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On three-dimensional magnetosonic waves in an isothermal atmosphere with a horizontal magnetic field

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Abstract. Magnetosonic–gravity waves in an isothermal non-dissipative atmosphere, with a uniform horizontal external magnetic field have been considered in the literature in two cases: (i) 'one-dimensional' magnetosonic–gravity waves, in the case of zero horizontal wavenumber and (ii) 'two-dimensional' magnetosonic–gravity waves, in which the horizontal wave vector lies in the plane of gravity and the external magnetic field. In the present paper, an extension of case (i) is considered that is distinct from case (ii). This case (iii) is that of magnetosonic–gravity waves with a horizontal wave vector orthogonal to the plane of gravity and the external magnetic field. Since the wave fields depend only on two spatial coordinates and time, the problem could be called 'two-and-half'-dimensional. The three-dimensional magnetosonic–gravity wave propagates a magnetic field perturbation parallel to the external magnetic field, and velocity perturbations transverse to it. Elimination for the vertical velocity perturbation leads to a second-order wave equation, with four regular singularities. Three regular singularities specify (a) the wave fields at high altitude, where there are two cut-off frequencies involving the acoustic cut-off frequency; (b) the wave fields in the deep layers, where another two cut-off frequencies appear, involving both the acoustic and gravity cut-off frequencies; and (c) the transition between the two regimes, occurring across a critical layer, where one solution of the wave equation vanishes and the other has a logarithmic singularity in the amplitude and also a phase jump. The whole altitude range can be covered using the three pairs of solutions of the wave equation, obtained by expanding in Frobenius–Fuchs series about each regular singularity. The power series solutions are used to plot the wave fields, for several values of the three dimensionless parameters of the problem, namely the plasma β, frequency and wavenumber. It is shown that the presence of a horizontal wave vector transverse to the plane of gravity and the external magnetic field, can change the properties of the waves significantly: first, the two cut-off frequencies may cease to exist, in which case the full wave frequency spectrum can propagate; secondly, the critical layer occurs at different altitudes for different frequencies, allowing gradual absorption of the waves (e.g. in the solar transition region).

1. Introduction

Magnetohydrodynamic (MHD) waves in a homogeneous, compressible, ionized fluid consist [1–6] of a decoupled Alfvén mode [7–9] with perturbations transverse to the wave vector and external magnetic field, plus two coupled, compressive slow and fast modes, in the latter plane. In all cases, the dispersion relation, which is of third degree for MHD waves and second degree for compressive MHD modes, can be factorized to give roots corresponding to slow and fast modes. In the particular case of a wave vector perpendicular to the external magnetic field, there is only one compressive mode, namely a magnetosonic wave, for which the square of the phase speed is the sum of squares of the sound and Alfvén speeds. In the case of a magnetosonic–gravity wave, propagating vertically in an isothermal atmosphere, with a uniform horizontal magnetic field [10, 11], the wave equation is similar to that for acoustic–gravity waves [12, 13], adding again to the square of the sound speed, the square of the Alfvén speed. This has led to the prediction [14–17] that the same substitution would apply to the cut-off frequencies of acoustic–gravity waves. In fact [11, 18–22], the cut-off frequencies for magnetosonic–gravity waves are not affected by a horizontal external magnetic field. A vertical external magnetic field [23–25] does not change either the acoustic or the gravity cut-off frequencies, which are modified only for an oblique magnetic field [26–31], and depend on the direction of the external magnetic field, but not on its strength. Magnetosonic waves have recently been observed in the solar corona [32]. In the solar application, the assumption of a horizontal magnetic field holds over an altitude range that is small compared with the solar radius.

Concerning magnetosonic–gravity waves in an isothermal atmosphere with a uniform horizontal magnetic field, three cases can be considered:

(i) the 'one-dimensional' case of vertical propagation, or zero horizontal wavenumber [10], in which the solution is specified by hypergeometric functions of two kinds [11], which show that the acoustic cut-off frequency is retained;

(ii) the 'two-dimensional' case, with a non-zero horizontal wave vector parallel to the external magnetic field [18, 19], in which the solution is specified by hypergeometric functions of the first kind, showing [20, 22] that both the acoustic and gravity cut-offs are preserved;

(iii) the 'three-dimensional' case, considered in the present paper, in which the horizontal wave vector is transverse to the external magnetic field (Fig. 1), and several new features arise.

In this classification 'three-dimensional' is taken to mean that gravity $\mathbf{g}$, the wave vector $\mathbf{k}$ and the external magnetic field $\mathbf{B}$ are not coplanar, and thus span a three-dimensional subspace. The wave fields will be assumed to depend on time and two spatial coordinates in the plane of gravity and the wave vector; thus the problem is 'two-dimensional' in terms of coordinates. The combination of 'three-dimensional' in terms of wave (or dependent) variables, and 'two-dimensional' in terms of co-ordinates (or independent variables) can be more accurately called 'two-and-half-dimensional'. The magnetosonic–gravity wave equation [14, 22, 33, 34] shows (Sec. 2.1) that the only propagating variables are the velocity perturbations transverse to the external magnetic field and the magnetic field perturbation along the external magnetic field (Sec. 2.2). Elimination of the vertical velocity perturbation

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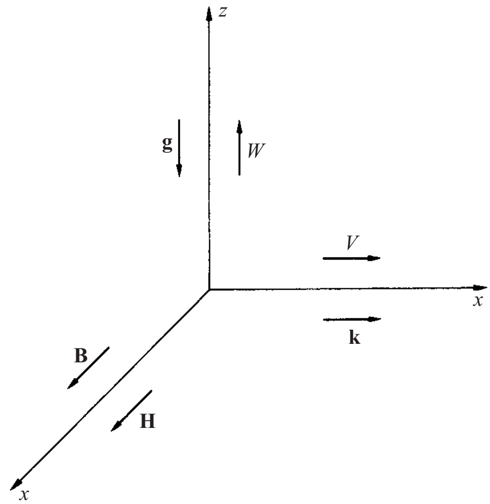

Figure 1. A three-dimensional magnetosonic–gravity wave, in an isothermal atmosphere, with uniform gravity **g** (vertically downwards) and uniform horizontal external magnetic field **B**, and with transverse horizontal wave vector **k**, propagates a magnetic field perturbation **H** parallel to the external magnetic field **B**, and transverse velocity perturbations, with vertical (W) and horizontal (V) components.

leads (Sec. 2) to a second-order wave equation, with (Sec. 2.3) four regular singularities:

(a) three singularities correspond to the critical layer (Sec. 3.3) and wave fields at high (Sec. 3.1) and low (Sec. 3.2) altitude, as in the one-dimensional case (i) and two-dimensional case (ii), for which the solution is specified by Gaussian hypergeometric functions [35–40];

(b) the fourth singularity, although it occurs for complex 'altitude', i.e. outside the physical region, implies that the wave equation cannot be reduced to the Gaussian hypergeometric type, and is in fact comparable in complexity to the Lamé or Heun equations, which have four regular singularities [41–44].

The solutions around the three regular singularities cover the whole physical region in all cases (Fig. 2), and allow the plotting of the wave fields (Figs 3–6), for several values (Sec. 4) of plasma β and dimensionless frequency and horizontal wavenumber.

In the case of '$2\frac{1}{2}$-dimensional' magnetosonic–gravity waves, there are two cut-off frequencies – either above (Sec. 3.1) or below (Sec. 3.2) the critical layer; they reduce to the acoustic cut-off frequency in the case of zero horizontal wavenumber, corresponding to vertical or 'one-dimensional' waves [10, 11]. In the case of '$2\frac{1}{2}$-dimensional' waves, the two cut-off frequencies separate non-propagating waves at intermediate frequencies, from propagating waves at low or high frequencies. The leading term of the amplitude varies like the inverse square root of the mass density, as for acoustic gravity waves, but only if the horizontal wavenumber is short; the condition under which this ceases to hold depends on (Sec. 3.1) the acoustic cut-off above the critical layer and also on the gravity cut-off below the critical layer (Sec. 3.2). The transition between these two regimes occurs across a critical layer (Sec. 3.3), where one solution of the wave equation vanishes and the other has a

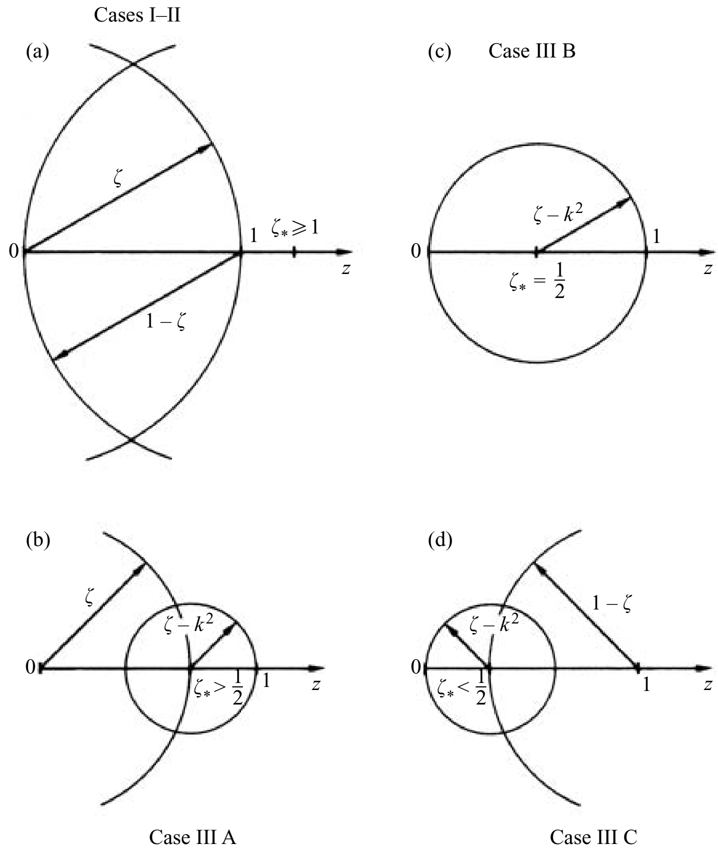

Figure 2. Since the magnetosonic–gravity wave equation has three regular singularities for finite ζ, namely (i) at the deep layers $\zeta = 1$, (ii) at high altitude $\zeta = 0$ and (iii) at the critical layer $\zeta = K^2$, depending on the value of $\zeta_* \equiv K^2$, four cases can arise (Table 3), concerning the solutions in powers of respectively (i) ζ, (ii) $1 - \zeta$ and (iii) $\zeta - K^2$ needed to cover the whole physical region $0 < \zeta < 1$, corresponding to the altitude range $-\infty < z < +\infty$.

logarithmic singularity for the amplitude and a phase jump. The existence of a critical layer [45], is common to the two-dimensional (ii) and three-dimensional (iii) linear, non-dissipative magnetosonic waves, whereas the one-dimensional case (i) corresponds to a transition layer [11, 30, 22]. No critical layer occurs for other non-dissipative magnetosonic–gravity waves; that is, Alfvén waves in an isothermal atmosphere have no critical layer [11, 21, 46–51], nor do compressive modes in a vertical or oblique magnetic field [22–25, 28, 29, 30, 52]. Critical layers do occur for Alfvén–gravity waves in two cases: (a) in the presence of the Hall effect [53, 54], if the ion gyrofrequency varies with altitude [55, 56]; (b) in the presence of viscous and resistive dissipation [57–60], using exact solutions [21, 61–65]. Since Alfvén waves are incompressible, they can be dissipated only by electrical resistance and shear viscosity [66], whereas magnetosonic–gravity waves can also be dissipated by bulk viscosity, and thermal conduction and radiation, leading to the existence of one or two critical layers [67, 68]; these are extensions of the critical layers that occur for dissipative acoustic–gravity waves [61–71].

