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A NOTE ON TWO RELIABILITY LOWER BOUNDS FOR MULTISTATE SYSTEMS

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Maymin proved that for binary systems, the path-cut lower bound is sharper than the minimax lower bound for highly reliable components, which was originally conjectured by Barlow and Proschan. In this note, an example is constructed to illustrate that this, in general, is not the case for multistate systems. However, an affirmative result is obtained under mild conditions we imposed. Examples are given to illustrate the applications of our results.

1. INTRODUCTION

First, we introduce some basic notations for binary coherent systems. A binary coherent system composed of n components is denoted by (C,ϕ), where $C=\{1,\dots,n\}$ designates the n components and $\phi:\{0,1\}^n \mapsto \{0,1\}$ denotes the nondecreasing structure function of the system. Assume that the system contains relevant components only; that is, for each $i \in C$, there exists a vector $(\cdot_i,\mathbf{x})$ such that $\phi(1_i,\mathbf{x}) = 1 - \phi(0_i,\mathbf{x}) = 1$, where $(\cdot_i,\mathbf{x}) = (x_1,\dots,x_{i-1},\cdot_i,x_{i+1},\dots,x_n)$. The reliability of the ith component is denoted by $p_i = \Pr\{X_i = 1\}$ $(i=1,\dots,n)$. Assume that the n components are independent. The reliability function of the system is denoted by $h(\mathbf{p}) = \Pr\{\phi(\mathbf{X})=1\}$, where $\mathbf{p}=(p_1,\dots,p_n)$ and $\mathbf{X}=(X_1,\dots,X_n)$. Denote by $P_1,\dots,P_p$ and $K_1,\dots,K_c$ the p minimal path sets and the c minimal cut sets, respectively, of the system. It is known that (see [1,5])

$$h(\mathbf{p}) \ge \max\{l_{\text{pc}}(\mathbf{p}), l_{\text{mm}}(\mathbf{p})\}, \tag{1}$$

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where the *path-cut lower bound* $l_{\text{pc}}(\mathbf{p}) = \prod_{j=1}^{c} \coprod_{i\in K_j} p_i$, the *minimax lower bound* $l_{\text{mm}}(\mathbf{p}) = \max_{1\le r\le p} \prod_{i\in P_r} p_i$, and $\coprod_{i\in A} p_i \equiv 1 - \prod_{i\in A}(1-\text{p}_\text{i})$ (see [1] for unspecified notations).

It is quite easy to see that if ϕ is in series, then $l_{\text{pc}}(\mathbf{p}) = l_{\text{mm}}(\mathbf{p}) = \prod_{1\le i\le n} p_i$; and if ϕ is parallel, then $l_{\text{pc}}(\mathbf{p}) \ge l_{\text{mm}}(\mathbf{p})$ for all $\mathbf{p} \in [0,1]^n$. Barlow and Proschan [1, p. 38] conjectured that, for general binary systems, if the components are all highly reliable, then the path-cut lower bound provides the better lower bound for $h(\mathbf{p})$. In formulation,

$$\prod_{j=1}^{c} \coprod_{i\in K_j} p_i \ge \max_{1\le r\le p} \prod_{i\in P_r} p_i \quad \text{as } p_1,\ldots,p_n \to 1. \tag{2}$$

The correctness of this conjecture has been confirmed by Maymin [7].

We now need to introduce some notations for multistate systems. Let ϕ: $S^n \mapsto S$ denote the nondecreasing structure function of a multistate system, where $S = \{0,1,\ldots,M\}$ represents the $M+1$ distinct levels of performance of the system and its n independent components, varying from perfect functioning (M) to total failure (0). We call a vector $\mathbf{x}$ a *critical cut vector* of ϕ for level k ($k > 0$) if $\phi(\mathbf{x}) < k$ and $\phi(\mathbf{y}) \ge k$ for all $\mathbf{y} > \mathbf{x}$; and we call a vector $\mathbf{z}$ a *critical path vector* of ϕ for level k if $\phi(\mathbf{z}) \ge k$ and $\phi(\mathbf{y}) < k$ for all $\mathbf{y} < \mathbf{z}$, where $\mathbf{y} > \mathbf{x}$ means $\text{y}_i \ge x_i$ for each i and strict inequality holds for some i. Denote by L_k (U_k) the set of all critical cut (critical path) vectors of ϕ for level k. If $\mathbf{x} \in L_k$, let

$$L_k(\mathbf{x}) = \{(i, x_i) | x_i < M\},$$

and if $\mathbf{z} \in U_k$, let

$$U_k(\mathbf{z}) = \{(i, z_i) | z_i > 0\}.$$

The set $L_k(\mathbf{x})$ ($U_k(\mathbf{z})$) is called a critical cut (path) set of ϕ for level k. The reliabilities of the ith component is denoted by a vector $\mathbf{p}_i = (p_{i,0},\ldots,p_{i,M-1})$, where $p_{i,s} = \Pr\{X_i = s\}$ for $s = 0,\ldots,M-1$ and $p_{i,M} = 1 - p_{i,0} - \cdots - p_{i,M-1}$. The reliability vector of the n components is denoted by $\mathbf{p} \equiv (\mathbf{p}_1,\ldots,\mathbf{p}_n)$. In the case that level $k > 0$ meets our minimum requirement for system performance, we are primarily concerned with the reliability function $h_k(\mathbf{p}) = \Pr\{\phi(\mathbf{X}) \ge k\}$. It is quite clear that

$$h_k(\mathbf{p}) = \Pr\left\{\bigcup_{\mathbf{z}\in U_k} \mathbf{X} \ge \mathbf{z}\right\} = \Pr\left\{\bigcap_{\mathbf{x}\in L_k} \mathbf{X} \not\le \mathbf{x}\right\}, \tag{3}$$

where $\mathbf{x} \not\le \mathbf{y}$ means that $x_i > \text{y}_i$ for at least one i.

