Proceedings of the Royal Society of Edinburgh, **142A**, 391–423, *2012*

On the motion of a rigid body with a cavity filled with a viscous liquid

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(MS received 14 June 2010; accepted 3 May 2011)

We study the motion of a rigid body with a cavity filled with a viscous liquid. The main objective is to investigate the well-posedness of the coupled system formed by the Navier–Stokes equations describing the motion of the fluid and the ordinary differential equations for the motion of the rigid part. To this end, appropriate function spaces and operators are introduced and analysed by considering a completely general three-dimensional bounded domain. We prove the existence of weak solutions using the Galerkin method. In particular, we show that if the initial velocity is orthogonal, in a certain sense, to all rigid velocities, then the velocity of the system decays exponentially to zero as time goes to infinity. Then, following a functional analytic approach inspired by Kato's scheme, we prove the existence and uniqueness of mild solutions. Finally, the functional analytic approach is extended to obtain the existence and uniqueness of strong solutions for regular data.

1. Introduction

The dynamics of bodies containing fluids is a subject of long-standing importance in many technical applications. The development of this theory was stimulated by a variety of applied problems, including the dynamics of fluid-filled missiles, rockets and artificial earth satellites. These challenging problems are a particular case of fluid–structure interactions and raise many interesting mathematical issues ranging from the well-posedness of the equations to their numerical analysis and simulation.

Let us describe the model that we will study in this paper. Consider a solid body $\mathcal{B}$ with a cavity completely filled with a viscous liquid $\mathcal{L}$ moving in the three-dimensional space. We begin by presenting the equations of motion of the system body liquid in an inertial reference frame, where we fix a Cartesian coordinate system $\mathcal{I} = \{o, a_1, a_2, a_3\}$. We denote by $\mathcal{F}(t)$ the domain occupied by the fluid at time t and by $\mathcal{S}(t)$ the interior of the region occupied by the rigid part of the system body-liquid; $\partial\mathcal{F}(t)$ is the boundary of $\mathcal{F}(t)$ which coincides with the interior boundary of $\mathcal{S}(t)$.

Concerning the motion of the rigid part of the system $\{\mathcal{B},\mathcal{L}\}$, let $\varpi = \varpi(t)$ be the angular velocity of $\mathcal{B}$ and $\eta = \eta(t)$ the velocity of its centre of mass, C. Then the velocity field of $\mathcal{B}$ is

$$V(t,y) = \eta(t) + \varpi(t) \times (y - y_C(t)) \quad \text{in } \bigcup_{t>0} \{t\} \times \overline{\mathcal{S}(t)},$$

where $y_C(t) = \int_0^t \eta(s)\,\mathrm{d}s$ gives the position of C, if we assume, without loss, that $y_C(0) = 0$. The centre of mass of $\mathcal{B}$ is also defined as

$$y_C(t) = \frac{\int_{\mathcal{S}(t)} \varrho_{\mathcal{B}}(t,y) y \,\mathrm{d}y}{m_{\mathcal{B}}},$$

where $\varrho_{\mathcal{B}}$ is the density of $\mathcal{B}$ and $m_{\mathcal{B}}$ is its mass. The regions $\mathcal{S}(t)$ and $\mathcal{F}(t)$ can be described in terms of η and ϖ in the following way. Let W be the matrix defined by

$$W_{ij}(t) = -\varepsilon_{ijk}\varpi_k(t), \tag{1.1}$$

where ε_{ijk} is the classical Levi–Civita symbol

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3) \text{ or } (2,3,1) \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) \text{ is } (1,3,2) \text{ or } (3,2,1) \text{ or } (2,1,3), \\ 0 & \text{otherwise.} \end{cases}$$

Denoting by Q the fundamental matrix of W, that is,

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = WQ, \quad Q(0) = \mathbb{I},$$

we have

$$\left.\begin{aligned} \mathcal{S}(t) &= \{y \in \mathbb{R}^3 \colon y = y_C(t) + Q(t)x,\ x \in \mathcal{S}(0)\},\ t \in (0,\infty), \\ \mathcal{F}(t) &= \{y \in \mathbb{R}^3 \colon y = y_C(t) + Q(t)x,\ x \in \mathcal{F}(0)\},\ t \in (0,\infty). \end{aligned}\right\} \tag{1.2}$$

Note that $QQ^{\mathrm{T}} = Q^{\mathrm{T}}Q = \mathbb{I}$. We also have $\varrho_{\mathcal{B}}(t,y) = \rho_{\mathcal{B}}(Q(t)^{\mathrm{T}}(y - y_C(t)))$, where $\rho_{\mathcal{B}}$ is a given positive integrable function over $\mathcal{S}(0)$.

Now let us consider the motion of the liquid inside the cavity, assuming that it is governed by the Navier–Stokes equations. In this case, the Eulerian velocity $v = v(t,y)$ and the pressure $q = q(t,y)$ of $\mathcal{L}$ obey the following equations:

$$\begin{aligned} \rho(\partial_t v + v \cdot \nabla v) &= \nabla \cdot T(v,q) + \rho f_{\mathcal{L}} && \text{in } \bigcup_{t>0} \{t\} \times \mathcal{F}(t), \\ \nabla \cdot u &= 0 && \text{in } \bigcup_{t>0} \{t\} \times \mathcal{F}(t), \\ v(t,y) &= V(t,y) && \text{on } \bigcup_{t>0} \{t\} \times \partial\mathcal{F}(t), \end{aligned}$$

https://doi.org/10.1017/S0308210510001034 Published online by Cambridge University Press

where ρ is the (constant) density of the fluid, where $f_{\mathcal{L}} = f_{\mathcal{L}}(t, y)$ represents an external body force acting on the fluid and where $T(v, q)$ is the stress tensor, defined as

$$T_{ij}(v, q) = \mu\left(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i}\right) - q\delta_{ij}, \quad i, j = 1, 2, 3,$$

with μ the viscosity coefficient of the fluid.

The equations of balance of linear momentum and angular momentum for the rigid body are

$$\begin{aligned} m_{\mathcal{B}}\frac{\mathrm{d}\eta}{\mathrm{d}t} &= -\int_{\partial\mathcal{F}(t)} T(v, q)N \,\mathrm{d}\sigma_y + f_{\mathcal{B}}, & t \in (0, \infty), \\ \frac{\mathrm{d}(J_{\mathcal{B}}\varpi)}{\mathrm{d}t} &= -\int_{\partial\mathcal{F}(t)} (y - y_C) \times T(v, q)N \,\mathrm{d}\sigma_y + t_{\mathcal{B}}, & t \in (0, \infty), \end{aligned}$$

where $N = N(t)$ is the external unit normal to $\mathcal{F}(t)$, $f_{\mathcal{B}} = f_{\mathcal{B}}(t)$ and $t_{\mathcal{B}} = t_{\mathcal{B}}(t)$ are the total force and torque acting on $\mathcal{B}$, respectively, and where $J_{\mathcal{B}} = J_{\mathcal{B}}(t)$ is the tensor of inertia of $\mathcal{B}$. Recall that $J_{\mathcal{B}}(t) = Q(t)I_{\mathcal{B}}Q(t)^{\mathrm{T}}$, where $I_{\mathcal{B}}$ is the tensor of inertia of $\mathcal{B}$ at time zero:

$$(I_{\mathcal{B}})_{ij} = \int_{\mathcal{S}} \rho_{\mathcal{B}}(x)(\delta_{ij}|x|^2 - x_i x_j) \,\mathrm{d}x \quad \text{and} \quad \mathcal{S} := \mathcal{S}(0).$$

Hence, the motion of the system $\{\mathcal{B}, \mathcal{L}\}$ is described by the following initial boundary-value problem:

$$\left.\begin{aligned} \rho(\partial_t v + v \cdot \nabla_y v) &= \nabla_y \cdot T(v, q) + \rho f_{\mathcal{L}} && \text{in } \bigcup_{t>0}\{t\} \times \mathcal{F}(t), \\ \nabla_y \cdot v &= 0 && \text{in } \bigcup_{t>0}\{t\} \times \mathcal{F}(t), \\ v(t, y) &= V(t, y) && \text{in } \bigcup_{t>0}\{t\} \times \partial\mathcal{F}(t), \\ m_{\mathcal{B}}\frac{\mathrm{d}\eta}{\mathrm{d}t} &= -\int_{\partial\mathcal{F}(t)} T(v, q)N \,\mathrm{d}\sigma_y + f_{\mathcal{B}} && \text{in } (0, \infty), \\ \frac{\mathrm{d}(J_{\mathcal{B}}\varpi)}{\mathrm{d}t} &= -\int_{\partial\mathcal{F}(t)} (y - y_C) \times T(v, q)N \,\mathrm{d}\sigma_y + t_{\mathcal{B}} && \text{in } (0, \infty), \\ \eta(0) = \eta_0, \quad \varpi(0) &= \varpi_0, \quad v(0, x) = v_0(x), \quad x \in \mathcal{F}, \end{aligned}\right\} \tag{1.3}$$

where $\mathcal{F} := \mathcal{F}(0)$ and the fluid domain is given by the second equation in (1.2).

The general problem associated to system (1.3) is the following. Given the external forces acting on the rigid body and on the liquid, and given the initial velocity of the system, find the velocity of the rigid body and the velocity and pressure of the liquid, satisfying equations (1.3). Several authors have studied the well-posedness of (1.3) using a different formulation and classical function spaces, as in [14], and related problems such as the instability of certain steady motions have been studied by Lyashenko and Friedlander [6, 16, 17].

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Besides the difficulties related to the coupling of the equations, in system (1.3) the fluid domain is time-dependent and unknown, since it is defined in terms of the velocity of the solid part. This is the main feature of fluid–structure interaction problems. As shown in [6, 10, 11, 16, 17, 19], and as we will recall in the next section, it is possible to overcome the latter difficulty by making an appropriate change of variables that leads to a system of equations defined in a fixed, known domain. However, this approach produces additional nonlinear terms in the equations and introduces additional unknowns, namely, the external forces and torque. The coupling of the equations can in turn be dealt with by formulating the problem in an appropriate functional setting, based on extended velocity fields whose restriction to $\mathcal{S}$ are rigid velocities.

Another interesting aspect of this problem is the fact that a Poincaré-type inequality is valid in function spaces composed of velocity fields which are orthogonal, with respect to a certain inner product, to the space of all rigid body velocities. On the other hand, the operator associated with the viscous term of the equations (1.3) has a non-trivial kernel which is precisely the space of all rigid velocities. These aspects have important consequences in the analysis of long time behaviour and decay of solutions and other related issues such as stability and attainability of steady states.

We are interested in the study of existence, uniqueness and regularity of solutions, with particular emphasis in functional analytic approaches which exploit the properties of the operators associated with the equations and are valid for *arbitrary fluid domains*. A primary goal is to understand the properties of the underlying function spaces and operators in a general domain, as in the works of Sohr [21] and Monniaux [18]. Having developed the abstract framework for equations (1.3), we prove existence and uniqueness results for weak, mild and strong solutions in the absence of external forces and torque, *without any assumptions on the regularity of the fluid domain*. In the case of strong solutions, we also consider the case of a regular fluid domain.

This paper is organized as follows. In § 2 we recall the formulation of the equations in a reference frame attached to the body. Then, in § 3 we introduce the notation and we recall some classical results. Section 4 is devoted to the functional setting for the equations. The existence of weak solutions is proved in § 5, and § 6 is devoted to existence and uniqueness of mild solutions. Finally, in § 7 we show the existence and uniqueness of strong solutions.

2. Formulation of the problem in a reference frame attached to the body

In system (1.3), the fluid domain is time-dependent and unknown because it is expressed in terms of η and ϖ. However, we can avoid this difficulty by reformulating the equations in the fixed domain $\mathcal{F}$. For this purpose, we consider a Cartesian coordinate system attached to the body, $\mathcal{Y} = \{C, e_1, e_2, e_3\}$, and the Euclidean transformation $x = Q^{\mathrm{T}}(y - y_C) := \mathbb{T}(y)$ representing the change of coordinate system $\mathcal{I} \to \mathcal{Y}$. It is clear that $\mathbb{T}(\mathcal{F}(t)) = \mathcal{F}$ and $\mathbb{T}(\mathcal{S}(t)) = \mathcal{S}$ for all $t \in [0, \infty[$, where $\mathcal{S} := \mathcal{S}(0)$ and $\mathcal{F} := \mathcal{F}(0)$ are fixed domains. Moreover, the external unit normal to

$\partial\mathcal{F}$ is related to N by $n(x) = Q^{\mathrm{T}}(t)N(t, y_C(t) + Q(t)x)$ and we have

$$\int_{\mathcal{S}} \rho_{\mathcal{B}}(x)x\,\mathrm{d}x = 0.$$

Let $x = x(t)$ be the position of a particle in $\mathcal{Y}$ at time t and let $y = y(t)$ be the position of the same particle in $\mathcal{I}$ at time t. Introducing the following transformed fields:

$$\begin{aligned} U(t,x) &:= \xi(t) + \omega(t)\times x \quad \text{with } \xi = Q^{\mathrm{T}}\eta \text{ and } \omega = Q^{\mathrm{T}}\varpi, \\ u(t,x) &= Q^{\mathrm{T}}(t)v(t, y_C(t) + Q(t)x), \\ p(t,x) &= q(t, y_C(t) + Q(t)x), \end{aligned}$$

where $v = v(t,y)$ is the velocity of the particle in $\mathcal{I}$ and $q = q(t,y)$ is the associated pressure. Using this change of variables, system (1.3) becomes

$$\left.\begin{aligned} &\rho\partial_t u = \nabla\cdot T(u,p) + \rho[(U-u)\cdot\nabla u - \omega\times u + g_{\mathcal{L}}] && \text{in } (0,\infty)\times\mathcal{F}, \\ &\nabla\cdot u = 0 && \text{in } (0,\infty)\times\mathcal{F}, \\ &u = U && \text{on } (0,\infty)\times\partial\mathcal{F}, \\ &m_{\mathcal{B}}\frac{\mathrm{d}\xi}{\mathrm{d}t} = -\int_{\partial\mathcal{F}} T(u,p)n\,\mathrm{d}\sigma_x - m_{\mathcal{B}}\omega\times\xi + g_{\mathcal{B}}, \\ &I_{\mathcal{B}}\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\int_{\partial\mathcal{F}} x\times T(u,p)n\,\mathrm{d}\sigma_x - \omega\times(I_{\mathcal{B}}\omega) + \tau_{\mathcal{B}}, \\ &\xi(0) = \xi_0, \qquad \omega(0) = \omega_0, \qquad u(0,x) = u_0(x),\ x\in\mathcal{F}, \end{aligned}\right\} \tag{2.1}$$

with $\xi_0 = \eta_0$, $\omega_0 = \varpi_0$, $u_0(x) = v_0(x)$, because $Q(0) = \mathbb{I}$. The quantities $g_{\mathcal{L}}$, $g_{\mathcal{B}}$ and $\tau_{\mathcal{B}}$ are related to $f_{\mathcal{L}}$, $f_{\mathcal{B}}$ and $t_{\mathcal{B}}$ by the following equations:

$$\left.\begin{aligned} g_{\mathcal{L}}(t,x) &= Q^{\mathrm{T}}(t)f_{\mathcal{L}}(t, y_C(t) + Q(t)x), \\ g_{\mathcal{B}}(t) &= Q^{\mathrm{T}}(t)f_{\mathcal{B}}(t), \\ \tau_{\mathcal{B}}(t) &= Q^{\mathrm{T}}(t)t_{\mathcal{B}}(t). \end{aligned}\right\} \tag{2.2}$$

Note that $f_{\mathcal{L}}$, $f_{\mathcal{B}}$ and $t_{\mathcal{B}}$ are known in the inertial frame $\mathcal{I}$ but, since the motion of the body is not prescribed, $g_{\mathcal{L}}$, $g_{\mathcal{B}}$ and $\tau_{\mathcal{B}}$ are unknown in $\mathcal{Y}$. Therefore, in that case, we have to append equations (2.2) to the system (2.1). We have defined Q and y_C in terms of η and ϖ, but they can be rewritten as functions of ξ and ω in the following way. Defining a matrix $\mathit{\Pi}$ by $\mathit{\Pi}_{ij} = \varepsilon_{ijk}\omega_k$, $i,j = 1,2,3$, we have $\mathit{\Pi} = -Q^{\mathrm{T}}WQ$, where W is defined by (1.1). Then

$$\frac{\mathrm{d}Q^{\mathrm{T}}}{\mathrm{d}t}Q = -Q^{\mathrm{T}}\frac{\mathrm{d}Q}{\mathrm{d}t} = -Q^{\mathrm{T}}WQ = \mathit{\Pi},$$

from which it follows that Q^{T} is the fundamental matrix of $\mathit{\Pi}$, that is,

$$\frac{\mathrm{d}Q^{\mathrm{T}}}{\mathrm{d}t} = \mathit{\Pi}Q^{\mathrm{T}} \quad \text{and} \quad Q^{\mathrm{T}}(0) = \mathbb{I}.$$

Since $\xi = Q^{\mathrm{T}}\eta$, it is clear that

$$y_C(t) = \int_0^t Q(s)\xi(s)\,\mathrm{d}s.$$

3. Notation and preliminary results

In addition to the notation introduced in the previous sections, we shall adopt the following. We denote by $L^s(\mathcal{O})$, $H^s(\mathcal{O})$ etc., the classical Lebesgue and Sobolev spaces on a domain $\mathcal{O}$, with norms $\|\cdot\|_{s,\mathcal{O}}$ and $\|\cdot\|_{s,2,\mathcal{O}}$, respectively. Whenever confusion does not arise, we shall omit the subscript $\mathcal{O}$ in the above norms. Classical properties and results related to these spaces can be found in, for example, [1, 9]. Likewise, if X is a Banach space, we denote by $L^r(a,b;X)$ and $C([a,b];X)$ the space of all measurable functions from $[a,b]$ to X such that

$$\int_a^b \|u(t)\|_X^r\,\mathrm{d}t < \infty,\ 1 \leqslant r < \infty, \quad \text{or} \quad \operatorname{ess\,sup} \|u(t)\|_X^r < \infty,\ r = \infty,$$

and the space of continuous functions from $[a,b]$ to X, respectively. Throughout the paper we shall use the same font style to denote scalar, vector and tensor-valued functions.

