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On the inference of the state of turbulence and mixing efficiency in stably stratified flows

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Scaling arguments are presented to quantify the widely used diapycnal (irreversible) mixing coefficient $\Gamma = \epsilon_{PE}/\epsilon$ in stratified flows as a function of the turbulent Froude number $Fr = \epsilon/Nk$. Here, N is the buoyancy frequency, k is the turbulent kinetic energy, ϵ is the rate of dissipation of turbulent kinetic energy and ϵ_{PE} is the rate of dissipation of turbulent potential energy. We show that for $Fr \gg 1$, $\Gamma \propto Fr^{-2}$, for $Fr \sim O(1)$, $\Gamma \propto Fr^{-1}$ and for $Fr \ll 1$, $\Gamma \propto Fr^{0}$. These scaling results are tested using high-resolution direct numerical simulation (DNS) data from three different studies and are found to hold reasonably well across a wide range of Fr that encompasses weakly stratified to strongly stratified flow conditions. Given that the Fr cannot be readily computed from direct field measurements, we propose a practical approach that can be used to infer the Fr from readily measurable quantities in the field. Scaling analyses show that $Fr \propto (L_T/L_O)^{-2}$ for $L_T/L_O > O(1)$, $Fr \propto (L_T/L_O)^{-1}$ for $L_T/L_O \sim O(1)$, and $Fr \propto (L_T/L_O)^{-2/3}$ for $L_T/L_O < O(1)$, where L_T is the Thorpe length scale and L_O is the Ozmidov length scale. These formulations are also tested with DNS data to highlight their validity. These novel findings could prove to be a significant breakthrough not only in providing a unifying (and practically useful) parameterization for the mixing efficiency in stably stratified turbulence but also for inferring the dynamic state of turbulence in geophysical flows.

Key words: stratified flows, stratified turbulence, turbulent mixing

1. Introduction

Diapycnal (irreversible) mixing is critical for maintaining the oceanic meridional overturning circulation and other important related issues such as global mass and heat budgets, ocean productivity etc. (Munk & Wunsch 1998). Hence, an accurate estimate of mixing in oceanic flows is essential but remains challenging due to complexities associated primarily with density stratification and limitations associated with direct measurements of pertinent turbulence quantities (Venayagamoorthy & Koseff 2016; Gregg *et al.* 2018). For instance, it is well known that internal wave motions are prevalent in density stratified flows and hence contaminate flux measurements. Thus, a number of indirect techniques are commonly used in practice in oceanography to

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infer turbulent heat and momentum fluxes. For example, the popular Osborn (1980) model recasts the diapycnal diffusivity for a homogeneous and stationary flow as

$$K_\rho = \Gamma \frac{\epsilon}{N^2}, \tag{1.1}$$

where $\Gamma = R_f/(1 - R_f)$ is a mixing coefficient and R_f is the mixing efficiency, ϵ is the rate of dissipation of turbulent kinetic energy and $N = \sqrt{(-g/\rho)(\mathrm{d}\langle\rho\rangle/\mathrm{d}z)}$ is the buoyancy frequency. Assuming that both ϵ and N are readily measurable in the ocean (albeit based on some underlying assumptions such as local isotropy of small scales and choices on how N is computed; see, for example, Arthur *et al.* (2017) for a discussion on N), then clearly the only remaining quantity that needs to be quantified for computing K_ρ is the mixing coefficient Γ. In practice, Γ is typically considered to have a canonical constant value of 0.2 ($R_f \approx 0.17$) (Osborn 1980). The constancy of Γ has been the subject of extensive debate over the last few decades (Gregg *et al.* 2018). In fact, a number of different parameterizations for the mixing efficiency have been proposed that are primarily based on one of three fundamental dimensionless parameters. These are the gradient Richardson number Ri, the buoyancy Reynolds number Re_b and the turbulent Froude number Fr. We note that it has been argued that a multi-parameter framework might be necessary to adequately parameterize mixing (Mater & Venayagamoorthy 2014; Monismith, Koseff & White 2018). However, from a practical standpoint, it is useful to identify an optimal single parameter that can be used to infer the state of turbulence and mixing efficiency. This is the primary impetus behind this current study.