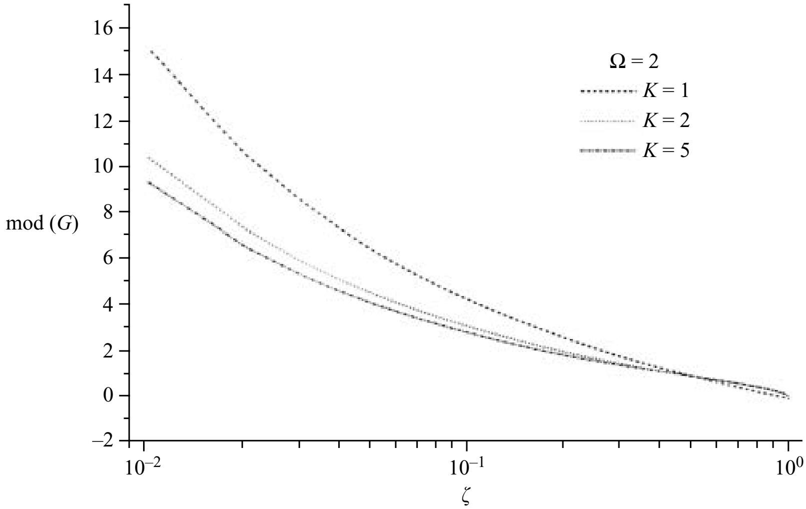

Figure 3. Modulus of the wave fields normalized to their value at the altitude of equal sound and Alfvén speeds, plotted over the physical region $0 < \zeta < 1$ for fixed dimensionless frequency $\Omega = 2$ and three values of the dimensionless wavenumber K.

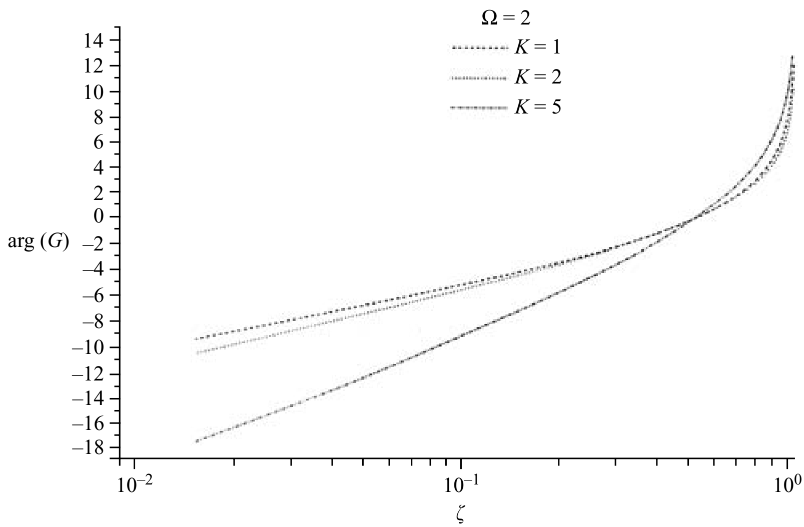

Figure 4. Phase difference between altitude z and the altitude of equal sound and Alfvén speeds, for an upward propagating wave (a downward-propagating wave has the same phase with opposite sign), plotted over the physical region $0 < \zeta < 1$ for fixed dimensionless frequency $\Omega = 2$ and three values of the dimensionless wavenumber K.

One area in which there has been more progress in Alfvén–gravity than in magnetosonic–gravity waves concerns solutions in a non-uniform external magnetic field [72–77]. Magnetosonic–gravity waves have been invoked in connection with heating of the solar chromosphere and corona [78–84], umbral oscillations [85–87] and other phenomena [88–91]. In all of these applications of magnetosonic–gravity waves to the solar atmosphere, it is assumed that Alfvén waves are decoupled at

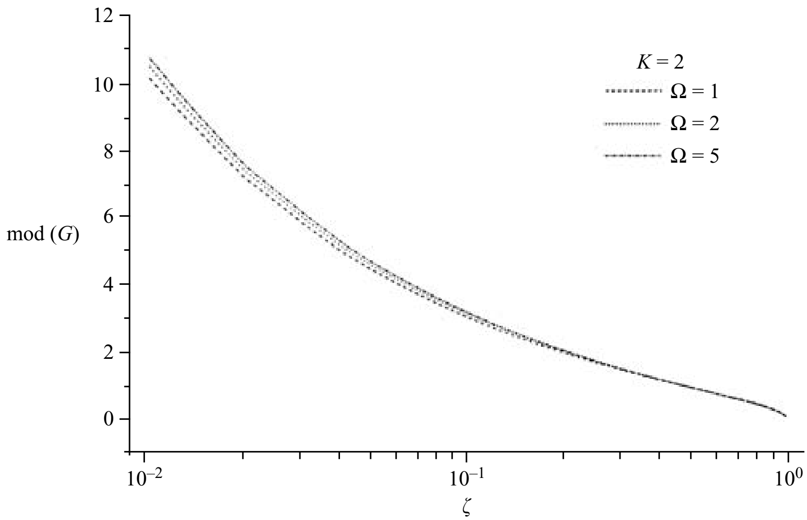

Figure 5. As in Fig. 3, but for fixed dimensionless wavenumber $K = 2$ and three values of the dimensionless frequency Ω.

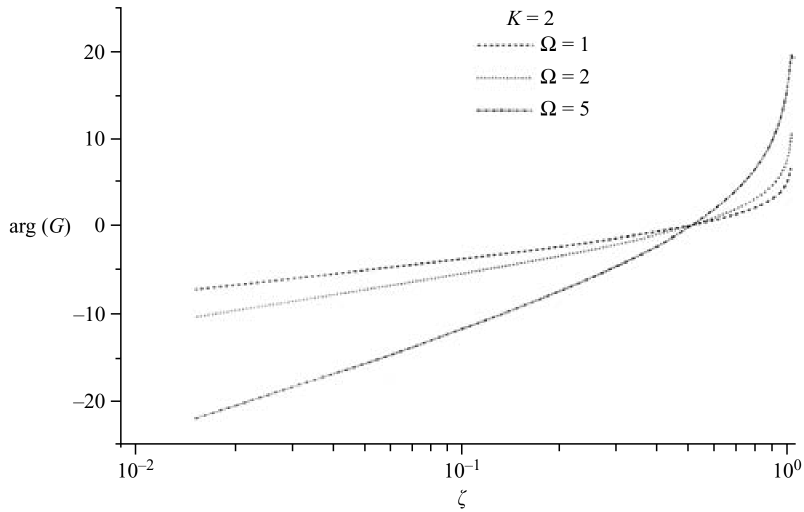

Figure 6. As in Fig. 4, but for fixed dimensionless wavenumber $K = 2$ and three values of the dimensionless frequency Ω.

a linear level from compressible slow and fast modes, as would be the case for magneto-acoustic waves in a homogeneous medium; for example, the coupling of Alfvén and compressible modes in an atmosphere has been studied as a nonlinear effect [92, 93]. In fact, Alfvén and slow and fast modes do not couple linearly in an atmosphere for 'two-dimensional' waves, for which gravity, the external magnetic field and the horizontal wave vector lie on the same plane: in this case, Alfvén waves have horizontal velocity perturbations that do not couple either to stratification or to compressibility. In contrast, for 'three-dimensional' waves with the horizontal wave vector, the external magnetic field and gravity not in the same plane, the Alfvén waves have a vertical velocity component, which couples to the compressive modes through the stratification. The linear coupling of Alfvén waves with compressive modes in an atmosphere is implied in some of the literature [14, 22, 28, 29, 91], but, to the best of our knowledge, this is the first time that this case has been addressed explicitly.

2. Wave equation for three-dimensional magnetosonic–gravity waves

The magnetosonic–gravity wave equation is considered, in an isothermal atmosphere, with a horizontal uniform magnetic field (Sec. 2.1), allowing for a transverse horizontal wave vector (Fig. 1). The propagating components of the velocity and magnetic field perturbations are determined (Sec. 2.2), and by elimination between them a second-order wave equation is obtained for the vertical velocity perturbation spectrum. It can be put into dimensionless form, involving the plasma β and a dimensionless frequency and wavenumber, allowing (Sec. 2.3) a discussion of cases of propagating and evanescent waves, and also of the conditions in which a critical layer exists.