The following result concerning reliability lower bounds for $h_k(\mathbf{p})$ extends Eq. (1) from binary systems to multistate systems (see [3, Lemma 3.1]):

$$h_k(\mathbf{p}) \ge \max\{l_{\text{pc}}^k(\mathbf{p}), l_{\text{mm}}^k(\mathbf{p})\}, \quad k = 1,\ldots,M, \tag{4}$$

where the *multistate path-cut lower bound*

$$l_{\text{pc}}^{k}(\mathbf{p}) = \prod_{\mathbf{x}\in L_k} \coprod_{(i,x_i)\in L_k(\mathbf{x})} \bar{p}_{i,x_i}, \tag{5}$$

the *multistate minimax lower bound*

$$l_{\text{mm}}^{k}(\mathbf{p}) = \max_{\mathbf{z}\in U_k} \prod_{(i,z_i)\in U_k(\mathbf{z})} \bar{p}_{i,z_i-1}, \tag{6}$$

and $\bar{p}_{i,s} \equiv \Pr\{X_i > s\}$ for $s = 0,\dots,M-1$.

A multistate system belongs to the Barlow–Wu class [2] if and only if its structure function ϕ can be represented as

$$\phi(\mathbf{x}) = \max_{1\le r\le p} \min_{i\in P_r} x_i = \min_{1\le j\le c} \max_{i\in K_j} x_i \quad \text{for all } \mathbf{x} \in S^n, \tag{7}$$

where $P_1,\dots,P_p$ and $K_1,\dots,K_c$ represent the p minimal path sets and the c minimal cut sets, respectively, of a binary coherent system. Natvig [8] introduced a class of multistate systems which is closely related to the Barlow–Wu class, namely the N2 class. A multistate system belongs to the N2 class if and only if there exist binary coherent structures ϕ_k $(k = 1,\dots,M)$ such that its structure function ϕ satisfies

$$\phi(\mathbf{x}) \ge k \Leftrightarrow \phi_k(\mathbf{I}_k(\mathbf{x})) = 1, \tag{8}$$

where $\mathbf{I}_k(\mathbf{x}) \equiv (I_k(x_1),\dots,I_k(x_n))$ and $I_k(x_i) = 1$ (0) if and only if $x_i \ge k$ $(< k)$. The N2 class reduces to the Barlow–Wu class when $\phi_1 = \cdots = \phi_M$.

Equation (2) for binary systems can be easily extended to the Barlow–Wu and the N2 classes of multistate systems as follows.

THEOREM 1.1: *Let ϕ be a multistate structure which belongs to the N2 class. Then, for each level $k > 0$,*

$$\prod_{\mathbf{x}\in L_k} \coprod_{(i,x_i)\in L_k(\mathbf{x})} \bar{p}_{i,x_i} \ge \max_{\mathbf{z}\in U_k} \prod_{(i,z_i)\in U_k(\mathbf{z})} \bar{p}_{i,z_i-1} \quad \text{as } \bar{p}_{1,k-1},\dots,\bar{p}_{n,k-1} \to 1. \tag{9}$$

PROOF: Let $\mathbf{x}$ be a critical cut vector of ϕ for level k. Then, from the result of Borges and Rodrigues [4], $\mathbf{x}$ must be of the form $((k-1)^{C'},\mathbf{M})$, where $C' \subset C$ and $(s^{C'},\mathbf{t})$ means that $x_i = s\,(t)$ for $i \in C'$ $(i \notin C')$. Consider the corresponding binary structure ϕ_k, defined in Eq. (8), with minimal cut sets $K_1,\dots,K_c$. Its component reliabilities are $p_i = \Pr\{I_k(X_i) = 1\} = \bar{p}_{i,k-1}$ $(i = 1,\dots,n)$. From the relationship between ϕ and ϕ_k defined in Eq. (8), it is easy to see that C' must be a minimal cut set for ϕ_k. Thus,

$$\prod_{\mathbf{x}\in L_k} \coprod_{(i,x_i)\in L_k(\mathbf{x})} \bar{p}_{i,x_i} = \prod_{\mathbf{x}\in L_k} \coprod_{(i,x_i)\in L_k(\mathbf{x})} \bar{p}_{i,k-1} = \prod_{j=1}^{c} \coprod_{i\in K_j} p_i.$$

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Applying a similar argument on the critical path vectors, we obtain that

$$\max_{\mathbf{z}\in U_k} \prod_{(i,z_i)\in U_k(\mathbf{z})} \bar{p}_{i,z_i-1} = \max_{\mathbf{z}\in U_k} \prod_{(i,z_i)\in U_k(\mathbf{z})} \bar{p}_{i,k-1} = \max_{1\le r\le p} \prod_{i\in P_r} p_i,$$

where $P_1,\ldots,P_p$ are the p minimal path sets for ϕ_k. The conclusion then follows from Eq. (2). ■

The main question we study in this note is that for more general classes of multistate systems, whether the path-cut lower bound $l^k_{\mathrm{pc}}(\mathbf{p})$ provides a better lower bound than the minimax lower bound $l^k_{\mathrm{mm}}(\mathbf{p})$ when the components are all *highly reliable*. We consider in this note a multistate component i as highly reliable if either $\Pr\{X_i = M\} \to 1$ or $\Pr\{X_i = 0\} \to 0$, the former one implies, but is not implied by, the latter one. In Section 2, we give an example to illustrate that the bound $l^k_{\mathrm{pc}}(\mathbf{p})$ is not always better in general, even when the components are all highly reliable. However, under mild conditions imposed, we obtain a main result of this note which states that the bounds $l^k_{\mathrm{pc}}(\mathbf{p})$ is indeed sharper than $l^k_{\mathrm{mm}}(\mathbf{p})$ when $\Pr\{X_i = M\} \to 1$ $(i = 1,\ldots,n)$. Section 3 gives some examples to illustrate our results.

2. RESULTS

A well-known and important property for binary coherent systems states that each minimal path set has a nonempty intersection with each minimal cut set (see [1]). This property is generalized for multistate systems in Lemma 2.1, which will be used repeatedly later in this note to derive further results.

