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Global convergence for two-pulse rest-to-rest learning for single-degree-of-freedom systems with stick-slip Coulomb friction

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SUMMARY
In this paper, we consider the problem of rest-to-rest maneuver learning, via iterative learning control (ILC), for single-degree-of-freedom systems with stick-slip Coulomb friction and input bounds. The static coefficient of friction is allowed to be as large as three times the kinetic coefficient of friction. The input is restricted to be a two-pulse one. The desired input's first pulse magnitude is required to be five times the largest possible kinetic (sliding) friction force. The theory therefore allows the stiction force to be as large as the desired second input pulse. Under these conditions, we prove global convergence of a simple iterative learning controller. To the best of our knowledge, such a global-convergence proof has not been presented previously in the literature for the rest-to-rest problem with stick-slip Coulomb friction.

KEYWORDS: Rest-to-rest maneuver; Iterative learning control (ILC); Coulomb friction; Convergence theory; Input bounds/limits; Single-degree-of-freedom system.

1. Introduction

Learning control is a method of control that repetitively feeds the system inputs for a specific task, and uses the actual on-line measured response of the system to evaluate the quality or goodness of the input. The actual responses are used in a feedback loop in which the inputs are adjusted to reduce the measured errors in the output. Example applications include robotics and manufacturing, where a certain output–tracking task is to be performed repeatedly. Usually the output is the position or velocity history of the robot's joints although sometimes it also includes measured forces at the end–effector (see Cheah and Wang[2]).

Learning control has a history dating back to 1984 (see Arimoto *et al.*[1]) when it was first applied to robot motion control. Horowitz[8] gives a proper history of the development and usage of learning controllers for (rigid) robot manipulators. He compares and contrasts different learning algorithms and also provides an experimental demonstration of a robot that learns to make its end–effector track a circular trajectory. He insightfully points out that an open area of research is in finding methods for robust *optimal* (e.g., minimum energy, minimum vibration, or minimum time) trajectory learning, as opposed to only finding a control history that meets the output requirements. Examples of works that have empirically investigated approaches to this problem include works of Gorinevsky and coworkers,[5–7] who consider the use of the Levenberg–Marquardt optimization method for least squares, and Sadegh and Driessen,[10] who consider the use of gradient-based algorithms for constrained optimization.

Cheng and Peng[3] consider learning control with input bounds and modeling error. However, the methodology and convergence theory was restricted to single-input/single-output systems.

While systems with unknown Coulomb friction [non-Lipschitz right-hand side (rhs) for the plant's dynamics] have been considered and convergence theory developed for repetitive/adaptive control utilizing linearity in the unknown coefficient of *sliding* friction, relatively little convergence theory has been developed for iterative learning control (ILC) with Coulomb friction. Longman and Chang[9] and Wang and Longman[11] proposed the few ILC theories for systems with Coulomb friction, and convergence theory was developed for single-input/single-output systems with one degree of freedom for the discrete-time trajectory-tracking problem. The present paper, however, is devoted to the learning of rest-to-rest maneuvers for single-degree-of-freedom systems. Longman and coworkers also address the problem case in which the output is a discontinuous function of the input, i.e., when the static coefficient of friction μ_s, is larger than the sliding coefficient of friction μ_k. Then, even for inputs without bounds, there may not exist an input to achieve zero output error due to the stick-slip/jump and overshoot property. (This property does not exist if the two coefficients of friction are the same.) In the present paper, for two-pulse rest-to-rest learning control, we allow $\mu_s \le 3\mu_k$.

The present paper considers the problem of using ILC to learn a rest-to-rest maneuver for a single-degree-of-freedom system with stick-slip Coulomb friction and input bounds. The static coefficient of friction is allowed to be three times as large as the kinetic coefficient. The desired input (and learning iterates) are restricted to be two-pulse, and we require that the desired first pulse magnitude be greater or equal to five times the largest possible kinetic friction force. The theory therefore allows the stiction force to be as large as the desired second

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input pulse. In Section 2, we present the details of the problem statement. In Section 3, we present the main theorem of global convergence and its proof. In Section 4, we present numerical examples, and in Section 5, we end with conclusions.