2.1. *Linear, non-dissipative magnetosonic–gravity wave equation*

The linear, non-dissipative magnetosonic–gravity wave equation in an atmosphere, not necessarily isothermal, with an external magnetic field $\mathbf{B}$ (possibly non-uniform but steady), is [22 (with correction of misprints), 91]

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} - \frac{1}{\rho}\nabla\cdot(\rho c^2 \nabla\cdot\mathbf{v}) - \frac{1}{\rho}\nabla(\rho\mathbf{g}\cdot\mathbf{v}) + \frac{\mathbf{g}}{\rho}\nabla\cdot(\rho\mathbf{v}) + \frac{\mu}{4\pi\rho}\nabla\left\{\mathbf{v}\cdot[\mathbf{B}\times(\nabla\times\mathbf{B})]\right\}$$
$$- \frac{\mu}{4\pi\rho}\left\{\mathbf{B}\times[\nabla\times\nabla\times(\mathbf{B}\times\mathbf{v})] + [\nabla\times(\mathbf{B}\times\mathbf{v})]\times(\nabla\times\mathbf{B})\right\} = 0, \tag{1}$$

where $\mathbf{v}$ is the velocity perturbation, ρ the mean state density, c the sound speed, $\mathbf{g}$ the acceleration due to gravity and μ is the magnetic permeability. In the case of a uniform external magnetic field, this simplifies to [6, 14, 15, 34]

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} - c^2\nabla(\nabla\cdot\mathbf{v}) - \nabla(\mathbf{v}\cdot\mathbf{g}) - (\gamma-1)\mathbf{g}(\nabla\cdot\mathbf{v})$$
$$= A^2[\nabla(\nabla\cdot\mathbf{v}) - \ell(\ell\cdot\nabla)\nabla\cdot\mathbf{v} - (\ell\cdot\nabla)\nabla(\mathbf{v}\cdot\ell) + (\ell\cdot\nabla)^2\mathbf{v}], \tag{2}$$

where the equation of state of a perfect gas has been used, $\gamma \equiv C_p/C_v$ denotes the ratio of specific heats at constant pressure C_p and volume C_v, ℓ is the unit vector along the external magnetic field,

$$\ell = \frac{\mathbf{B}}{B}, \tag{3a}$$

and only the modulus of the external magnetic field appears in the Alfvén speed A,

$$A^2 \equiv \frac{\mu B^2}{4\pi\rho}. \tag{3b}$$

Denote by z the altitude, so that gravity is vertically downwards,

$$\mathbf{g} = -g\mathbf{e}_z, \tag{4a}$$

and assume the external magnetic field to be horizontal, i.e. aligned with the x axis.

$$\ell = \mathbf{e}_x. \tag{4b}$$

In the case of an isothermal atmosphere,

$$T(z) = T_0, \tag{5a}$$

the sound speed is constant,

$$c = (\gamma R T_0)^{1/2} = \left(\frac{\gamma p_0}{\rho_0}\right)^{1/2}, \tag{5b}$$

where R denotes the gas constant. In this case, the background mass density decays exponentially with altitude,

$$\rho(z) = \rho_0 \, e^{-z/L}, \tag{6a}$$

on the scale height

$$L \equiv \frac{RT_0}{g}, \tag{6b}$$

and thus the Alfvén speed (3b) increases exponentially with altitude twice the scale height,

$$A(z) = a \, e^{z/2L}, \tag{7a}$$

from a value

$$a \equiv B\left(\frac{\mu}{4\pi\rho_0}\right)^{1/2} \tag{7b}$$

at zero altitude.

Since the properties of the atmosphere vary only with altitude z, it is convenient to use a Fourier decomposition in time and horizontal coordinate,

$$\mathbf{v}(\mathbf{x}, t) = \iint_{-\infty}^{+\infty} \mathbf{V}(z; k, \omega) \, e^{i(ky-\omega t)} \, dk \, d\omega, \tag{8}$$

corresponding to a plane wave $\exp[i(ky-\omega t)]$, with amplitude $\mathbf{V}(z; k, \omega)$ depending on altitude; $\mathbf{V}$ is the velocity perturbation spectrum for a wave of frequency ω and horizontal wavenumber k at altitude z. Note that a vertical wavenumber does not exist, because the properties of the atmosphere depend on altitude, and the waves cannot be sinusoidal in that direction. It is assumed that the horizontal wave vector lies (Fig. 1) in the direction transverse to the magnetic field,

$$\mathbf{k} = k\mathbf{e}_y, \tag{9a}$$

and the usual three components of the velocity perturbation are considered,

$$\mathbf{V} \equiv (U, V, W). \tag{9b}$$

In the literature on linear non-dissipative magnetosonic–gravity waves in an isothermal atmosphere, with a uniform horizontal magnetic field, the wave vector is taken to be parallel to the field, $\mathbf{k} \parallel \mathbf{B}$ [18, 19, 20], i.e. $\mathbf{k} = k\mathbf{e}_x$, leading to a 'two-dimensional' configuration. The present paper addresses the simplest 'three-dimensional' configuration, namely (9a), corresponding to $k_x = 0 \neq k_y$.

2.2. Propagating components of velocity and magnetic field perturbations

From the magnetosonic–gravity wave equation (2), in the present configuration given by (4) and (9a,b), it follows that the velocity perturbation along the external

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magnetic field does not propagate,

$$U = 0, \tag{10a}$$

whereas the components (V, W) transverse to the external magnetic field are generally coupled for $k \neq 0$:

$$[\omega^2 - k^2(c^2 + A^2)]V = ik[g\,W - (c^2 + A^2)W'], \tag{10b}$$

$$(c^2 + A^2)W'' - \gamma g W' + \omega^2\,W = ik[g(\gamma - 1)V - (c^2 + A^2)V']. \tag{10c}$$

In the case where $k = 0$, the horizontal velocity perturbation does not propagate, i.e. (10b) becomes

$$V = 0, \tag{11a}$$

and the vertical velocity perturbation satisfies a second-order wave equation given by (10c) with zero right-hand side, i.e.

$$(c^2 + A^2)W'' - \gamma g W' + \omega^2 W = 0, \tag{11b}$$

which is solvable in terms of hypergeometric functions [11]. This solution shows that the cut-off frequency for magnetosonic–gravity waves is the same as for acoustic–gravity waves, i.e. it is not affected by the magnetic field [22]. The application of a 'dispersion relation' to (11b) suggests that the cut-off frequency depends on the magnetic field strength [14–17]. However, a dispersion relation cannot be written for (11b), because the Alfvén speed depends on altitude, (7a). The Alfvén speed could be made constant by choosing a non-uniform external magnetic field [33] related to the mass density by $B(z) \sim [\rho(z)]^{1/2}$, but in this case the wave equation (1) does not reduce to (2). The conclusion that a horizontal external magnetic field does not change the cut-off frequencies also applies in the case of the horizontal wave vector being parallel to the external magnetic field [18, 19, 20], and will be shown subsequently to also hold in the present case where the horizontal wave vector is transverse to the external magnetic field. A vertical external magnetic field does not change the cut-off frequencies [23, 25], but an oblique magnetic field does [30, 31].

The linearized induction equation:

$$\frac{\partial \mathbf{h}}{\partial t} = B(\ell \cdot \nabla)\mathbf{v} - B\ell(\nabla \cdot \mathbf{v}), \tag{12a}$$

on substitution of the spectra of the velocity and magnetic field perturbations, respectively, with amplitude $\mathbf{V}$ in (8) and $\mathbf{H}$, yields

$$i\omega\mathbf{H} = B\ell(W' + ikV), \tag{12b}$$

which shows that only the magnetic field perturbation along the external magnetic field propagates:

$$H_y = 0 = H_z, \tag{13a,b}$$

$$H_x = -i\frac{B}{\omega}(W' + ikV), \tag{13c}$$

in contrast with the velocity perturbations (10a–c). In the case of zero horizontal wavenumber, $k = 0$, (13c) simplifies to

$$H_x = -i\frac{B}{\omega}W', \tag{14}$$

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and only one component each of the velocity $V_z \equiv W$ and magnetic field perturbation H_x propagates, i.e. it is sufficient to solve (11b), and substitute the solution into (14). In the case where $k \neq 0$, the components of the velocity perturbation transverse to the external magnetic field are coupled, (10b,c), and a single magnetosonic–gravity wave equation can be obtained by eliminating between them. Thus, it is sufficient to obtain a single wave equation for $V_z \equiv W$, and substitute its solution into (10b) to determine $V_y \equiv V$ and into (13c) to determine H_x, which are the propagating components of the velocity and magnetic field perturbations (the remaining three components (10a) and (13a,b) do not propagate). Since the external magnetic field (3a), (4b) and wave vector (9a) are horizontal and perpendicular, it could be argued that the transverse velocity component, which is vertical, $W \equiv V_z$, can correspond to an Alfvén wave, which thereby couples to stratification and compressibility. The other two propagating fields, namely the velocity perturbation $V = V_y$ along the wave vector (9a) and the magnetic field perturbation H_x along the external magnetic field (4b), correspond to the compressive, fast or magnetosonic mode. Since there is no magnetic field perturbation transverse to the external magnetic field, this property of Alfvén waves is not present. Thus, the present problem concerns three-dimensional magnetosonic (or fast) waves with some properties suggesting coupling to an incomplete form of Alfvén waves. Since the slow mode is not present, a second-order wave equation should be obtained, as shown next; the appearance of a second-rather than fourth-order wave equation suggests that the fast mode is not fully coupled to the Alfvén mode. From (8) and (10a), the dilatation $\nabla \cdot \mathbf{V} = ikV + W'$ is in general non-zero, so the waves are compressible; this excludes pure Alfvén waves, and justifies the designation magnetosonic (or magnetosonic–gravity) waves used henceforth.

The starting point to obtain a wave equation is (10b) in the form

$$\alpha V = ik[gW - (c^2 + A^2)W'] = ik\delta, \tag{15}$$

where

$$\alpha \equiv \omega^2 - k^2(c^2 + A^2), \tag{16a}$$

$$\delta \equiv gW - (c^2 + A^2)W'. \tag{16b}$$

Together with (10c), this allows all wave components to be expressed in terms of the vertical velocity perturbation spectrum W, so that it is sufficient to eliminate only for the latter, using

$$\alpha V' = ik\delta' - \alpha' V, \tag{17a}$$

$$\alpha' = -2k^2 AA', \tag{17b}$$

$$\delta' = gW' - (c^2 + A^2)W'' - 2AA'W', \tag{17c}$$

and noting that the acceleration due to gravity g and sound speed c are constant, but the Alfvén speed A is not. Using (15) and (17a) to express V' in terms of W and its derivatives yields

$$\alpha^2 V' = ik\alpha\delta' - \alpha'\alpha V = ik(\alpha\delta' - \alpha'\delta). \tag{18}$$

Substituting V and V' from (15) and (18) into (10c) yields

$$\alpha^2[(c^2 + A^2)W'' - \gamma g W' + \omega^2 W] = -k^2 g(\gamma - 1)\alpha\delta + k^2(c^2 + A^2)(\alpha\delta' - \alpha'\delta), \tag{19}$$

which involves, from (16a,b) and (17b,c), only W and its derivatives, namely

$$\omega^2(c^2+A^2)[\omega^2-k^2(c^2+A^2)]W''+[k^2\omega^2(c^2+A^2)(\gamma g+2AA')-\gamma g\,\omega^4]W' + \{\omega^2[\omega^2-k^2(c^2+A^2)]^2+\omega^2k^2g^2(\gamma-1) - k^4(c^2+A^2)g[(\gamma-1)g+2A'A]\}W=0, \tag{20}$$

is the wave equation for three-dimensional magnetosonic–gravity waves, which is next put into dimensionless form. Before doing so, it is checked that this equation describes magnetosonic waves as two simple particular decoupled cases. One case is $k=0$, when (20) simplifies to (11b) for vertical magnetosonic waves. The second case concerns a homogeneous medium, for which the Alfvén speed is uniform ($A'=0$), and stratification is suppressed by neglecting gravity ($g=0$):

$$(c^2+A^2)W''+[\omega^2-k^2(c^2+A^2)]W=0. \tag{21}$$

Acoustic waves cannot be eliminated by an incompressibility condition $c\to\infty$, because then the Alfvén speed would be omitted $A^2\ll c^2$, and, to $O(c^2)$, the result $W''-k^2W=0$ would specify horizontal evanescence or divergence. Instead, the low-plasma-β limit of the sound speed being small relative to the Alfvén speed ($c^2\ll A^2$) is taken, leading to

$$W''+\left(\frac{\omega^2}{A^2}-k^2\right)W=0, \tag{22}$$

which describes fast magnetoacoustic waves [94]. This corresponds to the fast MHD mode in a low-β plasma being an Alfvén wave propagating in all directions [6]. The derivation of (22) from (20) is less simple than that of (11b), because of the need to exclude stratification and compressibility.

