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A generalization of Krasnosel'skii compression fixed point theorem by using star convex sets

Cristina Lois-Prados and Rosana Rodríguez-López
Instituto de Matemáticas, Facultade de Matemáticas,
Universidade de Santiago de Compostela, Santiago
de Compostela 15782, Spain (cristina.lois.prados@usc.es;
rosana.rodriguez.lopez@usc.es)

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In the framework of fixed point theory, many generalizations of the classical results due to Krasnosel'skii are known. One of these extensions consists in relaxing the conditions imposed on the mapping, working with k-set contractions instead of continuous and compact maps. The aim of this work if to study in detail some fixed point results of this type, and obtain a certain generalization by using star convex sets.

Keywords: Krasnosel'skii fixed point theorem; k-set contraction; star convex set

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1. Introduction

Fixed point theorems are a fundamental tool to study the existence of solution to a wide range of different problems in mathematics, such as boundary value problems in the framework of differential equations.

For $T : D \subset X \longrightarrow X$ a mapping between sets, we say that an element $x \in D$ is a fixed point of T if $T(x) = x$. Here, we consider fixed point results that provide the existence and location of fixed points for mappings and sets satisfying certain hypotheses.

The usual technique followed to apply these results to boundary value problems consists in transforming the problem into an integral equation. In this way, it is obtained a mapping whose fixed points are the solutions to the boundary value problem.

Our main results generalize some classical fixed point theorems due to Krasnosel'skii. Two of these classical results can be found in [6] (dated in 1960), and they establish that, if $(X, ||\cdot||)$ is a Banach space, C a cone in X, $T : C \longrightarrow C$ a continuous and compact map such that $T(0) = 0$, and there exist $r, R \in \mathbb{R}^+$, $r < R$, satisfying some of the following conditions

$$\begin{aligned} x - T(x) &\notin C, \quad \forall\, x \in C, ||x|| \leqslant r, \\ T(x) - x &\notin C, \quad \forall\, x \in C, ||x|| \geqslant R, \end{aligned} \tag{1.1}$$

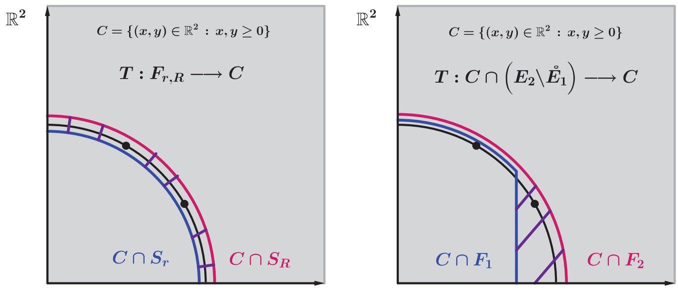

Figure 1. Motivation to work with more general sets.

or

$$
\begin{aligned}
T(x) - x \notin C, &\quad \forall\, x \in C, ||x|| \leqslant r, \\
x - T(x) \notin C, &\quad \forall\, x \in C, ||x|| \geqslant R,
\end{aligned} \tag{1.2}
$$

then T has a nontrivial fixed point in $\{x \in C : r \leqslant ||x|| \leqslant R\}$. Next, we mention the different directions in which these classical results have been generalized, some of these extensions can be found in [7].

The first direction of extension is to relax conditions (1.1) and (1.2), in such a way that it is only required that

$$
\begin{aligned}
x - T(x) \notin C, &\quad \forall\, x \in C, ||x|| = r, \\
\forall \varepsilon > 0,\ T(x) - (1+\varepsilon)x \notin C, &\quad \forall\, x \in C, ||x|| = R,
\end{aligned} \tag{1.3}
$$

or

$$
\begin{aligned}
\forall \varepsilon > 0,\ T(x) - (1+\varepsilon)x \notin C, &\ \forall x \in C, ||x|| = r, \\
x - T(x) \notin C, &\quad \forall\, x \in C, ||x|| = R.
\end{aligned} \tag{1.4}
$$

The second direction of extension is to seek more general regions, in which the theorem provides the fixed point. As an example, a result due to Güo and Lakshmikhantan [5] states that, if Ω_1, Ω_2 are bounded open sets in the Banach space X such that $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$ and $T : C \cap (\overline{\Omega_2} \backslash \Omega_1) \longrightarrow C$ is a continuous and compact mapping, then T has a fixed point in the region $C \cap (\overline{\Omega_2} \backslash \Omega_1)$, where $\overline{\Omega_i}$ is the closure of Ω_i, for $i = 1, 2$. We can find another example in [1], where the region is generalized using general functionals instead of the norm. Recently, Webb [11] provides new results for nonlinear integral operators by using fixed point index and modifying the underlying sets.

