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Spectral distribution of symmetrized circulant matrices

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Abstract. We provide a description of the spectrum and compute the eigenvalues distribution of circulant Hankel matrices obtained as symmetrization of classical Toeplitz circulant matrices. Other types of circulant matrices such as simple and Cesàro circulant matrices are also considered.

1 Introduction

Let $a_0,\ a_1,\dots,\ a_{n-1}$ be n real numbers. We denote by $\operatorname{circ}(a_0,\dots,a_{n-1})$ the circulant matrix defined as

$$\operatorname{circ}(a_0,\dots,a_{n-1}) = [a_{i-j\,(\mathrm{mod}\,n)}]_{i,j=1}^{n}.$$

More explicitly, we have

$$\operatorname{circ}(a_0,\dots,a_{n-1}) = \begin{bmatrix} a_0 & a_{n-1} & \cdots & a_1 \\ a_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1} \\ a_{n-1} & \cdots & a_1 & a_0 \end{bmatrix}. \tag{1.1}$$

It is well known that $\operatorname{circ}(a_0,\dots,a_{n-1})$ is diagonalizable by the (unitary) Fourier matrix

$$F_n = \left[\frac{\omega_n^{jk}}{\sqrt{n}}\right]_{j,k=0}^{n-1} \qquad (\omega_n = e^{2\pi i/n}).$$

That is,

$$\operatorname{circ}(a_0,\dots,a_{n-1}) = F_n^* \operatorname{diag}\left(a(1), a(\omega_n),\dots,a(\omega_n^{n-1})\right) F_n, \tag{1.2}$$

where $a(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1}$. In particular, $\operatorname{circ}(a_0,\dots,a_{n-1})$ is normal, and its eigenvalues are given by

$$a(1),\ a(\omega_n),\ a(\omega_n^2),\dots,\ a(\omega_n^{n-1}). \tag{1.3}$$

We refer the interested reader to the excellent books [2, 4, 6, 8] for the derivation of the aforementioned results and other facts about circulant matrices. A friendly exposition with some applications is also available in [7].

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The main purpose of this paper is to compute the spectral distribution of families of Hankel matrices associated to $\mathrm{circ}(a_0, \ldots, a_{n-1})$ and to other circulant matrices such as simple and Cesàro circulant matrices (see Section 3).

To this end, let $\mathcal{W}_+$ be the positive Wiener Algebra consisting of all functions on the unit circle $\mathbb{T} = \{z : |z| = 1\}$ with absolutely convergent Fourier series. That is, $a \in \mathcal{W}_+$ if it can be written as

$$a(z) = \sum_{k=0}^{\infty} \hat{a}_k z^k \quad (z \in \mathbb{T}) \qquad \text{with} \qquad \|a\|_{\mathcal{W}_+} = \sum_{k=0}^{\infty} |\hat{a}_k| < \infty. \tag{1.4}$$

We denote by a_n the nth partial sum of a, i.e.,

$$a_n(z) = \hat{a}_0 + \hat{a}_1 z + \cdots + \hat{a}_{n-1} z^{n-1}. \tag{1.5}$$

Throughout the rest of the paper, we adopt the shortened notation $\mathrm{circ}_n(a)$ for $\mathrm{circ}(\hat{a}_0, \ldots, \hat{a}_{n-1})$ and refer to a as the symbol of $\mathrm{circ}_n(a)$. More general symbols will be considered for special types of circulant matrices in Section 3.

The eigenvalues of $\mathrm{circ}_n(a)$ can now be written as

$$a_n(1),\ a_n(\omega_n), \ldots, a_n(\omega_n^{n-1}).$$

From these expressions, one can easily obtain the spectral distribution of $\mathrm{circ}_n(a)$. Indeed, a simple exercise using Riemann sums and the uniform convergence of a_n to a show that

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(a_n(\omega_n^k)) &= \lim_{n\to\infty} \frac{\mathrm{Trace}[\varphi(\mathrm{circ}_n(a))]}{n} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(a(e^{i\theta}))\, d\theta \end{aligned}$$

for every continuous function φ on the closed disk $\{z : |z| \le \|a\|_{\mathcal{W}_+}\}$.

Now, let J_n be the n-by-n antidiagonal matrix given by

$$J_n = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & .\cdot^{\cdot} & .\cdot^{\cdot} & 0 \\ 0 & .\cdot^{\cdot} & .\cdot^{\cdot} & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix},$$

and let $J_n \mathrm{circ}_n(a)$ and $\mathrm{circ}_n(a) J_n$ be the circulant Hankel matrices obtained from the multiplication of J_n and the classical circulant matrix $\mathrm{circ}_n(a)$. That is,

$$J_n \mathrm{circ}_n(a) = \begin{bmatrix} \hat{a}_{n-1} & \cdots & \hat{a}_1 & \hat{a}_0 \\ \vdots & .\cdot^{\cdot} & .\cdot^{\cdot} & \hat{a}_{n-1} \\ \hat{a}_1 & .\cdot^{\cdot} & .\cdot^{\cdot} & \vdots \\ \hat{a}_0 & \hat{a}_{n-1} & \cdots & \hat{a}_1 \end{bmatrix} = \left[\hat{a}_{1-i-j\,(\mathrm{mod}\, n)}\right]_{i,j=1}^{n} \tag{1.6}$$

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and

$$\text{(1.7)} \qquad \mathrm{circ}_n(a)J_n = \begin{bmatrix} \hat{a}_1 & \cdots & \hat{a}_{n-1} & \hat{a}_0 \\ \vdots & \iddots & \iddots & \hat{a}_1 \\ \hat{a}_{n-1} & \iddots & \iddots & \vdots \\ \hat{a}_0 & \hat{a}_1 & \cdots & \hat{a}_{n-1} \end{bmatrix} = \left[\hat{a}_{i+j-1\,(\mathrm{mod}\,n)}\right]_{i,j=1}^{n}.$$

Under the assumption that $\hat{a}_k \in \mathbb{R}$, for all k, the matrices $J_n\mathrm{circ}_n(a)$ and $\mathrm{circ}_n(a)J_n$ are symmetric, although $\mathrm{circ}_n(a)$ is generally not. For this reason, $J_n\mathrm{circ}_n(a)$ and $\mathrm{circ}_n(a)J_n$ have also been called symmetrized circulant matrices [5, 9]. These matrices are also often referred to as reversed or backward circulant matrices, and anticirculant matrices.