2.3. *Role of plasma β and dimensionless frequency and wavenumber*

Noting from (6b), (5b) and (7a) that

$$\gamma g=\frac{\gamma RT_0}{L}=\frac{c^2}{L}, \tag{23}$$

$$2A'A=(A^2)'=\frac{A^2}{L}, \tag{24}$$

and substituting in the wave equation (20), the latter becomes

$$\left(1+\beta^{-1}e^{z/L}\right)\left[1-K^2\left(1+\beta^{-1}e^{z/L}\right)\right]L^2W''+\left[K^2\left(1+\beta^{-1}e^{z/L}\right)^2-1\right]LW' + \left\{\Omega^2\left[1-K^2\left(1+\beta^{-1}e^{z/L}\right)^2\right]+\left(\frac{K}{\gamma}\right)^2(\gamma-1) - \frac{K^4}{\gamma}\left(1+\beta^{-1}e^{z/L}\right)\left(1+\beta^{-1}e^{z/L}-\frac{1}{\gamma}\right)\right\}W=0, \tag{25}$$

and involves three dimensionless parameters, namely the plasma β, defined as the ratio of squares of the sound and Alfvén speed at zero altitude,

$$\beta\equiv\frac{c^2}{a^2}, \tag{26a}$$

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the dimensionless frequency

$$\Omega \equiv \frac{\omega L}{c}, \tag{26b}$$

and the dimensionless inverse phase speed or dimensionless wavenumber

$$K \equiv \frac{kc}{\omega}.$$

It is clear that the wave equation (20) has a singularity when the coefficient of W'' vanishes, namely

$$\omega^2 = k^2\{c^2 + [A(z_*)]^2\}, \tag{27a}$$

$$A(z_*) = \left(\frac{\omega^2}{k^2} - c^2\right)^{1/2}, \tag{27b}$$

i.e. there is a critical layer at an altitude

$$\left[\frac{A(z_*)}{c}\right]^2 = \frac{1}{K^2} - 1, \tag{28a}$$

$$z_* = L\log[\beta(1/K^2 - 1)], \tag{28b}$$

which is real for $K^2 < 1$, i.e. the critical layer exists only if $k^2c^2 < \omega^2$, which implies that the 'local' vertical 'wavenumber' for an acoustic wave

$$\bar{k}^2 = \left(\frac{\omega^2}{c^2} - k^2\right) = \frac{\omega}{c}(1 - K^2)^{1/2} \tag{29}$$

is real, i.e. acoustic waves can propagate under these conditions. Thus, the criterion for the existence of a critical layer at real altitude for '$2\frac{1}{2}$-dimensional' magnetosonic–gravity waves is that acoustic waves would propagate under the same conditions.

The coefficients of the wave equation (25) are transformed from exponentials to polynomials by the change of independent variable

$$\frac{1}{\zeta} = 1 + \beta^{-1}e^{z/L} = 1 + \left[\frac{A(z)}{c}\right]^2, \tag{30a}$$

$$W(z; k, \omega) \equiv \Phi(\zeta; K, \Omega), \tag{30b}$$

which places the critical layer (28b) at the position

$$\zeta_* = K^2. \tag{31a}$$

Also, from (30a,b),

$$L\frac{d}{dz} = \zeta(\zeta - 1)\frac{d}{d\zeta}, \tag{31b}$$

and the plasma β no longer appears in the wave equation:

$$0 = \zeta^2(\zeta-1)^2(\zeta - K^2)\Phi'' + \zeta(\zeta-1)^2(\zeta - 2K^2)\Phi' + \left\{-K^2\left(\Omega^2 + \frac{K^2}{\gamma}\right) + \left(\frac{K^2}{\gamma}\right)^2\zeta + \left[\Omega^2 + \frac{(\gamma-1)K^2}{\gamma^2}\right]\zeta^2\right\}\Phi, \tag{32}$$

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Table 1. Singularities of the wave/differential equation.

Singularity	ζ	z	Type
High altitude	0	∞	Regular
Deep layers	1	$-\infty$	Regular
Critical layer	K^2	z_*	Regular
Complex	∞	z_∞	Regular

Table 2. Conditions for the existence of a critical layer.

	Case I	Case II	Case III
Acoustic waves as a 'reference'			
Spectrum	$\omega^2 < k^2c^2$	$\omega = \pm kc$	$\omega^2 > k^2c^2$
Type of wave	Vertically evanescent	Propagating horizontally	Propagating vertically
Condition	$\bar{k} = i\|\bar{k}\|$ (imaginary)	$\bar{k} = 0$ (zero)	$\bar{k} = \|\bar{k}\|$ (real)
Three-dimensional magnetosonic waves			
Condition	$K^2 > 1$	$K = \pm 1$	$K^2 < 1$
Critical layer	Does not exist	Exists at boundary	Exists in physical region
Altitude of critical layer	z^* complex	$z_* = -\infty$	z_* real

which involves only the dimensionless frequency (26b) and the horizontal wavenumber (26c). The wave equation (32) has four regular singularities, of which three specify the asymptotic wave field at high altitude $z \to \infty$ or $\zeta \to 0$ in (30a), or in the deep layers $z \to -\infty$ or $\zeta \to 1$, and also the wave field in the neighbourhood of the critical layer, (31a), (28b): see Table 1. The regular singularity for $\zeta = \infty$ corresponds, (30a), to a 'complex altitude':

$$z_\infty = L \log \beta \pm i\pi L, \tag{33}$$

which implies that it is of no physical interest, since it occurs outside the real altitude range $-\infty < z < \infty$, which is mapped by the change of variable (30a) into the unit interval $0 < \zeta < 1$. This singularity is of no mathematical interest either, because it does not limit the radius of convergence of the solutions around the other three singularities in the finite part of the z plane, as shown next.

The critical layer (28b) $\equiv$ (31a) lies in the physical region $|\zeta| < 1$ if $K^2 < 1$, i.e. the local vertical wavenumber for acoustic waves (29) is real, corresponding to vertically propagating acoustic waves (case III): see Table 2. In the case of horizontally propagating acoustic waves $\bar{k} = 0$ (case II), in (29), the critical layer lies at one end of the physical region $|\zeta| = 1$, and in the case of vertically evanescent acoustic waves (case I), it lies outside the physical region $|\zeta| > 1$. Note that it is the dispersion relation for 'local' acoustic waves that specifies whether magnetosonic–gravity waves have a critical layer at real or imaginary altitude. Note that although acoustic waves do not exist under the physical conditions indicated, they serve as a 'reference' to indicate whether there is a critical layer for three-dimensional magnetosonic–gravity waves, which do exist under the physical conditions stated.

The position of the critical layer is important for deciding which power series solutions are needed to cover the whole physical region, e.g. if (cases I and II) the

Table 3. Combination of solutions needed to cover the physical region.

Case	Fig.	Critical layer at	Expansions needed in powers of
I, II	2a	$\zeta_* = K^2 \geqslant 1$	ζ or $1-\zeta$
IIIA	2b	$\zeta_* = K^2 > \frac{1}{2}$	ζ and $\zeta - K^2$
IIIB	2c	$\zeta_* = K^2 = \frac{1}{2}$	$\zeta - K^2$
IIIC	2d	$\zeta_* = K^2 < \frac{1}{2}$	$1-\zeta$ and $\zeta - K^2$

critical layer $\zeta_* = K^2 \geqslant 1$ does not lie inside the physical region $0 < \zeta < 1$, then an expansion either in powers of ζ or $1-\zeta$ will cover the whole physical region (Table 3 and Fig. 2a). If the critical layer lies in the physical region (case III), then an expansion in powers of $\zeta - K^2 = \zeta - \zeta_*$, will cover the physical region $0 < \zeta < 1$ only (case IIIB) if $\zeta_* = \frac{1}{2}$ (Table 3 and Fig. 2c). Otherwise, if $\zeta_* \neq \frac{1}{2}$, depending on whether $\zeta_* > \frac{1}{2}$ (case IIIA) or $\zeta_* < \frac{1}{2}$ (case IIIC), then, in order to cover the whole physical region, an expansion in powers of ζ (Table 3 and Fig. 2b) or $1-\zeta$ (Table 3 and Fig. 2d), respectively, is also needed.

3. Wave fields in the neighbourhood of the three regular singularities

The solutions of the wave equation in powers of ζ, $1-\zeta$ and $\zeta - K^2$ cover the whole physical region, and specify the wave fields as follows: (Sec. 3.1) asymptotically at high altitude, where the constant magnetic pressure dominates the decaying gas pressure, so that the cut-off frequency for acoustic waves is involved; (Sec. 3.2) asymptotically in the deep layers of the atmosphere, where the increasing gas pressure dominates the magnetic pressure, so that the cut-off frequency for gravity waves is also involved; (Sec. 3.3) in the neighbourhood of the critical layer, where the transition between the predominance of gas and magnetic pressures occurs, and one wave component is singular, whereas the other has a phase jump.