LEMMA 2.1: *For each* $\mathbf{x} \in L_k$ *and each* $\mathbf{z} \in U_k$, *there exists a* $j \in \{1,\ldots,n\}$ *such that* $z_j > x_j$ *(hence,* $z_j > 0$ *and* $x_j < M$*).*

PROOF: Suppose this is not true. Then, $z_j \le x_j$ for all $(j, x_j) \in L_k(\mathbf{x})$. Hence, $\mathbf{z} \le \mathbf{x}$, which implies that $\phi(\mathbf{z}) < k$, a contradiction. ■

We call $\mathbf{x}$ a *cut vector* of ϕ for level k if $\phi(\mathbf{x}) < k$, and the components $\{i | x_i < M\}$ the *cut components* of $\mathbf{x}$. If there exists an $\mathbf{x} \in L_k$ with size $|L_k(\mathbf{x})| = |\{(i, x_i) | x_i < M\}| = 1$, we call the vector $\mathbf{x}$ a *singleton cut vector* of ϕ for level k, and the component i a *singleton cut component*. Denote the set of singleton cut vectors of ϕ for level k by A_S,

$$A_S = \{\mathbf{x} | \mathbf{x} \in L_k; |L_k(\mathbf{x})| = 1\}, \tag{10}$$

and by A_{IS} the index set of such singleton cut components,

$$A_{\mathrm{IS}} = \{i | (x_i, \mathbf{M}) \in A_S; x_i < M\}. \tag{11}$$

The following theorem is easily obtained, which states that if the critical cut vectors are all singleton cut vectors, then the two bounds are identical.

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THEOREM 2.2: *Suppose that* $L_k = \{(x_i, \mathbf{M}) | i \in A_{IS}\}$ *(i.e.,* $L_k = A_S$*), where* $A_{IS} \subset C$. *Then,* $l_{pc}^k(\mathbf{p}) = l_{mm}^k(\mathbf{p})$ *for all* $\mathbf{p}$.

PROOF: Clearly in this case, $l_{pc}^k(\mathbf{p}) = \prod_{\mathbf{x}\in L_k} \coprod_{(i,x_i)\in L_k(\mathbf{x})} \bar{p}_{i,x_i} = \prod_{i\in A_{IS}} \bar{p}_{i,x_i}$. Let $\mathbf{z}$ be a given critical path vector in U_k. Applying Lemma 2.1 on $\mathbf{z}$ and each $(x_i, \mathbf{M})$, we have that $z_i > x_i$ for each $i \in A_{IS}$. Thus, by the critical property, there exists only one critical path vector $\mathbf{z}$ and $U_k(\mathbf{z}) = \{(i, x_i + 1) | i \in A_{IS}\}$. Hence, $l_{mm}^k(\mathbf{p}) = l_{pc}^k(\mathbf{p}) = \prod_{i\in A_{IS}} \bar{p}_{i,x_i}$. ■

Example 2.3: Let a multistate structure $\phi: \{0,1,2\}^2 \mapsto \{0,1,2\}$ be defined as follows:

$$\phi(0,0) = 0, \quad \phi(0,1) = 1, \quad \phi(0,2) = 1,$$
$$\phi(1,0) = 0, \quad \phi(1,1) = 1, \quad \phi(1,2) = 1,$$
$$\phi(2,0) = 1, \quad \phi(2,1) = 2, \quad \phi(2,2) = 2.$$

Consider level $k = 2$. Then, $L_2 = \{(1,2),(2,0)\}$ and $U_2 = \{(2,1)\}$. It is easy to see that $l_{pc}^2(\mathbf{p}) = l_{mm}^2(\mathbf{p}) = p_{1,2}(p_{2,1} + p_{2,2})$. This fact can be directly deduced from Theorem 2.2, since $|L_2(1,2)| = |L_2(2,0)| = 1$.

The following theorem is similarly obtained, and the proof is omitted.

THEOREM 2.4: *Suppose that* $U_k = \{(z_i, \mathbf{0}) | i \in C'\}$, *where* $C' \subset C$. *Then,* $l_{pc}^k(\mathbf{p}) \geq l_{mm}^k(\mathbf{p})$ *for all* $\mathbf{p}$.

Example 2.5: Let a multistate structure $\phi: \{0,1,2\}^2 \mapsto \{0,1,2\}$ be defined as follows:

$$\phi(0,0) = 0, \quad \phi(0,1) = 0, \quad \phi(0,2) = 1,$$
$$\phi(1,0) = 1, \quad \phi(1,1) = 1, \quad \phi(1,2) = 2,$$
$$\phi(2,0) = 1, \quad \phi(2,1) = 1, \quad \phi(2,2) = 2.$$

Consider level $k = 1$. Then, $U_1 = \{(0,2),(1,0)\}$ and $L_1 = \{(0,1)\}$. It is easy to see that $l_{pc}^1(\mathbf{p}) = 1 - (1 - p_{2,2})(1 - p_{1,1} - p_{1,2})$, and $l_{mm}^1(\mathbf{p}) = \max\{p_{2,2}, p_{1,1} + p_{1,2}\}$. Hence, $l_{pc}^1(\mathbf{p}) \geq l_{mm}^1(\mathbf{p})$ for all $\mathbf{p}$.

Remark 1: A multistate structure that satisfies the condition stated in Theorem 2.2 (2.4) has been called, by Langseth and Lindqvist [6], a *generalized series* (*parallel*) *system*. In Theorem 2.2 (2.4), (i) if $A_{IS} = C$ ($C' = C$), then the system is a multistate series (parallel) system [i.e., $\phi(\mathbf{x}) = \min_{1\le i\le n} x_i$ ($\max_{1\le i\le n} x_i$)] and (ii) if component $i \notin A_{IS}$ (C'), then this component is irrelevant to the system for level k, but may be relevant for some other levels.

We now give an example to illustrate that for a general multistate system with *highly reliable* components ($p_{i,M} \to 1$ or $p_{i,0} \to 1$), without further assumption, neither one of the bounds $l_{pc}^k(\mathbf{p})$ and $l_{mm}^k(\mathbf{p})$ dominates the other one.