It might be argued that a popular dimensionless number for parameterization of mixing is the buoyancy Reynolds (or Gibson) number $Re_b = \epsilon/\nu N^2$, where ν is the kinematic viscosity. This is mainly because it is readily computable from quantities that are measurable in the ocean. It has been argued that in strongly stratified flows, Γ is generally constant (typically assumed to be ~ 0.2 in practice) up to some critical value of Re_b and then gradually decreases like $\Gamma \propto Re_b^{-1/2}$, as stratification weakens (Shih *et al.* 2005; Lozovatsky & Fernando 2013; Salehipour & Peltier 2015; Monismith *et al.* 2018). However, there is no consensus on a definitive critical value of Re_b at which the transition from a constant value to a functional dependence on Re_b occurs as evident from analysis of data obtained from different numerical simulations and field measurements. For example, Shih *et al.* (2005) and Salehipour & Peltier (2015) found a critical Re_b of order 10^2 while Lozovatsky & Fernando (2013) found this transition to occur around 10^4. Using different data sets, Monismith *et al.* (2018) noticed that for high value of Re_b, mixing efficiency is not constant and varies with Re_b. However, there is no unique relation of R_f with Re_b. On the other hand, parameterizations based on the gradient Richardson number $Ri = N^2/S^2$ (where S is the mean shear rate) suggest that R_f increases with Ri in the shear-dominated or weakly stratified flow up to a critical value of gradient Richardson number ($Ri \sim 0.25$) (Linden 1979) and approaches a constant value for high-Ri regime (Venayagamoorthy & Koseff 2016). However, it should be noted that Ri is restricted to shear-driven turbulence and hence is not practically useful for quantifying mixing in the absence of mean shear. Therefore, while Ri and Re_b may appear to be convenient parameters for quantifying mixing, they may not be useful for robust characterizations of the local state of turbulence and associated mixing in stratified flows for the aforementioned reasons.

The turbulent Froude number $Fr = \epsilon/Nk$ (where k is the turbulent kinetic energy) is perhaps a better-suited parameter (Ivey & Imberger 1991; Shih *et al.* 2000), but least investigated in the context of oceanography because it is difficult to compute directly from measurable quantities in the field. It is worth noting that Fr can be viewed as a competition of time scales (i.e. between the turbulence time scale $T_L = k/\epsilon$ and the buoyancy time scale N^{-1}). Hence, Fr is indicator of the local state of turbulence in a stably stratified flow. For high Fr (weak stratification), it has been shown using simple scaling analysis that $\Gamma \sim Fr^{-2}$, and this has been verified using direct numerical simulation (DNS) results (Maffioli, Brethouwer & Lindborg 2016). For strongly stratified flows (i.e. for $Fr < 1$), DNS data show that Γ approaches a constant value provided the irreversible definition is used (i.e. $\Gamma = \epsilon_{PE}/\epsilon$) (Peltier & Caulfield 2003; Venayagamoorthy & Stretch 2010). However, to our knowledge, no physically based quantitative arguments have been presented to support the constancy of Γ in the low-Fr limit as well as how Γ should vary in the transition region between the low-Fr and the high-Fr limits. In the present work, our key goal is to show that the turbulent Froude number is indeed likely to be an optimal single parameter of choice that can provide a basis for inferring the state of turbulence and for parameterizing the irreversible mixing efficiency in stratified flows.

In the light of the discussions presented above, it is clear that despite the numerous studies on this topic, there is still no universal consensus on what the optimal parameter of choice is for quantifying Γ (Mater & Venayagamoorthy 2014; Gregg *et al.* 2018), and, more importantly, a robust and practically useful parameterization of Γ (or equivalently the mixing efficiency R_f) remains elusive. In what follows, we present a new formulation for the mixing efficiency in stably stratified turbulence using scaling arguments in § 2. This is followed by a brief description of the DNS data that are used for testing the proposed formulations in § 3. Results to highlight the fidelity of the proposed scaling relationships are presented in § 4 and concluding remarks are made in § 5.