3.1. Asymptotic wave field at high altitude

Since the change of variable (30a) maps 'infinite' altitude $z \to \infty$ to the origin $\zeta \to 0$, the wave field above the critical layer is specified by a power series in ζ, and, besides, since $\zeta = 0$ is a regular singularity of the differential equation (32), the solution exists as a Frobenius–Fuchs expansion:

$$\Phi_\sigma(\zeta) = \sum_{n=0}^{\infty} a_n(\sigma)\zeta^{n+\sigma} \qquad (|\zeta| \leqslant 1, \quad K^2 \equiv \zeta_*), \tag{34}$$

with coefficients $a_n(\sigma)$ and σ to be determined (Sec. 4). The index σ alone specifies the leading term of the wave field, (30a,b), at high altitude:

$$\zeta \sim \beta e^{-z/L}, \qquad \Phi_\sigma(\zeta) \sim \zeta^\sigma \sim e^{-\sigma z/L} \quad (z \gg z_*). \tag{35a,b}$$

The index σ can be determined by taking the lowest powers of ζ in the coefficients of Φ, Φ' and Φ'' in the wave equation (32), namely

$$\zeta^2\Phi'' + 2\zeta\Phi' + \left(\Omega^2 + \frac{K^2}{\gamma}\right)\Phi = 0 \qquad (\zeta \to 0). \tag{36}$$

This is an Euler equation with power solutions of the type (35b), with exponent σ satisfying

$$\sigma(\sigma-1)+2\sigma+\Omega^2+\frac{K^2}{\gamma}=0, \tag{37a}$$

which will be shown (Sec. 4) to coincide with the indicial equation:

$$\sigma^2+\sigma+\Omega^2+\frac{K^2}{\gamma}=0, \tag{37b}$$

the roots of which are given by

$$\sigma_{\pm}=-\frac{1}{2}\pm\left(\frac{1}{4}-\Omega^2-\frac{K^2}{\gamma}\right)^{1/2}, \tag{38}$$

and will be discussed next.

The indices (38) have a real part $-\frac{1}{2}$, corresponding, (35b), to acoustic–gravity waves:

$$\sigma_{\pm}=-\tfrac{1}{2}\pm ik_{+}L. \tag{39a}$$

The imaginary part involves the effective vertical wavenumber, defined by

$$k_{+}\equiv\frac{\omega}{c}\left[\left(1-\frac{\omega_{+}^2}{\omega^2}\right)\left(1-\frac{\omega_{-}^2}{\omega^2}\right)\right]^{1/2}, \tag{39b}$$

where the cut-off frequencies are specified by

$$\omega_{\pm}^2\equiv\frac{\omega_a^2}{2}\left[1\pm\left(1-\frac{64k^2L^2}{\gamma}\right)^{1/2}\right], \tag{40a}$$

in terms of the acoustic cut-off frequency

$$\omega_a\equiv\frac{c}{2L}. \tag{40b}$$

For frequencies far above the acoustic cut-off $\omega^2\gg\omega_a^2$, and hence far above $\omega_{\pm}$, namely $\omega^2\gg\omega_{+}^2$, the effective vertical wavenumber (39b) simplifies to $k_{+}\sim\omega/c$. In general, it is real above the upper cut-off ($\omega>\omega_{+}$) and below the lower cut-off ($\omega<\omega_{-}$), i.e. in these two ranges there are propagating waves. The effective vertical wavenumber vanishes ($k_{+}=0$) at both cut-off frequencies ($\omega=\omega_{\pm}$), and in between ($\omega_{-}<\omega<\omega_{+}$) is imaginary, showing that only evanescent waves exist in this range of frequencies. In the low-wavenumber limit $kL\to 0$, of the cut-off frequencies (40a), only one remains ($\omega_{-}\to 0, \omega_{+}\to\omega_a$), and it coincides with the acoustic cut-off frequency (40b), i.e. the waves propagate above the acoustic cut-off frequency ($\omega>\omega_a$) and are evanescent below ($\omega<\omega_a$). The cut-off frequencies (40a) remain real if $k^2L^2<\frac{1}{64}\gamma$, which is equivalent to

$$\tfrac{1}{8}\gamma^{1/2}>kL=K\Omega=\frac{K\omega L}{c}=\frac{K\omega}{2\omega_a}, \tag{41}$$

where (26b,c) have been used. If this condition is not met, it is simpler to calculate the indices from (38), and the real part will no longer always be $-\frac{1}{2}$.

It can be confirmed that $\omega_{\pm}$ given by (40a,b) are the cut-off frequencies, by considering the following three cases.

(i) At either of the two cut-off frequencies, the exponents (39a) coincide

$$\sigma_{\pm}=-\tfrac{1}{2}\qquad(\omega=\omega_{\pm}) \tag{42}$$

and the two components of the wave field,

$$\Phi_+(\zeta) \sim \zeta^{-1/2}, \qquad \Phi_-(\zeta) \sim \zeta^{-1/2} \log \zeta, \tag{43}$$

correspond to a vertical velocity perturbation, (30b) and (35a),

$$W_+(z) \sim e^{z/2L}, \qquad W_-(z) \sim \frac{e^{z/L} z}{L} \quad (z \to \infty), \tag{44}$$

which grows exponentially with altitude as the inverse square root of the mass density (6a), $e^{z/2L} \sim [\rho(z)]^{-1/2}$ as is the case for acoustic–gravity waves.

(ii) Between the cut-off frequencies, the exponents (39a) are real and distinct,

$$\sigma_\pm = -\tfrac{1}{2} \pm |k_+|L \qquad (\omega_- < \omega < \omega_+), \tag{45a}$$

corresponding to non-propagating waves:

$$W_\pm(z) \sim e^{z/2L} \exp(\mp|k_+|z) \qquad (z \to \infty). \tag{45b}$$

(iii) Above the upper cut-off frequency and below the lower cut-off frequency, the exponents (39a) are complex-conjugate,

$$\sigma_\pm = -\tfrac{1}{2} \pm i|k_+|L \qquad (\omega > \omega_+ \text{ or } \omega < \omega_-) \tag{46a}$$

corresponding to propagating wave fields,

$$W_\pm(z) \sim e^{z/2L} \exp(\mp i|k_+|L). \tag{46b}$$

In these three cases, it has been assumed that $kL < \frac{1}{8}\gamma^{1/2}$, so that the cut-off frequencies (40a) remain real. Otherwise, if $kL \geqslant \frac{1}{8}\gamma^{1/2}$, it is preferable to calculate the indices directly from (38), and the asymptotic wave field (35b) scales as

$$W_\pm(z) \sim \exp\left[-\frac{\mathrm{Re}(\sigma_\pm)z}{L}\right] \exp\left[-i\frac{\mathrm{Im}(\sigma_\pm)z}{L}\right] \qquad (z \to \infty), \tag{47}$$

so that the real part determines the amplitude and the imaginary part determines the phase. Whereas the leading term of (35b) specifies the asymptotic wave field at high altitude ($z \to \infty$), the whole series (34) is needed to specify the wave field exactly at finite altitude above the critical layer ($z > z_*$). The atmospheric region below the critical layer is considered next.

3.2. *Initial wave field in the deep layers of the atmosphere*

In the deep layers of the atmosphere ($z \to -\infty$), since $\zeta \to 1$ in (30a), this suggests the following change of independent variable:

$$\xi = 1 - \zeta, \tag{48a}$$

$$\Phi(\zeta; K, \Omega) = \Psi(\xi; K, \Omega), \tag{48b}$$

which transforms the differential equation (32) to give

$$\begin{aligned} &\xi^2(1-\xi)^2(1-K^2-\xi)\Psi'' - \xi^2(1-\xi)(1-2K^2-\xi)\Psi' \\ &+ \left\{(1-K^2)\left[\Omega^2 + \frac{(\gamma-1)K^2}{\gamma^2}\right] - \xi\left[2\Omega^2 + \left(\frac{K}{\gamma}\right)^2(2\gamma+K^2-2)\right]\right. \\ &\left. + \xi^2\left[\Omega^2 + \frac{(\gamma-1)K^2}{\gamma^2}\right]\right\}\Psi = 0. \end{aligned} \tag{49}$$

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Since $\zeta = 1$ or $\xi = 0$ by (32) is a regular singularity, (49) has solutions as Frobenius–Fuchs series:

$$\Psi(\xi) = \sum_{n=0}^{\infty} b_n(\vartheta)\xi^{n+\vartheta} = \sum_{n=0}^{\infty} b_n(\vartheta)(1-\zeta)^{n+\vartheta} \qquad (|\xi| < 1, |1-K^2|), \tag{50}$$

where the coefficients satisfy the recurrence relation

$$(1-K^2)\left[(n+\vartheta)(n+\vartheta-1) + \Omega^2 + \frac{(\gamma-1)K^2}{\gamma^2}\right] b_n(\vartheta) = O(b_{n-1}, b_{n-2}, b_{n-3}). \tag{51}$$

It is sufficient to write explicitly just the coefficient $b_0(\vartheta)$, because this is all that is needed to specify the indicial equation

$$\vartheta(\vartheta-1) + \Omega^2 + \frac{(\gamma-1)K^2}{\gamma^2} = 0 \qquad (n = 0) \tag{52}$$

at low altitude, which can be compared with the high-altitude case (37a,b).