In recent papers [3, 5, 9], the spectral distribution of families of symmetrized Toeplitz matrices was obtained. Following the approach developed in [3], we extend those results to $J_n\mathrm{circ}_n(a)$ and $\mathrm{circ}_n(a)J_n$. We conclude with several applications to other types of circulant matrices in Section 3.

All the results presented in this paper are stated for circulant Hankel matrices of the form (1.6). Some obvious modifications of the proofs of those results show that the same results hold for circulant Hankel matrices of the form (1.7) as well.

2 Spectral distribution of $J_n\mathrm{circ}_n(a)$

We start by giving a description of the spectrum of $J_n\mathrm{circ}_n(a)$ in terms of the eigenvalues of $\mathrm{circ}_n(a)$.

Proposition 2.1 *Let $a \in \mathcal{W}_+$ with real Fourier coefficients. The eigenvalues of $J_n\mathrm{circ}_n(a)$ are given by*

$$-|a_n(\omega_n^k)| \;\; \textit{or} \;\; |a_n(\omega_n^k)|,$$

for $k = 0, \ldots, n-1$, with a_n as in (1.5). In particular, the spectrum of $J_n\mathrm{circ}_n(a)$ lies within the interval

$$\mathcal{I}_a := [-\|a\|_{\mathcal{W}_+}, \|a\|_{\mathcal{W}_+}].$$

Proof From the diagonalization (1.2), it is readily seen that the singular values of $\mathrm{circ}_n(a)$ are given by

$$|a_n(1)|, |a_n(\omega_n)|, \ldots, |a_n(\omega_n^{n-1})|.$$

From

$$(J_n\mathrm{circ}_n(a))^T(J_n\mathrm{circ}_n(a)) = (\mathrm{circ}_n(a))^T\mathrm{circ}_n(a),$$

we see that $J_n\mathrm{circ}_n(a)$ and $\mathrm{circ}_n(a)$ have the same singular values. The proposition then follows by observing that the eigenvalues of $J_n\mathrm{circ}_n(a)$ are in absolute value equal to its singular values as $J_n\mathrm{circ}_n(a)$ is symmetric. ∎

The main result of this section—Theorem 2.5 below—is concerned with the asymptotic distribution of the eigenvalues $J_n \mathrm{circ}_n(a)$. But first, we need some preliminary results.

Lemma 2.2 *Let $a, b \in \mathcal{W}_+$ with real Fourier coefficients. We have*

$$\left| \frac{\mathrm{Trace}[\varphi(J_n \mathrm{circ}_n(a))]}{n} - \frac{\mathrm{Trace}[\varphi(J_n \mathrm{circ}_n(b))]}{n} \right| = \mathcal{O}(\|a-b\|_{\mathcal{W}_+}),$$

for every $\varphi \in C(\mathcal{I}_a)$.

Proof We denote the eigenvalues of $J_n \mathrm{circ}_n(a)$ by

$$\lambda_{1;n}(a),\ \lambda_{2;n}(a),\ \ldots,\ \lambda_{n;n}(a),$$

and those of $J_n \mathrm{circ}_n(b)$ by

$$\lambda_{1;n}(b),\ \lambda_{2;n}(b),\ \ldots,\ \lambda_{n;n}(b).$$

By a standard density argument, we only need to consider φ to be Lipschitz on the compact interval $\mathcal{I}_a$. From Weyl's Perturbation Theorem [10], there exists a permutation σ of $\{1,\ldots,n\}$ for which

$$\max_{1\le k\le n} \left|\varphi(\lambda_{\sigma(k);n}(a)) - \varphi(\lambda_{k;n}(b))\right| \le \mathcal{O}\left(\|J_n \mathrm{circ}_n(a) - J_n \mathrm{circ}_n(b)\|_{op}\right).$$

By Gershgorin's Circle Theorem [10], we deduce

$$\|J_n \mathrm{circ}_n(a) - J_n \mathrm{circ}_n(b)\|_{op} \le \sum_{k=0}^{n-1} |\hat{a}_k - \hat{b}_k| \le \|a-b\|_{\mathcal{W}_+},$$

as desired. ∎

In order to prove Theorem 2.5, we compute the spectral distribution of $\varphi(J_n \mathrm{circ}_n(a))$ for φ even and φ odd. To this end, for every φ, we write

$$\varphi(x) = \frac{\varphi(x)+\varphi(-x)}{2} + \frac{\varphi(x)-\varphi(-x)}{2} := \varphi_e(x) + \varphi_o(x), \tag{2.1}$$

where φ_e and φ_o, respectively, denote the even and odd parts of φ.

Proposition 2.3 *Let $a \in \mathcal{W}_+$ with real Fourier coefficients. We have*

$$\mathrm{Trace}\left[\varphi_o(J_n \mathrm{circ}_n(a))\right] = o(n),$$

for every function $\varphi \in C(\mathcal{I}_a)$.