2. Theoretical analysis

2.1. *Parameterization of Γ as a function of Fr*

In the limit of high Fr, considering a balance between advection and background stratification terms in the buoyancy equation, Maffioli *et al.* (2016) have shown that $\Gamma \propto Fr^{-2}$. Their DNS data also suggest that Γ approaches a constant for low Fr. Here, we use different scaling arguments that consider dominant time scales governing the flow for different stratified conditions (i.e. from weak to strong stratification) to derive the functional dependence of Γ on Fr. It is important to clarify that a strongly stratified flow ($Fr \ll 1$) can be classified into two different regimes depending on the value of Re_b, *viz.*, viscosity-affected regime ($Re_b \ll 1$) and strongly stratified regime ($Re_b \gg 1$). Note the viscosity-affected regime is virtually non-turbulent (Brethouwer *et al.* 2007). Given that the focus is on turbulent flows that are affected by buoyancy, the viscosity-affected region can be neglected without loss of generality in the discussion that follows (i.e. flow with $Re_b < 1$). Hence, it might be instructive to classify stably stratified flows into roughly three flow regimes based on the turbulent Froude number. As such, it can be expected that the functional relationship of Γ and Fr should be different for each of these different regimes. This is because, the transition between weakly stratified regime and strongly stratified regime does not happen at one single value of Fr. There should be an intermediate regime in between which is influenced by buoyancy (i.e. moderately stratified), where both stratification and turbulence are important. We note that Rehmann (2004) has also suggested three different regimes for quantifying R_f as a function of grid Richardson number for weak, moderate and strong stratification, respectively.

2.1.1. *Weakly stratified regime* ($Fr \gg O(1)$)

For a weakly stratified fluid or in the limit of high Fr, it is clear that density acts as a passive scalar since the flow is nearly isotropic and hence there is very little to negligible diapycnal mixing (Holford & Linden 1999). The dominant time scale is therefore the turbulent time scale, $T_L = k/\epsilon$. Thus, the vertical displacement of fluid particles $L_{disp} \sim w'T_L$, where w' is the vertical velocity. From this, the density fluctuation $\rho' \sim w'T_L(\partial\langle\rho\rangle/\partial z)$, where $\partial\langle\rho\rangle/\partial z$ is the background density gradient. Hence, the buoyancy flux $B = -(g/\rho)\langle\rho'w'\rangle \sim -(g/\rho)w'^2T_L(\partial\langle\rho\rangle/\partial z) = w'^2N^2T_L$. Also for this case, the turbulent kinetic energy $k \sim w'^2$ and its rate of dissipation $\epsilon \sim w'^2/T_L$, by assuming that dissipation rate of k is independent of buoyancy effects. Based on these arguments, it follows that the mixing coefficient for the weakly stratified regime should scale as

$$\varGamma = \frac{B}{\epsilon} = \frac{\epsilon_{PE}}{\epsilon} \sim \frac{w'^2N^2T_L}{w'^2/T_L} = Fr^{-2}. \tag{2.1}$$

We note that the end result shown in (2.1) is identical to that found in the DNS study by Maffioli *et al.* (2016) for weakly stratified turbulent flows.

2.1.2. *Moderately stratified regime* ($Fr \sim O(1)$)

For a moderately stratified turbulent flow, it is reasonable to assume that buoyancy effects would begin to influence the dynamics during the active mixing period. Hence, both the buoyancy time scale N^{-1} and turbulent time scale T_L are important but are not necessarily equal. We assume that buoyancy time scale sets the vertical displacement of fluid particles, i.e. $L_{disp} \sim w'/N$ and the density fluctuation $\rho' \sim (w'/N)(\partial\langle\rho\rangle/\partial z)$. Hence, the buoyancy flux $B \sim -(g/\rho)(w'^2/N)(\partial\langle\rho\rangle/\partial z) = w'^2N$. By assuming independence of the dissipation rate of k from buoyancy effects (i.e. the relevant time scale is T_L), it follows that

$$\varGamma \sim \frac{w'^2N}{w'^2/T_L} = Fr^{-1}. \tag{2.2}$$

This is a new scaling result for the transition region that is sandwiched between the weakly stratified and strongly stratified flow regimes.