If X_1 is a Banach space and X_0 is a closed subspace of X_1, we will denote by $\mathbb{P}_{X_1,X_0}$ the projection operator from X_1 onto X_0, while $\mathbb{J}_{X_0,X_1}$ will denote the canonical injection $X_0 \hookrightarrow X_1$.

It is well known that, for an arbitrary domain $\mathcal{O}$, the *Helmholtz decomposition* $L^2(\mathcal{O})^3 = H(\mathcal{O}) \oplus G(\mathcal{O})$ holds with

$$H(\mathcal{O}) := \text{completion of } \{u \in C_0^\infty(\mathcal{O})^3; \nabla\cdot u = 0\} \text{ in the norm of } L^2(\mathcal{O})^3,$$
$$G(\mathcal{O}) := \{w \in L^2(\mathcal{O})^3; w = \nabla p \text{ for some } p \in L^2_{\mathrm{loc}}(\mathcal{O})\},$$

see [9, p. 107]. If $\mathcal{O}$ is a $C^{0,1}$-domain, then

$$H(\mathcal{O}) = \{u \in L^2(\mathcal{O})^3; \nabla\cdot u = 0 \text{ in } \mathcal{O} \text{ and } u\cdot n = 0 \text{ on } \partial\mathcal{O}\},$$

where $\nabla\cdot u = 0$ and $u\cdot n$ are understood in the weak sense [9, p. 119]. Another common space in hydrodynamics is

$$V(\mathcal{O}) := \text{completion of } \{u \in C_0^\infty(\mathcal{O})^3; \nabla\cdot u = 0\} \text{ in the norm of } H^1(\mathcal{O}).$$

If $\mathcal{O}$ is a $C^{0,1}$-domain then

$$V(\mathcal{O}) = \{u \in L^2(\mathcal{O})^3; \nabla\cdot u = 0 \text{ in } \mathcal{O} \text{ and } u = 0 \text{ on } \partial\mathcal{O}\}.$$

Throughout the text of this paper, absolute constants depending on $m_{\mathcal{B}}$ and/or $I_{\mathcal{B}}$ will be represented by $C(\mathcal{B})$, whereas the dependence on ρ and ν will be expressed in terms of a constant represented by $C(\mathcal{L})$. We will also use other notation such as $C(\mathcal{B},\mathcal{L})$ to express the dependence on physical properties of both $\mathcal{B}$ and $\mathcal{L}$.

4. Functional setting for the equations

From now on, we set $\Omega := \bar{\mathcal{F}} \cup \mathcal{S}$ and denote by $\mathcal{R}(\Omega)$ the space of all rigid velocity fields in Ω:

$$\mathcal{R}(\Omega) = \{u \in C^\infty(\Omega)^3; u(x) = \xi + \omega \times x,\ \xi, \omega \in \mathbb{R}^3\}.$$

Note that we have $u \in \mathcal{R}(\Omega)$ if and only if $D(u) = 0$, where $D(u)$ is the symmetric part of ∇u. Moreover, we define

$$C^\infty_R(\Omega) = \{\phi \in C^\infty(\Omega)^3; D(\phi) = 0 \text{ in a neighbourhood of } \bar{\mathcal{S}}\},$$
$$\mathcal{D}_R(\Omega) = \{\phi \in C^\infty_R(\Omega); \nabla \cdot \phi = 0\}.$$

For $1 \leqslant s \leqslant \infty$, we denote by $L^s_R(\Omega)$ and $H^s_R(\Omega)$ the spaces obtained by completion of $C^\infty_R(\Omega)$ in the norms of $L^s(\Omega; \tilde{\rho}\,\mathrm{d}x)^3$ and $H^s(\Omega; \tilde{\rho}\,\mathrm{d}x)^3$, respectively, where

$$\tilde{\rho}(x) = \begin{cases} \rho_{\mathcal{B}}(x) & \text{in } \mathcal{S}, \\ \rho & \text{in } \mathcal{F}. \end{cases}$$

In particular, $L^2_R(\Omega)$ is a Hilbert space with inner product

$$(u, v) := \int_\Omega u \cdot v \tilde{\rho}\,\mathrm{d}x = m_{\mathcal{B}} \xi_u \cdot \xi_v + \omega_u \cdot I_{\mathcal{B}} \omega_v + \rho \int_{\mathcal{F}} u \cdot v\,\mathrm{d}x, \tag{4.1}$$

whose induced norm will be denoted by $|\cdot|$. It is easy to show that

$$L^s_R(\Omega) = \{\phi \in L^s(\Omega); D(\phi) = 0 \text{ in } \mathcal{S}\}.$$

For $u \in L^s_R(\Omega)$, we will use the notation $\bar{u} = u|_{\mathcal{S}}$ and $\bar{u}(x) = \xi_u + \omega_u \times x$.

LEMMA 4.1. *For $1 < s < \infty$, $(L^s_R(\Omega))' = L^{s'}_R(\Omega)$ where s' is such that $1/s + 1/s' = 1$. This means that the dual space of $L^s_R(\Omega)$ is isometrically isomorphic to $L^{s'}_R(\Omega)$.*

Proof. To begin with, we observe that, for $1 < s < \infty$, $L^s_R(\Omega)$ is reflexive, because it is a closed subspace of $L^s(\Omega)^3$.

Now, we consider the mapping $\mathcal{T}\colon L^{s'}_R(\Omega) \to (L^s_R(\Omega))'$ defined by

$$\langle \mathcal{T}u, v\rangle = \int_\Omega u \cdot v \tilde{\rho}\,\mathrm{d}x, \quad v \in L^s_R(\Omega).$$

By a standard procedure (see, for example, [3, p. 60]), we show that

$$\|\mathcal{T}u\|_{(L^s_R(\Omega))'} = \|u\|_{L^{s'}_R(\Omega)} \quad \forall u \in L^{s'}_R(\Omega),$$

and, therefore, $\mathcal{T}$ is an isometry between $L^{s'}_R(\Omega)$ and $\mathcal{T}(L^{s'}_R(\Omega)) \subseteq (L^s_R(\Omega))'$.

It remains to show that $\mathcal{T}$ is surjective. This will be achieved by showing that $\mathcal{T}(L^{s'}_R(\Omega))$ is dense in $(L^s_R(\Omega))'$. Let $v \in (L^s_R(\Omega))'' = L^s_R(\Omega)$ be such that

$$\langle \mathcal{T}u, v\rangle = \int_\Omega u v \tilde{\rho}\,\mathrm{d}x = 0 \quad \forall u \in L^{s'}_R(\Omega).$$

Then, by considering u such that $\bar{u} = 0$, we have in particular that

$$\int_{\mathcal{F}} u \cdot v\,\mathrm{d}x = 0 \quad \forall u \in L^{s'}(\mathcal{F}),$$

which implies that $v|_{\mathcal{F}} \equiv 0$. Hence,

$$\langle \mathcal{T}u, v\rangle = m_{\mathcal{B}}\xi_u \cdot \xi_v + \omega_u \cdot I_{\mathcal{B}}\omega_v \quad \forall u \in L^{s'}_R(\Omega).$$

Choosing $u = e_i$, $i = 1, 2, 3$, yields $m_{\mathcal{B}}\xi_v = 0$, while, for $u = e_i \times x$, $i = 1, 2, 3$, we obtain $I_{\mathcal{B}}\omega_v = 0$, and therefore $\bar{v} \equiv 0$. Since $v \equiv 0$, by the Hahn–Banach theorem, $\mathcal{T}(L^{s'}_R(\Omega))$ is dense in $(L^s_R(\Omega))'$. □

We now introduce two subspaces of $L^2_R(\Omega)$: the space $\mathcal{H}(\Omega)$ obtained by completion of $\mathcal{D}_R(\Omega)$ in the norm induced by (4.1), and the space $\mathcal{V}(\Omega)$ obtained by completion of $\mathcal{D}_R(\Omega)$ with respect to the norm induced by the inner product

$$((u, v)) := (u, v) + 2\nu \int_{\mathcal{F}} D(u) : D(v)\,\mathrm{d}x, \tag{4.2}$$

whose associated norm in $\mathcal{V}(\Omega)$ will be denoted by $\|\cdot\|$. The notation $\mathbb{P}_{L^2_R,\mathcal{H}}$ will indicate the orthogonal projector in $L^2_R(\Omega)$ onto $\mathcal{H}(\Omega)$ with respect to the inner product (4.1). It is the adjoint of the embedding $\mathbb{J}_{\mathcal{H},L^2_R}$.

The spaces $\mathcal{H}(\Omega)$ and $\mathcal{V}(\Omega)$ are characterized as follows:

$$\mathcal{H}(\Omega) = \{\phi \in L^2(\Omega)^3; \nabla\cdot\phi = 0 \text{ in } \Omega, D(\phi) = 0 \text{ in } \mathcal{S}\},$$
$$\mathcal{V}(\Omega) = \{\phi \in H^1(\Omega)^3; \nabla\cdot\phi = 0 \text{ in } \Omega, D(\phi) = 0 \text{ in } \mathcal{S}\}.$$

Clearly, $\mathcal{D}_R(\Omega) \subset H^1_R(\Omega)$ and, since $\mathcal{D}_R(\Omega) \subset \mathcal{V}(\Omega)$, $\mathcal{V}(\Omega)$ is dense in $\mathcal{H}(\Omega)$. Moreover, due to the compact embedding of $H^1(\Omega)$ in $L^2(\Omega)$, we have the following.

Lemma 4.2. *$\mathcal{V}(\Omega)$ is compactly embedded in $\mathcal{H}(\Omega)$.*

We will use another subspace of $\mathcal{H}(\Omega)$ formed by more regular functions than those in $\mathcal{V}(\Omega)$: the space $\mathcal{W}(\Omega)$ obtained by completion of $\mathcal{D}_R(\Omega)$ in $H^2_R(\Omega)$. Denoting by $\mathcal{V}'(\Omega)$ and $\mathcal{W}'(\Omega)$ the dual spaces, with respect to the pivot space $\mathcal{H}(\Omega)$, of $\mathcal{V}(\Omega)$ and $\mathcal{W}(\Omega)$, respectively, the following injections are valid:

$$\mathcal{W}(\Omega) \hookrightarrow \mathcal{V}(\Omega) \hookrightarrow \mathcal{H}(\Omega) \hookrightarrow \mathcal{V}'(\Omega) \hookrightarrow \mathcal{W}'(\Omega).$$

For the orthogonal complement $\mathcal{H}^{\perp}(\Omega)$ of $\mathcal{H}(\Omega)$ in $L^2_R(\Omega)$, the following characterization holds.

Lemma 4.3. *We have*

$$\mathcal{H}^{\perp}(\Omega) = \biggl\{u \in L^2_R(\Omega);\ u|_{\mathcal{F}} = \nabla p,\ \xi_u = -\frac{\rho}{m_{\mathcal{B}}}\int_{\mathcal{F}} \nabla p\,\mathrm{d}x \\ \text{and } \omega_u = -\rho I_{\mathcal{B}}^{-1}\int_{\mathcal{F}} x \times \nabla p\,\mathrm{d}x \text{ for some } p \in L^2_{\mathrm{loc}}(\mathcal{F})\biggr\}.$$

If $\partial\mathcal{F}$ is locally Lipschitz, then

$$\mathcal{H}^{\perp}(\Omega) = \biggl\{u \in L^2_R(\Omega);\ u|_{\mathcal{F}} = \nabla p,\ \xi_u = -\frac{\rho}{m_{\mathcal{B}}}\int_{\partial\mathcal{F}} pn\,\mathrm{d}\sigma_x \\ \text{and } \omega_u = -\rho I_{\mathcal{B}}^{-1}\int_{\partial\mathcal{F}} px \times n\,\mathrm{d}\sigma_x \text{ for some } p \in H^1(\mathcal{F})\biggr\}.$$

https://doi.org/10.1017/S0308210510001034 Published online by Cambridge University Press

Proof. The arguments are similar to those used in [20]. Set

$$\mathcal{Z}(\Omega) := \Big\{ u \in L^2_R(\Omega);\ u|_{\mathcal{F}} = \nabla p,\ \xi_u = -\frac{\rho}{m_{\mathcal{B}}} \int_{\mathcal{F}} \nabla p \,\mathrm{d}x \\ \text{and } \omega_u = -\rho I_{\mathcal{B}}^{-1} \int_{\mathcal{F}} x \times \nabla p \,\mathrm{d}x \text{ for some } p \in L^2_{\mathrm{loc}}(\mathcal{F}) \Big\}.$$

Let $u \in \mathcal{Z}(\Omega)$ and $v \in \mathcal{D}_R(\Omega)$. Then we have

$$\begin{aligned} (u, v) &= m_{\mathcal{B}} \xi_v \cdot \Big(-\frac{\rho}{m_{\mathcal{B}}} \int_{\mathcal{F}} \nabla p \,\mathrm{d}x \Big) + \omega_v \cdot I_{\mathcal{B}} \Big(-\rho I_{\mathcal{B}}^{-1} \int_{\mathcal{F}} x \times \nabla p \,\mathrm{d}x \Big) \\ &\quad + \rho \int_{\mathcal{F}} v \cdot \nabla p \,\mathrm{d}x \\ &= \rho \int_{\mathcal{F}} (v - \bar{v}) \cdot \nabla p \,\mathrm{d}x = \rho \int_{\mathcal{F}} p \nabla \cdot (\bar{v} - v) \,\mathrm{d}x \\ &= 0 \end{aligned}$$

because $v - \bar{v}$ vanishes in a neighbourhood of $\partial\mathcal{F}$ and $\nabla \cdot (\bar{v} - v) = 0$. Hence $(u, v) = 0$ for all $v \in \mathcal{D}_R(\Omega)$ and, by density, $(u, v) = 0$ for all $v \in \mathcal{H}(\Omega)$, that is, $u \in \mathcal{H}^{\perp}(\Omega)$. Conversely, $u \in \mathcal{H}^{\perp}(\Omega)$ means that

$$(u, v) = 0 \quad \forall v \in \mathcal{H}(\Omega), \tag{4.3}$$

and, in particular, $(u, v) = 0$ for all $v \in \mathcal{H}(\Omega)$ such that $\bar{v} = 0$. This in turn implies that

$$\int_{\mathcal{F}} u \cdot v \,\mathrm{d}x = 0 \quad \forall v \in H(\mathcal{F}),$$

and therefore (recall the Helmholtz decomposition of the space $L^2(\mathcal{F})$) there exists $p \in L^2_{\mathrm{loc}}(\mathcal{F})$ such that $u|_{\mathcal{F}} = \nabla p$. Now, we take $v = e_i$ and $v = e_i \times x$, $i = 1, 2, 3$, in (4.3) and use the fact that $u|_{\mathcal{F}} = \nabla p$ to obtain

$$m_{\mathcal{B}} \xi_u + \rho \int_{\mathcal{F}} \nabla p \,\mathrm{d}x = I_{\mathcal{B}} \omega_u + \rho \int_{\mathcal{F}} x \times \nabla p \,\mathrm{d}x = 0.$$

Thus, $u \in \mathcal{Z}(\Omega)$.

