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Kadomtsev–Petviashvili equation for acoustic wave in quantum pair plasmas

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Abstract. The Kadomtsev–Petviashvili equation is derived for two-dimensional propagations of electrostatic solitons in unmagnetized dense pair-plasmas. The reductive perturbation method is employed and two-dimensional electrostatic potential hump structures are obtained. The conditions for a stable two-dimensional solitary structure are discussed using energy consideration method. The numerical results are also presented by considering the parameters for the outer layers of white dwarfs/neutron stars.

1. Introduction

The study of quantum electron–positron plasmas has become important due to its application in astrophysical objects such as compact stars and in-future laser plasma experiments (Haberland et al. 2001; Jung 2001; Manfredi and Haas 2001; Marklund 2005; Shukla and Eliasson 2010). The electrons and positrons are believed to exist around astrophysical plasma situations such as active galactic nuclei (AGN), quasars, neutron stars or pulsar magnetospheres, *etc*. (Miller and Wiita 1987; Iwamoto 1992; Zank and Greaves 1995; Reynolds et al. 1996; Hirotani et al. 1999). The dynamics of electron–positron (e–p) plasma consisting of same mass but oppositely charged particles is quite different from the usual electron–ion (e–i) dynamics, in which both fast and slow time scales exist. However, there is still a possibility of the existence of some fraction of ions around the atmosphere of astrophysical objects containing most of e–p plasmas. The e–p plasma symmetry breaks in the presence of ions and both fast and slow time scale can occur in the dynamics of electron–positron–ion (e–p–i) plasmas. In most of the astrophysical plasma conditions, the electrons and positrons exist in relativistic regimes and therefore most of the research work has been investigated for relativistic e–p plasmas (Iwamoto 1989; Berezhiani et al. 1993 and references therein). The classical study of collective behavior of e–p plasmas is also important to understand some aspects of astrophysical plasma situations because e–p plasma radiates effectively by emission of cyclotron radiations and eventually it cools down due to loss of energy from plasmas (Zank and Greaves 1995). The study of quantum plasma becomes important when the de-Broglie wave length associated with the charged particles becomes of the order or greater than the inter-particle distance of the system, and plasma behaves like a Fermi gas and quantum mechanical effects play significant role in the dynamics (Haas et al. 2003; Menfredi 2005; Ali et al. 2007; Moslem et al. 2007). The study of quantum e–p plasma remains in the non-relativisitic regime if the Fermi energies of electron and positron are much less than their rest mass energies, and this holds in our present study.

The quantum hydrodynamic (QHD) model is a useful approximation to study short-scale collective phenomenon such as waves, instabilities and nonlinear structures etc. in dense plasmas (Manfredi and Haas 2001; Haas et al. 2003; Menfredi 2005). The QHD model generalizes the fluid model with the inclusion of quantum statistical pressure and quantum diffraction (also known as the Bohm potential) terms. The validity of QHD is limited to those systems that are large compared to the Fermi lengths of the species in the system. The ion-acoustic solitary waves in quantum e–p–i plasma have been studied (Hass et al. 2003; Ali et al. 2007; Moslem et al. 2007; Sabry et al. 2007; Haque and Mahmood 2008; Khan et al. 2008; Jehan et al. 2009). It is found that the presence of positron plays a significant role in the formation of solitons in dense plasmas. The low-frequency waves, such as ion-acoustic waves, drift waves etc., have been studied in quantum e–p–i plasmas with their application to neutron stars/pulsars (Haque et al. 2008). However, the electrostatic fast frequency wave on electron (or positron) dynamic scale can also propagate in dense e–p–i plasmas in which ions are assumed to be stationary. The one-dimensional low and arbitrary amplitude nonlinear structures have been studied in dense e–p–i plasmas (Misra et al. 2008; Mahmmod et al. 2010).

The two-dimensional soliton solutions from the Kadomtsev–Petviashvili (KD) equation have been investigated in plasmas (Das and Sen 1993; Das et al.

1997; Duan 2002; Chakraborty and Das 2004; Masood et al. 2009). The KP equation is a nonlinear partial differential equation in two spatial (i.e. in horizontal and transverse directions) and temporal coordinates, which describes the evolution of nonlinear long waves of small amplitude with slow dependence on transverse coordinate. The KP soliton emerges in the asymptotic description of such systems in which only the leading order terms are retained and an asymptotic balance between weak dispersion, quadratic nonlinearity and diffraction is assumed.