2.1.3. *Strongly stratified regime* ($Fr \ll O(1)$)

In the limit of strong stratification, the buoyancy effects will be dominant and hence it is reasonable to assume that the dominant time scale will be only the buoyancy time scale N^{-1}. This implies that a bulk of the turbulent kinetic energy dissipates within one buoyancy period. From this, it follows that $\epsilon \sim w'^2N$. Also here, $L_{disp} \sim w'/N$, $\rho' \sim (w'/N)(\partial\langle\rho\rangle/\partial z)$ and $B \sim w'^2N$. Hence, the mixing efficiency scales as

$$\varGamma \sim \frac{w'^2NN^{-1}}{w'^2} = \text{const.} \sim Fr^0. \tag{2.3}$$

We note that this new scaling result provides a physically based underpinning for what has been observed from different data obtained from DNS, field and laboratory experiments concerning the constancy of the irreversible mixing efficiency in the strongly stratified regime.

2.2. *Inferring Fr from Ozmidov and Thorpe length scales*

As we have argued, the turbulent Froude number can be used to diagnose the state of turbulence and parameterize the mixing efficiency in stably stratified turbulent flows. However, its use is limited in practice due to reasons already mentioned. Here, we propose a novel approach to estimate Fr using measurable turbulent length scales in oceanic flows.

A fundamental length scale that provides a measure of overturning length scale in a stratified flow is the well-known Ellison length scale (Ellison 1957) defined as

$$L_E = \frac{\langle \rho'^2 \rangle^{1/2}}{\partial \langle \rho \rangle / \partial z}, \tag{2.4}$$

where ρ' is the turbulent density fluctuation, $\partial\langle\rho\rangle/\partial z$ is the mean background density gradient and $\langle\,\rangle$ represents ensemble average. Conceptually, L_E is obtained from three-dimensional resorting of instantaneous density fields to a reference state of minimum potential energy (Winters *et al.* 1995). However, in the context of oceanography, instantaneous density is measured using vertical profilers on moorings or dropped from research vessels. In the limit of available one-dimensional vertical profiles, statistically L_E represents the largest overturns in the flow and gives an indication of the available potential energy per unit mass. A similar and relatively simpler kinematic length scale obtained from instantaneous vertical density profiles is the Thorpe length scale, L_T (Thorpe 1977). In a statistical sense, both L_T and L_E represent a measure of the vertical distance travelled by fluid parcels in order to achieve an equilibrium position through adiabatic resorting. Thus, for one-dimensional vertical profiles, both L_T and L_E should be equivalent, a result that has been verified in previous studies (Itsweire *et al.* 1993; Mater, Schaad & Venayagamoorthy 2013). Another length scale that is often used to represent the size of a turbulent eddy in stratified turbulence was suggested by Ozmidov (1965) through a dimensionally constructed length scale commonly known as the Ozmidov length scale (L_O) that is defined as

$$L_O = (\epsilon/N^3)^{1/2}. \tag{2.5}$$

Here L_O is the length scale at which inertial forces balance the buoyancy forces, in such a way that L_O represents the largest (isotropic) eddy unaffected by buoyancy.

The ratio of these two length scales has been used to denote the age of a turbulent event (Smyth, Moum & Caldwell 2001). Mater *et al.* (2013) suggested that L_T and L_O are equivalent only for turbulent Froude number of order one. Here, we delve further to show that the ratio of L_T/L_O or, more explicitly, L_E/L_O is not only a signature of the age of turbulence but, more importantly, is a quantitative representation of the strength of stratification in a turbulent flow similar to the concept of a turbulent Froude number. A quantitative relationship between L_E/L_O and Fr can be derived as follows. The ratio of the Ellison length scale to the Ozmidov length scale can be written as

$$\frac{L_E}{L_O} = \frac{\langle \rho'^2 \rangle^{1/2} N^{3/2}}{(\partial\langle\rho\rangle/dz)\epsilon^{1/2}} = \frac{\langle \rho'^2 \rangle^{1/2}}{\epsilon^{1/2}} \frac{g}{\rho} N^{-1/2}. \tag{2.6}$$

In the limit of strong stratification, $(g/\rho)\langle\rho'^2\rangle^{1/2}$ represents a gravitational acceleration term which can be expected to scale with the velocity scale $k^{1/2}$ and time scale N^{-1}, such that $\langle\rho'^2\rangle^{1/2}(g/\rho) \sim wN \sim k^{1/2}N$. Thus, in the strongly stratified regime ($Fr < O(1)$), the length-scale ratio can be written as

$$\frac{L_E}{L_O} \sim \frac{k^{1/2}N}{\epsilon^{1/2}N^{1/2}} = (kN/\epsilon)^{1/2} = Fr^{-1/2}. \tag{2.7}$$

This can be rewritten to express Fr in terms of L_T/L_O as

$$Fr \sim (L_E/L_O)^{-2} \sim (L_T/L_O)^{-2}. \tag{2.8}$$

We note that this scaling also verifies the results by Mater *et al.* (2013) which show that for strongly stratified regime, $L_T \sim L_{kN} = (k/N^2)^{1/2}$, where L_{kN} is an overturning length scale in the buoyancy dominated regime such that $L_T/L_O \sim Fr^{-1/2}$.