2. Problem Statement

We are given a single-degree-of-freedom mechanical system with Coulomb friction as

$$m\ddot{q} = u + F(q, \dot{q}, u) \tag{1}$$

where $q \in R^1$ and $u \in R^1$, and with initial conditions

$$q(0) = 0 \tag{2}$$

$$\dot{q}(0) = 0 \tag{3}$$

and with input bounds/limits

$$-u_{\max} \leq u \leq u_{\max}. \tag{4}$$

The friction force F is given by

$$F(q, \dot{q}, u) = \left\{ \begin{array}{ll} -\mu_k \text{sign}(\dot{q}), & \dot{q} \neq 0 \\ -u, & \dot{q} = 0 \text{ and } |u| \leq \mu_s \geq \mu_k \\ -\mu_s \text{sign}(u), & \dot{q} = 0 \text{ and } |u| > \mu_s \end{array} \right\} \tag{5}$$

where μ_s is the static coefficient of friction, μ_k is the kinetic coefficient of friction, and $\mu_s \geq \mu_k$ but

$$\mu_s \leq 3\mu_k. \tag{6}$$

A two-pulse input $u(t)$ is to be applied to the system (1)

$$u(t) = u_1, \quad t \in [0, T] \tag{7}$$

$$u(t) = u_2, \quad t \in [T, 2T] \tag{8}$$

where $u_1 \in R^1$ and $u_2 \in R^1$ are constants. We have a desired terminal state

$$q_2 \equiv q(2T) \overset{\text{want}}{=} q_2^* \tag{9}$$

$$\dot{q}_2 \equiv \dot{q}(2T) \overset{\text{want}}{=} 0 \tag{10}$$

where superscript asterisks denote the desired quantity values. We assume that there exist u_1^* and u_2^* such that Eqs. (9) and (10) are achieved with

$$-u_{\max} \leq u_i^* \leq u_{\max}, \qquad (i = 1,\ 2) \tag{11}$$

and where the associated $q^*(t)$ satisfies

$$\dot{q}^*(t) > 0, \qquad \forall t \in (0, 2T) \tag{12}$$

and where

$$u_1^* \geq 5\mu_k. \tag{13}$$

Let

$$\bar{u} \equiv \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad \bar{u}^* \equiv \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} \tag{14}$$

$$e \equiv \begin{pmatrix} q_2 - q_2^* \\ \dot{q}_2 \end{pmatrix}. \tag{15}$$

A learning controller of the following form is to be used

$$\bar{u}^{i+1} = boxfcn\,(\bar{u}^i - \alpha G e^i, u_{\max}), \qquad \alpha > 0 \tag{16}$$

where superscripts denote learning trial index, and where $boxfcn\,(\hat{u}, u_{\max})$ means

$$(boxfcn\,(\hat{u}, u_{\max})) = \left\{ \begin{array}{ll} \hat{u}(i), & \text{if } -u_{\max} \leq \hat{u}(i) \leq u_{\max} \\ u_{\max}, & \text{if } \hat{u}(i) > u_{\max} \\ -u_{\max}, & \text{if } \hat{u}(i) < -u_{\max} \end{array} \right\} \tag{17}$$

and where $G \in R^{2\times 2}$ is a constant gain matrix. The objective is to choose G and provide an associated proof that $\bar{u} \to \bar{u}^*$ as $i \to \infty$ for any starting value of $\bar{u}$ and $\bar{u}^0$, provided the scalar gain $\alpha > 0$ is sufficiently small:

$$\lim_{i\to\infty} (\bar{u}^i) \overset{\text{want}}{=} \bar{u}^*, \quad \forall \bar{u}^0 : \quad |\bar{u}^0(j)| \leq u_{\max}, \quad (j = 1, 2), \quad \text{small enough } \alpha > 0. \tag{18}$$

3. Global Convergence Proof

Theorem 3.1: For

$$G = \bar{J}^{-1} \tag{19}$$

where

$$\bar{J} = \left(\frac{1}{\hat{m}}\right) \begin{bmatrix} \frac{3T^2}{2} & \frac{T^2}{2} \\ T & T \end{bmatrix} \tag{20}$$

and $\hat{m} > 0$ is a finite model-based estimate of the mass m in Eq. (1), the global convergence condition [Eq. (18)], for the problem defined in Section 2, is satisfied. [Eq. (18) could be changed to indicate a small enough $\hat{m}\alpha$, but since $\hat{m}$ is finite, the two statements are equivalent.]

Proof: Let

$$q_1 \equiv q(T) \tag{21}$$

$$\dot{q}_1 \equiv \dot{q}(T). \tag{22}$$

The $q(t)$ that results from application of (u_1, u_2) [as in Eqs. (7) and (8) to system (1)] can be broken up into nine possibilities,

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as indicated in (23)–(31):

$$\dot{q} > 0 \quad \forall t \in (0, 2T) \tag{23}$$

$$\dot{q} < 0 \quad \forall t \in (0, 2T) \tag{24}$$

$$\dot{q} > 0 \quad \forall t \in (0, t_s), \quad t_s \in (T, 2T); \quad \dot{q} < 0 \quad \forall t \in (t_s, 2T) \tag{25}$$

$$\dot{q} > 0 \quad \forall t \in (0, t_s), \quad t_s \in (T, 2T); \quad \dot{q} = 0 \quad \forall t \in (t_s, 2T) \tag{26}$$

$$\dot{q} = 0 \quad \forall t \in [0, T]; \quad \dot{q} > 0 \quad \forall t \in (T, 2T) \tag{27}$$