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Example 2.6: Let $\phi:\{0,1,2\}^2 \mapsto \{0,1,2\}$ be defined as follows:

$$\phi(0,0)=0,\quad \phi(0,1)=0,\quad \phi(0,2)=1,$$
$$\phi(1,0)=0,\quad \phi(1,1)=1,\quad \phi(1,2)=2,$$
$$\phi(2,0)=1,\quad \phi(2,1)=2,\quad \phi(2,2)=2.$$

Consider level $k=2$. Then, $U_2=\{(1,2),(2,1)\}$ and $L_2=\{(2,0),(0,2),(1,1)\}$. Assume that the two components have equal reliabilities: Let $p_{i,k}=a_k$ for $k=0,1,2$ and $i=1,2$. The path-cut and minimax lower bounds for $h_2(\mathbf{p})$ are

$$\begin{aligned} l_{\text{pc}}^2(\mathbf{p}) &= \coprod_{(i,x_i)\in L_2(2,0)} \bar{p}_{i,x_i} \coprod_{(i,x_i)\in L_2(0,2)} \bar{p}_{i,x_i} \coprod_{(i,x_i)\in L_2(1,1)} \bar{p}_{i,x_i} \\ &= (a_1+a_2)(a_1+a_2)(a_2+a_2-a_2^2) \\ &= a_2(a_1+a_2)^2(2-a_2); \end{aligned} \tag{12}$$

$$\begin{aligned} l_{\text{mm}}^2(\mathbf{p}) &= \max\{\bar{p}_{1,0}\bar{p}_{2,1}, \bar{p}_{1,1}\bar{p}_{2,0}\} \\ &= a_2(a_1+a_2). \end{aligned} \tag{13}$$

$$l_{\text{mm}}^2(\mathbf{p}) - l_{\text{pc}}^2(\mathbf{p}) = a_2(a_1+a_2)[1-(1-a_0)(2-a_2)]. \tag{14}$$

Hence,

$$\begin{aligned} l_{\text{mm}}^2(\mathbf{p}) \geq l_{\text{pc}}^2(\mathbf{p}) &\Leftrightarrow 1 \geq (1-a_0)(2-a_2) \\ &\Leftrightarrow a_0 \geq \frac{1-a_2}{2-a_2}. \end{aligned} \tag{15}$$

Clearly, $1 > a_2 + (1-a_2)(2-a_2)^{-1}$ holds for all $a_2 < 1$. Thus, for any $a_2 \to 1$, (i) $l_{\text{mm}}^2(\mathbf{p}) > l_{\text{pc}}^2(\mathbf{p})$ if $(1-a_2)(2-a_2)^{-1} < a_0 < 1-a_2$ and (ii) $l_{\text{mm}}^2(\mathbf{p}) < l_{\text{pc}}^2(\mathbf{p})$ if $a_0 < (1-a_2)(2-a_2)^{-1}$. Similarly, we obtain that, for any $a_0 \to 0$, $l_{\text{mm}}^2(\mathbf{p}) > (<)$ $l_{\text{pc}}^2(\mathbf{p})$ if $a_2 > (<)\ (1-2a_0)(1-a_0)^{-1}$.

In the following, without loss of generality and for convenience, we let $p_{i,M} = 1-\delta_i$ $(0<\delta_i<1)$ and $p_{i,s}=\delta_i b_{i,s}$ for $s=0,1,\ldots,M-1$, where $b_{i,0}+\cdots+b_{i,M-1}=1$ $(i=1,\ldots,n)$. Furthermore,

$$q_{i,s} \equiv 1-\bar{p}_{i,s} \equiv 1-\Pr\{X_i > s\}, \quad s=0,\ldots,M-1.$$

We now present a main result of this note, which states that for a general multistate system, if either (i) the system contains no *singleton cut vector* or (ii) there exists at least one critical cut vector $\mathbf{x} \in L_k \backslash A_S$ $(|L_k(\mathbf{x})| \geq 2)$ and $\mathbf{x}$ *contains no singleton cut component as its cut component*, then the path-cut lower bound $l_{\text{pc}}^k(\mathbf{p})$ is better than the minimax lower bound $l_{\text{mm}}^k(\mathbf{p})$ as $\Pr\{X_i = M\} \to 1$ for $i=1,\ldots,n$.

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Remark 2: Assumptions (i) and (ii) can be roughly interpreted as: If the structure contains some singleton cut components (i.e., any one of these components can cause system failure), then the other *nonsingleton cut components* must be relevant to the system in the sense that they can cause system failure even all the singleton cut components attain their maximum performance levels.

Remark 3: Assumption (ii) is equivalent to saying that for each critical path vector $\mathbf{z} \in U_k$, there exists an $(i, z_i) \in U_k(\mathbf{z})$ and i is not a singleton cut component ($i \notin A_{IS}$). This equivalent condition can be used in checking (ii).

THEOREM 2.7: *Suppose that either (i) $A_S = \{\emptyset\}$ or (ii) $A_S \neq \{\emptyset\}$ and there exists an $\mathbf{x} \in L_k \backslash A_S$ with $x_i = M$ for all $i \in A_{IS}$. Then,*

$$\max_{\mathbf{z}\in U_k} \prod_{(i,z_i)\in U_k(\mathbf{z})} \bar{p}_{i,z_i-1} \leq \prod_{\mathbf{x}\in L_k} \coprod_{(j,x_j)\in L_k(\mathbf{x})} \bar{p}_{j,x_j} \quad \text{as } \delta_1,\ldots,\delta_n \to 0.$$

PROOF: To prove the theorem, it suffices to show that, in either cases (i) or (ii), for each $\mathbf{z} \in U_k$,

$$\prod_{(i,z_i)\in U_k(\mathbf{z})} \bar{p}_{i,z_i-1} \leq \prod_{\mathbf{x}\in L_k} \coprod_{(j,x_j)\in L_k(\mathbf{x})} \bar{p}_{j,x_j} \quad \text{as } \delta_1,\ldots,\delta_n \to 0. \tag{16}$$

For each $i \in \{1,\ldots,n\}$, denote by $\mathbf{C}(i)$ the set of critical cut vectors of which each one contains i as a cut component; that is,

$$\mathbf{C}(i) = \{\mathbf{x} | \mathbf{x} \in L_k; x_i < M\}. \tag{17}$$

Consider a given $\mathbf{x} \in L_k$. Since, by Lemma 2.1, $z_i > x_i$ for some i, we see that $\mathbf{x} \in \mathbf{C}(i)$ for some $(i, z_i) \in U_k(\mathbf{z})$. Thus,

$$\prod_{\mathbf{x}\in L_k} \coprod_{(j,x_j)\in L_k(\mathbf{x})} \bar{p}_{j,x_j} \geq \prod_{(i,z_i)\in U_k(\mathbf{z})} \prod_{\mathbf{x}\in\mathbf{C}(i)} \coprod_{(j,x_j)\in L_k(\mathbf{x})} \bar{p}_{j,x_j}. \tag{18}$$

