Phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as follows:

$$\Phi(z) = \begin{cases} \varphi & \text{on } K_A, \\ \beta^{-1} \circ \alpha & \text{on } \widehat{\mathbb{C}} \setminus K_A. \end{cases}$$

The proof of Proposition 6.4 shows that the map $\Phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is holomorphic, hence it is a Möbius transformation. Since Φ conjugates P_A and $P_{A'}$, it fixes the parabolic fixed point $z = \infty$ and its preimage $z = 0$, and it can fix or interchange the critical points $z = 1$ and $z = -1$. Hence, either Φ is the identity or $\Phi(z)$ is the map $z \to -z$. Therefore, $[P_A] = \{P_A,\ \ P_{-A}\}$, and finally $A^2 = (A')^2$. □

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