For the moderately stratified regime where $Fr \sim O(1)$, it directly follows that the length-scale ratio, $L_E/L_O \sim Fr^{-1}$. Hence, Fr scales with L_E/L_O as

$$Fr \sim (L_E/L_O)^{-1} \sim (L_T/L_O)^{-1}. \tag{2.9}$$

Now, for a weakly stratified turbulent flow or in the limit of high Froude number $Fr > O(1)$, the Thorpe length scale as well as the Ellison length scale should scale with the isotropic turbulent length scale $L_{k\epsilon} = k^{3/2}/\epsilon$ (Luketina & Imberger 1989; Ivey & Imberger 1991). Thus, $L_E \sim L_T \sim L_{k\epsilon} = k^{3/2}/\epsilon$. With this information, the length-scale ratio, L_E/L_O, for a weakly stratified flow can be written as

$$\frac{L_E}{L_O} \sim \frac{k^{3/2}/\epsilon}{\epsilon^{1/2}/N^{3/2}} = (kN/\epsilon)^{3/2} = Fr^{-3/2}, \tag{2.10}$$

which translates to

$$Fr \sim (L_E/L_O)^{-2/3} \sim (L_T/L_O)^{-2/3}, \tag{2.11}$$

similar to the scaling provided by Ivey & Imberger (1991). Given that both L_T and L_O are directly measurable in the ocean using conductivity-temperature-depth (CTD) and microstructure profilers, equations (2.8), (2.9) and (2.11) are key scaling results that can be used to obtain direct estimates of Fr in the field. The scaling arguments presented for both Γ and Fr are tested using three independent DNS data sets in what follows.

3. Data sources

Three independent DNS data sets are considered to test the veracity of our proposed scaling arguments. The first data set is from our own set of DNS of decaying homogeneous stably stratified turbulence. These simulations were carried out using a pseudo-spectral DNS code developed by Riley, Metcalfe & Weissman (1981) for stably stratified homogeneous turbulent flows. A cubical periodic domain of dimension 2π with 512^3 grid points was considered for all the simulations. The turbulent flow was initialized with a Gaussian isotropic three-dimensional solenoidal velocity field and allowed to evolve and decay under the influence of a constant background stratification. Flow was characterized with an initial Reynolds number of 1000 defined as $Re_0 = u_0 L_0/\nu$, where $u_0 = 1$ is the initial velocity scale and $L_0 = 1$ is the initial length scale. Background stratification was characterized with an initial Richardson number $Ri_0 = (NL_0/u_0)^2$. Four DNS simulations were performed for the present study with $Ri_0 = 0.01$, 0.1, 1.0 and 10, respectively, for a duration of $5L_0/u_0$. Further details of the simulations are given in Garanaik & Venayagamoorthy (2018). The second data set was obtained from a high-resolution DNS study of stratified turbulence by Maffioli *et al.* (2016). A body force was included in their numerical simulations in order to achieve stationary turbulence. They used isotropic forcing for most of the simulations and two-dimensional vortical forcing was used for a few

