Proof: Let A have n digits. Then $A \geqslant 10^{n-1}$, so $\sqrt{A} \geqslant 10^{(n-1)/2}$. If $n \geqslant 5$, then, by the Lemma, $\sqrt{A} > 9n$. But the sum of the digits of A is at most $9n$ (reached when each digit is 9). Thus if $n \geqslant 5$, $\sqrt{A}$ exceeds the sum of the digits of A.

If $n = 4$, the digit sum of A is at most $9 \times 4 = 36$, so if $A > 1296 = 36^2$, then $\sqrt{A}$ exceeds the sum of the digits of A. But if $A \leqslant 1296$, then the sum of its digits is less than $1 + 2 + 9 + 9 = 21$, yet $\sqrt{A} \geqslant \sqrt{1000} > 21$.

It remains to consider the case $n \leqslant 3$. Now, by direct verification, it is easy to find that there are only two numbers, 1 and 81, that satisfy the problem:

number	(number)^2	sum of digits	number	(number)^2	sum of digits
1	1	1	16	256	13
2	4	4	17	289	19
3	9	9	18	324	9
4	16	7	19	361	10
5	25	7	20	400	4
6	36	9	21	441	9
7	49	13	22	484	16
8	64	10	23	529	16
9	81	9	24	576	18
10	100	1	25	625	13
11	121	4	26	676	19
12	144	9	27	729	18
13	169	16	28	784	19
14	196	16	29	841	13
15	225	9	30	900	9
			31	961	16

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108.39 A quick proof that π is less than 2ϕ

The golden ratio ϕ is $\frac{1}{2}(1 + \sqrt{5})$ and $\tan\frac{1}{12}\pi = 2 - \sqrt{3}$. The aim of this Note is to give a quick proof of the well known inequality $\pi < 2\phi$. Our proof is more elementary than Nelsen ([1]).

This proof uses the familiar inequality $\sin x < x < \tan x$ for $0 < x < \frac{1}{2}\pi$. An alternative to the standard proof is given by the following diagram:

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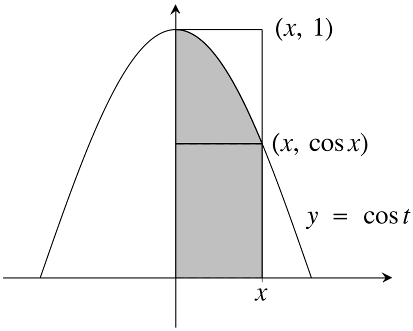

$$x \cdot \cos x < \int_0^x \cos t \, dt = \sin x < x \cdot 1.$$

With $x = \frac{1}{12}\pi$, we have $\frac{1}{12}\pi < \tan \frac{1}{12}\pi = 2 - \sqrt{3}$, so that

$$\pi < 12\left(2 - \sqrt{3}\right) < 2\phi = \sqrt{5} + 1,$$

since $23 < 12\sqrt{3} + \sqrt{5}$ (by squaring both sides).

This bound is equivalent to comparing the area of a unit circle with that of a circumscribing regular dodecagon where, as usual, a lower bound of $12 \sin \frac{1}{12}\pi < \pi$ comes from the inscribed dodecagon.

Furthermore, the actual computing yields a slightly sharper inequality (but perhaps less interesting): $\pi < 3.2154$.

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