A third way to extend the mentioned classical results consists in considering another type of mappings instead of compact ones, the maps that we consider will be referred to as k-set contractions.

Many of the generalizations of the classical results due to Krasnosel'skii have been proved using topological degree theory, such as those appearing in [1, 5]. It is important to mention that this theory is not the approach used here.

Now, we can explain properly the motivation of our work. Assume that $T : D \subset X \longrightarrow X$ has two fixed points with the same norm. We cannot prove their existence by using the mentioned fixed point results due to Krasnosel'skii. However, if we prove similar results working with more general sets, it could be possible to distinguish these two fixed points. Figure 1 illustrates an example of sets that allow us to separate two fixed points with the same norm.

2. Preliminaries

For the sake of completeness, we provide the following definitions, results and notations, which are useful to our procedure. We refer to [8, 10] for some basic monographs.

NOTATION 2.1. *Let $\varepsilon > 0$ and d a distance in X, we denote $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ and $\overline{B}_d(x, \varepsilon) = \{y \in X : d(x, y) \leqslant \varepsilon\}$. If there is no possibility of confusion, we fix the notation $B(x, \varepsilon) \equiv B_d(x, \varepsilon)$ and $\overline{B}(x, \varepsilon) \equiv \overline{B}_d(x, \varepsilon)$.*

NOTATION 2.2. *Let X be a set, A, B subsets of X and $\lambda \in \mathbb{R}$, we establish the notation:*

- *If X is a topological space, $\overline{A}$ denotes the closure of A in X.*
- *If X is a real vector space, we define $A + B = \{a + b : a \in A, b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$.*

DEFINITION 2.3. Let X be a real vector space and A a subset of X. The convex hull of A is the set

$$\operatorname{co}(A) := \left\{ \sum_{i=1}^{n} \lambda_i x_i \,:\, n \in \mathbb{N},\ \sum_{i=1}^{n} \lambda_i = 1,\ x_i \in A,\ \lambda_i \in [0, 1],\ \forall i \in \{1, \ldots, n\} \right\}.$$

Furthermore, if X is a topological space, we denote the closure of $\operatorname{co}(A)$ by $\overline{\operatorname{co}}(A)$.

DEFINITION 2.4. Let (X, d) be a metric space, $K \neq \emptyset$ a subset of X and $x \in X$. The distance between x and K is defined by

$$d_K(x) \equiv d(x, K) := \inf_{y \in K} d(x, y).$$

When $K = \emptyset$, we define $d(x, \emptyset) := +\infty$.

REMARK 2.5. If $A \subset X$ and $A \neq \emptyset$, it is satisfied that $d(x, A) < +\infty$, for all $x \in X$.

DEFINITION 2.6. Let X, Y be metric spaces and $D \subset X$. The mapping $T : D \subset X \longrightarrow Y$ is compact if, for all $A \subset D$ bounded, $\overline{T(A)}$ is a compact set.

The following concept allows us to relax the compactness hypothesis assumed in the mentioned classical results due to Krasnosel'skii. Some of its principal properties can be found in [9].

Definition 2.7. Let (X, d) be a metric space and $A \subset X$ a bounded set. The measure of noncompactness of A is the nonnegative real number

$$\alpha(A) := \inf\left\{\varepsilon > 0 \,:\, A \subset \bigcup_{i=1}^{n} A_i,\ \operatorname{diam}(A_i) \leqslant \varepsilon,\ \forall i \in \{1, \dots, n\}\right\},$$

where $\operatorname{diam}(A_i) := \sup\{d(x, y) : x, y \in A_i\}$, for all $i \in \{1, \dots, n\}$.