In this manuscript we will study the two-dimensional dynamics of degenerate electron and positron fluids, while ions are considered to be stationary to neutralize the plasma background only. The paper is organized in the following way. In Sec. 2 we describe the mathematical model and set of governing equations of the system. Using the reductive perturbation method, the KP equation is derived for acoustic wave in degenerate e–p–i plasmas in Sec. 3. In Sec. 4, the numerical plots are presented by taking into account the parameters of outer layers of white dwarfs/neutron stars (Moslem et al. 2007). The conclusion is presented in Sec. 5.

2. Model

In this section, two-dimensional electrostatic solitons are studied in unmagnetized quantum e–p plasmas in the presence of stationary ions. The quantum fluids of electron and positron are assumed to be dynamic while ions are taken to be stationary. The continuity and momentum equations for e–p plasmas are given by

$$\frac{\partial n_j}{\partial t} + \nabla.(n_j\mathbf{v}_j) = 0, \tag{2.1}$$

$$\frac{\partial \mathbf{v}_j}{\partial t} + (\mathbf{v}_j.\nabla)\mathbf{v}_j = \frac{q_j\mathbf{E}}{m} - \frac{1}{mn_j}\nabla p_j + \frac{\hbar^2}{2m^2}\nabla\left(\frac{1}{\sqrt{n_j}}\nabla^2\sqrt{n_j}\right). \tag{2.2}$$

The Poisson equation is described as follows:

$$\nabla.\mathbf{E} = 4\pi e(n_p - n_e + n_{i0}). \tag{2.3}$$

The equilibrium is defined as

$$n_{i0} + n_{p0} = n_{e0}, \tag{2.4}$$

where E is the electric field intensity, n_j and v_j (where $j = e, p$) are perturbed densities and velocities of electron and positron fluids, respectively and q_j is the charge $(-e, e)$ on electron and positron. The mass 'm' of positron and electron is the same. The equilibrium densities of electrons, positrons and ions are n_{e0}, n_{p0} and n_{i0}, respectively. Here $\hbar$ is Planck's constant divided by 2π, $p_j = (mv_{Fj}^2/5n_{j0}^{\frac{2}{3}})n_j^{\frac{5}{3}}$ is the Fermi pressure of jth species, the Fermi velocity and temperature of the species are related as $\frac{1}{2}mv_{Fj}^2 = k_BT_{Fj}$ and the Fermi velocity of jth species is defined as $v_{Fj} = \sqrt{2k_BT_{Fj}/m}$. In a quantum plasma, the Fermi temperature and equilibrium density of jth species are related as $k_BT_{Fj} = \frac{\hbar^2(3\pi^2 n_{j0})^{\frac{2}{3}}}{2m}$. The last two terms in the momentum equation are quantum terms, *i.e.* the Fermi pressure and the Bohm potential terms of the jth species, which appears due to the Fermi statistics and quantum tunneling effects in dense plasmas.

Now assuming the wave propagation in two dimensions, i.e., $\nabla = (\partial_x, \partial_y, 0)$, the normalized continuity and momentum equations for electron quantum fluid in the component form can be written as

$$\frac{\partial n_e}{\partial t} + \frac{\partial(n_e v_{ex})}{\partial x} + \frac{\partial(n_e v_{ey})}{\partial y} = 0, \tag{2.5}$$

$$\frac{\partial v_{ex}}{\partial t} + \left(v_{ex}\frac{\partial}{\partial x} + v_{ey}\frac{\partial}{\partial y}\right)v_{ex} = \frac{\partial \Phi}{\partial x} - \frac{1}{5n_e}\frac{\partial n_e^{5/3}}{\partial x} + \frac{H_e^2}{2}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{n_e}}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\sqrt{n_e}\right], \tag{2.6}$$

$$\frac{\partial v_{ey}}{\partial t} + \left(v_{ex}\frac{\partial}{\partial x} + v_{ey}\frac{\partial}{\partial y}\right)v_{ey} = \frac{\partial \Phi}{\partial y} - \frac{1}{5n_e}\frac{\partial n_e^{5/3}}{\partial y} + \frac{H_e^2}{2}\frac{\partial}{\partial y}\left[\frac{1}{\sqrt{n_e}}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\sqrt{n_e}\right]. \tag{2.7}$$