In the deep layers of the atmosphere, the variable (48a) scales as

$$\xi \sim \beta^{-1} e^{z/L} \qquad (z \to -\infty) \tag{53a}$$

and the wave field (50) scales as

$$\Psi(\xi) \sim \xi^{\vartheta} \sim e^{\vartheta z} \sim e^{\vartheta z/L}, \tag{53b}$$

where the index ϑ is given by the roots of

$$\vartheta^2 - \vartheta + \Omega^2 + \frac{(\gamma-1)K^2}{\gamma} = 0, \tag{54}$$

namely

$$\vartheta_{\pm} = \frac{1}{2} \pm \left[\frac{1}{4} - \Omega^2 - \frac{(\gamma-1)K^2}{\gamma^2}\right]^{1/2}, \tag{55}$$

in comparison with the high-altitude case (38). Similarly, the indices can be written as

$$\vartheta_{\pm} = \tfrac{1}{2} \pm ik_{-}L, \tag{56a}$$

where the effective vertical wavenumber is given by

$$k_{-} \equiv \frac{\omega}{c}\left\{\left[1 - \left(\frac{\omega^{+}}{\omega}\right)^2\right]\left[1 - \left(\frac{\omega^{-}}{\omega}\right)^2\right]\right\}^{1/2}, \tag{56b}$$

in terms of the cut-off frequencies $\omega^{\pm}$,

$$(\omega^{\pm})^2 \equiv \frac{\omega_a^2}{2}\left\{1 \pm \left[1 - \frac{64k^2L^2(\gamma-1)}{\gamma^2}\right]^{1/2}\right\}, \tag{57}$$

which involve the acoustic cut-off frequency (40b). For much higher frequencies $\omega^2 \gg \omega_a^2$, so that $\omega^2 \gg (\omega^{\pm})^2$, the effective wavenumber (56b) reduces to $k_{-} \sim \omega/c$. It is real for $\omega > \omega^{+}$ and $\omega < \omega^{-}$, corresponding to propagating waves; it is imaginary for $\omega^{-} < \omega < \omega^{+}$ corresponding to evanescent waves; and it vanishes at $\omega^{\pm}$. The cut-off frequencies (57) are real for

$$\frac{1}{8} > \frac{kL}{\gamma}(\gamma-1)^{1/2} = \frac{kL^2\omega_g}{c} = \frac{kL}{2}\frac{\omega_g}{\omega_a}, \tag{58a}$$

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where (40b) has been used and the gravity cut-off frequency is defined by

$$\omega_g \equiv \frac{c(\gamma-1)^{1/2}}{L\gamma}. \tag{58b}$$

Thus, the cut-off frequencies below the critical layer are specified, from (57) and (58b), by

$$(\omega^{\pm})^2 = \frac{\omega_a^2}{2}\left\{1 \pm \left[1 - \left(\frac{4kL\omega_g}{\omega_a}\right)^2\right]^{1/2}\right\}, \tag{59}$$

and are real if the condition (58a) is met. If this condition is not met, it is simpler to calculate the indices from (55), and in general $\mathrm{Re}(\vartheta_\pm) \neq \frac{1}{2}$.

In order to confirm that (59) are cut-off frequencies, the following three cases are considered.

(i) At one of the cut-off frequencies, the exponents (56a,b) coincide,

$$\vartheta_\pm = \tfrac{1}{2} \qquad (\omega = \omega^{\pm}), \tag{60}$$

and the wave fields

$$\Psi_+(\xi) \sim \xi^{1/2}, \qquad \Psi_-(\xi) \sim \xi^{1/2}\log\xi, \tag{61}$$

or alternatively

$$W_+(z) \sim e^{z/2L}, \qquad W_-(z) \sim \frac{z}{L}e^{z/2L} \qquad (z \to -\infty), \tag{62a,b}$$

decay exponentially with decreasing altitude, as for an acoustic–gravity wave, i.e. with the inverse square root of the mass density.

(ii) Above the upper cut-off frequency and below the lower cut-off frequency, the exponents (56a,b) are complex-conjugate,

$$\vartheta_\pm = \tfrac{1}{2} \pm i|k_-|L \qquad (\omega > \omega_+ \text{ or } \omega < \omega_-), \tag{63a}$$

corresponding to propagating waves,

$$W_\pm(z) \sim e^{z/2L}\exp(\pm i|k_-|z) \qquad (z \to -\infty). \tag{63b}$$

(iii) Between the cut-off frequencies, the exponents (56a,b) are real and distinct,

$$\vartheta_\pm = \tfrac{1}{2} \pm |k_-|L \qquad (\omega^- < \omega < \omega^+), \tag{64}$$

corresponding to non-propagating waves,

$$W_\pm(z) \sim e^{z/2L}\exp(\pm|k_-|z) \qquad (z \to -\infty). \tag{65}$$

These three cases assume that the condition (58a) is met. If it is not, then it is simpler to calculate the indices from (55), and the wave field (53b) scales as

$$W^{\pm}(z) \sim \exp\left[\frac{\mathrm{Re}(\vartheta_\pm)z}{L}\right]\exp\left[i\frac{\mathrm{Im}(\vartheta_\pm)z}{L}\right], \tag{66}$$

showing that the real part determines the amplitude and the imaginary part determine the phase. The exact wave fields are given by the full series (50) below the critical layer ($z < z_*$).

3.3. Wave fields in the neighbourhood of the critical level

When the critical layer exists, $K^2 \leqslant 1$ (Table 2), neither the high-altitude solution (34), for $|\zeta| < K^2$, nor the low-altitude solution (50), for $|1-\zeta| < |1-K^2|$, hold near the critical layer (31a), and the wave fields in its neighbourhood are specified by the following change of variable:

$$\eta \equiv \zeta - K^2, \tag{67a}$$

$$\Phi(\zeta; K, \Omega) \equiv F(\eta; K, \Omega). \tag{67b}$$

This transforms the wave equation (32) to

$$\begin{aligned} 0 = {} & \eta(\eta + K^2)^2(\eta + K^2 - 1)^2 F'' + (\eta^2 - K^4)(\eta + K^2 - 1)^2 F' \\ & + \left\{ -(1-K^2)K^2\left(\Omega^2 + \frac{K^2}{\gamma}\right) + \eta K^2\left[2\Omega^2 + \frac{(2\gamma-1)K^2}{\gamma^2}\right] \right. \\ & \left. + \eta^2\left[\Omega^2 + \frac{(\gamma-1)K^2}{\gamma^2}\right]\right\} F. \end{aligned} \tag{68}$$

Since $\zeta = K^2$ or $\eta = 0$ is a regular singularity of the differential equation (32) or (68), a solution exists as a Frobenius–Fuchs series:

$$F_\mu(\eta) = \sum_{n=0}^{\infty} d_n(\mu)\eta^{n+\mu}, \tag{69}$$

whose coefficients satisfy the recurrence relation

$$[(n+\mu)^2 - 1]K^4(K^2-1)^2 d_{n+1}(\mu) = O(d_n, d_{n-1}, d_{n-2}, d_{n-3}), \tag{70}$$

where only the terms needed to specify the indicial equation

$$\mu(\mu - 2)d_0(\mu) = 0 \qquad (n = -1) \tag{71}$$

have been written. This equation has roots $\mu = 0$ and 2.

The higher index $\mu = 2$ corresponds to a solution that vanishes at the critical layer:

$$F_2(\eta) = \sum_{n=0}^{\infty} d_n(2)\eta^{n+2} = O(\eta^2). \tag{72}$$

Recalling (67a), (31a) and (30a),

$$\eta = \frac{1}{1 + \beta^{-1}e^{z/L}} - \frac{1}{1 + \beta^{-1}e^{z_*/L}}, \tag{73}$$

which leads to

$$\eta = \frac{\beta^{-1}(e^{z_*/L} - e^{z/L})}{(1 + \beta^{-1}e^{z_*/L})(1 + \beta^{-1}e^{z/L})}. \tag{74}$$

In the neighbourhood of the critical layer, this scales as

$$\eta \sim \frac{K^2(1-K^2)(z_* - z)}{L} \qquad ((z_* - z)^2 \lll L^2), \tag{75}$$

and thus, (67b), (30b) and (72), the wave field vanishes as the square of the distance from the critical layer, divided by the scale height:

$$W_{-2}(z) \sim \left(\frac{z_* - z}{L}\right)^2 \qquad (z \to z_*). \tag{76}$$

The lower index differs from the higher one by an integer, and thus it is necessary to examine the recurrence relation (70) for the coefficients of the solution (69) of the differential equation (68) to the next order,

$$\begin{aligned}&[(n+\mu)^2 - 1]\, K^4 (K^2-1)^2 d_{n+1}(\mu)\\ &\quad + K^2(K^2-1)\left\{\Omega^2 + \frac{K^2}{\gamma} - 2K^2(n+\mu)[1-(2K^2-1)(n+\mu+1)]\right\} d_n(\mu)\\ &= O(d_{n-1}, d_{n-2}, d_{n-3}),\end{aligned} \tag{77}$$

so as to go one step beyond the indicial equation (71):

$$(\mu^2-1)K^2(K^2-1)d_1(\mu) + \left\{\Omega^2 + \frac{K^2}{\gamma} - 2K^2\mu[1-(\mu+1)(2K^2-1)]\right\} d_0(\mu) = 0, \qquad (n=0). \tag{78}$$

For the index $\mu = 2$, it follows from (78) that $d_1(2)$ is determined by $d_0(2)$, and likewise for $d_n(2)$ with $n = 2, 3, \ldots$. For the lower index $\mu = 0$, it follows from (78) that $d_1(0)$ is determined from $d_0(0)$, but (77) with $n = 1$ and $\mu = 0$ implies that $0 \cdot d_2(0) \neq 0$ so that $d_2(0) = \infty$. It is well known that in this case, a second solution is obtained [35, 95] by taking the limit

$$F_0(\eta) = \lim_{\mu\to 0} \frac{\partial}{\partial\mu} F_\mu(\eta) = \lim_{\mu\to 0}\frac{\partial}{\partial\mu}\sum_{n=0}^{\infty} d_n(\mu)\eta^{\mu+n}, \tag{79}$$

and involves a logarithmic term

$$F_0(\eta) = \log\eta \sum_{n=0}^{\infty} d_n(0)\eta^n + \sum_{n=0}^{\infty} d_n'(0)\eta^n. \tag{80}$$

Thus the wave field has a logarithmic singularity at the critical layer:

$$W_0(z) \sim d_0'(0) + d_0(0)\log\left(\frac{z_* - z}{L}\right) \qquad (z \to z_*) \tag{81}$$

and the logarithmic term also causes a phase jump.

Since the critical layer lies in the physical region for $K^2 < 1$ (Table 2), the logarithmic term in (81) is

$$\log\eta \sim \log\left(\frac{z_* - z}{L}\right) = \log\left(\frac{|z_* - z|}{L}\right) - \begin{cases} 0 & (z < z_*), \quad \text{(82a)} \\ \vartheta\pi & (z > z_*), \quad \text{(82b)}\end{cases}$$

where the phase is determined by the sign of $\operatorname{Im}(\eta)$, which is the same as the sign of $\operatorname{Im}(K^2)$, with K given by (26c):

$$\eta = K^2\left(\frac{\zeta}{K^2} - 1\right) = K^2\left(\frac{1+\beta^{-1}e^{z_*/L}}{1+\beta^{-1}e^{z/L}} - 1\right). \tag{83}$$

There are two possibilities.

(i) If the frequency is given a small positive imaginary part,

$$\omega = \bar{\omega} + i\delta, \tag{84a}$$

corresponding to slow growth in time,

$$e^{-i\omega t} = e^{-i\bar{\omega} t} e^{t\delta}, \tag{84b}$$

then

$$K^2 = \left(\frac{kc}{\omega}\right)^2 = \left(\frac{kc}{\bar{\omega}}\right)^2 \left(1 - \frac{2i\delta}{\bar{\omega}}\right) \tag{85}$$

has a negative imaginary part, $\mathrm{Im}(K^2) < 0$, justifying the presence of $-i\pi$ in (82b).