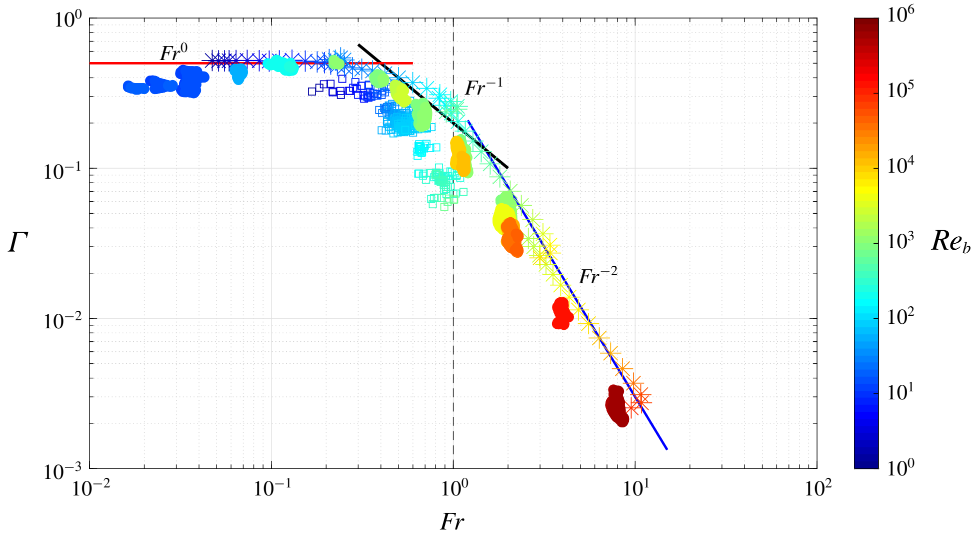

FIGURE 1. Mixing coefficient Γ as a function of turbulent Froude number Fr. The colour bar shows values of Re_b. Star: decaying DNS; circle: forced DNS of Maffioli *et al.* (2016); square: sheared DNS of Shih *et al.* (2005). Solid lines display the scaling relations derived in § 2.1.

simulations to achieve a strongly stratified regime with buoyancy Reynolds number greater than 10. The third data set was obtained from DNS study of homogeneous sheared stably stratified turbulence by Shih *et al.* (2005). These simulations are for temporally developing homogeneous turbulence with constant mean shear and constant background stratification. These simulations were performed at a relatively lower resolution (128^3) compared to the other two data sets but include the effects of background shear.

4. Results

4.1. *Mixing coefficient as a function of turbulent Froude number*

Figure 1 shows the mixing coefficient Γ as a function of turbulent Froude number for decaying, forced and sheared DNS data. The buoyancy Reynolds number Re_b is also shown in the colour bar for reference. There are two observations that are noteworthy. The first key observation that can be made from figure 1 is that all the data collapse reasonably well on to the lines given by the new scaling results for Γ proposed in § 2.1. These data show that in the strongly stratified flow regime ($Fr \ll 1$), Γ is approximately a constant. In the moderately stratified flow regime ($Fr \sim O(1)$), $\Gamma \propto Fr^{-1}$, and for the weakly stratified flow regime ($Fr \gg 1$), $\Gamma \propto Fr^{-2}$, which is also in agreement with what has been previously shown by Maffioli *et al.* (2016) in the limit of high Fr.

Second, it appears that there is no unique relationship between Γ and buoyancy Reynolds number Re_b. In order to see this point clearly, the mixing coefficient Γ as a function of Re_b is shown in figure 2 for all three data sets. The Fr is also shown in the colour bar to facilitate cross-referencing with figure 1. It can be seen that Γ is approximately constant for strongly stratified flows (i.e. low values of Fr). However,