Proof By the previous lemma, we only need to prove the proposition for $\mathrm{circ}_n(a_m)$ with $m < n$ large enough and a_m as in (1.5). In fact, we will prove the slightly stronger result

$$\mathrm{Trace}[\varphi_o(J_n \mathrm{circ}_n(a_m)] = \mathcal{O}(1).$$

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By Weierstrass Approximation Theorem, it suffices to consider odd functions of the form

$$\varphi_o(x) = x^{2p-1} \qquad (p \in \mathbb{N},\ |x| \le \|a\|_{\mathcal{W}_+}).$$

To do this, we break up $\mathrm{circ}_n(a_m)$ as the sum of $2m-1$ matrices $A_0, A_{\pm 1}, \dots, A_{\pm m-1}$ having at most one nonzero antidiagonal. More precisely, we write

$$J_n \mathrm{circ}_n(a_m) = \sum_{k=-m+1}^{m-1} A_k,$$

where

$$A_k = \begin{cases} \left[\hat{a}_{-k}\, \delta_{-k,1-i-j\,(\mathrm{mod}\, n)}\right]_{i,j=1}^{n} & \text{if } -m+1 \le k \le 0, \\ \left[\hat{a}_{n-k}\, \delta_{-k,1-i-j\,(\mathrm{mod}\, n)}\right]_{i,j=1}^{n} & \text{if } 1 \le k \le m-1. \end{cases}$$

It follows that

$$\mathrm{Trace}[(J_n \mathrm{circ}_n(a_m))^{2p-1}] = \sum \mathrm{Trace}[A_{k_1} \cdots A_{k_{2p-1}}],$$

where the last sum is taken over all $2p-1$-tuples $(k_1, \dots, k_{2p-1})$ with $-m+1 \le k_i \le m-1$. A simple induction shows that the product of an odd number of matrices of the form A_k is an antidiagonal matrix with at most one nonzero antidiagonal. It implies that

$$\mathrm{Trace}[(J_n \mathrm{circ}_n(a_m))^{2p-1}] = \mathcal{O}(1),$$

as desired. ∎

Proposition 2.4 *Let $a \in \mathcal{W}_+$ with real Fourier coefficients. We have*

$$\lim_{n\to\infty} \frac{\mathrm{Trace}\,[\varphi_e(J_n \mathrm{circ}_n(a))]}{n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_e\left(|a(e^{i\theta})|\right) d\theta,$$

for every function $\varphi \in C(\mathcal{I}_a)$.

Proof By Proposition 2.1, it follows that

$$\mathrm{Trace}\,[\varphi_e(J_n \mathrm{circ}_n(a))] = \sum_{k=0}^{n-1} \varphi_e(|a_n(\omega_n^k)|).$$

Because a_n converges uniformly to a on $\mathbb{T}$, the last equation implies that

$$\frac{\mathrm{Trace}\,[\varphi_e(J_n \mathrm{circ}_n(a))]}{n} = \frac{1}{n} \sum_{k=0}^{n-1} \varphi_e(|a(\omega_n^k)|) + o(1).$$

Now, the right-hand side is a Riemann sum for the continuous function $\varphi_e(|a(e^{i\theta})|)$ on the interval $[-\pi, \pi]$. Hence, we deduce

$$\lim_{n\to\infty} \frac{\mathrm{Trace}\,[\varphi_e(J_n \mathrm{circ}_n(a))]}{n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_e(|a(e^{i\theta})|)\, d\theta,$$

as desired. ∎

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Having all the necessary results at hand, we are now in position to state and prove the main result of this section.

Theorem 2.5 *Let $a \in \mathcal{W}_+$ with real Fourier coefficients. We have*

$$\lim_{n\to\infty} \frac{\operatorname{Trace}[\varphi(J_n \operatorname{circ}_n(a))]}{n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_e(|a(e^{i\theta})|)\,d\theta, \tag{2.2}$$

for every $\varphi \in C(\mathcal{I}_a)$.

Proof By Propositions 2.3 and 2.4, we have from (2.1) and the linearity of the trace

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\operatorname{Trace}[\varphi(J_n \operatorname{circ}_n(a))]}{n}\\
&\quad= \lim_{n\to\infty}\left[\frac{\operatorname{Trace}[\varphi_e(J_n \operatorname{circ}_n(a))]}{n} + \frac{\operatorname{Trace}[\varphi_o(J_n \operatorname{circ}_n(a))]}{n}\right]\\
&\quad= \lim_{n\to\infty} \frac{\operatorname{Trace}[\varphi_e(J_n \operatorname{circ}_n(a))]}{n}\\
&\quad= \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_e(|a(e^{i\theta})|)\,d\theta,
\end{aligned}$$

for every $\varphi \in C(\mathcal{I}_a)$. ∎

2.1 Some consequences

An obvious consequence of Theorem 2.5 is concerned with the asymptotic distribution of the singular values

$$|a_n(1)|, |a_n(\omega_n)|, \ldots, |a_n(\omega_n^{n-1})|$$

of $J_n \operatorname{circ}_n(a)$. Indeed, by (2.2) applied to $\varphi(|x|)$, we obtain

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \varphi(|a_n(\omega_n^k)|) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(|a(e^{i\theta})|)\,d\theta,$$

for every $\varphi \in C([0, \|a\|_{\mathcal{W}_+}])$.

In a recent paper [1], Angerer conjectured a formula for the spectral distribution of band Hankel matrices in terms of the Chebyshev polynomials $T_k(\cos\theta) = \cos(k\theta)$. His conjecture was proved in Proposition 3.1 of [3]. We now a derive a similar formula for $J_n C_n(a)$.