Proposition 2.8. *Let A, B be subsets of a metric space (X, d). The measure of noncompactness satisfies the following properties:*

(i) $A \subseteq B \Rightarrow \alpha(A) \leqslant \alpha(B)$.

(ii) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.

(iii) $\alpha(A) = \alpha(\overline{A})$.

Moreover, if $(X, \|\cdot\|)$ is a Banach space, then:

(iv) $\alpha(A + B) \leqslant \alpha(A) + \alpha(B)$.

(v) $\alpha(\lambda A) = |\lambda|\alpha(A)$, $\forall \lambda \in \mathbb{R}$.

(vi) $\alpha(co(A)) = \alpha(A)$.

(vii) $\overline{A}$ *is a compact set if and only if* $\alpha(A) = 0$.

The measure of noncompactness can be considered as a tool to determine how much a particular set differs from being compact. In this way, we will be able to define a concept close to compact mapping, this one will be known as k-set contraction. So, we give a formal definition of this kind of mappings and some results about them. This information can be found in [9].

Definition 2.9. Let X, Y be metric spaces and $D \subset X$. Assume that the mapping $T : D \subset X \longrightarrow Y$ is continuous. We say that T is a k-set contraction if there exists a constant $k \geqslant 0$ such that

$$\alpha(T(A)) \leqslant k\alpha(A), \quad \text{for all bounded } A \subset D.$$

Remark 2.10. If T is a k-set contraction, it is implicitly required that $T(A)$ is bounded when $A \subset D$ is bounded since it is a necessary condition to calculate the measure of noncompactness of $T(A)$.

Example 2.11. Continuous and compact mappings are in correspondence with 0-set contractions when X is a Banach space.

Proposition 2.12. *Let (X_i, d_i) be metric spaces for $i \in \{1, 2, 3\}$, and $(X, \|\cdot\|)$ a Banach space. The following properties are satisfied:*

(i) *If $T_1 : X_1 \longrightarrow X_2$, $T_2 : X_2 \longrightarrow X_3$ are, respectively, k_1, k_2-set contractions, then $T_2 \circ T_1 : X_1 \longrightarrow X_3$ is a k_1k_2-set contraction.*

(ii) *If* $S_1 : X_1 \longrightarrow X$, $S_2 : X_1 \longrightarrow X$ *are, respectively,* k_1, k_2*-set contractions, then* $S_1 + S_2 : X_1 \longrightarrow X$ *is a* $k_1 + k_2$*-set contraction.*

PROPOSITION 2.13. *Let* $D, \hat{D}$ *be closed subsets of a metric space* (X, d)*. Assume that* $T : D \longrightarrow X$, $\hat{T} : \hat{D} \longrightarrow X$ *are* k*-set contractions and* $T_{|D\cap\hat{D}} = \hat{T}_{|D\cap\hat{D}}$*. If we define another mapping by*

$$\begin{aligned}\tilde{T} : D \cup \hat{D} &\longrightarrow X \\ x &\longmapsto \tilde{T}(x) := \begin{cases} T(x), & x \in D, \\ \hat{T}(x), & x \in \hat{D}, \end{cases}\end{aligned}$$

then $\tilde{T}$ *is a* k*-set contraction.*

COROLLARY 2.14. *Let* $D, \hat{D}$ *be closed subsets of a metric space* (X, d)*. Suppose that* $T : D \longrightarrow X$ *is a* k*-set contraction and* $\hat{T} : \hat{D} \longrightarrow X$ *is a* $\hat{k}$*-set contraction such that* $T_{|D\cap\hat{D}} = \hat{T}_{|D\cap\hat{D}}$*. Define*

$$\begin{aligned}\tilde{T} : D \cup \hat{D} &\longrightarrow X \\ x &\longmapsto \tilde{T}(x) := \begin{cases} T(x), & x \in D, \\ \hat{T}(x), & x \in \hat{D}, \end{cases}\end{aligned}$$

then $\tilde{T}$ *is a* $\tilde{k}$*-set contraction with* $\tilde{k} = \max\{k, \hat{k}\}$*.*