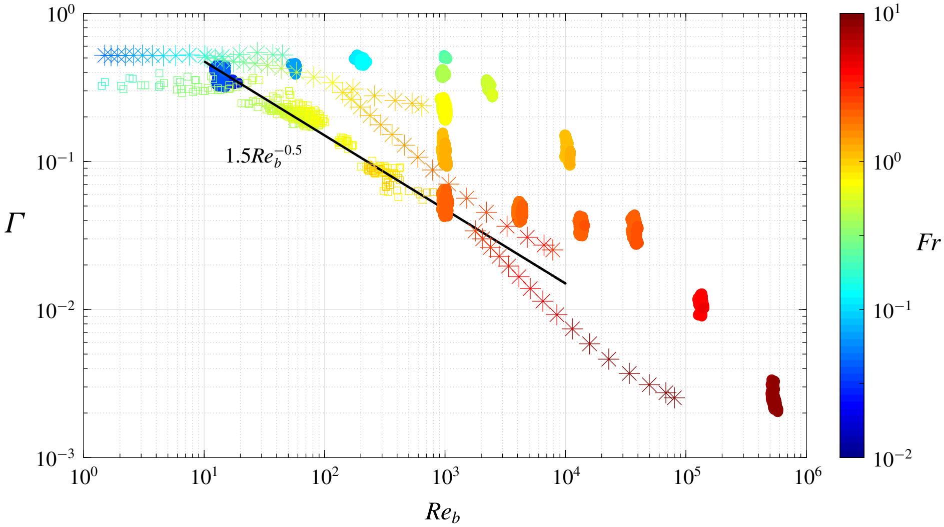

FIGURE 2. Mixing coefficient Γ as a function of buoyancy Reynold number Re_b for decaying DNS (star), forced DNS of Maffioli *et al.* (2016) (circle) and sheared DNS of Shih *et al.* (2005) (square). Solid line indicates the functional relation of Γ and Re_b proposed by Shih *et al.* (2005).

it is evident that the transition of Γ from a constant value to a functional dependence on Re_b is not unique. For example, for various flows which have $Re_b = 10^3$, the measured values of Γ can differ by over an order of magnitude as shown in figure 2. These results highlight how Re_b is an ambiguous parameter for quantifying mixing in stratified flows. We note that the ambiguity associated with Re_b has been discussed in previous studies (Mater & Venayagamoorthy 2014; Holleman, Geyer & Ralston 2016; Scotti & White 2016; Monismith *et al.* 2018). Based on the results shown in figures 1 and 2, it is plausible to suggest that Fr might be an optimal (single) parameter that can be used to parameterize the mixing efficiency.

4.2. *Inferring the state of turbulence*

We have seen in figure 1 that the mixing coefficient is well described by the scaling results in § 2.1. However, as discussed in § 2.2, there is the impending issue of how to estimate Fr from measurable quantities in the field. To this end, scaling arguments to find relationships between L_E/L_O and turbulent Froude number were presented in § 2.2. The results presented in figure 3 for decaying, sheared and forced DNS data validate the scaling results shown in (2.8), (2.9), and (2.11), respectively. The data indicate that $Fr \sim (L_E/L_O)^{-2/3}$ for $(L_E/L_O) < O(1)$, $Fr \sim (L_E/L_O)^{-1}$ for $(L_E/L_O) \sim O(1)$ and $Fr \sim (L_E/L_O)^{-2}$ for $(L_E/L_O) > O(1)$. By combining the scaling results provided for Γ and Fr (as shown in figures 1 and 3), the scaling relationship between Γ and the (field) measurable length-scale ratio L_E/L_O can be also established for practical purposes as follows: for weakly stratified regime, $\Gamma \sim (L_E/L_O)^{4/3}$; for moderately stratified regime, $\Gamma \sim (L_E/L_O)^1$; and for the strongly stratified regime where Γ is approximately constant, $\Gamma \sim (L_E/L_O)^0$. The DNS data are in good agreement with these results, as shown in figure 4. We note that, using a least squares fitted power law relationship, Smyth *et al.* (2001) found that $\Gamma \sim (L_T/L_O)^{0.6}$, which may be

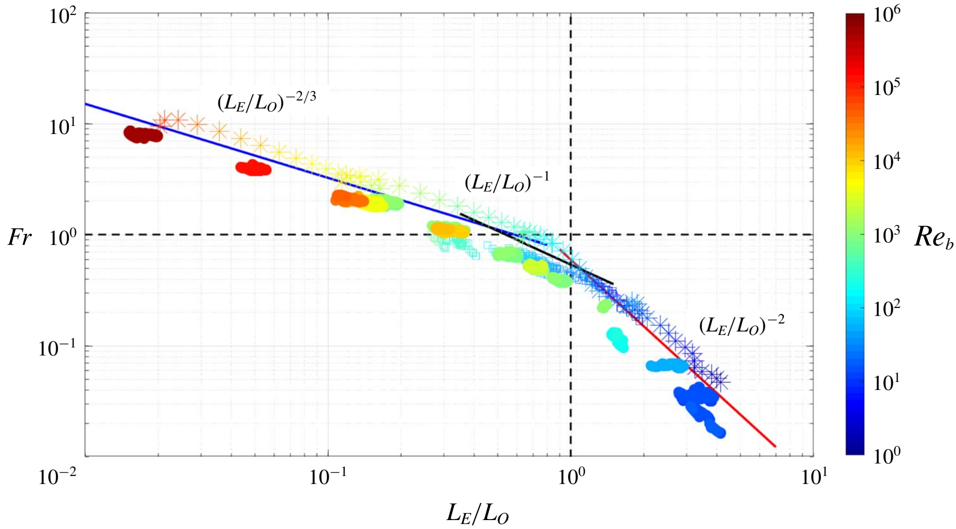

FIGURE 3. Froude number as a function of L_E/L_O for decaying DNS (star), forced DNS (circle) and sheared DNS (square) data with Re_b in colour bar. Solid lines display the scaling relations derived in § 2.2.