Corollary 2.6 *For every $\varphi \in C(\mathcal{I}_a)$, we have*

$$\lim_{n\to\infty} \frac{\operatorname{Trace}[\varphi(J_n C_n(a))]}{n} = \frac{1}{\pi}\int_{-1}^{1} \frac{\varphi_e(\alpha(x))}{\sqrt{1-x^2}}\,dx,$$

where

$$\alpha(x) = \sqrt{\sum_{k=0}^{\infty} \hat{a}_k^2 + 2\sum_{k=0}^{\infty}\sum_{m=1}^{\infty} \hat{a}_k \hat{a}_{m+k} T_k(x)} \qquad (-1 \le x \le 1).$$

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Proof By Theorem 2.5 and the substitution $x = \cos\theta$, we get

$$\lim_{n\to\infty} \frac{\mathrm{Trace}[\varphi(J_n \mathrm{circ}_n(a))]}{n} = \frac{1}{\pi}\int_{-1}^{1} \frac{\varphi_e(\alpha(x))}{\sqrt{1-x^2}}\, dx,$$

where

$$\begin{aligned}\alpha(x) &= |a(e^{i\theta})| \\ &= \sqrt{\sum_{k=0}^{\infty} \hat{a}_k^2 + 2\sum_{k=0}^{\infty}\sum_{m=1}^{\infty} \hat{a}_k \hat{a}_{m+k} \cos(k \arccos x)} \\ &= \sqrt{\sum_{k=0}^{\infty} \hat{a}_k^2 + 2\sum_{k=0}^{\infty}\sum_{m=1}^{\infty} \hat{a}_k \hat{a}_{m+k} T_k(x)},\end{aligned}$$

as desired. ∎

The next result is an extension of Theorem 2.5 to convergent sequences of symbols $\{a_n\}$ in $\mathcal{W}_+$. The proof is an immediate consequence of Lemma 2.2 and Theorem 2.5.

Corollary 2.7 *Let $\{a_n\}$ be a sequence in $\mathcal{W}_+$ for which every a_n has real Fourier coefficients. If $a_n \to a$ in $\mathcal{W}_+$, then we have*

$$\lim_{n\to\infty} \frac{\mathrm{Trace}\,[\varphi(J_n \mathrm{circ}_n(a_n))]}{n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_e(|a(e^{i\theta})|)\, d\theta, \tag{2.3}$$

for every $\varphi \in C_0(\mathbb{R})$, the space of compactly supported continuous functions on $\mathbb{R}$.

We conclude with a result on the signature, i.e., the number of positive and negative eigenvalues, of $J_n \mathrm{circ}_n(a)$. We denote by

$$N^+[J_n \mathrm{circ}_n(a)] = \#\{k : \lambda_{k;n}(a) > 0\}$$

and

$$N^-[J_n \mathrm{circ}_n(a)] = \#\{k : \lambda_{k;n}(a) < 0\}$$

the number of positive and negative eigenvalues of $J_n \mathrm{circ}_n(a)$.

Corollary 2.8 *Let $a \in \mathcal{W}_+$ with real Fourier coefficients. We have*

$$\Big| N^+[J_n \mathrm{circ}_n(a)] - N^-[J_n \mathrm{circ}_n(a)] \Big| = o(n),$$

as $n \to \infty$.

Proof This is an immediate consequence of the fact that

$$-J_n \mathrm{circ}_n(a) = J_n \mathrm{circ}_n(-a)$$

and $J_n \mathrm{circ}_n(a)$ have the same spectral distribution due to Theorem 2.5. ∎

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3 Other circulant matrices

In this section, we consider three special classes of circulant matrices. In each case, we obtain an extension of Theorem 2.5 by either allowing φ to be a function of bounded variation, or by allowing the symbol a to be in $L^2(\mathbb{T})$ or $L^1(\mathbb{T})$.

3.1 Laurent polynomials

First, we extend our definition of circulant matrices to allow symbols given by Laurent polynomials with nontrivial singular parts. Namely, we consider Laurent polynomial $b_{r,s}$ with real coefficients given by

$$b_{r,s}(z) = \sum_{j=-r}^{s} b_j z^j \qquad (r, s \in \mathbb{N},\ b_j \in \mathbb{R},\ z \in \mathbb{T}). \tag{3.1}$$

For $n \geq r+s+1$, we define as in [2] the $n \times n$ circulant matrix $L_n(b_{r,s})$ as the "periodization" of the Toeplitz matrix with symbol $b_{r,s}$, i.e.,

$$\begin{aligned} L_n(b_{r,s}) &:= \mathrm{circ}_n\left(\sum_{j=0}^{s} b_j z^j + \sum_{j=1}^{r} b_{-j} z^{n-j}\right) \\ &= \mathrm{circ}(b_0, \ldots, b_s, 0, \ldots, 0, b_{-r}, \ldots, b_{-1}). \end{aligned}$$

It follows from (1.3) that the eigenvalues of $L_n(b_{r,s})$ are given by

$$\sum_{j=0}^{s} b_j (\omega_n^k)^j + \sum_{j=1}^{r} b_{-j} (\omega_n^k)^{n-j} = \sum_{j=0}^{s} b_j (\omega_n^k)^j + \sum_{j=1}^{r} b_{-j} (\omega_n^k)^{-j} = b_{r,s}(\omega_n^k),$$

for $k = 0, \ldots, n-1$. Moreover, the matrix $L_n(b_{r,s})$ is normal and admits the diagonalization

$$L_n(b_{r,s}) = F_n^* \,\mathrm{diag}\left(b_{r,s}(1), b_{r,s}(\omega_n), \ldots, b_{r,s}(\omega_n^{n-1})\right) F_n.$$

The next result describing the spectrum of $J_n L_n(b_{r,s})$ is proved in the exact same manner as Proposition 2.1, i.e., by observing that $J_n L_n(b_{r,s})$ is symmetric and has the same singular values as $L_n(b_{r,s})$.