The normalized continuity and momentum equations for positron quantum fluid in the component form can be written as

$$\frac{\partial n_p}{\partial t} + \frac{\partial(n_p v_{px})}{\partial x} + \frac{\partial(n_p v_{py})}{\partial y} = 0, \tag{2.8}$$

$$\frac{\partial v_{px}}{\partial t} + \left(v_{px}\frac{\partial}{\partial x} + v_{py}\frac{\partial}{\partial y}\right)v_{px} = -\frac{\partial \Phi}{\partial x} - \frac{(1-\delta)^{2/3}}{5n_p}\frac{\partial n_p^{5/3}}{\partial x} + \frac{H_p^2(1-\delta)^{1/3}}{2}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{n_p}}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\sqrt{n_p}\right], \tag{2.9}$$

$$\frac{\partial v_{py}}{\partial t} + \left(v_{px}\frac{\partial}{\partial x} + v_{py}\frac{\partial}{\partial y}\right)v_{py} = -\frac{\partial \Phi}{\partial y} - \frac{(1-\delta)^{2/3}}{5n_p}\frac{\partial n_p^{5/3}}{\partial y} + \frac{H_p^2(1-\delta)^{1/3}}{2}\frac{\partial}{\partial y}\left[\frac{1}{\sqrt{n_p}}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\sqrt{n_p}\right]. \tag{2.10}$$

The Poisson equation in the normalized form is given by

$$\nabla^2\Phi = n_e - n_p(1-\delta) - \delta, \tag{2.11}$$

where the electric field intensity is defined as $\mathbf{E} = -\nabla\varphi$ (where φ is the electrostatic potential). The normalization $t \to t\omega_{pe}$, $\nabla \to \nabla\, \lambda_{Fe}$, $n_j \to n_j/n_{j0}$, $v_j \to v_j/v_{F_e}$ and $\Phi \to \left(e\varphi/2k_BT_{Fe}\right)$ have been defined, where $\omega_{pj} = \sqrt{4\pi n_{j0}e^2/m}$ and $\lambda_{Fj} = \sqrt{2k_BT_{Fj}/(4\pi n_{j0}e^2)}$ are plasma frequency and the Fermi length of the jth species. The dimensionless parameters, such as ion to electron equilibrium density ratio, *i.e.* $\delta = \frac{n_{i0}}{n_{e0}}$ (where $0 \leqslant \delta < 1$ must hold) and the positron to electron Fermi temperature ratio, *i.e.* $\sigma_T = T_{Fp}/T_{Fe} = (1-\delta)^{2/3}$ have been defined. The quantum parameters for electron and positron fluids are defined as $H_e = \hbar\omega_{pe}/(2k_B T_{Fe})$ and $H_p = \hbar\omega_{pp}/(2k_B T_{Fp})$, respectively.

3. Nonlinear solution

In order to derive the KP equation for electrostatic potential in quantum e–p–i plasmas, we define the stretching of independent variable (Das and Sen 1993; Das et al. 1997; Duan 2002; Chakraborty and Das 2004; Masood et al. 2009), such as

$$\xi = \epsilon^{1/2}(x - \lambda t), \quad \eta = \epsilon y, \quad \tau = \epsilon^{3/2} t,$$

where ϵ is a small $(0 < \epsilon \leqslant 1)$ expansion parameter characterizing the strength of nonlinearity and λ is the phase velocity of wave normalized with the Fermi velocity of electron. Now using the reductive perturbation method, we can expand the perturbed quantities about their equilibrium values in the powers of ϵ as follows:

$$\begin{aligned} n_j &= 1 + \epsilon n_j^{(1)} + \epsilon^2 n_j^{(2)} + \epsilon^3 n_j^{(3)} ..., \\ v_{jx} &= \epsilon v_{jx}^{(1)} + \epsilon^2 v_{jx}^{(2)} + \epsilon^3 v_{jx}^{(3)} ..., \\ v_{jy} &= \epsilon^{3/2} v_{jy}^{(1)} + \epsilon^{5/2} v_{jy}^{(2)} + \epsilon^{7/2} v_{jy}^{(3)} ..., \\ \Phi &= \epsilon \Phi^{(1)} + \epsilon^2 \Phi^{(2)} + \epsilon^3 \Phi^{(3)} .... \end{aligned} \tag{3.1}$$

Substituting perturbed quantities defined in (3.1) in (2.5)–(2.11) and collecting terms of the lowest order $\left(\sim \epsilon^{3/2}\right)$ from continuity and the momentum equations along x-direction and $\left(\epsilon^2\right)$ order terms from momentum equations along y-axis of electrons and positrons, respectively, we obtain

$$n_e^{(1)} = \frac{3}{(1 - 3\lambda^2)}\Phi^{(1)}, \quad v_{ex}^{(1)} = \frac{3\lambda}{(1 - 3\lambda^2)}\Phi^{(1)},$$

$$\frac{\partial v_{ey}^{(1)}}{\partial \xi} = -\frac{1}{\lambda}\frac{\partial \Phi^{(1)}}{\partial \eta} + \frac{1}{3\lambda}\frac{\partial n_e^{(1)}}{\partial \eta}, \tag{3.2}$$

$$n_p^{(1)} = -\frac{3}{((1 - \delta)^{2/3} - 3\lambda^2)}\Phi^{(1)},$$

$$v_{px}^{(1)} = -\frac{3\lambda}{((1 - \delta)^{2/3} - 3\lambda^2)}\Phi^{(1)},$$

$$\frac{\partial v_{py}^{(1)}}{\partial \xi} = \frac{1}{\lambda}\frac{\partial \Phi^{(1)}}{\partial \eta} + \frac{(1 - \delta)^{2/3}}{3\lambda}\frac{\partial n_p^{(1)}}{\partial \eta}. \tag{3.3}$$