acceptable for the moderately stratified regime. In a recent study, using measurements made in the ocean, Ijichi & Hibiya (2018) have also shown that $\Gamma \sim (L_T/L_O)^{4/3}$, similar to the scaling provided here for $Fr > O(1)$. The results presented here are indeed powerful because they show that with L_T and L_O known (i.e. measurable), it is possible to estimate Fr from field measurements and hence identify the state of turbulence in stratified flows in the field. This in turn permits the use of a more accurate parameterization to predict Γ which is crucial for obtaining robust estimates of diapycnal mixing.

5. Concluding remarks

In this paper we first derived new scaling results for Γ as a function of Fr as well as for Fr as a function of L_T/L_O. We then used direct numerical simulation data from three independent studies to validate the scaling results. Three significant findings can be noted from the above discussions. First, scaling results that are validated using high-resolution DNS data show that the mixing coefficient scales with the turbulent Froude number in a universal manner, noting, however, that the functional dependence on Fr is different for the three different flow regimes. Second, the results show that Re_b is an ambiguous parameter and hence parameterizations of Γ based on Re_b are not unique. The third key finding is that the ratio L_T/L_O not only is a representation of the age of turbulence, as has been previously suggested (i.e. young turbulence or old turbulence), but also represents the state of the flow (i.e. whether the flow is in a strongly stratified regime or in a weakly stratified regime). This finding will be useful in oceanography for inferring the state of turbulence and thereby facilitate the use of the more appropriate parameterizations for Γ that have been formulated in this study. To the knowledge of authors, this is the first time such an analysis has been carried out to both identify the state of turbulence and quantify the mixing efficiency using a single parameter in an unambiguous manner. The natural next steps are to

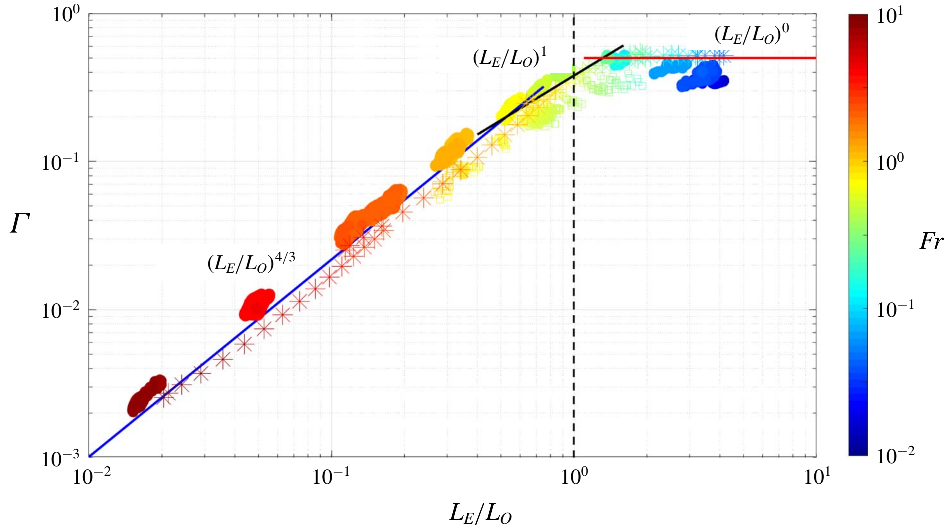

FIGURE 4. Mixing coefficient Γ as a function of L_E/L_O for decaying DNS (star), forced DNS (circle) and sheared DNS (square) data with Fr in colour bar. Solid lines display the scaling relations.

perform further tests with more complex forcing conditions and evaluate the utility of the proposed parameterizations in the field.

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