Proposition 3.1 *Let $b_{r,s}$ be a Laurent polynomial as in (3.1). For $n \geq r+s+1$, the eigenvalues of $J_n L_n(b_{r,s})$ consist of*

$$-|b_{r,s}(\omega_n^k)| \text{ or } |b_{r,s}(\omega_n^k)|,$$

for $k = 0, \ldots, n-1$. As a consequence, the spectrum of $J_n L_n(b_{r,s})$ is contained in the interval $\mathcal{I}_{r,s} = [-\|b_{r,s}\|_\infty, \|b_{r,s}\|_\infty]$.

Because $J_n L_n(b_{r,s})$ has at most $r+s+1$ nonzero antidiagonals, the proof of Proposition 2.3 implies that

$$\mathrm{Trace}[\varphi_o(J_n L_n(b_{r,s}))] = \mathcal{O}(1), \tag{3.2}$$

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for every $\varphi \in C(\mathcal{I}_{r,s})$. In addition, Proposition 2.4 together with the basic inequality

$$\left|\frac{1}{n}\sum_{k=0}^{n-1}\varphi_e(|b_{r,s}(\omega_n^k)|) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi_e(|b_{r,s}(e^{i\theta})|)\,d\theta\right| \le \frac{2\pi}{n}\operatorname{Var}_{\mathcal{I}_{r,s}}\varphi_e, \tag{3.3}$$

for every $\varphi \in BV(\mathcal{I}_{r,s})$, imply the following improvement of Theorem 2.5 in this present case.

Theorem 3.2 *Let $b_{r,s}$ be a Laurent polynomial as in (3.1). We have*

$$\lim_{n\to\infty}\frac{\operatorname{Trace}[\varphi(J_nL_n(b_{r,s}))]}{n} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi_e(|b_{r,s}(e^{i\theta})|)\,d\theta, \tag{3.4}$$

for every $\varphi \in C(\mathcal{I}_{r,s})$. If φ is even, then we can take $\varphi \in BV(\mathcal{I}_{r,s})$.

In that situation, we also have the minor improvement of Corollary 2.8 regarding the signature of $J_nL_n(b_{r,s})$.

Corollary 3.3 *Let $b_{r,s}$ be a Laurent polynomial as in (3.1). We have*

$$|N^+[J_nL_n(b_{r,s})] - N^-[J_nL_n(b_{r,s})]| = \mathcal{O}(1), \tag{3.5}$$

and

$$\lim_{n\to\infty}\frac{N^+[J_nL_n(b_{r,s})]}{n} = \lim_{n\to\infty}\frac{N^-[J_nL_n(b_{r,s})]}{n} = \frac{1}{2}. \tag{3.6}$$

Proof For the sake of simplicity, we denote $N^{\pm}[J_nL_n(b_{r,s})]$ by $N^{\pm}$. For every $\varepsilon > 0$, let N_ε^+, N_ε^-, and N_ε be the sets, respectively, defined by

$$N_\varepsilon^+ = \#\{k : \lambda_k(J_nL_n(b_{r,s})) > \varepsilon\},$$

$$N_\varepsilon^- = \#\{k : \lambda_k(J_nL_n(b_{r,s})) < -\varepsilon\},$$

and

$$N_\varepsilon = \#\{k : |\lambda_k(J_nL_n(b_{r,s}))| \le \varepsilon\}.$$

By the Fundamental Theorem of Algebra, for every $n \in \mathbb{N}$, there exists $\delta_n > 0$ such that

$$\operatorname{meas}\{\theta \in [-\pi,\pi] : |b_{r,s}(e^{i\theta})| < \delta_n\} = 1/n.$$

Let φ_n and ψ_n be the functions defined by

$$\varphi_n(x) = \begin{cases} 1 & \text{if } |x| \le \delta_n, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\psi_n(x) = \begin{cases} -1 & \text{if } -\|b_{r,s}\|_\infty \le x < -\delta_n, \\ \delta_n^{-1}x & \text{if } -\delta_n \le x \le \delta_n, \\ 1 & \text{if } \delta_n < x \le \|b_{r,s}\|_\infty. \end{cases}$$

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Clearly, φ_n is even and belongs to $BV(\mathcal{I}_{r,s})$. By (3.3), we obtain

$$N_{\delta_n} = \frac{n}{2\pi} \int_{-\pi}^{\pi} \varphi_n(|b_{r,s}(e^{i\theta})|)\, d\theta + \mathcal{O}(1) = \mathcal{O}(1). \tag{3.7}$$

Moreover, by (3.2) applied to ψ_n, we deduce

$$|N^+_{\delta_n} - N^-_{\delta_n}| \leq N_{\delta_n} + \mathcal{O}(1) = \mathcal{O}(1). \tag{3.8}$$

By combining (3.7) and (3.8), we obtain

$$|N^+ - N^-| \leq |N^+ - N^+_{\delta_n}| + |N^+_{\delta_n} - N^-_{\delta_n}| + |N^-_{\delta_n} - N^-| = \mathcal{O}(1).$$

Finally, we prove (3.6). By estimate (3.7), it follows that

$$\#\{k : \lambda_k(J_n L_n(b_{r,s})) = 0\} = \mathcal{O}(1),$$

and therefore,

$$\lim_{n\to\infty} \left[\frac{N^+}{n} + \frac{N^-}{n}\right] = 1.$$

On the other hand, (3.5) also implies that

$$\lim_{n\to\infty} \left[\frac{N^+}{n} - \frac{N^-}{n}\right] = 0.$$

By combining last two equations, one can easily deduce (3.6). ∎

3.2 Simple circulant matrices

In our second extension, the class symbol is enlarged to allow functions that are square-integrable on the unit circle. That is, let $a \in L^2(\mathbb{T})$ with

$$\|a\|_{L^2} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |a(e^{i\theta})|^2\, d\theta\right)^{1/2}.$$

We assume that a has real Fourier coefficients given by

$$\hat{a}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} a(e^{i\theta})\, e^{-ik\theta}\, d\theta \qquad (k \in \mathbb{Z}). \tag{3.9}$$