The lowest order $(\sim \epsilon)$ term from the Poisson equation gives

$$n_e^{(1)} - (1 - \delta)n_p^{(1)} = 0. \tag{3.4}$$

Now using the expressions of $n_e^{(1)}$ and $n_p^{(1)}$ from (3.2) and (3.3) in (3.4), the linear phase speed of the acoustic wave in e–p–i plasmas is obtained as follows:

$$\lambda = \sqrt{\frac{(1 - \delta) + (1 - \delta)^{2/3}}{3(2 - \delta)}}. \tag{3.5}$$

Now collecting the next higher order terms, *i.e.* $(\epsilon^{5/2})$ from continuity and the momentum equation along x-axis and (ϵ^3) order term of momentum equation along y-axis of electrons, we have

$$-\lambda\partial_\xi n_e^{(2)} + \partial_\xi v_{ex}^{(2)} = f_1, \tag{3.6}$$

$$-\lambda\partial_\xi v_{ex}^{(2)} - \partial_\xi \Phi^{(2)} + \frac{1}{3}\partial_\xi n_e^{(2)} = f_2,$$

$$-\lambda\partial_\xi v_{ey}^{(2)} - \partial_\eta \Phi^{(2)} + \frac{1}{3}\partial_\eta n_e^{(2)} = f_3, \tag{3.7}$$

where $f_1 = -\partial_\tau n_e^{(1)} - \partial_\xi(n_e^{(1)} v_{ex}^{(1)}) - \partial_\eta v_{ey}^{(1)}$, $f_2 = -\partial_\tau v_{ex}^{(1)} - v_{ex}^{(1)}\partial_\xi v_{ex}^{(1)} + \frac{n_e^{(1)}}{9}\partial_\xi n_e^{(1)} + \frac{H_e^2}{4}\partial_\xi^3 n_e^{(1)}$ and $f_3 = -\partial_\tau v_{ey}^{(1)} - v_{ex}^{(1)}\partial_\xi v_{ey}^{(1)} + \frac{n_e^{(1)}}{9}\partial_\eta n_e^{(1)} + \frac{H_e^2}{4}\partial\eta\partial_\xi^2 n_e^{(1)}$ have been defined.

Then collecting the next higher order terms, *i.e.* $(\epsilon^{5/2})$ from continuity and the momentum equation along x-axis and (ϵ^3) order term of momentum equation along y-axis of positrons we have

$$-\lambda\partial_\xi n_p^{(2)} + \partial_\xi v_{px}^{(2)} = f_4, \tag{3.8}$$

$$-\lambda\partial_\xi v_{px}^{(2)} + \partial_\xi \Phi^{(2)} + \frac{(1 - \delta)^{2/3}}{3}\partial_\xi n_p^{(2)} = f_5,$$

$$-\lambda\partial_\xi v_{py}^{(2)} + \partial_\eta \Phi^{(2)} + \frac{(1 - \delta)^{2/3}}{3}\partial_\eta n_p^{(2)} = f_6, \tag{3.9}$$

where $f_4 = -\partial_\tau n_p^{(1)} - \partial_\xi(n_p^{(1)} v_{px}^{(1)}) - \partial_\eta v_{py}^{(1)}$, $f_5 = -\partial_\tau v_{px}^{(1)} - v_{px}^{(1)}\partial_\xi v_{px}^{(1)} + \frac{(1-\delta)^{2/3} n_p^{(1)}}{9}\partial_\xi n_p^{(1)} + \frac{H_e^2}{4}\partial_\xi^3 n_p^{(1)}$ and $f_6 = -\partial_\tau v_{py}^{(1)} - v_{px}^{(1)}\partial_\xi v_{py}^{(1)} + \frac{(1-\delta)^{2/3} n_p^{(1)}}{9}\partial_\eta n_p^{(1)} + \frac{H_e^2}{4}\partial\eta\partial_\xi^2 n_p^{(1)}$ have been defined. The relation between quantum parameters of electron and positron fluids, *i.e.* $H_p = H_e(1 - \delta)^{-1/6}$ has been used in (2.9)–(2.10) to obtain (3.9).