PROPOSITION 2.15. *Let* $(X, \|\cdot\|)$ *be a Banach space,* $T : D \subset X \longrightarrow X$ *a* k*-set contraction and* $\lambda : D \longrightarrow \mathbb{R}^+ \cup \{0\}$ *a continuous function such that* $\sup_{x\in D} \lambda(x) = l < \infty$*. Define*

$$\begin{aligned}\hat{T} : D \subset X &\longrightarrow X \\ x &\longmapsto \hat{T}(x) := \lambda(x)T(x),\end{aligned}$$

then $\hat{T}$ *is a* kl*-set contraction.*

3. Main fixed point results

In this section, we include some known results proved by Potter [**9**] that generalize one of the fixed-point results due to Krasnosel'skii, in two of the mentioned directions. Besides, we show our contribution proving a result that generalizes the one proved by Potter in other of the directions mentioned working with more general sets which do not need to be convex.

The following result is basic in the proof of the fixed point theorem due to Potter. This can be found in [**4**].

PROPOSITION 3.1 (Fixed point theorem for k-set contractions). *Let* $(X, \|\cdot\|)$ *be a Banach space and* $B \subset X$ *a closed, convex and bounded set. Assume that* $T : B \longrightarrow B$ *is a* k*-set contraction with* $k < 1$*, then there exists* $x \in B$ *a fixed point of* T*.*

Now, we fix the notation about the subsets of a Banach space in which the main results will locate the fixed points.

Definition 3.2. Let $(X, \|\cdot\|)$ be a Banach space. A subset C of X is a cone if

- C is closed;
- for all $x, y \in C$, $a, b \in \mathbb{R}^+$, it is satisfied that $ax + by \in C$;
- $x \in C$, $-x \in C$ if and only if $x = 0$.

Example 3.3. Let us consider the Banach space $(\mathbb{R}^2, ||\cdot||)$, where

$$||\cdot|| : \mathbb{R}^2 \longrightarrow [0, +\infty), \quad (x, y) \longmapsto ||(x, y)|| := \sqrt{x^2 + y^2}.$$

The set $C := \{(x, y) \in \mathbb{R}^2 \ : \ x, y \geqslant 0\}$ is a cone in $\mathbb{R}^2$. We will make representations with this cone to illustrate the restrictions imposed in the statements of the main fixed point results.

Example 3.4. Let us consider the Banach space $(\mathcal{C}([0, 1], \mathbb{R}), ||\cdot||)$, where the elements in $\mathcal{C}([0, 1], \mathbb{R})$ are continuous functions on the interval $[0, 1]$ with values in $\mathbb{R}$ and

$$||\cdot|| : \mathcal{C}([0, 1], \mathbb{R}) \longrightarrow [0, +\infty), \quad x \longmapsto ||x|| := \sup_{t\in[0,1]} |x(t)|.$$

The set $C := \{x \in \mathcal{C}([0, 1], \mathbb{R}) \ : \ x \geqslant 0\}$ is a cone in $\mathcal{C}([0, 1], \mathbb{R})$.

Notation 3.5. *Let $(X, \|\cdot\|)$ be a Banach space and $C \subset X$ a cone. For $r, R \in \mathbb{R}$, with $0 < r < R$, let*

- $F_{r,R} = \{x \in C \ : \ r \leqslant \|x\| \leqslant R\}$;
- $B_r = \{x \in C \ : \ \|x\| \leqslant r\}$;
- $S_r = \{x \in C \ : \ \|x\| = r\}$.

We outline the mentioned result due to Potter. The following lemma is required to prove this fixed point theorem. It allows us to extend the domain of some k-set contractions, with $k < 1$, preserving this property of the mapping.

Lemma 3.6. *Assume that $T : S_r \longrightarrow C$ is a k-set contraction. Let us consider*

$$\tilde{T} : B_r \longrightarrow C, \ x \longmapsto \tilde{T}(x) := \begin{cases} \frac{\|x\|}{r} T\left(\frac{r}{\|x\|}x\right), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

then $\tilde{T}$ is a $\tilde{k}$-set contraction, with $\tilde{k} > k$, $\tilde{k}$ as near k as we please.