Following the notation introduced in [12], we define the n-by-n simple circulant matrix $S_n(a)$ by

$$S_n(a) := \mathrm{circ}_n\left(\sum_{j=0}^{\lceil n/2\rceil-1} \hat{a}_j z^j + \sum_{j=1}^{\lceil n/2\rceil-1} \hat{a}_{-j} z^{n-j}\right).$$

Equivalently, $S_n(a)$ is given for n even by

$$S_n(a) = \mathrm{circ}\left(\hat{a}_0, \ldots, \hat{a}_{n/2-1}, 0, \hat{a}_{-n/2+1}, \ldots, \hat{a}_{-1}\right),$$

and for n odd by

$$S_n(a) = \mathrm{circ}\left(\hat{a}_0, \ldots, \hat{a}_{\lceil n/2\rceil-1}, \hat{a}_{-\lceil n/2\rceil+1}, \ldots, \hat{a}_{-1}\right).$$

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As a consequence of (1.3), the eigenvalues of $S_n(a)$ are given by

$$\sum_{j=0}^{\lceil n/2\rceil-1} \hat{a}_j(\omega_n^k)^j + \sum_{j=1}^{\lceil n/2\rceil-1} \hat{a}_{-j}(\omega_n^k)^{n-j} = \sum_{j=-\lceil n/2\rceil+1}^{\lceil n/2\rceil-1} \hat{a}_j(\omega_n^k)^j =: a_{\lceil n/2\rceil}(\omega_n^k),$$

for $k = 0, \ldots, n-1$. It follows that $S_n(a)$ is normal with diagonalization

$$S_n(a) = F_n^* \operatorname{diag}\left(a_{\lceil n/2\rceil}(1), a_{\lfloor n/2\rfloor}(\omega_n), \ldots, a_{\lceil n/2\rceil}(\omega_n^{n-1})\right) F_n. \tag{3.10}$$

As a consequence, we have the following characterization of the spectrum of $S_n(a)$ whose proof follows the same argument as in Proposition 2.1.

Proposition 3.4 *Let $a \in L^2(\mathbb{T})$ with $\hat{a}_k \in \mathbb{R}$, for all $k \in \mathbb{Z}$. The eigenvalues of the Hankel matrix $J_nS_n(a)$ are given by*

$$-|a_{\lceil n/2\rceil}(\omega_n^k)| \qquad \text{or} \qquad |a_{\lceil n/2\rceil}(\omega_n^k)|,$$

for $k = 0, \ldots, n-1$. Moreover, the spectrum of $J_nS_n(a)$ is contained within the interval

$$[-\sqrt{n}\,\|a\|_{L^2}, \sqrt{n}\,\|a\|_{L^2}].$$

Proof The bound on the eigenvalues is obtained by using the Cauchy–Schwarz and Bessel inequalities, i.e.,

$$\begin{aligned} |a_{\lceil n/2\rceil}(\omega_n^k)| &\le \sum_{j=-\lceil n/2\rceil+1}^{\lceil n/2\rceil-1} |\hat{a}_j| \\ &\le \sqrt{n}\left(\sum_{j=-\lceil n/2\rceil+1}^{\lceil n/2\rceil-1} |\hat{a}_j|^2\right)^{1/2} \\ &\le \sqrt{n}\,\|a\|_{L^2}, \end{aligned}$$

for $k = 0, \ldots, n-1$. ∎

In order to show that $\operatorname{Trace}[\varphi_o(J_nS_n(a))] = \mathcal{O}(1)$, for every $\varphi \in C_0(\mathbb{R})$, we need to modify the perturbation Lemma 2.2 as follows.

Lemma 3.5 *Let a, b be two functions in $L^2(\mathbb{T})$ with real Fourier coefficients. For every $\varphi \in C_0(\mathbb{R})$, we have that*

$$\left|\frac{\operatorname{Trace}[\varphi(J_nS_n(a))]}{n} - \frac{\operatorname{Trace}[\varphi(J_nS_n(b))]}{n}\right| = \mathcal{O}(\|a-b\|_{L^2}).$$

Proof As before, we only need to consider φ to be Lipschitz with compact support. As a consequence of the Cauchy–Schwarz and Wielandt–Hoffman [10] inequalities, we get

$$\left|\frac{\operatorname{Trace}[\varphi(J_nS_n(a))]}{n} - \frac{\operatorname{Trace}[\varphi(J_nS_n(b))]}{n}\right| = \frac{1}{\sqrt{n}}\,\mathcal{O}\left(\|S_n(a) - S_n(b)\|_F\right).$$

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In addition, we have

$$\|S_n(a) - S_n(b)\|_F^2 \le n \sum_{k\in\mathbb{Z}} |\hat{a}_k - \hat{b}_k|^2 = n\,\|a-b\|_{L^2}^2,$$

where the last equality follows from Parseval's identity. The conclusion of the lemma follows by combining the previous two inequalities. ■

The next result is concerned with the spectral distribution of $J_n S_n(a)$.

Theorem 3.6 *Let $a \in L^2(\mathbb{T})$ with $\hat{a}_k \in \mathbb{R}$, for all $k \in \mathbb{Z}$. We have*

$$\lim_{n\to\infty} \frac{\mathrm{Trace}[\varphi(J_n S_n(a))]}{n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_e(|a(e^{i\theta})|)\,d\theta,$$

for every $\varphi \in C_0(\mathbb{R})$.