$$\dot{q} = 0 \quad \forall t \in [0, T]; \quad \dot{q} < 0 \quad \forall t \in (T, 2T) \tag{28}$$

$$\dot{q} < 0 \quad \forall t \in (0, t_s), \quad t_s \in (T, 2T); \quad \dot{q} > 0 \quad \forall t \in [t_s, 2T] \tag{29}$$

$$\dot{q} < 0 \quad \forall t \in (0, t_s), \quad t_s \in (T, 2T); \quad \dot{q} = 0 \quad \forall t \in [t_s, 2T] \tag{30}$$

$$\dot{q} = 0 \quad \forall t \in [0, 2T]. \tag{31}$$

Let

$$\Delta\bar{u} \equiv -\alpha G e. \tag{32}$$

For each of the nine cases [Eqs. (23)–(31)], we will show that

$$\bar{u} \neq \bar{u}^* \overset{?}{\Rightarrow} \Delta\bar{u}^T(\bar{u} - \bar{u}^*) < 0. \tag{33}$$

We have, by direction integration using Eqs. (1)–(3), (5), (7), (8), and (12)

$$q_1^* = \left(\frac{T^2}{2m}\right)(u_1^* - \mu_k) \tag{34}$$

$$\dot{q}_1^* = \left(\frac{T}{m}\right)(u_1^* - \mu_k) \tag{35}$$

$$\dot{q}_2^* = \dot{q}_1^* + \left(\frac{T}{m}\right)(u_2^* - \mu_k) \tag{36}$$

$$q_2^* = q_1^* + \dot{q}_1^* T + \left(\frac{T^2}{2m}\right)(u_2^* - \mu_k) \tag{37}$$

and since $\dot{q}_2^* = 0$

$$u_2^* = -u_1^* + 2\mu_k. \tag{38}$$

From Eq. (13), let

$$u_1^* = (5 + a)\mu_k, \quad a \geq 0. \tag{39}$$

From Eqs. (25), (26), (29), and (30), let

$$\Delta t_s = t_s - T \tag{40}$$

and let

$$\Delta t_s = rT, \qquad r \in [0, 1] \tag{41}$$

thus defining a and r. Since $\hat{m} \neq m$ does not affect the sign of $\Delta\bar{u}^T(\bar{u} - \bar{u}^*)$, and with absolutely no loss of generality as we will see, for simplicity we simply set $\hat{m} = m$ throughout.

For case (23), $u_1 > \mu_s$, and

$$q_1 = \left(\frac{T^2}{2m}\right)(u_1 - \mu_k) \tag{42}$$

$$\dot{q}_1 = \left(\frac{T}{m}\right)(u_1 - \mu_k) \tag{43}$$

$$\dot{q}_2 = \dot{q}_1 + \left(\frac{T}{m}\right)(u_2 - \mu_k) \tag{44}$$

$$q_2 = q_1 + \dot{q}_1 T + \left(\frac{T^2}{2m}\right)(u_2 - \mu_k). \tag{45}$$

Combining Eqs. (19), (20), (14), (15), (32), (38), (39), and (42)–(45), and simplifying gives

$$\Delta\bar{u}^T(\bar{u} - \bar{u}^*) = -2(17 + 8a + a^2)\mu_k^2 - u_1^2 - u_2^2 + 2\mu_k(5 + a)u_1 - 2\mu_k(3 + a)u_2. \tag{46}$$

Equation (46) is concave in u_2. Hence, an absolute worst-case value of u_2 occurs when the derivative on the rhs of Eq. (46) with respect to u_2, is zero, or, at $u_2 = -\mu_k(3 + a)$; and the associated value of $\Delta\bar{u}^T(\bar{u} - \bar{u}^*)$ is

$$\Delta\bar{u}^T(\bar{u} - \bar{u}^*) \leq -(u_1 - (5 + a)\mu_k)^2. \tag{47}$$

We conclude that, for case (23)

$$u_1 \neq (5 + a)\mu_k \Rightarrow \Delta\bar{u}^T(\bar{u} - \bar{u}^*) < 0. \tag{48}$$

If $u_1 = -(5 + a)\mu_k$, then $u_1 = u_1^*$ [from Eq. (39)] and (38) gives $u_2^* = -(3 + a)\mu_k$. Therefore, case (23) gives:

$$\bar{u} \neq \bar{u}^* \Rightarrow \Delta\bar{u}^T(\bar{u} - \bar{u}^*) < 0. \tag{49}$$