The conditions of the classical results due to Krasnosel'skii, which have been formulated in the Introduction (1.1), can be replaced by the ones in the following definition.

DEFINITION 3.7. Let $(X, ||\cdot||)$ be a Banach space and C a cone. A mapping $T : F_{r,R} \longrightarrow C$ is said to be a compression of the cone C if

- $x - Tx \notin C$ for all $x \in C$ with $\|x\| = r$;
- for all $\varepsilon > 0$ and $x \in C$ with $\|x\| = R$, $Tx - (1+\varepsilon)x \notin C$.

THEOREM 3.8. *Let $(X, ||\cdot||)$ be a Banach space, C a cone in X and $0 < r < R$ real numbers. Suppose that $T : F_{r,R} \longrightarrow C$ is a k-set contraction with $k < 1$ and a compression of the cone C. Then T has at least one fixed point in $F_{r,R} \subset C$.*

Now, we prove a more general result working with star convex sets that are not necessarily convex.

DEFINITION 3.9. Let $(X, \|\cdot\|)$ be a Banach space, $E \subset X$ and $x_0 \in E$ such that

$$\lambda x_0 + (1-\lambda)x \in E, \quad \text{for all } \lambda \in [0,1] \quad \text{and} \quad x \in E.$$

If $x_0 \neq 0$, E is said to be an x_0-star convex set. If $x_0 = 0$, it is called a star convex set.

In our main results, we will consider the following hypothesis.

CONDITION 3.10. *Let $(X, \|\cdot\|)$ be a Banach space, C a cone in X and E a star convex set (which trivially satisfies that $E \cap C \neq \emptyset$). We assume the following essential conditions for the star convex set E:*

- *E is bounded, closed and has nonempty interior.*
- *If F is the boundary of E in X, then $0 \notin F$.*
- *There exists a continuous mapping $\partial : E\backslash\{0\} \longrightarrow F$, $x \longmapsto \partial(x)$, such that (see figure 2):*

$$\partial(x) = \partial(\lambda x), \quad \forall x \in E, \forall \lambda \in (0,1],$$
$$\partial(x) = x, \quad \forall\, x \in F.$$

If the boundary F satisfies the appropriate conditions, there is a unique procedure valid to define the mapping ∂, related to the replication of its values through rays travelling to 0.

The last property in condition 3.10 is too strong since the interesting properties arise in $C \cap E\backslash\{0\}$.

REMARK 3.11. Suppose that $(X, \|\cdot\|)$ is a Banach space, C a cone in X and E a star convex set satisfying condition 3.10. As E is a closed and star convex set such

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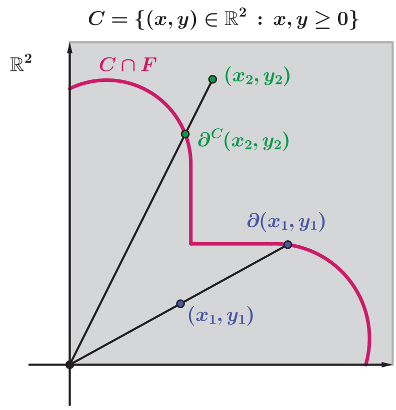

Figure 2. An example of mappings ∂ and ∂^C.

that $0 \in \mathring{E}$, it is possible to extend the domain of $\partial_{|C\cap(E\backslash\{0\})}$ to $C\backslash\{0\}$ by

$$\partial^C : C\backslash\{0\} \longrightarrow F$$
$$x \longmapsto \partial^C(x) := \begin{cases} \partial(x), & x \in E\backslash\{0\}, \\ \partial\left(\frac{d(0,F)}{||x||}x\right), & x \in C\backslash(C \cap \mathring{E}). \end{cases}$$

Besides, ∂^C is a continuous function. Figure 2 illustrates the behaviour of ∂^C in a particular case.