Proof From the proof of Proposition 2.3, we know that

$$\mathrm{Trace}[\varphi_o(J_n S_n(b))] = \mathcal{O}(1),$$

when b is a Laurent polynomial. Because those are dense in $L^2(\mathbb{T})$, Lemma 3.5 implies that

$$\mathrm{Trace}[\varphi_o(J_n S_n(a))] = o(n). \tag{3.11}$$

By Theorem 4.2 in [11], the singular values $|a_{\lceil n/2\rceil}(\omega_n^k)|$, for $k = 0,\dots,n-1$, of $S_n(a)$, and hence of $J_n S_n(a)$, satisfy

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \varphi(|a_{\lceil n/2\rceil}(\omega_n^k)|) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(|a(e^{i\theta})|)\,d\theta. \tag{3.12}$$

It then follows by Proposition 3.4 that

$$\lim_{n\to\infty} \frac{\mathrm{Trace}[\varphi_e(J_n S_n(a))]}{n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_e(|a(e^{i\theta})|)\,d\theta. \tag{3.13}$$

The conclusion of the theorem can easily be deduced from (3.11) and (3.13). ■

As for $J_n \mathrm{circ}_n(a)$, we have the following two results regarding sequences of symbols $\{a_n\}$ in $L^2(\mathbb{T})$ and the signature of $J_n S_n(a)$. These results are proved in a similar manner as Corollaries 2.7 and 2.8.

Corollary 3.7 *Let $\{a_n\}$ be a sequence in $L^2(\mathbb{T})$ for which every a_n has real Fourier coefficients. If $a_n \to a$ in $L^2(\mathbb{T})$, then we have*

$$\lim_{n\to\infty} \frac{\mathrm{Trace}\,[\varphi(J_n S_n(a_n))]}{n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_e(|a(e^{i\theta})|)\,d\theta, \tag{3.14}$$

for every $\varphi \in C_0(\mathbb{R})$.

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Corollary 3.8 *Let $a \in L^2(\mathbb{T})$ with real Fourier coefficients. We have*

$$\left|N^+[J_n S_n(a)] - N^-[J_n S_n(a)]\right| = o(n),$$

as $n \to \infty$.

3.3 Cesàro circulant matrices

In our last example, we consider circulant matrices usually referred to as Cesàro circulant matrices. The symbol a is assumed to be in $L^1(\mathbb{T})$ with

$$\|a\|_{L^1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |a(e^{i\theta})|\, d\theta.$$

These circulant matrices are defined as

$$C_n(a) := \operatorname{circ}_n\left(\sum_{k=0}^{n-1} a_{k;n}\, z^k\right) = \operatorname{circ}(a_{0;n}, a_{1;n}, \ldots, a_{n-1;n}),$$

where the coefficients $a_{k;n}$ are given by

$$a_{k;n} = \frac{(n-k)\hat{a}_k + k\hat{a}_{k-n}}{n} \qquad (k = 0, \ldots, n-1),$$

with $\hat{a}_k \in \mathbb{R}$, for all $k \in \mathbb{Z}$. These matrices are also known as optimal circulant matrices as they are the best circulant matrices approximating the Toeplitz matrix

$$T_n(a) = [\hat{a}_{i-j}]_{i,j=1}^{n}.$$

We refer the reader to [12] for more details.

These matrices are called Cesàro circulant for the reason that their eigenvalues are by (1.3) given by

$$\sum_{k=0}^{n-1} a_{k;n}\omega_n^k := \sigma_n(\omega_n^k) \qquad (k = 0, \ldots, n-1)$$

with σ_n being the Cesàro sum

$$\sigma_n(z) = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{j=-k}^{k} \hat{a}_j z^j.$$

Consequently, $C_n(a)$ is normal with diagonalization given by

$$C_n(a) = F_n^* \operatorname{diag}\left(\sigma_n(1), \sigma_n(\omega_n), \ldots, \sigma_n(\omega_n^{n-1}\right)\, F_n.$$

Proposition 3.9 *Let $a \in L^1(\mathbb{T})$ with $\hat{a}_k \in \mathbb{R}$, for all $k \in \mathbb{Z}$. The eigenvalues of $J_n C_n(a)$ are given by*

$$-|\sigma_n(\omega_n^k)| \qquad \text{or} \qquad |\sigma_n(\omega_n^k)|,$$

for $k = 0, \ldots, n-1$. Moreover, the spectrum of $J_n C_n(a)$ is contained in the interval

$$[-n\|a\|_{L^1},\ n\|a\|_{L^1}].$$

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Proof As observed before, the first part is proved in a similar fashion as Proposition 2.1. By Fejer's representation, we can express the eigenvalues as

$$\sigma_n(\omega_n^k) = (f_{n-1} \star (a-b))(\omega_n^k) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f_{n-1}\left(\frac{2k\pi}{n} - \theta\right) a(e^{i\theta})\, d\theta,$$

where f_{n-1} denotes Fejer's kernel given by

$$f_{n-1}(\theta) = \frac{1}{n}\frac{\sin^2(n\theta/2)}{\sin^2(\theta/2)}.$$

From the basic inequality $|\sin(n\theta)| \le n|\sin\theta|$, one can easily deduce that $\|f_{n-1}\|_\infty \le n$, and therefore,

$$|\sigma_n(\omega_n^k)| \le \frac{n}{2\pi}\int_{-\pi}^{\pi} |a(e^{i\theta})|\, d\theta = n\,\|a\|_{L^1}. \qquad \blacksquare$$

In order to compute the spectral distribution of $C_n(a)$, we need the following perturbation lemma.

Lemma 3.10 *For every $a, b \in L^1(\mathbb{T})$ with real Fourier coefficients, we have*

$$\left|\frac{\mathrm{Trace}[\varphi(J_nC_n(a))]}{n} - \frac{\mathrm{Trace}[\varphi(J_nC_n(b))]}{n}\right| = \mathcal{O}\left(\|a-b\|_{L^1}\right),$$

for every $\varphi \in C_0(\mathbb{R})$.