However, the next higher order (ϵ^2) terms of the Poisson equation gives

$$n_e^{(2)} - (1 - \delta)n_p^{(2)} = f_7, \tag{3.10}$$

where $f_7 = \partial_\xi^2 \Phi^{(1)}$ has been defined.

On solving (3.6)–(3.10), we find the expression after some simplification as follows:

$$f_1 + \frac{f_2}{\lambda} + f_4 + \frac{f_5}{\lambda} - \left(\frac{1 - 3\lambda^2}{3\lambda}\right)\partial_\xi f_7 = 0, \tag{3.11}$$

where (3.5) has been used to obtain the above relation.

Now using relations of f_s and (3.2–3.10) in (3.11) and after some simplification, we obtain the KP equation for electrostatic waves in e–p–i quantum plasmas in terms of electrostatic potential $\Phi^{(1)}$ as follows:

$$\partial_\xi\left(\partial_\tau \Phi + A\Phi\partial_\xi \Phi + B\partial_\xi^3 \Phi\right) + C\partial_\eta^2 \Phi = 0, \tag{3.12}$$

where $\Phi^{(1)}$ has been replaced by Φ, A is the coefficient of nonlinearity and B and C are the coefficients of weak dispersion and diffraction effects in the system, respectively, defined as

$$A = \frac{\left((1-\delta)^{4/3} + 9\lambda^4 - 6(1-\delta)^{2/3}\lambda^2\right)\left(1-27\lambda^2\right) + \left(1+9\lambda^4-6\lambda^2\right)\left((1-\delta)^{2/3}-27\lambda^2\right)}{6\lambda\left(1-(1-\delta)^{2/3}\right)\left(1-3\lambda^2\right)\left((1-\delta)^{2/3}-3\lambda^2\right)},$$

$$B = \frac{9H_e^2(1-\delta)\left((1-\delta)^{2/3}-3\lambda^2\right) - 9H_e^2(1-\delta)\left(1-3\lambda^2\right) - 4(1-\delta)\left(1-3\lambda^2\right)\left((1-\delta)^{2/3}-3\lambda^2\right)}{72\lambda(1-\delta)\left(1-(1-\delta)^{2/3}\right)},$$

$$C = \sqrt{\frac{(1-\delta)+(1-\delta)^{2/3}}{12(2-\delta)}}. \tag{3.13}$$

There are two different approaches, *i.e.* the *Tanh* method and the pseudo-potential approach to find a solution to the KP equation (3.12). We have defined the transformed coordinated ζ of the comoving frame such that $\zeta = \xi + \eta - u\tau$, where u is the speed of the nonlinear structure and using the boundary conditions, *i.e.* $\Phi \to 0$ and $\partial_\zeta \Phi$, $\partial_\zeta^2 \Phi$, $\partial_\zeta^3 \Phi \to 0$ as $\zeta \to \infty$ for localized solution.

The analytical solution of the KP equation (3.12) is derived using the *Tanh* method (the details are given in Appendix A), which is given below as

$$\Phi(\zeta) = \frac{12B}{A}\left[1 - \tanh^2(\xi + \eta - (4B + C)\tau)\right]. \tag{3.14}$$

The speed of the co-moving frame is related with the weak dispersive and diffraction coefficients such as $u = (4B + C)$, which has been obtained using the boundary conditions, *i.e.* as $\zeta \to \infty$, $\Phi(\zeta) \to 0$ and $\tanh^2(\zeta) \to 1$.

The solution using the pseudo-potential approach (whose details are described in Appendix B) is written as

$$\Phi = \phi_m \sec h^2\left(\frac{\xi + \eta - u\tau}{W}\right), \tag{3.15}$$

where maximum amplitude $\phi_m = \frac{3(u-C)}{A}$ and width $W = \sqrt{4B/(u-C)}$ of the soliton has been defined. Here it should be noted that both solutions have the same form with same numerical results.

In quantum fluid the Fermi temperature and density are related through the relation $T_{Fj} \propto (n_{j0})^{\frac{2}{3}}$ (where T_{Fj} and n_{j0} are the Fermi temperature and equilibrium density of the jth species, respectively), therefore in the presence of ions in quantum e–p–i plasmas $T_{Fe} \neq T_{Fp}$, *i.e.* $(0 < \delta < 1)$ remains hold, which is a necessary condition for the propagation of acoustic wave in e–p–i dense plasmas.