In the case when $\partial\mathcal{F}$ is locally Lipschitz, we only have to apply the divergence theorem to the integrals

$$\int_{\mathcal{F}} \nabla p \,\mathrm{d}x \quad \text{and} \quad \int_{\mathcal{F}} x \times \nabla p \,\mathrm{d}x.$$

□

Since $\mathcal{R}(\Omega) \subset \mathcal{H}(\Omega)$ has finite dimension, it is a closed subspace of $\mathcal{H}(\Omega)$. Consider the following closed linear subspace of $\mathcal{H}(\Omega)$:

$$\mathcal{H}_{\sharp}(\Omega) = \{ u \in \mathcal{H}(\Omega); (u, \bar{v}) = 0 \text{ for all } \bar{v} \in \mathcal{R}(\Omega) \}.$$

Then we have the following decomposition of $\mathcal{H}(\Omega)$:

$$\mathcal{H}(\Omega) = \mathcal{H}_{\sharp}(\Omega) \oplus \mathcal{R}(\Omega).$$

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Since $\mathcal{R}(\Omega) = \operatorname{span}\{e_i|_\Omega, e_i \times x|_\Omega;\ i = 1, 2, 3\}$, we have

$$(u, \bar{v}) = 0 \quad \text{for all } \bar{v} \in \mathcal{R}(\Omega)$$
$$\iff \begin{cases} m_{\mathcal{B}}\xi_u \cdot e_i + \rho \displaystyle\int_{\mathcal{F}} u \cdot e_i \,\mathrm{d}x = 0, & i = 1, 2, 3, \\ (\omega_u I_{\mathcal{B}}) \cdot e_i + \rho \displaystyle\int_{\mathcal{F}} u \cdot (e_i \times x) \,\mathrm{d}x = 0, & i = 1, 2, 3, \end{cases}$$

and, therefore,

$$\mathcal{H}_\sharp(\Omega) = \left\{ u \in \mathcal{H}(\Omega);\, m_{\mathcal{B}}\xi_u + \rho \int_{\mathcal{F}} u \,\mathrm{d}x = I_{\mathcal{B}}\omega_u + \rho \int_{\mathcal{F}} x \times u \,\mathrm{d}x = 0 \right\}.$$

LEMMA 4.4. *Let $u \in \mathcal{H}_\sharp(\Omega) \cap \mathcal{V}(\Omega)$. Then there exists a positive constant $C = C(\mathcal{B}, \mathcal{L})$ such that*

$$|\xi_u| + |\omega_u| + \|u\|_{2,\mathcal{F}} + \|\nabla u\|_{2,\mathcal{F}} \leqslant C\|D(u)\|_{2,\mathcal{F}}.$$

Proof. By the Poincaré inequality,

$$\|u - \bar{u}\|_{2,\mathcal{F}} \leqslant C\|\nabla(u - \bar{u})\|_{2,\mathcal{F}} = 2C\|D(u)\|_{2,\mathcal{F}}.$$

Since $u \in \mathcal{H}_\sharp(\Omega)$, we get

$$\begin{aligned} \|u - \bar{u}\|_{2,\mathcal{F}}^2 &= \|u\|_{2,\mathcal{F}}^2 - 2\int_{\mathcal{F}} u \cdot \bar{u} + \|\bar{u}\|_{2,\mathcal{F}}^2 \\ &= \|u\|_{2,\mathcal{F}}^2 + \frac{2m_{\mathcal{B}}}{\rho}|\xi_u|^2 + 2\omega_u \cdot \left(\frac{I_{\mathcal{B}}}{\rho}\omega_u\right) + \|\bar{u}\|_{2,\mathcal{F}}^2. \end{aligned}$$

□

Consider the symmetric bilinear form $a\colon \mathcal{V}(\Omega) \times \mathcal{V}(\Omega) \to \mathbb{R}$ defined by

$$a(u, v) = 2\nu \int_{\mathcal{F}} D(u) \colon D(v) \,\mathrm{d}x.$$

Then a satisfies $|a(u, v)| \leqslant \|u\|\|v\|$ for all $u, v \in \mathcal{V}(\Omega)$ and, for each $\lambda > 0$, we have $a(u, u) \geqslant \min\{1, \lambda\}\|u\|^2 - \lambda|u|^2$ for all $u \in \mathcal{V}(\Omega)$.

The form a induces a bounded linear operator $\mathcal{A}\colon \mathcal{V}(\Omega) \to \mathcal{V}'(\Omega)$ defined as

$$\langle \mathcal{A}u, v\rangle_{\mathcal{V}'(\Omega),\mathcal{V}(\Omega)} = a(u, v), \quad u, v \in \mathcal{V}(\Omega),$$

which satisfies

$$\|\mathcal{A}(u)\|_{\mathcal{V}'(\Omega)} \leqslant \|u\| \quad \forall u \in \mathcal{V}(\Omega). \tag{4.4}$$

It is easy to show that $\ker(\mathcal{A}) = \mathcal{R}(\Omega)$, and that $\mathcal{A} + \lambda\mathbb{J}_{\mathcal{V}(\Omega),\mathcal{V}'(\Omega)}$ is an isomorphism from $\mathcal{V}(\Omega)$ onto $\mathcal{V}'(\Omega)$ when $\lambda > 0$.

The operator $\mathcal{A}$ has a restriction to $\mathcal{H}(\Omega)$ defined as

$$\begin{gathered} D(A) = \{u \in \mathcal{V}(\Omega);\, \mathcal{A}u \in \mathcal{H}(\Omega)\}, \\ (Au, v) = a(u, v), \quad u \in D(A),\ v \in \mathcal{V}(\Omega). \end{gathered}$$

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By direct calculations, we can show that if $u \in \mathcal{D}_R(\Omega)$ then $a(u,v) = (-\nu\Delta u, v)$. Since $D(\Delta u) = 0$ in a neighbourhood of $\mathcal{S}$ and $\nabla \cdot (\Delta u) = 0$ in Ω, it follows that $\mathcal{D}_R(\Omega) \subset D(A)$, and therefore $D(A)$ is dense in $\mathcal{H}(\Omega)$.

The operator A is symmetric because a is a symmetric form, and A is accretive because $(Au,u) = a(u,u) = 2\nu\|D(u)\|^2_{2,\mathcal{F}} \geqslant 0$ for all $u \in D(A)$.

In order to show that $\mathrm{Ran}(\mathbb{I}_{\mathcal{H}(\Omega)} + A) = \mathcal{H}(\Omega)$, where $\mathbb{I}_{\mathcal{H}(\Omega)}$ is the identity operator in $\mathcal{H}(\Omega)$, we consider the following problem. Given $w \in \mathcal{H}(\Omega)$, find $u \in D(A)$ such that $u + Au = w$. By the Lax–Milgram theorem, the variational problem

$$(u,\varphi) + a(u,\varphi) = (w,\varphi) \quad \forall \varphi \in \mathcal{V}(\Omega), \tag{4.5}$$

has a unique solution $u \in \mathcal{V}(\Omega)$. It follows from (4.5) that $\mathcal{A}u = u - w$ in $\mathcal{V}'(\Omega)$, and since $u - w \in \mathcal{H}(\Omega)$, we conclude that $u \in D(A)$ and $u + Au = w$.

From the above properties of the operator A, we conclude that $-A$ generates an analytic semigroup of contractions in $\mathcal{H}(\Omega)$.

Let us henceforth write $\Delta_D^{\mathcal{F}}$ to denote the Dirichlet–Laplacian on $H_0^1(\mathcal{F})$. The next theorem gives a characterization of the operator A in terms of $\Delta_D^{\mathcal{F}}$.

THEOREM 4.5. *Let $u \in D(A)$. Then there exists $p \in L^2_{\mathrm{loc}}(\mathcal{F})$ such that*

$$Au = \begin{cases} -\dfrac{\nu}{\rho}\Delta_D^{\mathcal{F}}(u - \bar{u}) + \dfrac{1}{\rho}\nabla p \quad \text{in } \mathcal{F}, \\ \dfrac{1}{m_{\mathcal{B}}}\displaystyle\int_{\mathcal{F}} (\nu\Delta_D^{\mathcal{F}}(u - \bar{u}) - \nabla p)\,\mathrm{d}y \\ \qquad\qquad + \left(I_{\mathcal{B}}^{-1}\displaystyle\int_{\mathcal{F}} y \times (\nu\Delta_D^{\mathcal{F}}(u - \bar{u}) - \nabla p)\,\mathrm{d}y\right) \times x \quad \text{in } \mathcal{S}. \end{cases}$$

Proof. Let $u \in D(A)$. We have

$$(Au,v) = 2\nu\int_{\mathcal{F}} D(u) : D(v)\,\mathrm{d}x = 2\nu\int_{\mathcal{F}} D(u - \bar{u}) : D(v)\,\mathrm{d}x \quad \forall v \in \mathcal{V}(\Omega). \tag{4.6}$$

For $v \in \mathcal{V}(\Omega)$ with $\bar{v} = 0$, we obtain

$$\rho\int_{\mathcal{F}} Au|_{\mathcal{F}} \cdot v\,\mathrm{d}x = 2\nu\int_{\mathcal{F}} D(u - \bar{u}) : D(v)\,\mathrm{d}x,$$

with $u - \bar{u} \in V(\mathcal{F})$. This means that

$$\langle \rho Au|_{\mathcal{F}} + \nu\Delta_D^{\mathcal{F}}(u - \bar{u}), v\rangle_{(C_0^\infty(\mathcal{F})^3)', C_0^\infty(\mathcal{F})^3} = 0 \quad \forall v \in C_0^\infty(\mathcal{F})^3 \text{ with } \nabla \cdot v = 0,$$

and, therefore, there exists $p \in L^2_{\mathrm{loc}}(\mathcal{F})$ (with $\nabla p \in H^{-1}(\mathcal{F})^3$) such that

$$\rho Au|_{\mathcal{F}} = -\nu\Delta_D^{\mathcal{F}}(u - \bar{u}) + \nabla p \in L^2(\mathcal{F})^3.$$

Taking $v = e_i$ and then $v = e_i \times x$, $i = 1,2,3$, in (4.6) yields

$$\rho\int_{\mathcal{F}} Au|_{\mathcal{F}}\,\mathrm{d}x + m_{\mathcal{B}}\xi_{Au} = 0, \quad \text{i.e. } \xi_{Au} = \frac{1}{m_{\mathcal{B}}}\int_{\mathcal{F}} (\nu\Delta_D^{\mathcal{F}}(u - \bar{u}) - \nabla p)\,\mathrm{d}x,$$

$$\rho\int_{\mathcal{F}} x \times Au|_{\mathcal{F}}\,\mathrm{d}y + I_{\mathcal{B}}\omega_{Au} = 0, \quad \text{i.e. } \omega_{Au} = I_{\mathcal{B}}^{-1}\int_{\mathcal{F}} x \times (\nu\Delta_D^{\mathcal{F}}(u - \bar{u}) - \nabla p)\,\mathrm{d}x.$$

□

When $\mathcal{F}$ is a $C^{1,1}$-domain, we have $D(A) = \{u \in \mathcal{V}(\Omega); u|_{\mathcal{F}} \in H^2(\mathcal{F})^3\}$. Indeed, in that case,

$$-\frac{\nu}{\rho}\Delta_D^{\mathcal{F}}(u-\bar{u}) + \frac{1}{\rho}\nabla p \in L^2(\mathcal{F})^3$$

and $u = \bar{u}$ on $\partial\mathcal{F}$ in the trace sense, so that, by classical results for the Stokes equations [2], we conclude that $u \in H^2(\mathcal{F})^3$ and $p \in H^1(\mathcal{F})$. We also have

$$\int_{\mathcal{F}} (\nu\Delta_D^{\mathcal{F}}(u-\bar{u}) - \nabla p)\,\mathrm{d}x = 2\nu\int_{\partial\mathcal{F}} D(u)n\,\mathrm{d}\sigma_x - \int_{\partial\mathcal{F}} pn\,\mathrm{d}\sigma_x,$$

$$\int_{\mathcal{F}} x\times(\nu\Delta_D^{\mathcal{F}}(u-\bar{u}) - \nabla p)\,\mathrm{d}x = 2\nu\int_{\partial\mathcal{F}} x\times D(u)n\,\mathrm{d}\sigma_x - \int_{\partial\mathcal{F}} x\times pn\,\mathrm{d}\sigma_x.$$

Therefore, A is given by

$$Au = \begin{cases} -\dfrac{\nu}{\rho}\Delta u + \dfrac{1}{\rho}\nabla p \quad \text{in } \mathcal{F}, \\ \dfrac{2\nu}{m_{\mathcal{B}}}\displaystyle\int_{\partial\mathcal{F}} D(u)n\,\mathrm{d}\sigma_y + \left(2\nu I_{\mathcal{B}}^{-1}\int_{\partial\mathcal{F}} y\times D(u)n\,\mathrm{d}\sigma_y\right)\times x \\ \qquad\qquad - \dfrac{1}{m_{\mathcal{B}}}\displaystyle\int_{\partial\mathcal{F}} pn\,\mathrm{d}\sigma_y - \left(I_{\mathcal{B}}^{-1}\int_{\partial\mathcal{F}} y\times pn\,\mathrm{d}\sigma_y\right)\times x \quad \text{in } \mathcal{S}. \end{cases}$$

For an arbitrary $\mathcal{F}$, we introduce the operator $\mathcal{L}\colon H_R^1(\Omega) \to (H_R^1(\Omega))'$ defined by

$$\langle \mathcal{L}u, v\rangle_{(H_R^1(\Omega))', H_R^1(\Omega)} = \nu\int_{\mathcal{F}} \nabla(u-\bar{u})\colon \nabla(v-\bar{v})\,\mathrm{d}x, \quad u, v \in H_R^1(\Omega).$$

Furthermore, we define

$$D(L) = \{u \in \mathcal{V}(\Omega); \Delta_D^{\mathcal{F}}(u-\bar{u}) \in L^2(\mathcal{F})\},$$

$$Lu = \begin{cases} \dfrac{\nu}{\rho}\Delta_D^{\mathcal{F}}(u-\bar{u}) \text{ in } \mathcal{F}, \\ -\dfrac{\nu}{m_{\mathcal{B}}}\displaystyle\int_{\mathcal{F}} \Delta_D^{\mathcal{F}}(u-\bar{u})\,\mathrm{d}y - \left(\nu I_{\mathcal{B}}^{-1}\int_{\mathcal{F}} y\times\Delta_D^{\mathcal{F}}(u-\bar{u})\,\mathrm{d}y\right)\times x \text{ in } \mathcal{S}, \end{cases}$$

which takes the form

$$D(L) = \{u \in \mathcal{V}(\Omega); u|_{\mathcal{F}} \in H^2(\mathcal{F})\},$$

$$Lu = \begin{cases} \dfrac{\nu}{\rho}\Delta u \quad \text{in } \mathcal{F}, \\ -\dfrac{2\nu}{m_{\mathcal{B}}}\displaystyle\int_{\mathcal{F}} D(u)n\,\mathrm{d}\sigma_y - \left(2\nu I_{\mathcal{B}}^{-1}\int_{\partial\mathcal{F}} y\times D(u)n\,\mathrm{d}\sigma_y\right)\times x \quad \text{in } \mathcal{S}, \end{cases}$$

when the domain $\mathcal{F}$ is $C^{1,1}$. The next lemma gives a characterization of $\mathcal{A}$ and A in terms of the operators $\mathcal{L}$ and L, respectively.