To prove Eq. (16), it thus suffices to show that

$$\prod_{(i,z_i)\in U_k(\mathbf{z})} \bar{p}_{i,z_i-1} \leq \prod_{(i,z_i)\in U_k(\mathbf{z})} \prod_{\mathbf{x}\in\mathbf{C}(i)} \coprod_{(j,x_j)\in L_k(\mathbf{x})} \bar{p}_{j,x_j}. \tag{19}$$

Case (i): Suppose that $A_S = \{\emptyset\}$. Let a vector $\mathbf{x} \in L_k$ be given. By assumption, the cardinality $|L_k(\mathbf{x})| \geq 2$. Hence,

$$\begin{aligned} \coprod_{(j,x_j)\in L_k(\mathbf{x})} \bar{p}_{j,x_j} &= 1 - \prod_{(j,x_j)\in L_k(\mathbf{x})} q_{j,x_j} \\ &\geq 1 - q_{i,x_i} q_{j,x_j}, \quad \forall ((i,x_i),(j,x_j)) \subset L_k(\mathbf{x}). \end{aligned} \tag{20}$$

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Applying Lemma 2.1 to $\mathbf{z}$ and $\mathbf{x}$, we have that $z_w > x_w$ for some $(w, z_w) \in U_k(\mathbf{z})$ (and, hence, $q_{w,z_w-1} \geq q_{w,x_w}$). Furthermore, let q^* be the maximum of such q_{w,x_w}'s over all $\mathbf{x} \in L_k$; that is,

$$q^* = \max_{\mathbf{x} \in L_k} q_{w,x_w}.$$

Then, clearly, $1 - q^* \geq 1 - q_{w,z_w-1}$ for some $(w, z_w) \in U_k(\mathbf{z})$ and, hence,

$$\prod_{(i,z_i)\in U_k(\mathbf{z})} \bar{p}_{i,z_i-1} = \prod_{(i,z_i)\in U_k(\mathbf{z})} (1 - q_{i,z_i-1}) \leq 1 - q^*. \quad \textbf{(21)}$$

Also, let

$$\delta^* \equiv \max_{1\leq i\leq n} \delta_i.$$

Then, $q_{i,s} \leq 1 - \Pr\{x_i = M\} \leq \delta^*$ for all i, and $s < M$. Hence, $q^* \leq \delta^*$, and following Eq. (20),

$$\begin{aligned} \coprod_{(j,x_j)\in L_k(\mathbf{x})} \bar{p}_{j,x_j} &\geq 1 - q_{w,x_w}\delta^* \\ &\geq 1 - q^*\delta^*, \quad \forall \mathbf{x} \in L_k. \end{aligned} \quad \textbf{(22)}$$

From Eq. (21), to prove relation (19) we show that

$$1 - q^* \leq \prod_{(i,z_i)\in U_k(\mathbf{z})} \prod_{\mathbf{x}\in \mathbf{C}(i)} \coprod_{(j,x_j)\in L_k(\mathbf{x})} \bar{p}_{j,x_j} \quad \text{as } \delta_1, \ldots, \delta_n \to 0. \quad \textbf{(23)}$$

The right-hand side of relation (23), from relation (22), is

$$\begin{aligned} \prod_{(i,z_i)\in U_k(\mathbf{z})} \prod_{\mathbf{x}\in \mathbf{C}(i)} \coprod_{(j,x_j)\in L_k(\mathbf{x})} \bar{p}_{j,x_j} &\geq \prod_{(i,z_i)\in U_k(\mathbf{z})} (1 - q^*\delta^*)^{|\mathbf{C}(i)|} \\ &= (1 - q^*\delta^*)^{Q(\mathbf{z})}, \end{aligned} \quad \textbf{(24)}$$

where $Q(\mathbf{z}) = \sum_{(i,z_i)\in U_k(\mathbf{z})} |\mathbf{C}(i)|$.

We now show that

$$1 - q^* \leq (1 - q^*\delta^*)^{Q(\mathbf{z})} \quad \text{as } \delta_1, \ldots, \delta_n \to 0 \quad \textbf{(25)}$$

or, equivalently,

$$\begin{aligned} q^* &\geq 1 - (1 - q^*\delta^*)^{Q(\mathbf{z})} \\ &= Q(\mathbf{z})q^*\delta^* + g(\cdot) \quad \text{as } \delta_1, \ldots, \delta_n \to 0, \end{aligned} \quad \textbf{(26)}$$

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where

$$g(\cdot) = (-1)\binom{Q(\mathbf{z})}{2}(q^*\delta^*)^2 + \cdots + (-1)^{Q(\mathbf{z})+1}q^{*Q(\mathbf{z})}\delta^{*Q(\mathbf{z})}.$$

It is thus easy to see that Eq. (26) holds as $q^* \to 0$ and $\delta^* < Q(\mathbf{z})^{-1}$. Since $q^* \leq \delta^*$, we see that Eq. (26) holds as $\delta^* \to 0$ (i.e., as $\delta_1, \ldots, \delta_n \to 0$).

Case (ii): Suppose that $A_S \neq \{\emptyset\}$. Write the index set of the singleton cut components as $A_{\mathrm{IS}} = A_1 \cup A_2$, where any component in A_1 is not a cut component of any other critical cut vector outside A_S; that is,

$$A_1 = \{i \mid i \in A_{\mathrm{IS}}; (i, x_i) \notin L_k(\mathbf{x})\ \forall \mathbf{x} \in L_k \backslash A_S\}, \tag{27}$$

$$A_2 = \{i \mid i \in A_{\mathrm{IS}}; (i, x_i) \in L_k(\mathbf{x}) \text{ for some } \mathbf{x} \in L_k \backslash A_S\}, \tag{28}$$

where $A \backslash B$ denotes set A with elements in B removed.