4. Stability of KP solitons

The perturbation scheme (3.1) used to derive the KP equation shows more slowness in y-axis. The variations in y-axis shouldn't be too weak, otherwise one will simply obtain the Korteweg-de Vries (KdV) equation. On the other hand, if these variations are too strong, it will not be possible for a soliton-type solution to exist. The stability of soliton, perpendicular to its direction of propagation, was first discussed by Kadomtsev and Petviashvili (1970). Kako and Rowlands (1976) have considered three distinct generalizations of the KdV equation by considering three different types of perturbation schemes to study the two-dimensional stability of ion-acoustic soliton. The authors have used a perturbation scheme similar to (3.1), *i.e.* $\eta = \epsilon y$ and $v_y = \epsilon^{3/2} v_{y1} + \epsilon^{5/2} v_{y2} + \cdots$, to describe the variations in y-axis and have shown that the soliton solution is stable against such perturbations.

In order to discuss the stability of the solution (3.15), we use the method based on the energy consideration (Krall and Travelpiece 1973). The nonlinear equation (B2) leads to pseudo-potential (B3), which is given by

$$V(\Phi) = -\frac{s}{2B}\Phi^2 + \frac{A}{6B}\Phi^3, \tag{4.1}$$

where $s = (u - \lambda/2)$ has been defined. It is clear that $V(\Phi) = 0$, $|dV(\Phi)|/d\Phi = 0$ at $|\Phi| = 0$. The necessary conditions for a stable solitonic solution are

(i) $\left|\frac{d^2V(\Phi)}{d\Phi^2}\right|_{\Phi=0} < 0$,

(ii) a non-zero crossing point $\Phi = \phi_m$ must exist such that $V(\Phi = \phi_m) = 0$,

(iii) there must exist a Φ value between $\Phi = 0$ and $\Phi = \phi_m$ to make $V(\Phi) < 0$.

So according to condition (i), we find

$$\frac{d^2V(\Phi)}{d\Phi^2}\,|_{\Phi=0} = -\frac{s}{B}. \tag{4.2}$$

Equation (4.2) demands that s/B should be positive for the existence of soliton solution, which means $s > 0$ and $B > 0$ or $s < 0$ and $B < 0$ holds, otherwise there will be a shock wave. Since $0 < \delta < 1$ holds, which shows that $\lambda > 0$, and for a stable soliton solution we have,

$$u > C = \frac{\lambda}{2}. \tag{4.3}$$

Therefore, s must be positive, which requires that B should also be positive. The critical value of H_e in terms of δ from stability condition $B > 0$ can be obtained as follows:

$$H_e > \left[\frac{4}{9}\frac{(1-\delta)\left(1-(1-\delta)^{2/3}\right)}{(2-\delta)^2}\right]^{1/2}.$$

Hence, the stability of the KP soliton of acoustic wave in quantum pair plasma is associated with δ.

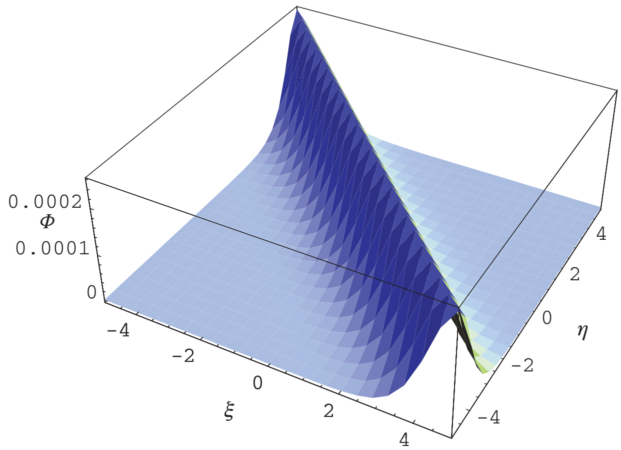

Figure 1. (Colour online) The two-dimensional electrostatic potential hump structure is shown for electron density $n_{e0} = 10^{28}$ cm^{-3} and ion density $n_{i0} = 0.2 \times 10^{28}$ cm^{-3}, $\delta = 0.2$, $C = 0.277$ for $u = 0.28$.

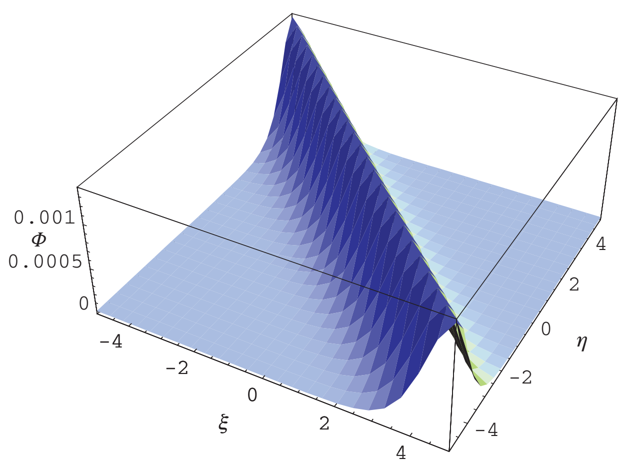

Figure 2. (Colour online) The two-dimensional electrostatic potential hump structure is shown for electron density $n_{e0} = 10^{28}$ cm^{-3} and ion density $n_{i0} = 0.3 \times 10^{28}$ cm^{-3}, $\delta = 0.3$, $C = 0.270$ for $u = 0.28$.