LEMMA 4.6. *For an arbitrary domain $\mathcal{F}$, we have $\mathcal{A} = -\mathbb{P}_{(H_R^1)',\mathcal{V}'}\mathcal{L}$. If $\mathcal{F}$ is a $C^{1,1}$ domain, then $A = -\mathbb{P}_{L_R^2,\mathcal{H}}L$.*

https://doi.org/10.1017/S0308210510001034 Published online by Cambridge University Press

Proof. To prove that $\mathcal{A} = -\mathbb{P}_{(H^1_R)',\mathcal{V}'}\mathcal{L}$ we just have to observe that $\mathbb{P}_{(H^1_R)',\mathcal{V}'}f$ means the restriction of $f \in (H^1_R(\Omega))'$ to the test space $\mathcal{V}(\Omega)$ and that

$$\int_{\mathcal{F}} \nabla u \colon \nabla v \,\mathrm{d}x = 2\int_{\mathcal{F}} D(u)\colon D(v)\,\mathrm{d}x, \quad u, v \in V(\mathcal{F}).$$

The second result follows from the previous properties of the operators A and L and lemma 4.3. □

REMARK 4.7. We recall the embedding $\mathbb{J}_{\mathcal{H},L^2_R}$ and note that the restriction of $\mathbb{J}_{\mathcal{H},L^2_R}$ to $\mathcal{V}(\Omega)$ has its range in $H^1_R(\Omega)$ so that, by using a classical result [25, proposition 2.9.3], $\mathbb{P}_{(H^1_R)',\mathcal{V}'}$ is the extension of $\mathbb{P}_{L^2_R,\mathcal{H}}$ to $(H^1_R(\Omega))'$.

To conclude this section, note that, since A is a non-negative operator, there exists a uniquely determined non-negative, self-adjoint operator $A^{1/2}\colon D(A^{1/2}) \to \mathcal{H}(\Omega)$ such that

$$D(A^{1/2}) = \mathcal{V}(\Omega), \qquad \langle A^{1/2}u, A^{1/2}v\rangle = 2\nu\int_{\mathcal{F}} D(u)\colon D(v)\,\mathrm{d}x = a(u,v).$$

More generally, since the resolvent set of A contains the negative real ray $(-\infty, 0)$ and e^{-tA} is an analytic semigroup in $\mathcal{H}(\Omega)$, according to [13], the fractional powers A^δ ($\delta > 0$) can be constructed, and the following estimates are valid for the semigroup $\{\mathrm{e}^{-tA}\}_{t\geqslant 0}$ generated by $-A$ [7]:

$$\|t^\delta A^\delta \mathrm{e}^{-tA}u\| \leqslant M(\delta)\|u\|, \quad u \in \mathcal{H}(\Omega),$$

where $M(\delta)$ is a positive constant.

5. Weak solutions

We will assume that the external forces and torque in (2.1) are zero. For $T > 0$ and u_0 in a suitable space, a *weak solution* of problem (2.1) should satisfy

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\biggl(m_{\mathcal{B}}\xi_u\cdot\xi_\varphi + (I_{\mathcal{B}}\omega_u)\cdot\omega_\varphi + \rho\int_{\mathcal{F}} u\cdot\varphi\,\mathrm{d}x\biggr) + 2\nu\int_{\mathcal{F}} D(u)\colon D(\varphi)\,\mathrm{d}x \\ + m_{\mathcal{B}}\omega_u\times\xi_u\cdot\xi_\varphi + \omega_u\times(I_{\mathcal{B}}\omega_u)\cdot\omega_\varphi + \rho\int_{\mathcal{F}}(u-\bar{u})\cdot\nabla u\cdot\varphi\,\mathrm{d}x \\ + \rho\int_{\mathcal{F}}\omega_u\times u\cdot\varphi\,\mathrm{d}x = 0 \quad \forall\varphi\in\mathcal{V}(\Omega) \text{ and almost everywhere (a.e.) in } [0,T],\end{aligned}$$

and $u(0) = u_0$ in some sense. In order to make the above integrals and the initial condition meaningful (and by analogy with the classical theory for the Navier–Stokes equations), we expect that if $u_0 \in \mathcal{H}(\Omega)$, then the function u satisfies $u \in L^2(0,T;\mathcal{V}(\Omega)) \cap C_w([0,T];\mathcal{H}(\Omega))$ with $u' \in L^\alpha(0,T;\mathcal{V}'(\Omega))$ for some $\alpha \geqslant 1$.

In order to show the existence of weak solutions, we will prove some auxiliary results, namely, the existence of a special basis of the space $\mathcal{H}(\Omega)$ and some properties of the trilinear form associated with the nonlinear terms in equations (2.1).

THEOREM 5.1.

(i) *The spectral problem*

$$(w, v)_{H^2_R(\Omega)} = \lambda(w, v), \quad \forall v \in \mathcal{W}(\Omega),$$

admits a sequence $\{w_k\}_{k\in\mathbb{N}} \subset \mathcal{W}(\Omega)$ *of non-zero solutions corresponding to a sequence* $\{\lambda_k\}_{k\in\mathbb{N}}$ *of eigenvalues* $0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$, *which satisfy* $\lambda_k \to \infty$ *as* $k \to \infty$.

(ii) *We can choose* $\{w_k\}_{k\in\mathbb{N}}$ *so that it forms an orthonormal basis of* $\mathcal{H}(\Omega)$ *and such that* $\{w_k/\sqrt{\lambda_k}\}_{k\in\mathbb{N}}$ *is an orthonormal basis of* $\mathcal{W}(\Omega)$.

Proof. Given $w \in \mathcal{H}(\Omega)$, by the Lax–Milgram theorem, the variational problem

$$(u, \varphi)_{H^2_R(\Omega)} = (w, \varphi) \quad \forall \varphi \in \mathcal{W}(\Omega)$$

has a unique solution $u \in \mathcal{W}(\Omega)$. The operator $\Lambda\colon \mathcal{H}(\Omega) \to \mathcal{W}(\Omega)$, $w \mapsto u$ is linear and continuous. Since the injection of $\mathcal{W}(\Omega)$ into $\mathcal{H}(\Omega)$ is compact, $S := \mathbb{J}_{\mathcal{W},\mathcal{H}} \circ \Lambda$ is a compact operator in $\mathcal{H}(\Omega)$.

Let $w, \tilde{w} \in \mathcal{H}(\Omega)$ and set $u := Sw$ and $\tilde{u} := S\tilde{w}$. Then

$$(Sw, \tilde{w}) = (u, \tilde{w}) = (\tilde{w}, u) = (\tilde{u}, u)_{H^2_R(\Omega)} = (u, \tilde{u})_{H^2_R(\Omega)} = (w, \tilde{u}) = (w, S\tilde{w}),$$

and therefore S is symmetric. On the other hand, S is a positive operator: $(Sw, w) = (u, u)_{H^2_R(\Omega)} \geqslant 0$ for all $w \in \mathcal{H}(\Omega)$ and $u := Sw$, and if $(Sw, w) = 0$, we deduce that $u = 0$ and, thus, that $w = 0$. As a consequence, $I + S$ is one-to-one and, since S is compact, we deduce that $I + S$ is surjective. Hence, S is self-adjoint and $\mathcal{H}(\Omega)$ admits an orthonormal basis of eigenfunctions w_k of S with corresponding eigenvalues μ_k verifying $\mu_k > 0$ for all $k \in \mathbb{N}$, and $\mu_k \to 0$ as $k \to \infty$. Therefore, we have

$$(w_k, \varphi)_{H^2_R(\Omega)} = \lambda_k(w_k, \varphi) \quad \forall \varphi \in \mathcal{W}(\Omega), \tag{5.1}$$

where $\lambda_k = 1/\mu_k$.

Finally, we show that $\{w_k/\sqrt{\lambda_k}\}_{k\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{W}(\Omega)$. Suppose that $u \in \mathcal{W}(\Omega)$ satisfies $(w_k, u)_{H^2_R(\Omega)} = 0$ for all $k \in \mathbb{N}$. Since

$$(w_k, u) = (w_k, u)_{H^2_R(\Omega)}/\lambda_k$$

and $\{w_k\}_{k\in\mathbb{N}}$ is a basis of $\mathcal{H}(\Omega)$, it follows that $u \equiv 0$, and therefore $\{w_k/\sqrt{\lambda_k}\}_{k\in\mathbb{N}}$ is a basis of $\mathcal{W}(\Omega)$. This basis is orthonormal since it follows from (5.1) that

$$\begin{aligned}
\left(\frac{w_k}{\sqrt{\lambda_k}}, \frac{w_j}{\sqrt{\lambda_j}}\right)_{H^2_R(\Omega)} &= \frac{1}{\sqrt{\lambda_k}\sqrt{\lambda_j}}(w_k, w_j)_{H^2_R(\Omega)} \\
&= \frac{\lambda_k}{\sqrt{\lambda_k}\sqrt{\lambda_j}}(w_k, w_j) \\
&= \frac{\lambda_k}{\sqrt{\lambda_k}\sqrt{\lambda_j}}\delta_{kj} = \delta_{kj} \quad \forall k, j \in \mathbb{N}.
\end{aligned}$$

□

Now we investigate the properties of the trilinear form

$$b\colon \mathcal{V}(\Omega)\times\mathcal{V}(\Omega)\times\mathcal{V}(\Omega)\to\mathbb{R}$$

defined as

$$\begin{aligned} b(u,v,w) &= m_{\mathcal{B}}\omega_v\times\xi_u\cdot\xi_w+\omega_v\times(I_{\mathcal{B}}\omega_u)\cdot\omega_w \\ &\qquad + \rho\int_{\mathcal{F}}(v-\bar{v})\cdot\nabla u\cdot w\,\mathrm{d}x+\rho\int_{\mathcal{F}}\omega_v\times u\cdot w\,\mathrm{d}x \\ &= -\int_{\Omega} g(u,v)\cdot w\tilde{\rho}\,\mathrm{d}x, \end{aligned}$$

where $g(u,v)\in L^{3/2}_R(\Omega)$ is given by

$$g(u,v)=\begin{cases}(\bar{v}-v)\cdot\nabla u-\omega_v\times u & \text{in } \mathcal{F},\\ -\omega_v\times\xi_u-[I_{\mathcal{B}}^{-1}(\omega_v\times(I_{\mathcal{B}}\omega_u))]\times x & \text{in } \mathcal{S}.\end{cases} \tag{5.2}$$

Using an interpolation inequality for Lebesgue norms, Sobolev and Korn inequalities, we get the following estimates for $b(u,v,w)$:

$$\begin{aligned} |b(u,v,w)| &\leqslant m_{\mathcal{B}}|\omega_v||\xi_u||\xi_w|+|I_{\mathcal{B}}||\omega_v||\omega_u||\omega_w| \\ &\qquad + \|v-\bar{v}\|_{3,\mathcal{F}}\|\nabla u\|_{2,\mathcal{F}}\|w\|_{6,\mathcal{F}}+|\omega_v|\|u\|_{2,\mathcal{F}}\|w\|_{2,\mathcal{F}} \\ &\leqslant C(\mathcal{B},\mathcal{L})(|v|\|u\|\|w\|+\|v-\bar{v}\|^{1/2}_{2,\mathcal{F}}\|D(v)\|^{1/2}_{2,\mathcal{F}}\|u\|\|w\|) \\ &\leqslant C(\mathcal{B},\mathcal{L})(|v|\|u\|\|w\|+|v|^{1/2}\|v\|^{1/2}\|u\|\|w\|) \quad \forall w\in\mathcal{V}(\Omega). \end{aligned}$$

Hence, defining $\mathcal{G}\colon \mathcal{V}(\Omega)\times\mathcal{V}(\Omega)\to\mathcal{V}'(\Omega)$ as

$$\langle\mathcal{G}(u,v),w\rangle_{\mathcal{V}'(\Omega),\mathcal{V}(\Omega)}=b(u,v,w),\quad w\in\mathcal{V}(\Omega),$$

we have

$$\|\mathcal{G}(u,v)\|_{\mathcal{V}'(\Omega)}\leqslant C(\mathcal{B},\mathcal{L})(|v|\|u\|+|v|^{1/2}\|v\|^{1/2}\|u\|). \tag{5.3}$$

Actually, $b(u,v,w)$ is also defined for $w\in H^1_R(\Omega)$ and therefore

$$\mathcal{G}(u,v)=\mathbb{P}_{(H^1_R)',\mathcal{V}'}g(u,v).$$

On the other hand, since $\mathcal{V}'(\Omega)\hookrightarrow\mathcal{W}'(\Omega)$, we also have $\mathcal{G}(u,v)\in\mathcal{W}'(\Omega)$, whenever $u,v\in\mathcal{V}(\Omega)$, with

$$\langle\mathcal{G}(u,v),w\rangle_{\mathcal{W}'(\Omega),\mathcal{W}(\Omega)}=b(u,v,w),\quad w\in\mathcal{W}(\Omega).$$

Using the density of $\mathcal{D}_R(\Omega)$ in $\mathcal{V}(\Omega)$, we show that

$$\int_{\mathcal{F}}(v-\bar{v})\cdot\nabla u\cdot w\,\mathrm{d}x=-\int_{\mathcal{F}}(v-\bar{v})\cdot\nabla w\cdot u\,\mathrm{d}x$$

and, therefore,

$$\begin{aligned} |b(u,v,w)| &\leqslant m_{\mathcal{B}}|\omega_v||\xi_u||\xi_w|+|I_{\mathcal{B}}||\omega_v||\omega_u||\omega_w| \\ &\qquad + \|v-\bar{v}\|_{6,\mathcal{F}}\|\nabla w\|_{3,\mathcal{F}}\|u\|_{2,\mathcal{F}}+|\omega_v||\|u\|_{2,\mathcal{F}}\|w\|_{2,\mathcal{F}} \\ &\leqslant C(\mathcal{B},\mathcal{L})\|v\||u|\|w\|_{2,2,\Omega} \quad \forall w\in\mathcal{W}(\Omega). \end{aligned}$$

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Hence

$$\|\mathcal{G}(u,v)\|_{\mathcal{W}'(\Omega)} \leqslant C(\mathcal{B},\mathcal{L})\|v\||u|. \tag{5.4}$$

With a similar reasoning, we also show that

$$|b(u,v,w)| \leqslant C(\mathcal{B},\mathcal{L})|v|\|u\|\|w\|_{2,2,\Omega} \quad \forall w \in \mathcal{W}(\Omega).$$

From the inequalities (4.4), (5.3) and (5.4), we obtain the following lemma.

Lemma 5.2. *Let $T > 0$. If $u \in L^2(0,T;\mathcal{V}(\Omega)) \cap L^\infty(0,T;\mathcal{H}(\Omega))$, then*

$$\mathcal{A}u \in L^2(0,T;\mathcal{V}'(\Omega)),$$
$$\mathcal{G}(u,u) \in L^{4/3}(0,T;\mathcal{V}'(\Omega)) \cap L^2(0,T;\mathcal{W}'(\Omega))$$

and the following estimates hold

$$\int_0^T \|\mathcal{A}u\|^2_{\mathcal{V}'(\Omega)} \leqslant C(\mathcal{B},\mathcal{L})\|u\|^2_{L^2(0,T;\mathcal{V}(\Omega))},$$
$$\int_0^T \|\mathcal{G}(u,u)\|^{4/3}_{\mathcal{V}'(\Omega)} \leqslant C(\mathcal{B},\mathcal{L})\|u\|^{2/3}_{L^\infty(0,T;\mathcal{H}(\Omega))}\|u\|^2_{L^2(0,T;\mathcal{V}(\Omega))},$$
$$\int_0^T \|\mathcal{G}(u,u)\|^2_{\mathcal{W}'(\Omega)} \leqslant C(\mathcal{B},\mathcal{L})\|u\|^2_{L^\infty(0,T;\mathcal{H}(\Omega))}\|u\|^2_{L^2(0,T;\mathcal{V}(\Omega))},$$

where $C(\mathcal{B},\mathcal{L})$ is a positive constant.