The right-hand side of Eq. (19) can be decomposed as

$$\prod_{(i,z_i)\in U_k(\mathbf{z})} \prod_{\mathbf{x}\in \mathbf{C}(i)} \coprod_{(j,x_j)\in L_k(\mathbf{x})} \bar{p}_{j,x_j} = C_1 C_2 C_3 C_4; \tag{29}$$

$$C_1 = \prod_{(i,z_i)\in U_k(\mathbf{z}); i\notin A_{\mathrm{IS}}} \prod_{\mathbf{x}\in \mathbf{C}(i)} \coprod_{(j,x_j)\in L_k(\mathbf{x})} \bar{p}_{j,x_j}, \tag{30}$$

$$C_2 = \prod_{(i,z_i)\in U_k(\mathbf{z}); i\in A_1} \prod_{\mathbf{x}\in \mathbf{C}(i), (i,x_i)\in L_k(\mathbf{x})} \bar{p}_{i,x_i}, \tag{31}$$

$$C_3 = \prod_{(i,z_i)\in U_k(\mathbf{z}); i\in A_2} \prod_{\mathbf{x}\in \mathbf{C}(i)\cap A_S, (i,x_i)\in L_k(\mathbf{x})} \bar{p}_{i,x_i}, \tag{32}$$

$$C_4 = \prod_{(i,z_i)\in U_k(\mathbf{z}); i\in A_2} \prod_{\mathbf{x}\in \mathbf{C}(i)\backslash A_S} \coprod_{(j,x_j)\in L_k(\mathbf{x})} \bar{p}_{j,x_j}. \tag{33}$$

Note that in Eqs. (31) and (32), the cut vectors are all singleton cut vectors, and the operator $\coprod$ is removed since the cardinalities $|\mathbf{C}(i)|$, $|\mathbf{C}(i) \cap A_S|$, and $|L_k(\mathbf{x})|$ all equal 1.

By assumption, there is an $\mathbf{x} \in L_k \backslash A_S$ with $x_i = M$ for all $i \in A_{\mathrm{IS}}$. We denote this vector by $\tilde{\mathbf{x}}$ in the sequel. By Lemma 2.1, we have that $z_w > \tilde{x}_w$ for at least one $w \in \{1, \ldots, n\} \backslash A_{\mathrm{IS}}$. Assume that there are d such components and denote them by $z_{w_t} > \tilde{x}_{w_t}$, $t = 1, \ldots, d$. Then, $q_{w_t, z_{w_t}} \geq q_{w_t, \tilde{x}_{w_t}}$ for each $1 \leq t \leq d$. Let

$$q^* = \max_{1\leq t\leq d} q_{w_t, \tilde{x}_{w_t}}.$$

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Then, $1 - q^* \geq \bar{p}_{w_t, z_{w_t}}$ for some $1 \leq t \leq d$ and, hence,

$$\begin{aligned}\prod_{(i,z_i)\in U_k(\mathbf{z})} \bar{p}_{i,z_i-1} &= \prod_{t=1}^{d} \bar{p}_{w_t,z_{w_t}-1} \prod_{(i,z_i)\in U_k(\mathbf{z});\, i\in A_{\mathrm{IS}}} \bar{p}_{i,z_i-1} \prod_{(i,z_i)\in U_k(\mathbf{z});\, i\notin A_{\mathrm{IS}}\cup\{\cup_{t=1}^{d} w_t\}} \bar{p}_{i,z_i-1} \\ &\leq \prod_{t=1}^{d} \bar{p}_{w_t,z_{w_t}-1} \prod_{(i,z_i)\in U_k(\mathbf{z});\, i\in A_{\mathrm{IS}}} \bar{p}_{i,z_i-1} \\ &\leq (1-q^*) \prod_{(i,z_i)\in U_k(\mathbf{z});\, i\in A_1} \bar{p}_{i,z_i-1} \prod_{(i,z_i)\in U_k(\mathbf{z});\, i\in A_2} \bar{p}_{i,z_i-1}. \end{aligned} \quad \textbf{(34)}$$

From Eqs. (29) and (34), it suffices to show that

$$(1-q^*) \prod_{(i,z_i)\in U_k(\mathbf{z});\, i\in A_1} \bar{p}_{i,z_i-1} \prod_{(i,z_i)\in U_k(\mathbf{z});\, i\in A_2} \bar{p}_{i,z_i-1} \leq C_1 C_2 C_3 C_4. \quad \textbf{(35)}$$

We shall respectively prove that

$$\prod_{(i,z_i)\in U_k(\mathbf{z});\, i\in A_1} \bar{p}_{i,z_i-1} \leq C_2, \quad \textbf{(36)}$$

$$\prod_{(i,z_i)\in U_k(\mathbf{z});\, i\in A_2} \bar{p}_{i,z_i-1} \leq C_3, \quad \textbf{(37)}$$

$$1 - q^* \leq C_1 C_4. \quad \textbf{(38)}$$

As stated, for each $(i, z_i) \in U_k(\mathbf{z})$ and $i \in A_1$, the cardinalities $|\mathbf{C}(i)| = 1$ and $|L_k(\mathbf{x})| = 1$ for the vector $\mathbf{x} \in L_k$. Clearly then, $x_i < z_i$ and, hence, $\bar{p}_{i,z_i-1} \leq \bar{p}_{i,x_i}$. Equation (36) is thus obtained. Equation (37) holds by the same reason.

Proving that Eq. (38) holds is quite similar to that in proving Case (i). However, for completeness, we prove it in detail. Since $|L_k(\mathbf{x})| \geq 2$ for all $\mathbf{x}$ in Eqs. (30) and (33), it holds that for any $((i, x_i), (j, x_j)) \subset L_k(\mathbf{x})$,

$$\begin{aligned}\coprod_{(j,x_j)\in L_k(\mathbf{x})} \bar{p}_{j,x_j} &\geq \bar{p}_{i,M-1} \coprod \bar{p}_{j,M-1} \\ &\geq 1 - \delta^{*2}. \end{aligned} \quad \textbf{(39)}$$

Thus,

$$\begin{aligned} C_4 &\geq \prod_{(i,z_i)\in U_k(\mathbf{z});\, i\in A_2} \prod_{\mathbf{x}\in \mathbf{C}(i)\setminus A_S} (1-\delta^{*2}) \\ &= \prod_{(i,z_i)\in U_k(\mathbf{z});\, i\in A_2} (1-\delta^{*2})^{|\mathbf{C}(i)\setminus A_S|}; \end{aligned} \quad \textbf{(40)}$$