For case (24), $u_1 < \mu_s$, and

$$q_1 = \left(\frac{T^2}{2m}\right)(u_1 + \mu_k) \tag{50}$$

$$\dot{q}_1 = \left(\frac{T}{m}\right)(u_1 + \mu_k) \tag{51}$$

$$\dot{q}_2 = \dot{q}_1 + \left(\frac{T}{m}\right)(u_2 + \mu_k) \tag{52}$$

$$q_2 = q_1 + \dot{q}_1 T + \left(\frac{T^2}{2m}\right)(u_2 + \mu_k). \tag{53}$$

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Combining Eqs. (19), (20), (14), (15), (32), (38), (39), and (50)–(53), and simplifying gives

$$\Delta\bar{u}^T(\bar{u}-\bar{u}^*) = -2(15+8a+a^2)\mu_k^2 - u_1^2 + 2(4+a)\mu_k(u_1-u_2) - u_2^2. \quad (54)$$

Equation (54), also concave in u_2 has an absolute worst-case value for u_2 of $\hat{u}_2$

$$\hat{u}_2 \equiv -(4+a)\mu_k \quad (55)$$

with $u_2 = \hat{u}_2$ giving

$$\Delta\bar{u}^T(\bar{u}-\bar{u}^*) \le -(14+8a+a^2)\mu_k^2 + 2(4+a)\mu_k u_1 - u_1^2, \quad \text{if } u_2 = \hat{u}_2. \quad (56)$$

Since $u_1 < -\mu_s$, we conclude

$$\mu_s \ne 0 \text{ or } \mu_k \ne 0 \Rightarrow \Delta\bar{u}^T(\bar{u}-\bar{u}^*) < 0. \quad (57)$$

If $\mu_s = \mu_k = 0$, Eqs. (38) and (39) give $u_1^* = u_2^* = 0$, which violates hypothesis (12). Thus, case (24) gives:

$$\bar{u} \ne \bar{u}^* \Rightarrow \Delta\bar{u}^T(\bar{u}-\bar{u}^*) < 0. \quad (58)$$

For case (25), $u_1 > \mu_s$, $u_2 < -\mu_s$, (42), (43), and

$$\dot{q}_s = \dot{q}_1 + \left(\frac{\Delta t_s}{m}\right)(u_2 - \mu_k) \quad (59)$$

$$q_s = q_1 + \dot{q}_1\,\Delta t_s + \left(\frac{\Delta t_s^2}{2m}\right)(u_2 - \mu_k) \quad (60)$$

$$\dot{q}_2 = \dot{q}_s + \left(\frac{T-\Delta t_s}{m}\right)(u_2 + \mu_k) \quad (61)$$

$$q_2 = q_s + \dot{q}_s(T-\Delta t_s) + \left(\frac{(T-\Delta t_s)^2}{2m}\right)(u_2+\mu_k) \quad (62)$$

where

$$u_1 = \mu_k + \left(\frac{\Delta t_s}{T}\right)(-u_2 + \mu_k) \quad (63)$$

where Eq. (63) just expresses $\dot{q}_s = 0$.
Combining Eqs. (19), (20), (14), (15), (32), (38), (43), and (59)–(63), and simplifying gives

$$\Delta\bar{u}^T(\bar{u}-\bar{u}^*) = -\mu_k^2(31+2a^2-7r-7r^2+r^3-2a(-8+r+r^2)) + \mu_k(1+r)(-8-2a+r+r^2)u_2 - (1+r^2)u_2^2. \quad (64)$$

The worst-case value of u_2 is $\hat{u}_2$

$$\hat{u}_2 \equiv \mu_k(1+r)(-8-2a+r+r^2)/2/(1+r^2) \quad (65)$$

giving

$$\Delta\bar{u}^T(\bar{u}-\bar{u}^*) \le \mu_k^2(-1+r)^2(-60-4a^2+20r+21r^2 + 2r^3 + r^4 + 4a(-8+r+r^2))/4/(1+r^2). \quad (66)$$

The -60 term dominates all others in Eq. (66). Thus

$$r \ne 1 \Rightarrow \Delta\bar{u}^T(\bar{u}-\bar{u}^*) < 0. \quad (67)$$

For $r=1$, case (25) reduces to case (23), and the proof for case (23) can be used to conclude that case (25) gives:

$$\bar{u} \ne \bar{u}^* \Rightarrow \Delta\bar{u}^T(\bar{u}-\bar{u}^*) < 0. \quad (68)$$

For case (26), $u_1 > \mu_s$, $|u_2| \le \mu_s$, and we have Eqs. (42), (43), $q_2 =$ rhs of Eq. (60), $\dot{q}_2 =$ rhs of Eq. (59) and (63). Combining Eqs. (19), (20), (14), (15), (32), (38)–(41), (42), and (43), $q_2 =$ rhs of Eq. (60), $\dot{q}_2 =$ rhs of Eqs. (59) and (63), and simplifying gives

$$2\Delta\bar{u}^T(\bar{u}-\bar{u}^*) = -\mu_k^2(56+4a^2-15r-6r^2+r^3-2a(-15+2r+r^2)) - 2\mu_k(4+a-r)(1+r^2)u_2 - r(1+r^2)u_2^2. \quad (69)$$