Now we recall a well-known compactness result, which can be found, for example, in [24, p. 271]. Let X_0, X, X_1 be three Banach spaces such that $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$ and X_0, X_1 are reflexive. For $T > 0$ and $\alpha, \beta > 1$, let

$$\mathcal{Y}(0,T;\alpha,\beta;X_0,X_1) := \{u \in L^\alpha(0,T;X_0);\ u' \in L^\beta(0,T;X_1)\}$$

with norm $\|\cdot\|_{\mathcal{Y}} = \|\cdot\|_{L^\alpha(0,T;X_0)} + \|\cdot\|_{L^\beta(0,T;X_1)}$.

Theorem 5.3. *Under the above assumptions, the injection of $\mathcal{Y}(0,T;\alpha,\beta;X_0,X_1)$ into $L^\alpha(0,T;X)$ is compact.*

Another useful result is the following [24, p. 263].

Theorem 5.4. *Let X and Y be two Banach spaces such that $X \hookrightarrow Y$. If $u \in L^\infty(0,T;X) \cap C_w([0,T];Y)$, then $u \in C_w([0,T];X)$.*

In terms of the forms previously introduced and their properties, we can give the following definition of weak solution of problem (2.1).

Definition 5.5. For $T > 0$, we say that $u \in L^2(0,T;\mathcal{V}(\Omega)) \cap C_w([0,T];\mathcal{H}(\Omega))$, with $u' \in L^{4/3}(0,T;\mathcal{V}'(\Omega))$, is a *weak solution* of (2.1) provided

$$\langle u',\varphi\rangle + a(u,\varphi) + b(u,u,\varphi) = 0 \quad \forall\varphi \in \mathcal{V}(\Omega) \text{ and a.e. in } (0,T), \tag{5.5}$$
$$u(0) = u_0, \tag{5.6}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality product between $\mathcal{V}(\Omega)$ and $\mathcal{V}'(\Omega)$.

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We are now in a position to prove the main result of this section.

THEOREM 5.6. *Let $u_0 \in \mathcal{H}(\Omega)$. There exists a weak solution*

$$u \in L^\infty(0,\infty;\mathcal{H}(\Omega)) \cap C_w([0,\infty);\mathcal{H}(\Omega)) \cap L^2_{\mathrm{loc}}([0,\infty);\mathcal{V}(\Omega))$$

of (2.1) satisfying $u' \in L^{4/3}_{\mathrm{loc}}([0,\infty);\mathcal{V}'(\Omega))$ and the energy inequality

$$|u(t)|^2 + 4\nu \int_0^t \|D(u)(s)\|^2_{2,\mathcal{F}}\,\mathrm{d}s \leqslant |u_0|^2 \quad \forall t \geqslant 0.$$

Moreover, if $u_0 \in \mathcal{H}_\sharp(\Omega)$, then $u(t) \in \mathcal{H}_\sharp(\Omega)$ for all $t \geqslant 0$ and there exists a positive constant $C = C(\mathcal{B},\mathcal{L})$ such that

$$|u(t)| \leqslant |u_0|\mathrm{e}^{-t/C} \quad \forall t \geqslant 0.$$

Proof. We will use the Galerkin method implemented with the basis of $\mathcal{H}(\Omega)$ introduced in theorem 5.1. For each $k \in \mathbb{N}$, we define an approximate solution u_k of the form

$$u_k(t,x) = \sum_{i=1}^n c_{ik}(t) w_i(x)$$

with the coefficients c_{ik} obtained from the system

$$\left.\begin{aligned} (u_k'(t),w_j) + a(u_k(t),w_j) + b(u_k(t),u_k(t),w_j) = 0, \quad j = 1,\dots,k,\\ u_k(0) = u_{0k}, \end{aligned}\right\} \tag{5.7}$$

where u_{0k} is the projection of u_0 onto $\operatorname{span}\{w_1,\dots,w_k\}$ in $\mathcal{H}(\Omega)$. For each k, this is a quadratic, constant coefficient $k \times k$ ordinary differential equation system, which has a unique solution defined in some interval $[0,T_k)$ with $T_k > 0$. We will see that $T_k = \infty$ for all $k \in \mathbb{N}$, as a consequence of the next uniform estimates for the approximate solutions.

The first estimates are obtained from the relation

$$(u_k'(t),u_k(t)) + a(u_k(t),u_k(t)) + b(u_k(t),u_k(t),u_k(t)) = 0 \quad \forall t \in [0,T_k),\ \forall k \in \mathbb{N},$$

which follows from (5.7) by multiplying the first equation by c_{jk} and summing from 1 to k. By density of $\mathcal{D}_R(\Omega)$ in $\mathcal{W}(\Omega)$, we deduce $b(u_k,u_k,u_k) = 0$. This identity, combined with the above equation, yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_k|^2 + a(u_k,u_k) = 0 \quad \text{in } (0,T_k)\ \forall k \in \mathbb{N}, \tag{5.8}$$

and, integrating in $[0,t]$ with $t < T_k$,

$$|u_k(t)|^2 + 2\int_0^t a(u_k(s),u_k(s))\,\mathrm{d}s \leqslant |u_0|^2 \quad \forall t \in [0,T_k).$$

This inequality implies that $T_k = \infty$ for all $k \in \mathbb{N}$, and that the sequence $\{u_k\}$ remains in a bounded set of $L^\infty(0,\infty;\mathcal{H}(\Omega))$. Then, integrating (5.8) in $(0,t)$ and letting $t \to \infty$ yields

$$2\nu\int_0^\infty \|D(u_k)(s)\|^2_{2,\mathcal{V}}\,\mathrm{d}s = \int_0^\infty a(u_k(s),u_k(s))\,\mathrm{d}s \leqslant \tfrac{1}{2}|u_0|^2,$$

and, therefore, the sequence $\{D(v_k)\}$ remains in a bounded set of $L^2(0,\infty;L^2(\Omega))$. Consequently, for each $T>0$, $\{u_k\}$ is bounded in $L^2(0,T;\mathcal{V}(\Omega))$.

For each $k\in\mathbb{N}$, let $\mathbb{P}_k$ be the orthogonal projector onto

$$\operatorname{span}\left\{\frac{w_1}{\sqrt{\lambda_1}},\dots,\frac{w_k}{\sqrt{\lambda_k}}\right\} \quad \text{in } \mathcal{W}(\Omega).$$

Then, by theorem 5.1, we have, for each $w\in\mathcal{W}(\Omega)$,

$$\|\mathbb{P}_k w\|_{H^2_R(\Omega)} \leqslant \|w\|_{H^2_R(\Omega)} \quad \text{and} \quad \mathbb{P}_k w\to w \text{ in } H^2(\Omega).$$

The last estimate that we will derive is obtained from the relation

$$\begin{aligned}(u_k'(t),w) &= (u_k'(t),\mathbb{P}_k w)\\ &= -a(u_k(t),\mathbb{P}_k w) - b(u_k(t),u_k(t),\mathbb{P}_k w) \quad \forall t\in(0,\infty),\ \forall w\in\mathcal{W}(\Omega),\end{aligned}$$

from which it follows that

$$\langle u_k',w\rangle_{\mathcal{W}'(\Omega),\mathcal{W}(\Omega)} = -\langle \mathcal{A}u_k+\mathcal{G}(u_k,u_k),\mathbb{P}_k w\rangle_{\mathcal{W}'(\Omega),\mathcal{W}(\Omega)} \quad \text{in } (0,\infty)\ \forall w\in\mathcal{W}(\Omega).$$

Using lemma 5.2, we conclude that $\{u_k'\}$ remains in a bounded set of

$$L^2_{\mathrm{loc}}([0,\infty);\mathcal{W}'(\Omega)).$$

The previous estimates enable us to assert the existence of an element

$$u\in L^\infty(0,\infty;\mathcal{H}(\Omega))\cap L^2_{\mathrm{loc}}([0,\infty);\mathcal{V}(\Omega))$$

with $u'\in L^2_{\mathrm{loc}}([0,\infty);\mathcal{W}'(\Omega))$, and a subsequence $\{u_{k'}\}$ of $\{u_k\}$ such that

$$\begin{aligned} u_{k'} &\to u && \text{in } L^2_{\mathrm{loc}}([0,\infty);\mathcal{V}(\Omega)) \text{ weakly},\\ u_{k'} &\to u && \text{in } L^\infty(0,\infty;\mathcal{H}(\Omega)) \text{ weak-star},\\ u_{k'}' &\to u' && \text{in } L^2_{\mathrm{loc}}([0,\infty);\mathcal{W}'(\Omega)) \text{ weakly},\\ u_{k'} &\to u && \text{in } L^2_{\mathrm{loc}}([0,\infty);\mathcal{H}(\Omega)) \text{ strongly}.\end{aligned}$$

The latter convergence result follows from theorem 5.3. These convergence results and the uniform bounds for the approximate solutions will allow us to pass to the limit $k'\to\infty$ and find a weak solution for our problem. In particular, for each $w\in\mathcal{W}(\Omega)$ and $\psi\in C_0^\infty(0,\infty)$, we have

$$\begin{aligned}&\left|\int_0^{\mathrm{T}}[a(u_{k'}(t),\psi(t)\mathbb{P}_{k'}w)-a(u(t),\psi(t)w)]\,\mathrm{d}t\right|\\ &\quad\leqslant \left|\int_0^{\mathrm{T}}\psi(t)a(u_{k'}(t),\mathbb{P}_{k'}w-w)\,\mathrm{d}t\right| + \left|\int_0^{\mathrm{T}} a(u_{k'}(t)-u(t),\psi(t)w)\,\mathrm{d}t\right|\\ &\quad\leqslant \|\mathbb{P}_{k'}w-w\|\int_0^{\mathrm{T}}|\psi(t)|\|u_{k'}(t)\|\mathrm{d}t + \left|\int_0^{\mathrm{T}} a(u_{k'}(t)-u(t),\psi(t)w)\,\mathrm{d}t\right|\\ &\quad\leqslant C(\mathcal{B},\mathcal{L})\|\psi\|_{L^2(0,T)}\|u_{k'}\|_{L^2(0,T;\mathcal{V}(\Omega))}\|\mathbb{P}_{k'}w-w\|_{2,2,\Omega}\\ &\qquad + \left|\int_0^{\mathrm{T}} a(u_{k'}(t)-u(t),\psi(t)w)\,\mathrm{d}t\right|,\end{aligned}$$

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where $T > 0$ is such that $\operatorname{supp}(\psi) \subset (0, T)$. With regard to the nonlinear terms, for each $w \in \mathcal{W}(\Omega)$ and $\psi \in C_0^\infty(0, \infty)$, we find that

$$
\begin{aligned}
&\left| \int_0^{\mathrm{T}} [b(u_{k'}, u_{k'}, \psi(t)\mathbb{P}_{k'} w) - b(u, u, \psi(t) w)] \,\mathrm{d}t \right| \\
&\quad = \left| \int_0^{\mathrm{T}} \psi(t) [b(u_{k'}, u_{k'}, \mathbb{P}_{k'} w - w) + b(u_{k'} - u, u_{k'}, w) + b(u, u_{k'} - u, w)] \,\mathrm{d}t \right| \\
&\quad \leqslant C \|\psi\|_\infty \int_0^{\mathrm{T}} [|u_{k'}| \|u_{k'}\| \|\mathbb{P}_{k'} w - w\|_{2,2,\Omega} + |u_{k'} - u| (\|u_{k'}\| + \|u\|) \|w\|_{2,2,\Omega}] \,\mathrm{d}t \\
&\quad \leqslant C \|\psi\|_\infty \|w\|_{2,2,\Omega} \int_0^{\mathrm{T}} |u_{k'} - u| (\|u_{k'}\| + \|u\|) \,\mathrm{d}t \\
&\qquad + C \|\psi\|_\infty \|\mathbb{P}_{k'} w - w\|_{2,2,\Omega} \int_0^{\mathrm{T}} |u_{k'}| \|u_{k'}\| \mathrm{d}t \\
&\quad \leqslant C \|\psi\|_\infty \|w\|_{2,2,\Omega} \|u_{k'} - u\|_{L^2(0,T;\mathcal{H}(\Omega))} (\|u_{k'}\|_{L^2(0,T;\mathcal{V}(\Omega))} + \|u\|_{L^2(0,T;\mathcal{V}(\Omega))}) \\
&\qquad + C \|\psi\|_\infty \|\mathbb{P}_{k'} w - w\|_{2,2,\Omega} \|u_{k'}\|_{L^2(0,T;\mathcal{H}(\Omega))} \|u_{k'}\|_{L^2(0,T;\mathcal{V}(\Omega))},
\end{aligned}
$$

where $C = C(\mathcal{B}, \mathcal{L})$. Letting $k' \to \infty$ in

$$
\begin{aligned}
- \int_0^{\mathrm{T}} (u_{k'}(t), \psi'(t) w) \,\mathrm{d}t + \int_0^{\mathrm{T}} a(u_{k'}(t), \psi(t) \mathbb{P}_{k'} w) \,\mathrm{d}t \\
+ \int_0^{\mathrm{T}} b(u_{k'}(t), u_{k'}(t), \psi(t) \mathbb{P}_{k'} w) \,\mathrm{d}t = 0
\end{aligned}
$$

yields

$$
\int_0^\infty (u(t), \psi'(t) w) \,\mathrm{d}t = \int_0^\infty a(u(t), \psi(t) w) \,\mathrm{d}t + \int_0^\infty b(u(t), u(t), \psi(t) w) \,\mathrm{d}t,
$$

for all $\psi \in C_0^\infty(0, \infty)$ and all $w \in \mathcal{W}(\Omega)$. Hence, the solution we have found satisfies

$$
u \in L^\infty(0, \infty; \mathcal{H}(\Omega)) \cap L^2_{\mathrm{loc}}([0, \infty); \mathcal{V}(\Omega)) \quad \text{and} \quad u' \in L^2_{\mathrm{loc}}([0, \infty); \mathcal{W}'(\Omega)).
$$

Actually, $u' \in L^{4/3}_{\mathrm{loc}}([0, \infty); \mathcal{V}'(\Omega))$. To see this, we use the density of $\mathcal{W}(\Omega)$ in $\mathcal{V}(\Omega)$ to obtain

$$
\int_0^\infty (u(t), \varphi) \psi'(t) \,\mathrm{d}t = \int_0^\infty a(u(t), \varphi) \psi(t) \,\mathrm{d}t + \int_0^\infty b(u(t), u(t), \varphi) \psi(t) \,\mathrm{d}t
$$

for all $\varphi \in \mathcal{V}(\Omega)$ and $\psi \in C_0^\infty(0, \infty)$. Therefore, $u' \in (C_0^\infty)'(0, \infty; \mathcal{V}'(\Omega))$ and

$$
\left\langle \int_0^\infty u(t) \psi'(t) \,\mathrm{d}t, \varphi \right\rangle = \left\langle \int_0^\infty \mathcal{A} u(t) \psi(t) \,\mathrm{d}t, \varphi \right\rangle + \left\langle \int_0^\infty \mathcal{G}(u(t), u(t)) \psi(t) \,\mathrm{d}t, \varphi \right\rangle,
$$

for all $\varphi \in \mathcal{V}(\Omega)$ and $\psi \in C_0^\infty(0, \infty)$. By lemma 5.2, we have

$$
\mathcal{A} u + \mathcal{G}(u, u) \in L^{4/3}_{\mathrm{loc}}([0, \infty); \mathcal{V}'(\Omega)),
$$

which implies that $u' \in L^{4/3}_{\mathrm{loc}}([0, \infty); \mathcal{V}'(\Omega))$ and satisfies (5.5). Hence,

$$
u, u' \in L^1_{\mathrm{loc}}([0, \infty); \mathcal{V}'(\Omega))
$$

and, therefore (see, for example, [24, p. 250]), $u \in C([0, \infty); \mathcal{V}'(\Omega))$.