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and since the vector $\tilde{\mathbf{x}}$ is counted d_0 ($d_0 \geq d$) times in Eq. (30),

$$C_1 \geq (1-\delta^{*2})^{d_0-d} \prod_{t=1}^{d} (1-\delta^* q_{w_t,\tilde{x}_{w_t}}) \prod_{(i,z_i)\in U_k(\mathbf{z});\, i\notin A_{\mathrm{IS}}} \prod_{\mathbf{x}\in \mathbf{C}(i)\backslash\{\tilde{\mathbf{x}}\}} (1-\delta^{*2})$$

$$\geq (1-\delta^{*2})^{d_0-d}(1-\delta^* q^*)^d \prod_{(i,z_i)\in U_k(\mathbf{z});\, i\notin A_{\mathrm{IS}}} (1-\delta^{*2})^{|\mathbf{C}(i)\backslash\{\tilde{\mathbf{x}}\}|}. \quad \mathbf{(41)}$$

To show Eq. (38) holds, it thus suffices to show

$$(1-\delta^* q^*)^d (1-\delta^{*2})^{Q(\mathbf{z})} \geq 1-q^* \quad \text{as } \delta_1,\ldots,\delta_n \to 0, \quad \mathbf{(42)}$$

where

$$Q(\mathbf{z}) = d_0 - d + \sum_{(i,z_i)\in U_k(\mathbf{z});\, i\notin A_{\mathrm{IS}}} |\mathbf{C}(i)\backslash\{\tilde{\mathbf{x}}\}| + \sum_{(i,z_i)\in U_k(\mathbf{z});\, i\in A_2} |\mathbf{C}(i)\backslash A_S|.$$

Again, we show that

$$q^* \geq 1-(1-\delta^* q^*)^d (1-\delta^{*2})^{Q(\mathbf{z})} \quad \text{as } \delta_1,\ldots,\delta_n \to 0. \quad \mathbf{(43)}$$

Proceeding along the same line as that in proving Eq. (26), we obtain that $q^* \geq 1 - (1-\delta^* q^*)^d$ holds as $\delta_1,\ldots,\delta_n \to 0$. Hence, the conclusion. ■

3. ILLUSTRATIVE EXAMPLES

A binary coherent system (except series systems) automatically satisfies conditions (i) or (ii) of Theorem 2.7, since a minimal cut set is not properly contained in any such set for the binary system. Hence, Theorem 2.7 represents a more general version of Eq. (2), obtained by Maymin [7] for binary systems. In this section, we give some examples to further illustrate our results. First, we show that the N2 class of multistate systems satisfies either condition (i) or (ii) of Theorem 2.7.

THEOREM 3.1: *Suppose that $\phi \in$ N2 class. Then, for each level $k > 0$, ϕ satisfies either (i) or (ii) of Theorem 2.7, if the corresponding binary structure ϕ_k is not series.*

PROOF: Suppose that $A_S \neq \{\varnothing\}$. Let $A_{\mathrm{IS}} = \{i_1,\ldots,i_s\}$ be the s singleton cut components of ϕ for level k. Since $\phi_k(\mathbf{I}_k(\mathbf{x}))$ is not series, from Eq. (8) we see that there exists a critical cut vector $\mathbf{x} \in L_k$ with size $|L_k(\mathbf{x})| \geq 2$. We then show that $x_i = M$ for each $i \in A_{\mathrm{IS}}$ to prove the theorem. Let $C' \subset \{1,\ldots,n\}$ denote the cut components of $\mathbf{x}$ (i.e., $x_i < M$ if and only if $i \in C'$). From the relationship defined in Eq. (8) between ϕ and ϕ_k, clearly $\{i_1\},\ldots,\{i_s\}$ and C' are all minimal cut sets for the binary structure ϕ_k. Hence, $i_1,\ldots,i_s \notin C'$ and, hence, $x_{i_1} = \cdots = x_{i_s} = M$. ■

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The structure in the following example belongs to a larger than N2 class of multistate systems, namely the N1 class, also introduced by Natvig [8].

Example 3.2: Let a structure $\phi:\{0,1,2\}^3 \mapsto \{0,1,2\}$ be defined as follows:

$$\begin{aligned}
&\phi(0,0,0) = 0, \quad \phi(0,0,1) = 0, \quad \phi(0,0,2) = 0;\\
&\phi(0,1,0) = 0, \quad \phi(0,1,1) = 0, \quad \phi(0,1,2) = 0;\\
&\phi(0,2,0) = 0, \quad \phi(0,2,1) = 0, \quad \phi(0,2,2) = 0;\\
&\phi(1,0,0) = 0, \quad \phi(1,0,1) = 0, \quad \phi(1,0,2) = 0;\\
&\phi(1,1,0) = 0, \quad \phi(1,1,1) = 1, \quad \phi(1,1,2) = 1;\\
&\phi(1,2,0) = 1, \quad \phi(1,2,1) = 1, \quad \phi(1,2,2) = 2;\\
&\phi(2,0,0) = 1, \quad \phi(2,0,1) = 1, \quad \phi(2,0,2) = 2;\\
&\phi(2,1,0) = 1, \quad \phi(2,1,1) = 1, \quad \phi(2,1,2) = 2;\\
&\phi(2,2,0) = 2, \quad \phi(2,2,1) = 2, \quad \phi(2,2,2) = 2.
\end{aligned}$$

It is easy to verify that the structure $\phi \in$ N1 class (see Natvig [8] for definitions). The critical cut and critical path vectors of ϕ for level $k = 2$ are

$$\begin{aligned}
L_2 &= \{(0,2,2),(2,1,1),(1,1,2),(1,2,1)\},\\
U_2 &= \{\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3\} = \{(2,0,2),(2,2,0),(1,2,2)\}.
\end{aligned}$$

The structure meets condition (ii) of Theorem 2.7, noting that $A_S = \{(0,2,2)\}$ and the vector (2,1,1) satisfies the condition stated in (ii). Assume that the three components possess equal reliabilities $\mathbf{p}_i = (a_0, a_1, a_2)$ for $i = 1,2,3$.

$$\begin{aligned}
h_2(\mathbf{p}) &= \sum_{r=1}^{3} \Pr\{\mathbf{X} \ge \mathbf{z}_r\} - \sum_{(r,r')\subset\{1,2,3\}} \Pr\{\mathbf{X} \ge \mathbf{z}_r \vee \mathbf{z}_{r'}\} + \Pr\{\mathbf{X} \ge \mathbf{z}_1 \vee \mathbf{z}_2 \vee \mathbf{z}_3\}\\
&= a_2^2(2 + a_1 - a_2),
\end{aligned}$$

where the maximum of two vectors $\mathbf{x} \vee \mathbf{y} \equiv (x_1 \vee y_1, \ldots, x_n \vee y_n)$.