(ii) If the horizontal wavenumber is given a small negative imaginary part,

$$k = \bar{k} - i\vartheta, \tag{86a}$$

corresponding to slow spatial growth,

$$e^{iky} = e^{i\bar{k}y} e^{\nu y}, \tag{86b}$$

then

$$K^2 = \left(\frac{kc}{\omega}\right)^2 = \left(\frac{kc}{\omega}\right)^2 \left(1 - \frac{2i\vartheta}{\bar{k}}\right) \tag{87}$$

again has a negative imaginary part, $\mathrm{Im}(K^2) < 0$, leading once more to the presence of $-i\pi$ in (82b).

4. Effects of dimensionless frequency and horizontal wavenumber on waveforms

In cases I and II (Table 2), in which the critical layer does not lie inside the physical region $K^2 \geqslant 1$, the solution (34) covers the whole physical region, and its coefficients satisfy a recurrence relation, obtained by substitution into (32):

$$\begin{aligned} &K^2[(n+\sigma)(n+\sigma+1) + \Omega^2 + K^2 lc^2/\gamma] a_n(\sigma) \\ &= \left\{(n+\sigma-1)[1+4K^2+(1+2K^2)(n+\sigma-2)] + \frac{K^4}{\gamma^2}\right\} a_{n-1}(\sigma) \\ &\quad - \left\{(n+\sigma-2)[2(1+K^2)+(2+K^2)(n+\sigma-3)] - \Omega^2 - \left(\frac{K}{\gamma}\right)^2(\gamma-1)\right\} \\ &\quad \times a_{n-2}(\sigma) + (n+\sigma-3)^2 a_{n-3}(\sigma). \end{aligned} \tag{88}$$

Setting $n = 0$ yields the indicial equation

$$\left[\sigma(\sigma+1) + \Omega^2 + \frac{K^2}{\gamma}\right] a_0(\sigma) = 0, \tag{89}$$

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Table 4. Values of dimensionless parameters for plots of wave fields.

Wave field	$\Omega = 2$; $K = 1, 2, 5$	$K = 2$; $\Omega = 1, 2, 5$
$\lvert G\rvert \equiv \lvert G_\pm\rvert$	Fig. 3	Fig. 5
$\arg(G) = \pm\arg(G_\pm)$	Fig. 4	Fig. 6

which coincides with (37b), and has roots (38) corresponding to the solutions

$$\Phi_\pm(\zeta) = \sum_{n=0}^{\infty} a_n(\sigma_\pm)\zeta^{n+\sigma_\pm}. \tag{90}$$

The complete wave field is given by

$$\Phi(\zeta) = C_+\Phi_+(\zeta) + C_-\Phi_-(\zeta), \qquad a_0(\sigma_\pm) = 1, \tag{91}$$

where $C_\pm$ replace $a_0(\sigma_\pm)$ as arbitrary constants. They are determined from boundary conditions. For example, specifying the wave field at two altitudes, or its scaling at high altitude (Sec. 3.1), low altitude (Sec. 3.2) or in the neighbourhood of the critical layer (Sec. 3.3).

The plotting over the physical region concerns both wave fields:

$$G_\pm(\zeta) \equiv \frac{\Phi_\pm(\zeta)}{\Phi_\pm(\frac{1}{2})} \qquad (0 \leqslant \zeta < 1), \tag{92a,b}$$

normalized to their value at the altitude corresponding to $\zeta = \frac{1}{2}$ in (30a), for which the sound and Alfvén speeds are equal, $A(z_2) = c$:

$$z_2 = 2L\log\left(\frac{c}{a}\right) = L\log\beta, \tag{93}$$

where the plasma β was introduced in (26a). The choice $K \geqslant 1$ ensures that the critical layer is outside the physical region, and K is given three values (Table 4), and Ω is also given three values. The adiabatic exponent $\gamma = \frac{5}{3}$ is taken for a monatomic gas, such as ionized hydrogen in the solar corona. Of the three dimensionless parameters (26a–c) specifying the propagation of three-dimensional magnetosonic–gravity waves in an isothermal atmosphere with a horizontal magnetic field, the plasma β (26a) is included in the variable (30a), which replaces altitude; thus, in the Table 4, only the dimensionless frequency (26b) and dimensionless horizontal wavenumber (26c) are specified. The values chosen in Table 4 imply that the indices (38) are complex-conjugate, and hence, from (88), so are the coefficients $a_n(\sigma_\pm)$ in the solutions (90):

$$(\sigma_+)^* = \sigma_-, \tag{94a}$$

$$[a_n(\sigma_+)]^* = a_n(\sigma_-), \tag{94b}$$

$$[\Phi_+(\zeta)]^* = \Phi_-(\zeta). \tag{94c}$$

Thus the wave fields (92a,b) to be plotted are complex-conjugates, $G(\zeta) \equiv G_+(\zeta) = [G_-(\zeta)]^*$, i.e. they have the same modulus or amplitude,

$$|G(\zeta)| \equiv |G_+(\zeta)| = |G_-(\zeta)|, \tag{95a}$$

and opposite phases

$$\arg[G(\zeta)] = \arg[G_+(\zeta)] = -\arg[G_-(\zeta)], \tag{95b}$$

so that it is sufficient to plot $|G|$ and $\arg(G)$ in each figure.

When interpreting Figs 3–6, it should be borne in mind that ζ is (30a) plotted on a logarithmic scale

$$\log\zeta = -\log(1+\beta^{-1}e^{z/L}) \sim \log\beta - \frac{z}{L}$$

for large altitude $z \gg L\log\beta \equiv z_1$ or small $\zeta \ll 1$, so that in this case it becomes a linear function of altitude z divided by the scale height L. The range $10^{-2} \leqslant \zeta \leqslant 1$ extends from $\zeta_a = 1$ at altitude $z_a = -\infty$ to $\zeta_b = 10^{-2}$ at altitude $z_2 = z_1 + 4.60L$; for example, for $\beta = 1$, this extends from $-\infty$ up to an altitude of about five scale heights, $z_2 = 4.6L$. Note that if the dimensionless altitude z/L was used as an ordinate, then there would be a third parameter, namely the plasma β from (26a); the latter is included in the modified altitude variable ζ in (30a), leaving only two parameters Ω and K in (26b,c), of which the latter alone specifies, through (31a), the location of the critical layer. The other advantage of using ζ rather than z is that an infinite altitude range $-\infty < z < z_2$ corresponds to a finite interval $-2 < \log\zeta < 0$. There is a correspondence (30a) between the original z and modified ζ altitude variables:

$$z = z_1 + L\log\left(\frac{1}{\zeta} - 1\right), \tag{96a}$$

$$z_1 \equiv L\log\beta; \tag{96b}$$

for example, for the range of values of the ordinate in Figs 3–6, this correspondence is as follows:

$$\begin{array}{ccccc} \zeta = 1 & 0.5 & 0.1 & 0.05 & 0.01 \\ z = -\infty & z_1 & z_1 + 2.20L & z_1 + 2.94L & z_1 + 4.60L. \end{array} \tag{97}$$

The amplitude of the wave fields (95a) increases with altitude more rapidly for shorter dimensionless horizontal wavenumber K, as can be seen in Fig. 3, where K is given three values, and the dimensionless frequency is kept fixed at $\Omega = 2$. Keeping the dimensionless wavenumber fixed at $K = 2$ and giving three values to the dimensionless frequency, it can be seen from Fig. 5 that the growth of wave amplitude with altitude is slightly larger for higher dimensionless frequency, but the effect is much weaker than that of increasing the dimensionless horizontal wavenumber. Concerning the phase, it has opposite signs for the two wave fields (95b), i.e. one (G_+) corresponds to upward propagation, and the other (G_-) to downward propagation. The variation of phase with altitude is more rapid for increasing dimensionless horizontal wavenumber (Fig. 4) and increasing dimensionless frequency (Fig. 6), with the latter having a slightly more profound effect. This contrasts with the amplitude growth with altitude, which was mostly affected by increasing dimensionless horizontal wavenumber (Fig. 3), and was not too sensitive to dimensionless frequency. The critical layer corresponds through (31a) to $\zeta = K^2$, and thus lies in the physical region $0 < \zeta < 1$ only if $K < 1$. The plots in Figs 3–6 concern cases ($K \geqslant 1$) in which the critical layer does not exist. The properties of the wave fields at the critical layer (Sec. 3.3) when the horizontal wavenumber is orthogonal to the horizontal external magnetic field are broadly similar to the well-known case when they are parallel [18, 19, 20]. The common instance is the case of zero wavenumber [10, 11], when there is a transition layer where the wave field is not singular; across this transition layer, the magnetosonic–gravity wave evolves gradually from an 'almost-acoustic' wave dominated by the gas pressure to an 'almost-longitudinal hydromagnetic' wave dominated by the magnetic pressure [22, 91]. The main difference in the present case of a wave vector not in the plane of gravity and the external magnetic field is that the magnetosonic–gravity wave couples to some properties of the Alfvén–gravity wave, with significant implications, for example, as for the solar atmosphere, which are discussed qualitatively next.

5. Discussion

It has been argued in the literature for about a quarter of a century that compressive MHD modes in an atmosphere (e.g. in the solar case) cannot transport significant energy over several scale heights because propagation is possible only above the cut-off frequency $\omega_b = (c^2 + a^2)^{1/2}/2L$ and there is not enough energy in this range of the spectrum. Actually, the cut-off frequency for magnetosonic–gravity waves in a horizontal magnetic field is the same as for acoustic–gravity waves, namely $\omega_a = c/2L$, also assuming that the horizontal wave vector lies in the plane of gravity and the external magnetic field [18, 20]. In the case of a vertical external magnetic field, there is no cut-off frequency [23, 25]. These two extremes are particular cases of [31] the cut-off frequency $\omega_c = (c/2L)\cos\theta$ for an external magnetic field making an angle θ to the vertical, i.e. $\omega_c = 0$ for a vertical magnetic field ($\theta = 0$), and $\omega_c = c/2L = \omega_a$ for a horizontal external magnetic field ($\theta = \frac{1}{2}\pi$). For an oblique external magnetic field ($0 < \theta < \frac{1}{2}\pi$), the cut-off frequency for compressive (slow and fast) magnetosonic–gravity waves is lower than the acoustic–gravity wave cut-off ($\omega_c < c/2L \equiv \omega_a$), allowing propagation of a larger part of the spectrum. All of these results, appearing in the literature on linear, non-dissipative magnetosonic–gravity waves in an isothermal atmosphere assume that the direction of stratification (or gravity $\mathbf{g}$), the external magnetic field $\mathbf{B}$ and the horizontal wave vector $\mathbf{k}$ lie in the same plane, $\mathbf{k} \cdot (\mathbf{B} \times \mathbf{g}) = 0$, in which case the Alfvén mode is decoupled from slow and fast waves.