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Since $u \in L^\infty(0,\infty;\mathcal{H}(\Omega)) \cap C([0,\infty);\mathcal{V}'(\Omega))$ then, by theorem 5.4, it follows that $u \in C_w([0,\infty);\mathcal{H}(\Omega))$. Using standard arguments, as in [24, pp. 288–289], for example, it is easy to show that $(u(0),\varphi) = (u_0,\varphi)$ for all $\varphi \in \mathcal{H}(\Omega)$, and, therefore, the initial condition (5.6) is also satisfied.

In order to show the energy inequality and to analyse the long-time behaviour of the weak solution, we follow [21, p. 334]. Let $\phi \in C^1([0,\infty);[0,\infty))$. From (5.7), we obtain

$$((\phi u_k)', \phi u_k) + \phi^2 a(u_k,u_k) + \phi^2 b(u_k,u_k,u_k) = \phi'\phi(u_k,u_k) \quad \text{in } (0,\infty)\ \forall k \in \mathbb{N},$$

which simplifies to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\phi u_k|^2 + \phi^2 a(u_k,u_k) = \phi'\phi|u_k|^2 \quad \text{in } (0,\infty)\ \forall k \in \mathbb{N}.$$

Integrating in $[0,t]$ and letting $k \to \infty$, we obtain the 'weighted energy inequality'

$$\begin{aligned}\tfrac{1}{2}\phi^2(t)|u(t)|^2 &+ \int_0^t \phi^2(s)a(u(s),u(s))\,\mathrm{d}s \\ &\leqslant \tfrac{1}{2}\phi^2(0)|u_0|^2 + \int_0^t \phi'(s)\phi(s)|u(s)|^2\,\mathrm{d}s \quad \forall t \geqslant 0. \end{aligned} \tag{5.9}$$

In particular, for $\phi(t) := 1$, we obtain the 'classical energy inequality'

$$\tfrac{1}{2}|u(t)|^2 + \int_0^t a(u(s),u(s))\,\mathrm{d}s \leqslant \tfrac{1}{2}|u_0|^2 \quad \forall t \geqslant 0.$$

Next, we will consider the case of initial velocities belonging to $\mathcal{H}_\sharp(\Omega)$. Taking $\varphi = e_i$, $i = 1,2,3$, in (5.5) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(m_{\mathcal{B}}\xi_u \cdot e_i + \rho\int_{\mathcal{F}} u \cdot e_i\,\mathrm{d}x\right) &= \frac{\mathrm{d}}{\mathrm{d}t}(u,e_i) \\ &= \langle u', e_i\rangle \\ &= -a(u,e_i) - b(u,u,e_i) \\ &= -m_{\mathcal{B}}(\omega_u \times \xi_u)\cdot e_i - \rho\int_{\mathcal{F}} \omega_u \times u \cdot e_i\,\mathrm{d}x,\end{aligned}$$

because $a(u,e_i) = 0$ and

$$\begin{aligned} b(u,u,e_i) &= m_{\mathcal{B}}\omega_u \times \xi_u \cdot e_i + \rho\int_{\mathcal{F}} (u - \bar{u})\cdot\nabla u \cdot e_i\,\mathrm{d}x + \rho\int_{\mathcal{F}} \omega_u \times u \cdot e_i\,\mathrm{d}x \\ &= m_{\mathcal{B}}\omega_u \times \xi_u \cdot e_i + \rho\int_{\mathcal{F}} \omega_u \times u \cdot e_i\,\mathrm{d}x. \end{aligned}$$

Hence, the following relation is valid for a weak solution:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m_{\mathcal{B}}\xi_u + \rho\int_{\mathcal{F}} u\,\mathrm{d}x\right) = -\omega_u \times \left(m_{\mathcal{B}}\xi_u + \rho\int_{\mathcal{F}} u\,\mathrm{d}x\right),$$

which implies

$$\left|m_{\mathcal{B}}\xi_u(t) + \rho\int_{\mathcal{F}} u(t)\,\mathrm{d}x\right| = \left|m_{\mathcal{B}}\xi_{u_0} + \rho\int_{\mathcal{F}} u_0\,\mathrm{d}x\right| \quad \forall t \geqslant 0. \tag{5.10}$$

Analogously, taking $\varphi = e_i \times x$, $i = 1, 2, 3$, in (5.5) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\biggl(I_{\mathcal{B}}\omega_u \cdot e_i + \rho\int_{\mathcal{F}} u \cdot (e_i \times x)\,\mathrm{d}x\biggr) &= \frac{\mathrm{d}}{\mathrm{d}t}(u, e_i \times x) \\ &= \langle u', e_i \times x\rangle \\ &= -a(u, e_i \times x) - b(u, u, e_i \times x) \\ &= -\omega_u \times (I_{\mathcal{B}}\omega_u) \cdot e_i - \rho\int_{\mathcal{F}} (u - \bar{u}) \cdot \nabla u \cdot (e_i \times x)\,\mathrm{d}x \\ &\quad - \rho\int_{\mathcal{F}} (\omega_u \times u) \cdot (e_i \times x)\,\mathrm{d}x.\end{aligned}$$

By direct calculations, and using the density of $\mathcal{D}_R(\Omega)$ in $\mathcal{V}(\Omega)$, we find

$$\begin{aligned}\int_{\mathcal{F}} (u - \bar{u}) \cdot \nabla u \cdot (e_i \times x)\,\mathrm{d}x &= \int_{\mathcal{F}} \bar{u} \cdot \nabla(e_i \times x) \cdot u\,\mathrm{d}x \\ &= (e_i \times \xi_u) \cdot \int_{\mathcal{F}} u\,\mathrm{d}x + \int_{\mathcal{F}} e_i \times (\omega_u \times x) \cdot u\,\mathrm{d}x \\ &= \biggl(\xi_u \times \int_{\mathcal{F}} u\,\mathrm{d}x\biggr) \cdot e_i + \int_{\mathcal{F}} u \times (x \times \omega_u) \cdot e_i\,\mathrm{d}x,\end{aligned}$$

$$\int_{\mathcal{F}} (\omega_u \times u) \cdot (e_i \times x)\,\mathrm{d}x = \int_{\mathcal{F}} \omega_u \times (x \times u) \cdot e_i\,\mathrm{d}x - \int_{\mathcal{F}} u \times (x \times \omega_u) \cdot e_i\,\mathrm{d}x,$$

so that a weak solution also satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\biggl(I_{\mathcal{B}}\omega_u + \rho\int_{\mathcal{F}} x \times u\,\mathrm{d}x\biggr) = -\omega_u \times \biggl(I_{\mathcal{B}}\omega_u + \rho\int_{\mathcal{F}} x \times u\,\mathrm{d}x\biggr) - \rho\xi_u \times \int_{\mathcal{F}} u\,\mathrm{d}x.$$

Therefore, we have

$$\begin{aligned}\frac{1}{2}\biggl|I_{\mathcal{B}}\omega_u(t) + \rho\int_{\mathcal{F}} x \times u(t)\,\mathrm{d}x\biggr|^2 &= \frac{1}{2}\biggl|I_{\mathcal{B}}\omega_{u_0} + \rho\int_{\mathcal{F}} x \times u_0\,\mathrm{d}x\biggr|^2 \\ &\quad - \rho\int_0^t \biggl(\xi_u(s) \times \int_{\mathcal{F}} u(s)\,\mathrm{d}x\biggr) \cdot \biggl(I_{\mathcal{B}}\omega_u(s) + \rho\int_{\mathcal{F}} x \times u(s)\,\mathrm{d}x\biggr)\,\mathrm{d}s \quad \forall t \geqslant 0.\end{aligned} \tag{5.11}$$

Now suppose that $u_0 \in \mathcal{H}_\sharp(\Omega)$, that is,

$$m_{\mathcal{B}}\xi_{u_0} + \rho\int_{\mathcal{F}} u_0\,\mathrm{d}x = I_{\mathcal{B}}\omega_{u_0} + \rho\int_{\mathcal{F}} x \times u_0\,\mathrm{d}x = 0.$$

Then, from (5.10) and (5.11) we conclude that $u(t) \in \mathcal{H}_\sharp(\Omega)$ for all $t \geqslant 0$, and by lemma 4.4, there exists a positive constant $C = C(\mathcal{B}, \mathcal{L})$ such that $|u(t)|^2 \leqslant Ca(u(t), u(t))$ for all $t \geqslant 0$. Choosing $\phi(t) := \mathrm{e}^{t/C}$ in (5.9) yields

$$\mathrm{e}^{2t/C}|u(t)|^2 \leqslant |u_0|^2 \quad \text{for all } t \geqslant 0$$

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and, therefore,

$$|u(t)|^2 \leqslant |u_0|^2 \mathrm{e}^{-2t/C} \quad \text{for all } t \geqslant 0.$$

□

Let $\{\mathrm{e}^{-tA}\}_{t\geqslant 0}$ be the semigroup generated by $-A$. We have the following integral representation for weak solutions.

THEOREM 5.7. *Let $u_0 \in \mathcal{H}(\Omega)$ and let $u \in L^\infty(0,\infty;\mathcal{H}(\Omega)) \cap L^2_{\mathrm{loc}}([0,\infty);\mathcal{V}(\Omega))$ be a weak solution to (2.1). Then*

$$u(t) = \mathrm{e}^{-tA}u_0 + (\delta I + A)^{1/2}\int_0^t \mathrm{e}^{-(t-s)A}(\delta I + \mathcal{A})^{-1/2}\mathcal{G}(u(s),u(s))\,\mathrm{d}s \quad \text{a.e. in } (0,\infty)$$

for $\delta > 0$.

Proof. A weak solution satisfies

$$u' + \mathcal{A}u = \mathcal{G}(u,u) \quad \text{in } L^{4/3}_{\mathrm{loc}}([0,\infty);\mathcal{V}'(\Omega)).$$

Since

$$(\delta I + \mathcal{A})^{-1/2}\mathcal{V}'(\Omega) = (\delta I + \mathcal{A})^{-1/2}(D(A^{1/2}))' = \mathcal{H}(\Omega),$$

we have

$$(\delta I + \mathcal{A})^{-1/2}u' + (\delta I + \mathcal{A})^{-1/2}\mathcal{A}u = (\delta I + \mathcal{A})^{-1/2}\mathcal{G}(u,u) \quad \text{in } \mathcal{H}(\Omega) \text{ and a.e. in } (0,\infty).$$

Setting

$$z := (\delta I + \mathcal{A})^{-1/2}u = (\delta I + A)^{-1/2}u$$

and using the property

$$(\delta I + \mathcal{A})^{-1/2}\mathcal{A}v = A(\delta I + \mathcal{A})^{-1/2}v, \quad v \in \mathcal{V}(\Omega),$$

we can write

$$z' + Az = (\delta I + \mathcal{A})^{-1/2}\mathcal{G}(u,u) \quad \text{in } \mathcal{H}(\Omega) \text{ and a.e. in } (0,\infty).$$

By the Duhamel formula, z satisfies

$$z(t) = \mathrm{e}^{-tA}z_0 + \int_0^t \mathrm{e}^{-(t-s)A}(\delta I + \mathcal{A})^{-1/2}\mathcal{G}(u(s),u(s))\,\mathrm{d}s \quad \text{a.e. in } (0,\infty)$$

with $z_0 = (\delta I + A)^{-1/2}u_0$, and, therefore,

$$u(t) = \mathrm{e}^{-tA}u_0 + (\delta I + A)^{1/2}\int_0^t \mathrm{e}^{-(t-s)A}(\delta I + \mathcal{A})^{-1/2}\mathcal{G}(u(s),u(s))\,\mathrm{d}s \quad \text{a.e. in } (0,\infty).$$

□

In the next section, we will consider a more regular initial velocity and investigate the existence of a more regular solution, the so-called mild solution, for system (2.1).

https://doi.org/10.1017/S0308210510001034 Published online by Cambridge University Press

6. Mild solutions

We now focus on the investigation of mild solutions. The idea is to write system (2.1) in the form

$$u'(t) + Au(t) = G(u(t), u(t)), \quad t > 0, \qquad u(0) = u_0, \tag{6.1}$$

where $-A$ is the generator of a semigroup and G is a bilinear map and, following the approach of [8] (see also [18]), to use the abstract semigroup theory for linear operators and the Picard fixed point theorem to solve (6.1). To this end, we need the following general hypotheses on A and on G.

(H_1) The operator $-A\colon D(A) \to H$ is the generator of an analytic semigroup in the Hilbert space H and the resolvent set of A contains the negative real ray $(-\infty, 0)$.

(H_2) There exist four non negative constants α, β, γ, κ with $\beta > \alpha$ such that

$$G(D(A^\beta) \times D(A^\beta)) \subset D(A^\gamma)',$$
$$G(D(A^\alpha) \times D(A^\beta)) \subset D(A^\kappa)',$$
$$G(D(A^\beta) \times D(A^\alpha)) \subset D(A^\kappa)',$$

and

$$\forall (u,v) \in D(A^\beta) \times D(A^\beta), \qquad \|(\delta + A)^{-\gamma} G(u,v)\|_H \leqslant C\|u\|_\beta \|v\|_\beta,$$
$$\forall (u,v) \in D(A^\alpha) \times D(A^\beta), \qquad \|(\delta + A)^{-\kappa} G(u,v)\|_H \leqslant C\|u\|_\alpha \|v\|_\beta,$$
$$\forall (u,v) \in D(A^\beta) \times D(A^\alpha), \qquad \|(\delta + A)^{-\kappa} G(u,v)\|_H \leqslant C\|u\|_\beta \|v\|_\alpha.$$

Here we have used the notation $\|u\|_l := \|u\|_H + \|A^l u\|_H$ for the norm of the space $D(A^l)$, $l \geqslant 0$.

(H_3) The constants α, β, γ, κ satisfy $\beta + \gamma < 1$, $2\beta - \alpha + \gamma = 1$ and $\beta + \kappa = 1$.

Let us introduce some more notation. For all $\alpha, \beta > 0$, with $\beta > \alpha$, and for all $T > 0$, we set

$$\mathcal{E}_T(\alpha,\beta) = \Big\{ u \in C([0,T]; D(A^\alpha)) \cap C^1((0,T]; D(A^\alpha)) \cap C((0,T]; D(A^\beta)); \\ \|u\|_{\mathcal{E}_T(\alpha,\beta)} < \infty, \ \lim_{s\to 0} (\|s^{\beta-\alpha} u(s)\|_\beta + \|su'(s)\|_\alpha) = 0 \Big\}$$

with

$$\|u\|_{\mathcal{E}_T(\alpha,\beta)} := \sup_{0<s<T} (\|u(s)\|_\alpha + \|s^{\beta-\alpha} u(s)\|_\beta + \|su'(s)\|_\alpha).$$

We are now in a position to give a definition of a mild solution for problem (6.1).

Definition 6.1. Let $T > 0$. We say that $u \in C([0,T]; D(A^\alpha))$ is a mild solution of (6.1) provided $s \mapsto \mathrm{e}^{-(t-s)A} G(u(s), u(s))$ is in $L^1_{\mathrm{loc}}(0,T; D(A^\alpha))$ and

$$u(t) = \mathrm{e}^{-tA} u_0 + \int_0^t \mathrm{e}^{-(t-s)A} G(u(s), u(s))\, \mathrm{d}s, \quad t \in [0,T].$$

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We have the following result of existence of mild solutions for small data.