Proof As we observed earlier, it suffices to take φ to be Lipschitz with compact support on $\mathbb{R}$. In such case, the Wielandt–Hoffman inequality [10] implies that

$$\begin{aligned}|\mathrm{Trace}[\varphi(J_nC_n(a))] - \mathrm{Trace}[\varphi(J_nC_n(b))]| &= \mathcal{O}\left(\|J_nC_n(a) - J_nC_n(b)\|_{tr}\right)\\ &= \mathcal{O}\left(\|C_n(a-b)\|_{tr}\right),\end{aligned}$$

where the second equality follows from the fact that $J_nC_n(\cdot)$ and $C_n(\cdot)$ have the same singular values. Therefore, we only need to prove that

$$\|C_n(a-b)\|_{tr} = \mathcal{O}(n\,\|a-b\|_{L^1}).$$

Using Fejer's kernel to express the singular values of $C_n(a)$ as in Proposition 3.9, we have

$$\begin{aligned}\|C_n(a-b)\|_{tr} &= \sum_{k=0}^{n-1} |\sigma_n(\omega_n^k; a-b)|\\ &\le \left[\sup_{\theta\in[-\pi,\pi]} \sum_{k=0}^{n-1} f_{n-1}\left(\frac{2k\pi}{n} - \theta\right)\right] \|a-b\|_{L^1}.\end{aligned}$$

Therefore, it remains to show that

$$\sup_{\theta\in[-\pi,\pi]} \sum_{k=0}^{n-1} f_{n-1}\left(\frac{2k\pi}{n} - \theta\right) = \mathcal{O}(n).$$

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To do this, we follow the argument given in [12]. From the properties of the sine function, one can easily deduce that

$$\sup_{\theta\in[-\pi,\pi]} \sum_{k=0}^{n-1} f_{n-1}\left(\frac{2k\pi}{n} - \theta\right) \le 2 \sup_{t\in[0,1]} \sum_{k=0}^{\lceil (n-1)/2 \rceil} f_{n-1}\left(\frac{2\pi(k+t)}{n}\right).$$

Using the basic inequality $\sin\theta \ge 2/\pi\,\theta$, for $0 \le \theta \le \pi/2$, we obtain

$$\left|\sin\left(\frac{\pi(k+t)}{n}\right)\right| \ge \frac{2}{\pi}\left(\frac{\pi(k+t)}{n}\right),$$

for $1 \le k \le \lceil (n-1)/2 \rceil$. It follows that

$$\begin{aligned}
\sup_{t\in[0,1]} &\sum_{k=0}^{\lceil (n-1)/2 \rceil} f_{n-1}\left(\frac{2\pi(k+t)}{n}\right) \\
&\le \sup_{t\in[0,1]} \left[f_{n-1}\left(\frac{2\pi t}{n}\right) + \sum_{k=1}^{\lceil (n-1)/2 \rceil} f_{n-1}\left(\frac{2\pi(k+t)}{n}\right)\right] \\
&\le n + \sup_{t\in[0,1]} \frac{1}{n} \sum_{k=1}^{\lceil (n-1)/2 \rceil} \frac{\sin^2(\pi t)}{\sin^2(\pi(k+t)/n)} \\
&\le n + \frac{n}{2\pi} \sum_{k=1}^{\lceil (n-1)/2 \rceil} \frac{1}{k^2} \\
&\le n + \frac{\pi n}{12},
\end{aligned}$$

as desired. ∎

Theorem 3.11 *Let $a \in L^1(\mathbb{T})$ with $\hat{a}_k \in \mathbb{R}$, for all $k \in \mathbb{Z}$. If the Fourier series of a converges in the L^1-norm, then we have*

$$\lim_{n\to\infty} \frac{\operatorname{Trace}[\varphi(J_n C_n(a))]}{n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_e(|a(e^{i\theta})|)\,d\theta,$$

for every $\varphi \in C_0(\mathbb{R})$.

Proof From the proof of Proposition 2.3, we know that

$$\operatorname{Trace}[\varphi_o(J_n C_n(b))] = \mathcal{O}(1)$$

when b is Laurent polynomial. Because the latter are dense in $L^1(\mathbb{T})$, Lemma 3.10 implies that

$$\operatorname{Trace}[\varphi_o(J_n C_n(a))] = o(n). \tag{3.15}$$

From the remark on page 20 of [11], the singular values $|\sigma_n(\omega_n^k)|$ of $J_n C_n(a)$ satisfy

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \varphi(|\sigma_n(\omega_n^k)|) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(|a(e^{i\theta})|)\,d\theta. \tag{3.16}$$

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It then follows by Proposition 3.9 that

$$\lim_{n\to\infty} \frac{\mathrm{Trace}[\varphi_e(J_nC_n(a))]}{n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_e(|a(e^{i\theta})|)\,d\theta. \tag{3.17}$$

The conclusion of the theorem can easily be deduced from (3.11) and (3.13). ∎

We conclude with the following two consequences of the previous result. They are proved in the exact same manner as Corollaries 2.7 and 2.8.

Corollary 3.12 *Let $\{a_n\}$ be a sequence in $L^1(\mathbb{T})$ for which every a_n has real Fourier coefficients with convergent Fourier series in $L^1(\mathbb{T})$. If $a_n \to a$ in $L^1(\mathbb{T})$, then we have*

$$\lim_{n\to\infty} \frac{\mathrm{Trace}\,[\varphi(J_nC_n(a_n))]}{n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_e(|a(e^{i\theta})|)\,d\theta, \tag{3.18}$$

for every $\varphi \in C_0(\mathbb{R})$.

Corollary 3.13 *Let $a \in L^1(\mathbb{T})$ with real Fourier coefficients and convergent Fourier series in L^1. We have*

$$\Big|N^+[J_nC_n(a)] - N^-[J_nC_n(a)]\Big| = o(n),$$

as $n \to \infty$.

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