The absolute worst-case value of u_2 is $\hat{u}_2$

$$\hat{u}_2 \equiv \frac{-\mu_k(4+a-r)}{r} < -(3+a)\mu_k \quad (70)$$

and $u_2 = \hat{u}_2$ produces $\Delta\bar{u}^T(\bar{u}-\bar{u}^*) > 0$; however, it violates the condition $|u_2| \le \mu_s$. The worst-case valid value of u_2 is then $\tilde{u}_2$

$$\tilde{u}_2 \equiv -3\mu_k \quad (71)$$

and $u_2 = \tilde{u}_2$ gives

$$2\Delta\bar{u}^T(\bar{u}-\bar{u}^*) \le -2\mu_k^2(a^2-2a(-3+2r+r^2) + 4(2-3r+r^3)). \quad (72)$$

The values of r that make zero the derivative of the rhs of Eq. (72) with respect to r are

$$r_1 = -1, \quad r_2 = \left(\frac{3+a}{3}\right). \quad (73)$$

Both values in Eq. (73) are outside [0, 1]. So, we need only consider $r=0$

$$2\Delta\bar{u}^T(\bar{u}-\bar{u}^*) \le -2\mu_k(a^2+6a+8). \quad (74)$$

Thus

$$\mu_k \ne 0 \Rightarrow \Delta\bar{u}^T(\bar{u}-\bar{u}^*) < 0. \quad (75)$$

If $\mu_k = 0$, Eqs. (38) and (39) give $u_1^* = u_2^* = 0$, which violates hypothesis (12). Thus, case (26) gives:

$$\bar{u} \ne \bar{u}^* \Rightarrow \Delta\bar{u}^T(\bar{u}-\bar{u}^*) < 0. \quad (76)$$

For case (27), $|u_1| \leq \mu_s$, $u_2 > \mu_s$, $q_1 = 0$, $\dot{q}_1 = 0$, Eqs. (44) and (45).

Combining Eqs. (19), (20), (14), (15), (32), (38), (39), $q_1 = 0, \dot{q}_1 = 0$, and (44) and (45), and simplifying gives

$$\Delta\bar{u}^T(\bar{u}-\bar{u}^*) = -(29+15a+2a^2)\mu_k^2 - u_2^2 + \mu_k((4+a)u_1 - 2(3+a)u_2) \quad (77)$$

from which, since $|u_1| < \mu_s$ and $u_2 > \mu_s$

$$\mu_k \neq 0 \Rightarrow \Delta\bar{u}^T(\bar{u}-\bar{u}^*) < 0. \quad (78)$$

If $\mu_k = 0$, Eqs. (38) and (39) give $u_1^* = u_2^* = 0$, which violates hypothesis (12). Thus, case (27) gives:

$$\bar{u} \neq \bar{u}^* \Rightarrow \Delta\bar{u}^T(\bar{u}-\bar{u}^*) < 0. \quad (79)$$

For case (28), $|u_1| \leq \mu_s$, $u_2 < -\mu_s$, $q_1 = 0$, $\dot{q}_1 = 0$, Eqs. (52), and (53).

Combining Eqs. (19), (20), (14), (15), (32), (38), (39), (52) and (53), and simplifying gives

$$\Delta\bar{u}^T(\bar{u}-\bar{u}^*) = -(35+17a+2a^2)\mu_k^2 + (4+a)\mu_k(u_1-2u_2) - u_2^2 \quad (80)$$

but since

$$(u_2+(4+a)\mu_k)^2 = u_2^2 + 2(4+a)\mu_k u_2 + (4+a)^2\mu_k^2 \quad (81)$$

Eq. (80) can be written as

$$\Delta\bar{u}^T(\bar{u}-\bar{u}^*) = -(35+17a+2a^2)\mu_k^2 + (4+a)^2\mu_k^2 + (4+a)\mu_k u_1 - (u_2+(4+a)\mu_k)^2 \quad (82)$$

or

$$\Delta\bar{u}^T(\bar{u}-\bar{u}^*) = -(19+9a+a^2)\mu_k^2 + (4+a)\mu_k u_1 - (u_2+(4+a)\mu_k)^2 \quad (83)$$

but with Eq. (6)

$$|u_1| \leq \mu_s \Rightarrow (4+a)\mu_k u_1 \leq (12+3a)\mu_k^2 \quad (84)$$

so that

$$\Delta\bar{u}^T(\bar{u}-\bar{u}^*) \leq (-7-6a-a^2)\mu_k^2 - (u_2+(4+a)\mu_k)^2. \quad (85)$$