5. Numerical solutions

In this section, the numerical solutions of two-dimensional propagation of soliton in dense magnetized e–p–i plasmas are obtained using (3.15). The parameters of the outside layers of white dwarf/neutron star (a compact star) have been used for numerical studies such as magnetic field $B_0 \sim 10^9 - 10^{12}$ *Gauss* , the minimum electron–positron density $n_{0e,p} \sim 10^{28}$ cm^{-3} (on which pair annihilation effects in dense plasmas can be ignored) and temperature on its surface ranges from $8000{-}40,000\ K$ (Ali et al. 2007; Moslem et al. 2007). The two-dimensional electrostatic potential hump structures are obtained in the presence of stationary ions in e–p unmagnetized quantum plasmas. In Fig. 1, the two-dimensional soliton is plotted for electron and ion densities $n_{e0} = 10^{28}$ cm^{-3} and $n_{i0} = 0.2 \times 10^{28}$ cm^{-3}, respectively. The Fermi temperatures of electron, positron and ion at such densities are turned out to be $T_{Fe} = 1.96 \times 10^8$ K, $T_{Fp} = 1.69 \times 10^8$ K and $T_{Fi} = 3.65 \times 10^4$ K, respectively. The quantum parameters for electrons and positrons at these densities are $H_e = 0.1097$ and $H_p = 0.1139$, while the Fermi lengths are $\lambda_{Fe} = 1.36 \times 10^{-9}$ cm , $\lambda_{Fp} = 1.44 \times 10^{-9}$ cm and plasma frequencies are of the order of 10^{18}sec^{-1}, respectively. The average particle distance (d) at these densities is of the order of 10^{-10} cm for each of the species. Therefore, $\lambda_{Fj} \gtrsim d$ holds to study collective effects in quantum plasmas. The two-dimensional soliton is plotted in Fig. 2 for ion density $n_{i0} = 0.3 \times 10^{28}$ cm^{-3} and for same electron density as described in Fig. 1. The Fermi temperatures of positron and ion at such densities are turned out to be $T_{Fp} = 1.54 \times 10^8$ K and $T_{Fi} = 4.79 \times 10^4$ K, respectively. It is found that wave amplitude and width of the nonlinear structure are increased with increase in ion density (or decrease in positron density) in unmagnetized quantum e–p–i plasmas. The Figs 1 and 2 are plotted on those parameters for which stability condition of the two-dimensional soliton remains hold, *i.e.* $u > C$ and $B > 0$. The Fermi temperatures of the jth species T_{Fj} for plasma densities in the outer layers of white dwarfs are turned out to be greater than system temperature, *i.e.* $T = 8,000$ K, *i.e.* $T_{Fj} \gg T$, so that the study of quantum effects becomes important for e–p–i plasmas around white dwarfs or compact stars.

6. Conclusion

To conclude, we have studied the two-dimensional electrostatic waves in unmagnetized quantum electron–positron plasmas in the presence of ions, which are assumed to be stationary to neutralize plasma background only. The KP equation is derived for two-dimensional propagation of nonlinear acoustic waves in quantum e–p plasmas. It is found that the presence of stationary ions in dense e–p plasmas are necessary for propagation of these solitons. Increase in ion concentration in dense e–p plasmas increases the amplitude and width of soliton. From stability analysis it is shown that the soliton solution is stable against the variations in transverse direction. The conditions for stable soliton via pseudo-potential approach have also been discussed. Our findings are general and may be applicable in the outer layers of compact stars such as white dwarfs, neutron stars, magnetars etc., where dense e–p plasmas exist.

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Appendix A. Tanh method for the solution of the KP equation.