Theorem 6.2. *Assume the above hypotheses.*

(i) *Let $T > 0$. Then, for all sufficiently small $u_0 \in D(A^\alpha)$ in the norm $\|\cdot\|_\alpha$, there exists a unique mild solution $u \in \mathcal{E}_T(\alpha,\beta)$ of (6.1).*

(ii) *For all $u_0 \in D(A^\alpha)$, there exist $T > 0$ and a unique mild solution $u \in \mathcal{E}_T(\alpha,\beta)$ of (6.1).*

(iii) *The mild solutions are unique in the space $C((0,T]; D(A^\beta))$.*

In order to prove theorem 6.2, analysing the nonlinear term is an important step. For $u, v \in \mathcal{E}_T(\alpha,\beta)$, we define

$$\Phi(u,v)(t) = \frac{1}{2}\int_0^t \mathrm{e}^{-(t-s)A}(G(u(s),v(s)) + G(v(s),u(s)))\,\mathrm{d}s.$$

Lemma 6.3. *The bilinear map Φ is continuous from $\mathcal{E}_T \times \mathcal{E}_T$ to $\mathcal{E}_T$ and the norm of Φ is an increasing function of T.*

Proof. By definition of Φ, we have

$$\begin{aligned}
&\|A^\alpha \Phi(u,v)(t)\|_H \\
&\quad\leqslant \frac{1}{2}\int_0^t \|A^\alpha(\delta+A)^\gamma \mathrm{e}^{-(t-s)A}\|_{\mathcal{L}(H)}\|(\delta+A)^{-\gamma}G(u(s),v(s))\|_H\,\mathrm{d}s \\
&\qquad + \frac{1}{2}\int_0^t \|A^\alpha(\delta+A)^\gamma \mathrm{e}^{-(t-s)A}\|_{\mathcal{L}(H)}\|(\delta+A)^{-\gamma}G(v(s),u(s))\|_H\,\mathrm{d}s.
\end{aligned}$$

Using the fact that $-A$ is the generator of an analytic semigroup and the hypotheses on G, we deduce from the above inequality that

$$\begin{aligned}
&\|A^\alpha \Phi(u,v)(t)\|_H \\
&\quad\leqslant C\int_0^t \frac{1}{(t-s)^{\alpha+\gamma}}\frac{1}{s^{2(\beta-\alpha)}}\,\mathrm{d}s \sup_{0<s<t}\|s^{\beta-\alpha}u(s)\|_\beta \sup_{0<s<t}\|s^{\beta-\alpha}v(s)\|_\beta \\
&\quad= C\int_0^1 \frac{1}{(1-\sigma)^{\alpha+\gamma}}\frac{1}{\sigma^{2(\beta-\alpha)}}\,\mathrm{d}\sigma \sup_{0<s<t}\|s^{\beta-\alpha}u(s)\|_\beta \sup_{0<s<t}\|s^{\beta-\alpha}v(s)\|_\beta \\
&\quad= C\|u\|_{\mathcal{E}_T(\alpha,\beta)}\|v\|_{\mathcal{E}_T(\alpha,\beta)},
\end{aligned}$$

where we have used $\alpha+\gamma+2(\beta-\alpha) = 1$. The estimate of $\|\Phi(u,v)(t)\|_H$ is obtained in a similar way:

$$\begin{aligned}
\|\Phi(u,v)(t)\|_H &\leqslant C\int_0^t \frac{1}{(t-s)^\gamma}\frac{1}{s^{2(\beta-\alpha)}}\,\mathrm{d}s\|u\|_{\mathcal{E}_T(\alpha,\beta)}\|v\|_{\mathcal{E}_T(\alpha,\beta)} \\
&= Ct^\alpha\int_0^1 \frac{1}{(1-\sigma)^\gamma}\frac{1}{\sigma^{2(\beta-\alpha)}}\,\mathrm{d}\sigma\|u\|_{\mathcal{E}_T(\alpha,\beta)}\|v\|_{\mathcal{E}_T(\alpha,\beta)} \\
&= CT^\alpha\|u\|_{\mathcal{E}_T(\alpha,\beta)}\|v\|_{\mathcal{E}_T(\alpha,\beta)}.
\end{aligned}$$

The fact that $\Phi(u,v) \in C([0,T]; D(A^\alpha))$ can then be deduced from the continuity of u and v. We proceed similarly for the second norm:

$$\begin{aligned}
&\|A^\beta \Phi(u,v)(t)\|_H \\
&\quad \leqslant \frac{1}{2}\int_0^t \|A^\beta(\delta+A)^\gamma \mathrm{e}^{-(t-s)A}\|_{\mathcal{L}(H)} \|(\delta+A)^{-\gamma} G(u,v)\|_H \,\mathrm{d}s \\
&\qquad + \frac{1}{2}\int_0^t \|A^\beta(\delta+A)^\gamma \mathrm{e}^{-(t-s)A}\|_{\mathcal{L}(H)} \|(\delta+A)^{-\gamma} G(v,u)\|_H \,\mathrm{d}s \\
&\quad \leqslant C\int_0^t \frac{1}{(t-s)^{\beta+\gamma}} \frac{1}{s^{2(\beta-\alpha)}} \,\mathrm{d}s \sup_{0<s<t} \|s^{\beta-\alpha} u(s)\|_\beta \sup_{0<s<t} \|s^{\beta-\alpha} v(s)\|_\beta \\
&\quad = C\frac{1}{t^{\beta-\alpha}} \int_0^1 \frac{1}{(1-\sigma)^{\beta+\gamma}} \frac{1}{\sigma^{2(\beta-\alpha)}} \,\mathrm{d}\sigma \sup_{0<s<t} \|s^{\beta-\alpha} u(s)\|_\beta \sup_{0<s<t} \|s^{\beta-\alpha} v(s)\|_\beta \\
&\quad = C\frac{1}{t^{\beta-\alpha}} \|u\|_{\mathcal{E}_T(\alpha,\beta)} \|v\|_{\mathcal{E}_T(\alpha,\beta)}.
\end{aligned}$$

We also deduce from the above calculations that

$$\lim_{t\to 0} t^{\beta-\alpha} \|A^\beta \Phi(u,v)(t)\|_H = 0.$$

Finally,

$$\begin{aligned}
\Phi(u,v)(t) &= \frac{1}{2}\int_0^{t/2} \mathrm{e}^{-(t-s)A}(G(u(s),v(s)) + G(v(s),u(s)))\,\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^{t/2} \mathrm{e}^{-sA}(G(u(t-s),v(t-s)) + G(v(t-s),u(t-s)))\,\mathrm{d}s
\end{aligned}$$

so that

$$\begin{aligned}
\Phi(u,v)'(t) &= \tfrac{1}{2}\mathrm{e}^{-(t/2)A}(G(u(\tfrac{1}{2}t),v(\tfrac{1}{2}t)) + G(v(\tfrac{1}{2}t),u(\tfrac{1}{2}t))) \\
&\quad - \frac{1}{2}\int_0^{t/2} A\mathrm{e}^{-(t-s)A}(G(u(s),v(s)) + G(v(s),u(s)))\,\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^{t/2} \mathrm{e}^{-sA}(G(u'(t-s),v(t-s)) + G(u(t-s),v'(t-s)) \\
&\quad + G(v'(t-s),u(t-s)) + G(v(t-s),u'(t-s)))\,\mathrm{d}s,
\end{aligned}$$

and thus

$$\begin{aligned}
\Phi(u,v)'(t) &= \tfrac{1}{2}\mathrm{e}^{-(t/2)A}(G(u(\tfrac{1}{2}t),v(\tfrac{1}{2}t)) + G(v(\tfrac{1}{2}t),u(\tfrac{1}{2}t))) \\
&\quad - \frac{1}{2}\int_0^{t/2} A\mathrm{e}^{-(t-s)A}(G(u(s),v(s)) + G(v(s),u(s)))\,\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^{t/2} \mathrm{e}^{-sA}(G(u'(t-s),v(t-s)) + G(u(t-s),v'(t-s)) \\
&\quad + G(v'(t-s),u(t-s)) + G(v(t-s),u'(t-s)))\,\mathrm{d}s,
\end{aligned}$$

which implies

$$
\begin{aligned}
&\|A^{\alpha}\Phi(u,v)'(t)\|_H \\
&\qquad \leqslant \frac{C}{t}\sup_{0<s<t}\|s^{\beta-\alpha}u(s)\|_{\beta}\sup_{0<s<t}\|s^{\beta-\alpha}v(s)\|_{\beta} \\
&\qquad\quad + C\int_0^{t/2}\frac{1}{(t-s)^{\alpha+\gamma+1}}\frac{1}{s^{2(\beta-\alpha)}}\,\mathrm{d}s\sup_{0<s<t}\|s^{\beta-\alpha}u(s)\|_{\beta}\sup_{0<s<t}\|s^{\beta-\alpha}v(s)\|_{\beta} \\
&\qquad\quad + CM(t)\int_0^{t/2}\frac{1}{s^{\kappa+\alpha}}\frac{1}{(t-s)^{\beta-\alpha+1}}\,\mathrm{d}s,
\end{aligned}
$$

where

$$M(t) := \sup_{0<s<t}\|su'(s)\|_{\alpha}\sup_{0<s<t}\|s^{\beta-\alpha}v(s)\|_{\beta} + \sup_{0<s<t}\|s^{\beta-\alpha}u(s)\|_{\beta}\sup_{0<s<t}\|sv'(s)\|_{\alpha}.$$

The above inequality yields

$$\|A^{\alpha}\Phi(u,v)'(t)\|_H \leqslant \frac{C}{t}\|u\|_{\mathcal{E}_T(\alpha,\beta)}\|v\|_{\mathcal{E}_T(\alpha,\beta)}$$

and

$$\lim_{t\to 0}\|tA^{\alpha}\Phi(u,v)'(t)\|_H = 0.$$

□

REMARK 6.4. In [18], a norm of Φ that is independent of T is obtained. We would obtain the same result by replacing the norm of $D(A^l)$ in (H_2) with the semi-norm $\|A^l\cdot\|_H$. However, in our case, this semi-norm is not sufficient to bound the nonlinear term.

We are now in a position to prove theorem 6.2.

Proof of theorem 6.2. First we set $\Psi(u_0)(t) := \mathrm{e}^{-tA}u_0$. Since $u_0 \in D(A^{\alpha})$, $\Psi(u_0) \in \mathcal{E}_T$. Indeed (see, for example, [7]), using the fact that $-A$ is the generator of an analytic semigroup, we have

$$\Psi(u_0) \in C([0,T];D(A^{\alpha})) \quad \text{and} \quad \|\Psi(u_0)(t)\|_{\alpha} \leqslant M\|u_0\|_{\alpha} \text{ for some } M>0.$$

We also have $t^{\beta-\alpha}A^{\beta}\Psi(u_0)(t) = t^{\beta-\alpha}A^{\beta-\alpha}A^{\alpha}\Psi(u_0)$ and, again, since $-A$ is the generator of an analytic semigroup,

$$\|t^{\beta-\alpha}A^{\beta}\Psi(u_0)(t)\|_H \leqslant M\|A^{\alpha}u_0\|_H.$$

If $u_0 \in D(A^{\beta})$, then

$$\|t^{\beta-\alpha}A^{\beta}\Psi(u_0)(t)\|_H \leqslant Mt^{\beta-\alpha}\|A^{\beta}u_0\|_H \to 0$$

as $t \to 0$ and, using the density of $D(A^{\beta})$ into $D(A^{\alpha})$, we obtain that

$$\lim_{t\to 0}\|t^{\beta-\alpha}A^{\beta}\Psi(u_0)(t)\|_H = 0$$

for $u_0 \in D(A^{\alpha})$ (this is a classical argument [8, lemma 2.10]). On the other hand,

$$\|t^{\beta-\alpha}\Psi(u_0)(t)\|_H \leqslant MT^{\beta-\alpha}\|u_0\|_H.$$

Finally, $\Psi(u_0)'(t) = A\Psi(u_0)(t)$ and, thus, again using the fact that $-A$ is the generator of an analytic semigroup,

$$\|tA^{\alpha}\Psi(u_0)'(t)\|_H \leqslant M\|A^{\alpha}u_0\|_H.$$

As above, using the density of $D(A^{\alpha+1})$ into $D(A^{\alpha})$, we deduce that

$$\lim_{t\to 0}\|tA^{\alpha}\Psi(u_0)'(t)\|_H = 0.$$

Let us now prove the global existence result for small data. Assume $u_0 \in D(A^{\alpha})$. We seek a fixed point of the map

$$\mathcal{N}\colon \mathcal{E}_T \to \mathcal{E}_T, \qquad u \mapsto \Psi(u_0) + \Phi(u,u).$$

From lemma 6.3, we know that there exists a positive constant C_T such that $\|\Phi\|_{\mathcal{L}(\mathcal{E}_T\times\mathcal{E}_T,\mathcal{E}_T)} \leqslant C_T$. Then we are going to show that, for sufficiently small R,

$$\mathcal{N}(\{u \in \mathcal{E}_T; \|u\|_{\mathcal{E}_T} \leqslant R\}) \subset \{u \in \mathcal{E}_T; \|u\|_{\mathcal{E}_T} \leqslant R\},$$

and that the restriction of $\mathcal{N}$ to the closed subset $\{u \in \mathcal{E}_T; \|u\|_{\mathcal{E}_T} \leqslant R\}$ of the Banach space $\mathcal{E}_T$ is a contraction, which will conclude the proof.

From the estimates on Φ and Ψ we can deduce that there exists a constant $C = C(T)$ such that if $\|u\|_{\mathcal{E}_T} \leqslant R$, then

$$\|\mathcal{N}(u)\|_{\mathcal{E}_T} \leqslant C\|u_0\|_{\alpha} + C_T R^2. \tag{6.2}$$

Let us assume that $\|u_0\|_{\alpha} \leqslant 1/(8CC_T)$, and let us set $R = 2C\|u_0\|_{\alpha}$. Then, from the two above inequalities and from (6.2), we deduce

$$\|\mathcal{N}(u)\|_{\mathcal{E}_T} \leqslant R \quad \text{and} \quad \|\mathcal{N}(u) - \mathcal{N}(v)\|_{\mathcal{E}_T} \leqslant 2C_T R\|u-v\|_{\mathcal{E}_T} \leqslant \tfrac{1}{2}\|u-v\|_{\mathcal{E}_T}.$$

The proof is similar for the local existence. Assume that $u_0 \in D(A^{\alpha})$. Then we consider the space

$$\mathcal{X}_T(\alpha,\beta) = \Big\{u \in C^1((0,T];D(A^{\alpha})) \cap C((0,T];D(A^{\beta})); \\ \lim_{s\to 0}(\|s^{\beta-\alpha}u(s)\|_{\beta} + \|su'(s)\|_{\alpha}) = 0\Big\}$$

with the norm

$$\|u\|_{\mathcal{X}_T(\alpha,\beta)} := \sup_{0<s<T}(\|s^{\beta-\alpha}u(s)\|_{\beta} + \|su'(s)\|_{\alpha}).$$

From the above calculation, $\mathcal{N}(\mathcal{X}_T) \subset \mathcal{X}_T$. Moreover, for all $u \in \mathcal{X}_T$ with $\|u\|_{\mathcal{X}_T(\alpha,\beta)} \leqslant R$, we have obtained

$$\|\mathcal{N}(u)\|_{\mathcal{X}_T} \leqslant \|\Psi(u_0)\|_{\mathcal{X}_T} + C_T R^2. \tag{6.3}$$

We note that $\lim_{T\to 0}\|\Psi(u_0)\|_{\mathcal{X}_T} = 0$ so that, for sufficiently small T, there exists $R > 0$ such that

$$\|\mathcal{N}(u)\|_{\mathcal{X}_T} \leqslant R \quad \text{and} \quad \|\mathcal{N}(u) - \mathcal{N}(v)\|_{\mathcal{X}_T} \leqslant \tfrac{1}{2}\|u-v\|_{\mathcal{E}_T},$$

which implies the existence and uniqueness of a solution in $\mathcal{X}_T$. By following the proof of lemma 6.3, one can check that $u \in \mathcal{E}_T$. To prove the uniqueness of the mild solutions of (6.1) in $C((0,T];D(A^\beta))$, we have to follow the idea of Brezis [4,12] to show that a mild solution of (6.1) in $C((0,T];D(A^\beta))$ is in the space

$$\hat{\mathcal{X}}_T(\alpha,\beta) = \Big\{u \in C((0,T];D(A^\beta));\lim_{s\to 0}\|s^{\beta-\alpha}u(s)\|_\beta = 0\Big\}$$

with the norm

$$\|u\|_{\hat{\mathcal{X}}_T(\alpha,\beta)} := \sup_{0<s<T}\|s^{\beta-\alpha}u(s)\|_\beta.$$

Then we can see that $\mathcal{N}$ is a contraction in $\hat{\mathcal{X}}_T(\alpha,\beta)$ for small time. □