Thus

$$\mu_k \neq 0 \Rightarrow \Delta\bar{u}^T(\bar{u}-\bar{u}^*) < 0. \quad (86)$$

If $\mu_k = 0$, $u_1^* = u_2^* = 0$ and Eq. (12) is violated. Thus, case (28) gives:

$$\bar{u} \neq \bar{u}^* \Rightarrow \Delta\bar{u}^T(\bar{u}-\bar{u}^*) < 0. \quad (87)$$

For case (29), $u_1 < -\mu_s$, $u_2 > \mu_s$, Eqs. (50) and (51), and

$$\dot{q}_s = \dot{q}_1 + \left(\frac{\Delta t_s}{m}\right)(u_2+\mu_k) \quad (88)$$

$$q_s = q_1 + \dot{q}_1\Delta t_s + \left(\frac{\Delta t_s^2}{2m}\right)(u_2+\mu_k) \quad (89)$$

$$\dot{q}_2 = \dot{q}_s + \left(\frac{T-\Delta t_s}{m}\right)(u_2-\mu_k) \quad (90)$$

$$q_2 = q_s + \dot{q}_s(T-\Delta t_s) + \left(\frac{(T-\Delta t_s)^2}{2m}\right)(u_2-\mu_k) \quad (91)$$

where

$$u_1 = -\mu_k + (-u_2-\mu_k)\left(\frac{\Delta t_s}{T}\right) \quad (92)$$

where Eq. (92) simply expresses $\dot{q}_s = 0$.

Combining Eqs. (19), (20), (14), (15), (32), (38)–(41), (50), (51), and (88)–(92), and simplifying gives

$$\Delta\bar{u}^T(\bar{u}-\bar{u}^*) = -\mu_k^2(33+2a^2+7r+9r^2+r^3 + 2a(8+r+r^2)) - \mu_k(6+11r+2r^2 + r^3 + 2a(1+r))u_2 - (1+r^2)u_2^2. \quad (93)$$

Since $r \in [0, 1]$, the first term on the rhs of Eq. (93) is less than or equal to $-\mu_k^2(33+2a^2+16a)$. By $r \in [0, 1]$ and $u_2 > \mu_s$, the second term on the rhs is less than or equal to $-(6+2a)\mu_k\mu_s$. The last term is less than or equal to $-\mu_s^2$. Thus

$$\Delta\bar{u}^T(\bar{u}-\bar{u}^*) \leq -\mu_k^2(33+2a^2+16a) - (6+2a)\mu_k\mu_s - \mu_s^2. \quad (94)$$

Thus

$$\mu_k \neq 0 \quad \text{or} \quad \mu_s \neq 0 \Rightarrow \Delta\bar{u}^T(\bar{u}-\bar{u}^*) < 0. \quad (95)$$

If $\mu_k = \mu_s = 0$, Eqs. (38) and (39) give $u_1^* = u_2^* = 0$, which violates Eq. (12). Thus, case (29) gives:

$$\bar{u} \neq \bar{u}^* \Rightarrow \Delta\bar{u}^T(\bar{u}-\bar{u}^*) < 0. \quad (96)$$

For case (30), $u_1 < -\mu_s$, $|u_2| \leq \mu_s$, (50), (51), $\dot{q}_2 =$ rhs of Eq. (88), $q_2 =$ rhs of Eq. (89), and (92).

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Combining Eqs. (19), (20), (14), (15), (32), (38)–(41), (50), (51), $\dot{q}_2$ = rhs of Eq. (88), q_2 = rhs of Eq. (89), and (92), and simplifying gives

$$2\Delta\bar{u}^T(\bar{u}-\bar{u}^*) = -\mu_k^2(72+4a^2+17r+10r^2+r^3 + 2a(17+2r+r^27r^2)) - 2\mu_k(1+r^2)(4+a+r)u_2 - r(1+r^2)u_2^2. \quad (97)$$

The absolute worst-case value of u_2 is $\hat{u}_2$

$$\hat{u}_2 \equiv \left(\frac{-\mu_k(4+a+r)}{r}\right) < -4\mu_k < -\mu_s \quad (98)$$

and $u_2 = \hat{u}_2$ gives $\Delta\bar{u}^T(\bar{u}-\bar{u}^*) > 0$; however, $u_2 = \hat{u}_2$ violates the condition $|u_2| \leq \mu_s$. So, the worst-case valid value of u_2 is $\tilde{u}_2$

$$\tilde{u}_2 \equiv -3\mu_k \quad (99)$$

with $u_2 = \tilde{u}_2$ giving

$$\Delta\bar{u}^T(\bar{u}-\bar{u}^*) \leq -2\mu_k^2(12+a^2-7r-2r^2 + r^3 - a(-7+2r+r^2)). \quad (100)$$