The minimax and path-cut bounds are

$$\begin{aligned}
l_{\text{mm}}^2(\mathbf{p}) &= \max\{a_2^2, a_2^2, (a_1 + a_2)a_2^2\} = a_2^2,\\
l_{\text{pc}}^2(\mathbf{p}) &= (a_1 + a_2)(a_2 + a_2 - a_2^2)^3 = (a_1 + a_2)a_2^3(2 - a_2)^3.
\end{aligned}$$

Thus,

$$l_{\text{pc}}^2(\mathbf{p}) \ge l_{\text{mm}}^2(\mathbf{p}) \quad \text{if } a_2(2 - a_2)^3 \ge \frac{1}{a_1 + a_2}.$$

Approximately, $l_{\text{pc}}^2(\mathbf{p}) \ge l_{\text{mm}}^2(\mathbf{p})$ when $a_2(2 - a_2)^3 \ge (a_2)^{-1}$. The polynomial $f(x) = x^2(2 - x)^3 - 1$ is decreasing (increasing) in x for $x > 0.8$ (< 0.8). Also, note that $f(1) = 0$ and $f(0.62) \approx 0.01$. It is concluded that the bounds $l_{\text{pc}}^2(\mathbf{p}) \ge l_{\text{mm}}^2(\mathbf{p})$ as long as $a_2 > 0.62$. Table 1 presents some numerical values of the two lower bounds.

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TABLE 1. Numerical Values of the Two Lower Bounds

	$h_2(\mathbf{p})$	$l^2_{pc}(\mathbf{p})$	$l^2_{mm}(\mathbf{p})$
$\mathbf{p} = (0.05, 0.05, 0.9)$	0.9315	0.9217	0.81
$\mathbf{p} = (0.15, 0.15, 0.7)$	0.7105	0.6405	0.49
$\mathbf{p} = (0.3, 0.05, 0.65)$	0.5915	0.4729	0.4225
$\mathbf{p} = (0.4, 0.2, 0.4)$	0.288	0.12	0.16

Example 3.3: Consider the offshore electrical power generation system given in Natvig, Sørmo, Hølen, and Høgåsen [9]. In the example, the amount of power that can be supplied to platform 1 is represented by a multistate structure function ϕ_1, which depends on a control unit and two generators. The functioning levels of the system and its three components were represented by the set $\{0,2,4\}$, but in the following, we shall denote it by $S = \{0,1,2\}$ for simplicity. Then, the structure function $\phi_1 : \{0,1,2\}^3 \mapsto \{0,1,2\}$ can be represented as

$$\phi_1(\mathbf{x}) = I(x_1 > 0) \min\{x_2 + x_3 I(x_1 = 2), 2\},$$

where x_1 denotes the control unit, x_2 and x_3 denote the two generators, and $I(\cdot)$ is the indicator function (for a detailed case study about the system, refer to Natvig et al. [9]).

Consider system performance level $k = 1$. Then,

$$L_1 = \{(2,0,0), (1,0,2), (0,2,2)\}, \qquad U_1 = \{(1,1,0), (2,0,1)\}.$$

It is clear that the structure ϕ_1 meets condition (ii) of Theorem 2.7, noting that $A_S = \{(0,2,2)\}$ and the vector (2,0,0) satisfies the condition stated in (ii). The minimax and the path-cut lower bounds for $h_1(\mathbf{p})$ are

$$l^1_{mm}(\mathbf{p}) = \max\{(1 - p_{1,0})(1 - p_{2,0}), p_{1,2}(1 - p_{3,0})\},$$

$$l^1_{pc}(\mathbf{p}) = (1 - p_{2,0}p_{3,0})[p_{1,2} + (1 - p_{2,0}) - p_{1,2}(1 - p_{2,0})](1 - p_{1,0}).$$

If components 2 and 3 possess equal reliabilities, then $l^1_{mm}(\mathbf{p}) = (1 - p_{1,0})(1 - p_{2,0})$, and, hence, $l^1_{pc}(\mathbf{p}) \geq l^1_{mm}(\mathbf{p})$ if $p_{1,2} \geq p_{3,0}(1 - p_{2,0})(1 - p_{3,0}p_{2,0})^{-1}$. Thus, $l^1_{pc}(\mathbf{p}) \geq l^1_{mm}(\mathbf{p})$ as long as $p_{1,2} \geq p_{3,0}$.

Use the data provided by Natvig et al. [9]:

$$p_{1,2} = 0.246; \quad p_{1,1} + p_{1,2} = 0.818,$$

$$p_{i,2} = 0.054; \quad p_{i,1} + p_{i,2} = 0.862 \quad \text{for } i = 2,3.$$

Then, $l^1_{mm}(\mathbf{p}) = 0.705116$ and $l^1_{pc}(\mathbf{p}) = 0.71892$.

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References

1. Barlow, R.E. & Proschan, F. (1981). *Statistical theory of reliability and life testing*. Silver Spring, MD: To Begin With.
2. Barlow, R.E. & Wu, A.S. (1978). Coherent systems with multistate components. *Mathematics of Operations Research* 4: 275–281.
3. Block, H.W. & Savits, T.H. (1982). A decomposition for multistate monotone systems. *Journal of Applied Probability* 19: 391–402.
4. Borges, W. & Rodrigues, F. (1983). Axiomatic characterization of multistate coherent structures. *Mathematics of Operations Research* 8: 435–438.
5. Esary, J.D. & Proschan, F. (1970). A reliability bound for systems of maintained, interdependent components. *Journal of the American Statistics Association* 65: 329–338.
6. Langseth, H. & Lindqvist, B.H. (1998). Uncertainty bounds for a monotone multistate system. *Probability in the Engineering and Informational Sciences* 12: 239–260.
7. Maymin, Z. (1987). On a conjecture of Barlow and Proschan concerning reliability bounds. *Journal of Statistical Planning Inference* 16: 337–344.
8. Natvig, B. (1982). Two suggestions of how to define a multistate coherent systems. *Advances in Applied Probability* 14: 434–455.
9. Natvig, B., Sørmo, S., Hølen, T., & Høgåsen, G. (1986). Multistate reliability theory—A case study. *Advances in Applied Probability* 18: 921–932.

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