We now apply theorem 6.2 to our problem and prove that a mild solution exists. For $(u,v) \in D(A^{1/4}) \times D(A^{1/2})$ or $(u,v) \in D(A^{1/2}) \times D(A^{1/4})$, we set

$$\tilde{g}(u,v) = \begin{cases} -\nabla\cdot[(\bar{v}-v)\otimes u] - \omega_v \times u & \text{in } \mathcal{F},\\ -\omega_v \times \xi_u - [I_{\mathcal{B}}^{-1}(\omega_v \times (I_{\mathcal{B}}\omega_u))] \times x & \text{in } \mathcal{S}. \end{cases}$$

Since $D(A^0) = \mathcal{H}(\Omega) \hookrightarrow L^2_R(\Omega)$ and $D(A^{1/2}) = \mathcal{V}(\Omega) \hookrightarrow L^6_R(\Omega)$, by interpolation, we deduce that $D(A^{1/4}) \hookrightarrow L^3_R(\Omega)$. Assume $u \in D(A^{1/4})$, $v \in D(A^{1/2})$ and let $w \in H^1_R(\Omega)$. Then

$$\begin{aligned}\langle \tilde{g}(u,v), w\rangle = -m_{\mathcal{B}}\omega_v \times \xi_u \cdot \xi_w - \omega_v \times (I_{\mathcal{B}}\omega_u)\cdot\omega_w \\ + \rho\int_{\mathcal{F}}(\bar{v}-v)\cdot\nabla w\cdot u\,\mathrm{d}x - \rho\int_{\mathcal{F}}\omega_v \times u \cdot w\,\mathrm{d}x\end{aligned}$$

defines $\tilde{g}(u,v)$ as an element of $(H^1_R(\Omega))'$, and its norm satisfies

$$\begin{aligned}\|\tilde{g}(u,v)\|_{(H^1_R(\Omega))'} \leqslant C(\mathcal{B})|\omega_v|(|\xi_u| + |\omega_u|) + C(\mathcal{L})\|\bar{v}-v\|_{6,\mathcal{F}}\|u\|_{3,\mathcal{F}} \\ + C(\mathcal{L})|\omega_v|\|u\|_{2,\mathcal{F}} \leqslant C(\mathcal{B},\mathcal{L})\|u\|_{1/4}\|v\|_{1/2}.\end{aligned}$$

If $u \in D(A^{1/2})$ and $v \in D(A^{1/4})$, we also have $\tilde{g}(u,v) \in (H^1_R(\Omega))'$. Defining

$$G(u,v) = \mathbb{P}_{(H^1_R)',\mathcal{V}'}\tilde{g}(u,v),$$

it follows that $G(u,v) \in D(A^{1/2})'$ for $(u,v) \in D(A^{1/4}) \times D(A^{1/2})$ or $(u,v) \in D(A^{1/2}) \times D(A^{1/4})$.

Now, if $u,v \in D(A^{1/2})$, then $\tilde{g}(u,v) = g(u,v) \in L^{3/2}_R(\Omega)$, where $g(u,v)$ is defined by (5.2), and

$$\begin{aligned}\|\tilde{g}(u,v)\|_{L^{3/2}_R(\Omega)} \leqslant C(\mathcal{L})(\|\bar{v}-v\|_{6,\mathcal{F}}\|\nabla u\|_{2,\mathcal{F}} + |\omega_v|\|u\|_{2,\mathcal{F}}) \\ + C(\mathcal{B})|\omega_v|(|\xi_u| + |\omega_u|) \leqslant C(\mathcal{B},\mathcal{L})\|u\|_{1/2}\|v\|_{1/2}.\end{aligned}$$

Since $D(A^{1/4}) \hookrightarrow L^3_R(\Omega)$, by duality and lemma 4.1, we conclude that $L^{3/2}_R(\Omega) \hookrightarrow D(A^{1/4})'$. With the help of figure 1, we conclude that if $u,v \in D(A^{1/2})$, then

$$G(u,v) = \mathbb{P}_{(H^1_R)',\mathcal{V}'}\tilde{g}(u,v) = \mathbb{P}_{L^{3/2}_R,D(A^{1/4})'}g(u,v) \in D(A^{1/4})'.$$

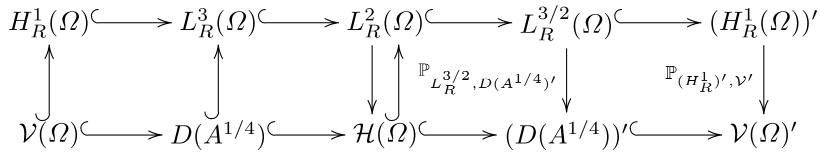

Figure 1. Functional spaces.

Hence, G satisfies the hypotheses of theorem 6.2 with $\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{4}$, $\kappa = \frac{1}{2}$, which allows us to obtain the existence and uniqueness of mild solutions for our system.

Theorem 6.5. *Let $u_0 \in D(A^{1/4})$. Then, for each $T > 0$, there exists a constant $C = C(\mathcal{B}, \mathcal{L}, T)$ such that if $\|u_0\|_{1/4} \leqslant C$, then the system (2.1) admits a unique mild solution on $[0, T]$:*

$$u \in C([0,T]; D(A^{1/4})) \cap C^1((0,T]; D(A^{1/4})) \cap C((0,T]; D(A^{1/2}))$$

with

$$\sup_{0<s<T} (\|s^{1/4} u(s)\|_{1/2} + \|s u'(s)\|_{1/4}) < \infty.$$

The mild solution obtained in theorem 6.5 is a strong solution in the following sense.

Theorem 6.6. *Let $u_0 \in D(A^{1/4})$ and let*

$$u \in C([0,T]; D(A^{1/4})) \cap C^1((0,T]; D(A^{1/4})) \cap C((0,T]; D(A^{1/2}))$$

be a mild solution to system (2.1). Then there exists a scalar function p such that $\nu \Delta^{\mathcal{F}}_D (u - \bar{u}) - \nabla p \in C((0,T]; L^{3/2}_R(\Omega))$ and

$$\begin{aligned} \rho \frac{\partial u}{\partial t} &= \nu \Delta^{\mathcal{F}}_D (u - \bar{u}) - \nabla p + \rho (\bar{u} - u) \cdot \nabla u + \rho \omega_u \times u \qquad \text{in } (0,T] \times \mathcal{F}, \\ m_{\mathcal{B}} \frac{\mathrm{d}\xi_u}{\mathrm{d}t} + m_{\mathcal{B}} \omega_u \times \xi_u &= -\int_{\mathcal{F}} (\nu \Delta^{\mathcal{F}}_D (u - \bar{u}) - \nabla p) \, \mathrm{d}x, \\ I_{\mathcal{B}} \frac{\mathrm{d}\omega_u}{\mathrm{d}t} + (I_{\mathcal{B}} \omega_u) \times \omega_u &= -\int_{\mathcal{F}} (\nu \Delta^{\mathcal{F}}_D (u - \bar{u}) - \nabla p) \times x \, \mathrm{d}x \quad \text{in } (0,T]. \end{aligned}$$

Proof. A mild solution satisfies $u \in C^1((0,T]; D(A^{1/4})) \cap C((0,T]; D(A^{1/2}))$ and therefore $u' \in C((0,T]; L^3_R(\Omega))$ and $g(u,u) \in C((0,T]; L^{3/2}_R(\Omega))$. Since

$$u'(t) + \mathcal{A} u(t) = \mathcal{G}(u(t), u(t)) \quad \text{in } C((0,T]; \mathcal{V}'(\Omega)),$$

it follows that

$$\langle \mathcal{A} u(t), v \rangle = \int_{\Omega} (g(u(t), u(t)) - u'(t)) \cdot v \tilde{\rho}(x) \, \mathrm{d}x \quad \forall v \in \mathcal{V}(\Omega),\ \forall t \in (0,T].$$

Combining the above relation with the fact that

$$\langle \mathcal{A} u, v \rangle = 2\nu \int_{\mathcal{F}} D(u) : D(v) \, \mathrm{d}x = 2\nu \int_{\mathcal{F}} D(u - \bar{u}) \colon D(v) \, \mathrm{d}x \quad \text{for } v \in \mathcal{V}(\Omega),$$

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we deduce

$$2\nu\int_{\mathcal{F}} D(u-\bar{u}) : D(v)\,\mathrm{d}x = \int_{\Omega}(g(u,u)-u')\cdot v\tilde{\rho}(x)\,\mathrm{d}x \quad \forall v\in V(\mathcal{F}) \text{ in } (0,T]. \tag{6.4}$$

In particular,

$$2\nu\int_{\mathcal{F}} D(u-\bar{u}) : D(v)\,\mathrm{d}x = \rho\int_{\mathcal{F}}(g(u,u)-u')\cdot v\,\mathrm{d}x \quad \forall v\in V(\mathcal{F}) \text{ in } (0,T].$$

From classical results for the Stokes problem (see, for example, [24, p. 14]), for each $t\in T$, there exists $p(t)\in L^2_{\mathrm{loc}}(\mathcal{F})$ such that

$$\rho(g(u,u)-u')|_{\mathcal{F}} = -\nu\Delta_D^{\mathcal{F}}(u-\bar{u}) + \nabla p \in C((0,T];L^{3/2}(\mathcal{F})). \tag{6.5}$$

and, therefore,

$$\rho\frac{\partial u}{\partial t} = \nu\Delta_D^{\mathcal{F}}(u-\bar{u}) - \frac{1}{\rho}\nabla p + \rho(\bar{u}-u)\cdot\nabla u + \rho\omega_u\times u \quad \text{in } (0,T]\times\mathcal{F}.$$

Now we take $v=e_i$, $i=1,2,3$, in (6.4) to get

$$\int_{\Omega}(g(u,u)-u')\tilde{\rho}(x)\,\mathrm{d}x = 0,$$

i.e.

$$\int_{\mathcal{S}}(g(u,u)-u')\rho_{\mathcal{B}}(x)\,\mathrm{d}x = -\rho\int_{\mathcal{F}}(g(u,u)-u')\mathrm{d}x.$$

Observing that

$$\rho\int_{\mathcal{F}}(g(u,u)-u')\,\mathrm{d}x = -\int_{\mathcal{F}}(\nu\Delta_D^{\mathcal{F}}(u-\bar{u})-\nabla p)\,\mathrm{d}x,$$

$$\int_{\mathcal{S}}(g(u,u)-u')\rho_{\mathcal{B}}(x)\,\mathrm{d}x = -m_{\mathcal{B}}\omega_u\times\xi_u - m_{\mathcal{B}}\frac{\mathrm{d}\xi_u}{\mathrm{d}t},$$

we find

$$m_{\mathcal{B}}\frac{\mathrm{d}\xi_u}{\mathrm{d}t} + m_{\mathcal{B}}\omega_u\times\xi_u = -\int_{\mathcal{F}}(\nu\Delta_D^{\mathcal{F}}(u-\bar{u})-\nabla p)\,\mathrm{d}x \quad \text{in } (0,T].$$

Finally, taking $v=e_i\times x$, $i=1,2,3$, in (6.4) gives

$$\int_{\Omega}(g(u,u)-u')\times x\tilde{\rho}(x)\,\mathrm{d}x = 0,$$

i.e.

$$\int_{\mathcal{S}}(g(u,u)-u')\times x\rho_{\mathcal{B}}(x)\,\mathrm{d}x = -\rho\int_{\mathcal{F}}(g(u,u)-u')\times x\,\mathrm{d}x.$$

By direct calculations, and using (6.5), we obtain

$$\rho\int_{\mathcal{F}}(g(u,u)-u')\times x\,\mathrm{d}x = -\int_{\mathcal{F}}(\nu\Delta_D^{\mathcal{F}}(u-\bar{u})-\nabla p)\times x\,\mathrm{d}x,$$

$$\int_{\mathcal{S}}(g(u,u)-u')\times x\rho_{\mathcal{B}}(x)\,\mathrm{d}x = -I_{\mathcal{B}}\omega_u\times\xi_u - I_{\mathcal{B}}\frac{\mathrm{d}\omega_u}{\mathrm{d}t},$$

https://doi.org/10.1017/S0308210510001034 Published online by Cambridge University Press

which yields

$$I_{\mathcal{B}}\frac{\mathrm{d}\omega_u}{\mathrm{d}t} + (I_{\mathcal{B}}\omega_u) \times \omega_u = -\int_{\mathcal{F}} (\nu\Delta_D^{\mathcal{F}}(u-\bar{u}) - \nabla p) \times x \,\mathrm{d}x.$$

Hence, (u, p) is a strong solution to (2.1) if we consider $\nu\Delta_D^{\mathcal{F}}(u-\bar{u}) - \nabla p$ undecoupled. □

In the next section we will construct strong solutions for more regular data.

7. Strong solutions

In this section we assume that the domain $\mathcal{F}$ is of class $C^{1,1}$. Then, using classical results (see, for example, [2, 5, 15]), we have

$$D(A) = \{u \in \mathcal{V}(\Omega);\; u|_{\mathcal{F}} \in H^2(\mathcal{F})\} \tag{7.1}$$

In that case, we can obtain the existence and uniqueness of strong solutions by using exactly the same arguments as those used in [22, 23].

THEOREM 7.1. *Assume that $\mathcal{F}$ is a $C^{1,1}$-domain. Let $u_0 \in \mathcal{V}(\Omega)$. Then there exist $T > 0$ and a strong solution (u, ξ, ω, p) of problem (2.1) satisfying*

$$u \in L^2(0, T; H^2(\mathcal{F})) \cap C([0, T]; H^1(\mathcal{F})) \cap H^1(0, T; L^2(\mathcal{F})),$$
$$\nabla p \in L^2(0, T; L^2(\mathcal{F})),$$
$$\xi \in H^1(0, T), \qquad \omega \in H^1(0, T).$$

Proof. First we write (2.1) as

$$u'(t) + Au(t) = G(u(t), u(t)), \quad t > 0, \tag{7.2 a}$$
$$u(0) = u_0 \in D(A^{1/2}), \tag{7.2 b}$$

where $G(u, u) := \mathbb{P}_{L^2_R,\mathcal{H}}\, g|_{D(A)\times D(A)}(u, u)$. We observe that, since $-A$ is the generator of an analytic semigroup, then, for any $F \in L^2(0, T; \mathcal{H}(\Omega))$, the system

$$\left.\begin{aligned} u'(t) + Au(t) &= F(t), \quad t > 0,\\ u(0) &= u_0 \in D(A^{1/2}) \end{aligned}\right\} \tag{7.3}$$

admits a unique solution

$$u \in L^2(0, T; D(A)) \cap C([0, T]; D(A^{1/2})) \cap H^1(0, T; \mathcal{H}(\Omega)).$$

Then, by using some calculations and (7.1), we obtain that

$$G(u, u) \in L^{5/2}(0, T; \mathcal{H}(\Omega)).$$

Consequently, we can consider the following mapping:

$$\mathcal{M}\colon L^2(0, T; \mathcal{H}(\Omega)) \to L^2(0, T; \mathcal{H}(\Omega)), \quad F \mapsto G(u, u),$$

where u is the solution of (7.3) associated with F. By using the Hölder inequality, we deduce that, for small time, there exists a closed ball $\mathcal{B}(0, R)$ of $L^2(0, T; \mathcal{H}(\Omega))$

invariant by $\mathcal{M}$ and such that the restriction of $\mathcal{M}$ to $\mathcal{B}(0,R)$ is a contraction, from which it follows that (7.2) has a unique solution in $\mathcal{B}(0,R)$. Finally, we observe that, by lemma 4.6, (7.2 *a*) is equivalent to $u' = \mathbb{P}_{L^2_R,\mathcal{H}}(L + g|_{D(A)\times D(A)}(u,u))$ and, therefore, there exists $p \in L^2(0,T;L^2(\mathcal{F}))$ such that (2.1) holds true. □

REMARK 7.2. It is important to note that a strong solution is a mild solution and a weak solution. Moreover, since a mild solution satisfies $u(\tau) \in D(A^{1/2})$ for all $\tau > 0$, then a mild solution is also a strong solution on the interval $[\tau, T]$ for all $\tau > 0$.

Acknowledgements

This work was partly supported by GRICES-EGIDE, Programa Pessoa 2007. A.S. was partly supported by the FCT through Project PTDC/MAT/105475/2008, while T.T. was partly supported by ANR Grant BLAN07-2 202879 and by ANR Grant 09-BLAN-0213-02.

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(*Issued 6 April 2012*)