Using $r \in [0,1]$, we conclude

$$\Delta\bar{u}^T(\bar{u}-\bar{u}^*) \leq -2\mu_k^2(3+a^2+4a). \quad (101)$$

Therefore

$$\mu_k \neq 0 \Rightarrow \Delta\bar{u}^T(\bar{u}-\bar{u}^*) < 0. \quad (102)$$

If $\mu_k = 0$, Eqs. (38) and (39) give $u_1^* = u_2^* = 0$, which violates Eq. (12). Thus, case (30)

$$\bar{u} \neq \bar{u}^* \Rightarrow \Delta\bar{u}^T(\bar{u}-\bar{u}^*) < 0. \quad (103)$$

For case (31), $|u_1| < \mu_s$, $|u_2| < \mu_s$, $q_2 = 0$, and $\dot{q}_2 = 0$. Combining Eqs. (19), (20), (14), (15), (32), (38), and (39), $q_2 = 0$, $\dot{q}_2 = 0$, and simplifying gives

$$\Delta\bar{u}^T(\bar{u}-\bar{u}^*) = -(4+a)\mu_k(2(4+a)\mu_k - u_1 + u_2). \quad (104)$$

Since $-u_1 + u_2 \geq -6\mu_k$

$$\Delta\bar{u}^T(\bar{u}-\bar{u}^*) \leq -(4+a)\mu_k^2(2+2a) \quad (105)$$

so that

$$\mu_k \neq 0 \Rightarrow \Delta\bar{u}^T(\bar{u}-\bar{u}^*) < 0. \quad (106)$$

If $\mu_k = 0$, Eqs. (38) and (39) give $u_1^* = u_2^* = 0$, which violates Eq. (12). Thus, case (31)

$$\bar{u} \neq \bar{u}^* \Rightarrow \Delta\bar{u}^T(\bar{u}-\bar{u}^*) < 0. \quad (107)$$

Thus, the direction

$$\Delta\bar{u} = -\alpha G e \quad (108)$$

is a descent direction on $z \equiv 1/2(\bar{u}-\bar{u}^*)^T(\bar{u}-\bar{u}^*)$, which is clearly a smooth function of $\bar{u}$. Therfore, if $u_{\max} = \infty$, then for small enough $\alpha > 0$, the learning controller [Eq. (16)] produces $\lim_{i\to\infty}(\bar{u}^i - \bar{u}^*) = 0$. Finally, application of the *boxfcn*$(\cdot)$ in controller [Eq. (16)] cannot decrease the magnitude of the reduction in $\|\bar{u}-\bar{u}^*\|$ from one learning trial to the next, since $|\bar{u}^*(i)| \leq u_{\max}$, $i = 1, 2$, (see Driessen *et al.*[4]). Thus, we have

$$\lim_{i\to\infty}(\bar{u}^i) = \bar{u}^*, \quad \forall\bar{u}^0 : \left|\bar{u}^0(j)\right| \leq u_{\max}, \quad (j = 1, 2), \quad \text{small enough} \quad \alpha > 0. \quad (109)$$

That is, the objective [Eq. (18)] of Section 2 is met. Hence, the proof of Theorem 3.1 is complete.

Remark: The above results do not extend, to our knowledge, to point-to-point rest-to-rest maneuver problems of a revolute-jointed direct-drive robot arm. However, they do apply to a multi-degree-of-freedom prismatic-jointed robot; and, possibly for a very highly geared revolute-jointed robot, for which the joints become very close to decoupled second-order systems with stiction, the results may also have some relevance.

4. Numerical Example

In this section, we present an ILC example for the problem described in Sections 2 and 3. The parameters in the example are from Eqs. (1), (4), (5), (7), (8), (20), (32), (18), and (9), respectively

$$m = 1.0 \quad (110)$$

$$u_{\max} = 5.5 \quad (111)$$

$$\mu_k = 1 \quad (112)$$

$$\mu_s = 2 \quad (113)$$

$$T = 1.0\,\text{s} \quad (114)$$

$$\hat{m} = 1.5 \quad (115)$$

$$\alpha = 0.3 \quad (116)$$

$$\bar{u}^0 = (0, 0)^T \quad (117)$$

$$q_2^* = 4. \quad (118)$$

Equations (110), (112), (114), (118), and (32)–(38) imply $u_1^* = 5$ and $u_2^* = -3$. Figures 1 and 2 show plots of the input learning error $\|\bar{u}-\bar{u}^*\|_2$ [from Eq. (14)] and the

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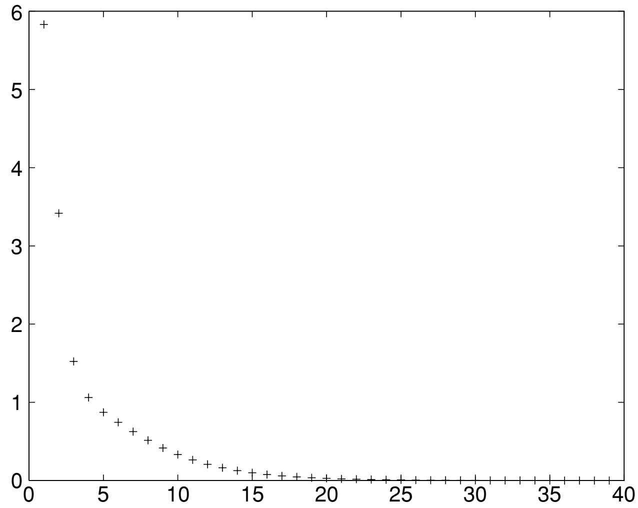

Fig. 1. Input learning error $\|\bar{u} - \bar{u}^*\|_2$ vs. learning iteration number.