The solution of nonlinear partial differential equation, *i.e.* the KP equation (3.12) is obtained using the well-known Tanh method (Malfliet 1992). The traveling wave solution requires the transformation defined as

$\zeta = \xi + \eta - u\tau$. Now integrating once and using the boundary conditions, *i.e.* $\Phi \to 0$, $\partial_\zeta \Phi$, $\partial_\zeta^2 \Phi$, $\partial_\zeta^3 \Phi \to 0$, we have

$$-(u-C)\frac{\partial \Phi}{\partial \zeta} + A\Phi\frac{\partial \Phi}{\partial \zeta} + B\frac{\partial^3 \Phi}{\partial \zeta^3} = 0. \qquad \text{(A 1)}$$

After using the transformation, the partial differential equation becomes an ordinary differential equation. Now we have to introduce a new independent variable $Y = \tanh\zeta$. The corresponding derivatives are given as

$$\frac{\partial}{\partial \zeta} = (1-Y^2)\frac{\partial}{\partial Y},$$

$$\frac{\partial^2}{\partial \zeta^2} = (1-Y^2)\frac{\partial}{\partial Y}\left((1-Y^2)\frac{\partial}{\partial Y}\right),$$

$$\frac{\partial^3}{\partial \zeta^3} = (1-Y^2)\frac{\partial}{\partial Y}\left[(1-Y^2)\frac{\partial}{\partial Y}\left((1-Y^2)\frac{\partial}{\partial Y}\right)\right].$$

Now we must look for solution in terms of Y. Since there is no general procedure for the solution, therefore the solution is defined in terms of the following series expansion, which is described as

$$\Phi(Y) = \sum_{s=0}^{M} a_s Y^s. \qquad \text{(A 2)}$$

A balancing procedure determines degree M of power series in which the linear term of the highest order is balanced by the nonlinear term. Using expression described in (A2) in (A1), then balancing of these terms leads to $2M + 1 = M + 3$, which gives $M = 2$. Therefore, the series expansion can be written as

$$\Phi(Y) = a_0 + a_1 Y + a_2 Y^2, \qquad \text{(A 3)}$$

which gives

$$-(u-C)\frac{\partial \Phi}{\partial \zeta} = -(u-C)a_1 - 2(u-C)a_2 Y + (u-C)a_1 Y^2 + 2(u-C)a_2 Y^3, \qquad \text{(A 4)}$$

$$A\Phi\frac{\partial \Phi}{\partial \zeta} = A\left[a_0 a_1 + \left(2a_0 a_2 + a_1^2\right)Y + (3a_1 a_2 - a_0 a_1)Y^2 + \left(2a_2^2 - a_1^2 - 2a_0 a_2\right)Y^3 - 3a_1 a_2 Y^4 - 2a_2^2 Y^5\right], \qquad \text{(A 5)}$$

$$B\frac{\partial^3 \Phi}{\partial \zeta^3} = -2a_1 B - 16a_2 BY + 8a_1 BY^2 + 40a_2 BY^3 - 6a_1 BY^4 - 24a_2 BY^5. \qquad \text{(A 6)}$$

Using these expressions in (A1) and comparing the terms with coefficients from the highest powers of $Y^{\prime s}$ and after solving the set of equations obtained from different powers of $Y^{\prime s}$ we have

$$a_0 = \frac{8B}{A} + \frac{(u-C)}{A}, \quad a_1 = 0 \quad a_2 = -\frac{12B}{A},$$

$$\Phi(\zeta) = \frac{8B}{A} + \frac{(u-C)}{A} - \frac{12B}{A}\tanh^2(\xi + \eta - u\tau). \qquad \text{(A 7)}$$

Appendix B. Pseudo-potential approach

The solution to (3.12) can also be obtained using the pseudo-potential approach. Using the transformation $\zeta = \xi + \eta - u\tau$ and integrating twice with boundary conditions, *i.e.* $\Phi \to 0$, $\partial_\zeta \Phi$, $\partial_\zeta^2 \Phi$, $\partial_\zeta^3 \Phi \to 0$ for localized solution, we have

$$-(u-C)\Phi + \frac{A}{2}\Phi^2 + B\frac{\partial^2 \Phi}{\partial \zeta^2} = 0. \qquad \text{(B1)}$$

Multiplying the above relation with $\frac{\partial \Phi}{\partial \zeta}$ and integrating it using the boundary conditions defined above, we have

$$\frac{1}{2}\left(\frac{\partial \Phi}{\partial \zeta}\right)^2 + V(\Phi) = 0, \qquad \text{(B2)}$$

where pseudo-potential is defined as

$$V(\Phi) = -\frac{(u-C)}{2B}\Phi^2 + \frac{A}{6B}\Phi^3. \qquad \text{(B3)}$$

Using (B3) in (B2) and after integration, we obtained

$$\Phi = \phi_m \sec h^2\left(\frac{\zeta}{W}\right), \qquad \text{(B4)}$$

where ϕ_m is the amplitude and W is the width of the nonlinear structure.

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