The observation that Alfvén waves can couple linearly to slow and fast modes in an atmosphere is implicit or explicit in some of the literature, for example.

(i) The local dispersion relation for magnetosonic–gravity waves [15] shows that Alfvén waves couple to slow and fast modes if the horizontal wave vector has a component $k_\perp \neq 0$ out of the plane of $\mathbf{B}$ and $\mathbf{g}$.

(ii) The second-order vector equation for magnetosonic–gravity waves can be eliminated generally [28] as a sixth-order scalar wave equation, implying that, in general, all three modes (Alfvén, slow and fast) are coupled.

(iii) It has been mentioned explicitly [22] that the condition $\mathbf{k} \cdot (\mathbf{B} \times \mathbf{g}) = 0$ allows decoupling of the slow and fast modes from Alfvén waves in an atmosphere.

It is well known that for magnetosonic waves in a homogeneous medium, Alfvén waves are always decoupled from slow and fast modes at a linear level, even in the presence of viscous and resistive dissipation [66]; their coupling is a nonlinear effect. This nonlinear coupling has also been considered for MHD waves in an atmosphere

[92, 93]. However, in an atmosphere, *linear* coupling of Alfvén waves to slow and fast modes is also possible.

The cause of linear coupling or decoupling of Alfvén waves and compressive MHD modes in homogeneous media and atmospheres can be readily understood with the help of diagrams. In the simplest case I of magneto-acoustic waves in a homogeneous medium, there are only two vectors, the external magnetic field **B** and the wave vector **k**, which have arbitrary directions (Fig. 7a); the velocity (and magnetic field) perturbations of compressible (slow and fast) modes lie in the plane of **k** and **B** and hence are decoupled from Alfvén waves, which have perturbations transverse to **k** and **B**. In the case of magneto-acoustic gravity waves in an atmosphere there is a third direction, namely that of stratification, identified with the direction of the gravitational field **g**; besides, in this case, there exists only a horizontal wave vector **k**, transverse to the direction of stratification ($\mathbf{k} \cdot \mathbf{g} = 0$), since sinusoidal oscillations cannot exist in the vertical direction, owing to the non-uniform wave speed. In case II, when **k**, **g** and **B** are coplanar (Fig. 7b), the Alfvén waves have horizontal perturbations orthogonal to this plane, and thus do not couple to stratification or compressive modes, whose perturbations lie in the plane of **k**, **g**, and **B**. In the general case III when **k** and **B** do not lie in a vertical plane, the velocity perturbation of the Alfvén waves, which is transverse to **k** and **B**, has (Fig. 7c) a vertical component, which couples through the stratification to the compressive slow and fast modes.

To the best of our knowledge, this paper contains the first solution of the wave equation for linear, non-dissipative magnetosonic–gravity waves in which **k**, **g** and **B** are *not* coplanar. The simplest three-dimensional (or '$2\frac{1}{2}$-dimensional') configuration was chosen (Fig. 1), with the three vectors forming an orthogonal triad (4a,b). In spite of the 'simple' geometry, the scalar wave equation has *four* singularities, compared, in the case of coplanar **k**, **g** and **B**, with (i) two singularities for a non-horizontal magnetic field (namely altitude plus or minus infinity) and (ii) three singularities for a horizontal magnetic field, because there is a critical layer in this case. In case (ii), the scalar wave equation is of second order, and since it has three regular singularities, the solution can be obtained in terms of Gaussian hypergeometric functions [18, 19, 20]. In the present case, the scalar wave equation (32) is also of second order, but it has a fourth singularity due to the three-dimensional effect. Since the wave equation in the present case of three-dimensional magnetosonic–gravity waves has four singularities, its solution could at best be obtained in terms of Lamé [41, 43] or Heun [42, 44] functions. It is simpler to solve the scalar wave equation (32) using the Frobenius–Fuchs method directly [35, 41, 43, 95], and this leads to cut-off frequencies that are distinct at high and low altitude ((40a) and (59), respectively). They simplify to the acoustic cut-off if the component of the horizontal wave vector orthogonal to the plane of **g** and **B** is zero ($k = 0$); if k is small ((41) at high altitude and (58a) at low altitude), the cut-off frequencies differ from the acoustic cut-off frequency. For larger k, there are no real cut-off frequencies to hinder wave propagation.

The above results have significant implications concerning energy transport by waves in an atmosphere, since they remove several of the constraints on the energy flux, for example those associated with cut-off frequencies or decoupled modes. The coupling of Alfvén and compressive modes shows that it is not possible to exclude one in isolation; the presence of Alfvén waves will generally imply the presence of compressive modes as well. The absence of cut-off frequencies will allow

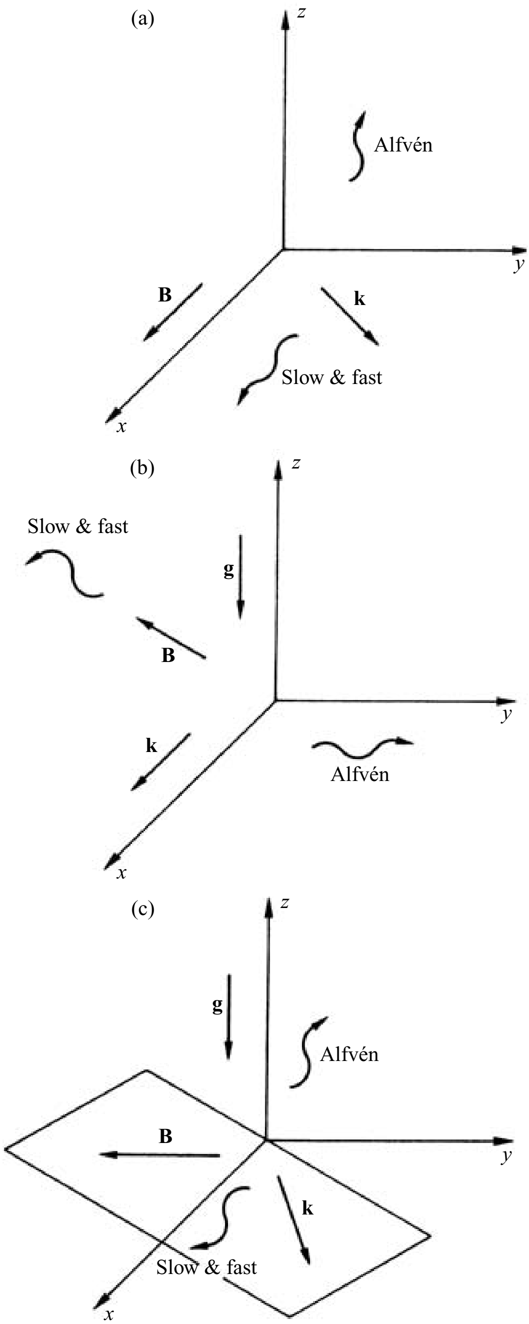

Figure 7. Cases of linear decoupling or coupling of Alfvén waves with compressive (slow and fast) modes: (a) decoupling for magneto-acoustic waves in a homogeneous medium for arbitrary direction of the wave vector $\mathbf{k}$ and the external magnetic field $\mathbf{B}$; (b) decoupling for magneto-acoustic gravity waves in an atmosphere stratified in the direction of gravity $\mathbf{g}$, if the horizontal wave vector $\mathbf{k}$ lies in the plane of $\mathbf{g}$ and $\mathbf{B}$; (c) coupling for magneto-acoustic gravity waves with the three vectors not coplanar ($\mathbf{k} \cdot (\mathbf{g} \times \mathbf{B}) \neq 0$).

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Table 5. Wave period for occurrence of a critical layer.

	Chromosphere: $T = 5\times10^3$ K, $c = 8.3\times10^5$ cm s^{-1}	Transition region: $T = 1\times10^5$ K $c = 3.7\times10^6$ cm s^{-1}	Corona: 1.8×10^6 K 1.6×10^7 cm s^{-1}
Flux tube ($\lambda = 10^7$ cm)	1.2×10 s	2.7 s	0.63 s
Granule ($\lambda = 3\times10^9$ cm)	3.6×10^3 s	8.1×10^2 s	1.9×10^2 s
Supergranule ($\lambda = 3\times10^{10}$ cm)	3.6×10^4 s	8.1×10^3 s	1.9×10^3 s

energy propagation over the full frequency spectrum. Another important result is the occurrence of a critical layer, where wave reflection and/or absorption can occur, at an altitude such that $K_* = 1$ in (26c) or $\omega_* = k_* c$. Thus, the wave period for which a critical layer occurs,

$$\tau_* = \frac{2\pi}{\omega_*} = \frac{2\pi}{kc} = \frac{\lambda}{c}, \tag{98}$$

is indicated in Table 5 for sound speeds c corresponding to temperatures T in the solar chromosphere, transition region and corona, and for wavelengths λ corresponding to the scales of a flux tube, granule and supergranule.

It is clear that a critical layer would occur in the chromosphere for large scales only for long periods; however, in the transition region and corona, there is a critical layer for most of the spectrum 30–8000 s where solar energy is concentrated in the form of hydromagnetic waves.

Earlier studies of magnetosonic–gravity waves in an atmosphere have assumed – in most cases implicitly rather than explicitly – that gravity, the external magnetic field and the horizontal wave vector are coplanar; this is an unnatural restriction (for example, there is no obvious reason why the horizontal wave vector should always lie in the plane of gravity and the magnetic field in the case of the solar atmosphere). The consideration of general geometries, with non-coplanar gravity, external magnetic field and wave vector implies coupling of all three MHD modes: Alfvén, slow and fast. It follows that energy transport and dissipation by hydromagnetic waves can be enhanced in three ways: (i) by removing cut-off frequencies, which apply to slow and fast modes when decoupled from Alfvén waves; (ii) by adding to viscous and resistive dissipation of Alfvén modes, the thermal conductive and radiative damping of the coupled compressive modes; (iii) by causing the occurrence of critical layers, where waves can be absorbed, at different altitudes for different wave frequencies, for most of the energy spectrum of hydromagnetic waves in the solar atmosphere.

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