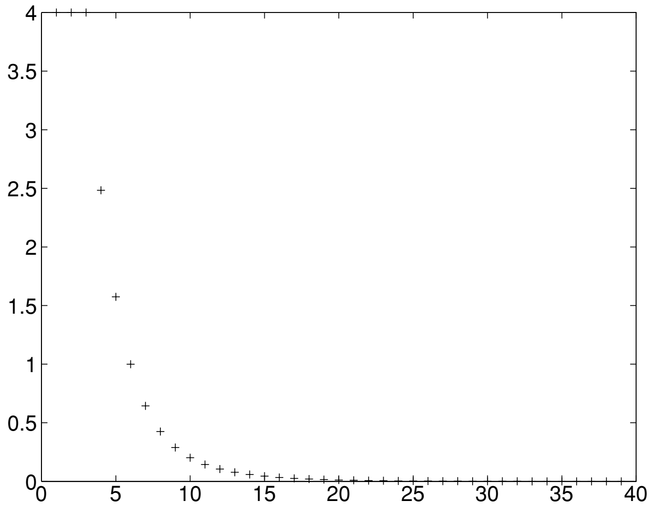

Fig. 2. Output learning error $||e||_2$ vs. learning iteration number.

output learning error $\|e\|_2$ [from Eq. (15)] versus learning iterate number, respectively. Iterations were stopped when the output error $\|e\|_2$ was less than or equal to 10^{-4}, giving a total of 38 iterations.

5. Conclusion

In this paper, we presented a global convergence proof for a simple iterative learning controller for learning a rest-to-rest maneuver of a single-degree-of-freedom system with stick-slip Coulomb friction. The input is restricted to be a two-pulse one. The sufficient conditions for global convergence included a static friction coefficient not larger than three times the kinetic coefficient of friction, and a desired first-pulse magnitude greater than or equal to five times the largest possible kinetic friction force. The theory, therefore, allows the stiction force to be as large as the desired second input pulse.

References

1. S. Arimoto, S. Kawamura and F. Miyazaki, "Bettering operation of robots by learning," *J. Robot. Syst.* **1**(2), 123–140 (1984).
2. C. Cheah and D. Wang, "Learning control for a class of nonlinear differential-algebraic systems with application to constrained robots," *J. Robot. Syst.* **13**(3), 141–151 (1996).
3. C. Chen and S. Peng, "Learning control of process systems with hard input constraints," *J. Process Control* **9**, 151–160 (1999).
4. B. Driessen, N. Sadegh and K. Kwok, "Multi-input square iterative learning control with bounded inputs," *ASME J. Dyn. Syst. Meas. Control* **124**, 582–584 (2002).
5. D. Gorinevsky, "An algorithm for on-line parametric nonlinear least square optimization," *Proceedings of the 33rd IEEE Conference on Decision and Control*, Lake Buena Vista, FL, **3** (Dec. 1994) pp. 2221–2226.
6. D. Gorinevsky, "An application of on-line parametric optimization to task-level learning control," *Proceedings of the American Control Conference*, Seattle, WA (Jun. 1995) pp. 862–866.
7. D. Gorinevsky, D. Torfs and A. A. Goldenberg, "Learning approximation of feedforward dependence on the task parameters: Experiments in direct-drive manipulator tracking," *Proceedings of the American Control Conference*, Seattle, WA (Jun. 1995) pp. 883–887.
8. R. Horowitz, "Learning control of robot manipulators," *ASME J. Dyn. Syst. Meas. Control* **115**, 402–411 (Jun. 1993).
9. R. W. Longman and C.-K. Chang, "Discrete time learning control in nonlinear systems," *Proceedings of the AIAA /AAS Astrodynamics Conference* (1992) pp. 501–511.
10. N. Sadegh and B. Driessen, "Minimum time trajectory learning," *ASME J. Dyn. Syst. Meas. Control* **121**, 213–217 (Jun. 1999).
11. Y. Wang and R. W. Longman, "Limit cycle behavior and convergence to zero error in learning control with stick-slip friction," *Proceedings of the IEEE Conference on Systems, Man, and Cybernetics* (1994) pp. 2774–2779.

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