Proposition 3.12. *Let $(X, \|\cdot\|)$ be a Banach space, C a cone in X and E a star convex set satisfying condition 3.10. Then, for all $x \in C \cap (E\backslash\{0\})$, there exists a unique number $\beta_x \in \mathbb{R}^+$ such that $\beta_x x \in C \cap F$.*

Proof. The existence of such a number is clear since C is a cone and E is closed, bounded, with nonempty interior and a star convex set. Suppose that there exist $\beta_x^1, \beta_x^2 \in \mathbb{R}^+$ such that $\beta_x^1 \neq \beta_x^2$ and $\beta_x^1 x, \beta_x^2 x \in C \cap F$. Assume that $\beta_x^2 > \beta_x^1$, then $0 < ((\beta_x^1)/(\beta_x^2)) < 1$ and, therefore,

$$\partial(\beta_x^1 x) = \partial\left(\frac{\beta_x^1}{\beta_x^2}\beta_x^2 x\right) = \partial(\beta_x^2 x) = \beta_x^2 x \in F.$$

Besides, since $\beta_x^1 x \in F$, $\partial(\beta_x^1 x) = \beta_x^1 x \in F$. As a consequence, $\beta_x^1 x = \beta_x^2 x$. Taking the norm, we get $\beta_x^1\|x\| = \beta_x^2\|x\|$, and, since $x \neq 0$, then $\beta_x^1 = \beta_x^2$. We conclude that the element β_x in the statement is unique and the proof is finished. □

Lemma 3.13. *Let $(X, \|\cdot\|)$ be a Banach space, C a cone in X and E a star convex set satisfying condition 3.10. Then*

$$\beta : C \cap (E\backslash\{0\}) \longrightarrow [1, +\infty)$$
$$x \longmapsto \beta(x) := \beta_x,$$

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where β_x is the unique $\beta_x \in \mathbb{R}^+$ such that $\beta_x x \in F \cap C$, satisfies the following properties:

(a) *β is a continuous function.*

(b) $\lim_{x\to 0} \beta(x) = +\infty$.

Proof. First of all, we prove that the image of β is a subset of $[1, +\infty)$. Let $x \in C \cap\ (E\backslash\{0\})$, then it is satisfied that $\beta_x x = \partial(x)$, where

$$\beta_x = \frac{\|\partial(x)\|}{\|x\|} \geqslant 1.$$

Secondly, we prove the properties (a) and (b). Indeed:

(a) The function

$$d_0 : X \longrightarrow [0, \infty),\ x \longmapsto d_0(x) := d(0, x)$$

is continuous because of the properties of the distance. By hypothesis, $\partial : E\backslash\{0\} \longrightarrow F$ is continuous too. Since β can be expressed as

$$\beta : C \cap (E\backslash\{0\}) \longrightarrow [1, +\infty)$$
$$x \longmapsto \beta(x) = \frac{d(0, \partial(x))}{d(0, x)} = \frac{(d_0 \circ \partial)(x)}{d_0(x)},$$

then β is a continuous function.

(b) For all $M \in \mathbb{R}^+$, we look for $\delta \in \mathbb{R}^+$ such that

$$\beta(x) > M, \quad \text{for all } x \in C \cap (E\backslash\{0\}) \quad \text{with } ||x|| < \delta.$$

If $M \in (0, 1)$, using that $\beta \in [1, +\infty)$, then $\beta(x) > M$ is trivially satisfied for all $x \in C \cap (E\backslash\{0\})$. If $M \geqslant 1$, let $0 < \delta = ((d(0, F))/(M)) \leqslant d(0, F) < +\infty$. We prove that, if $x \in C \cap (E\backslash\{0\})$ with $||x|| < \delta$, then $\beta(x) > M$:

$$\beta(x) = \frac{d(0, \partial(x))}{d(0, x)} \geqslant \frac{d(0, F)}{d(0, x)} = \frac{d(0, F)}{||x||} > \frac{d(0, F)}{\delta} = M.$$

The proof is concluded.

□

Remark 3.14. Assume that E is a star convex set satisfying condition 3.10. Since E is a bounded and closed set and F is its boundary, then there exists $L \in \mathbb{R}^+$ such that $d(0, \partial(x)) = \|\partial(x)\| \leqslant L$ for all $x \in E\backslash\{0\}$. It is enough to take $L = \sup\{d(0, x)\ :\ x \in F\}$.

We have stated sufficient conditions on the sets. The next step will be to reformulate definition 3.7 dealing with these more general sets.

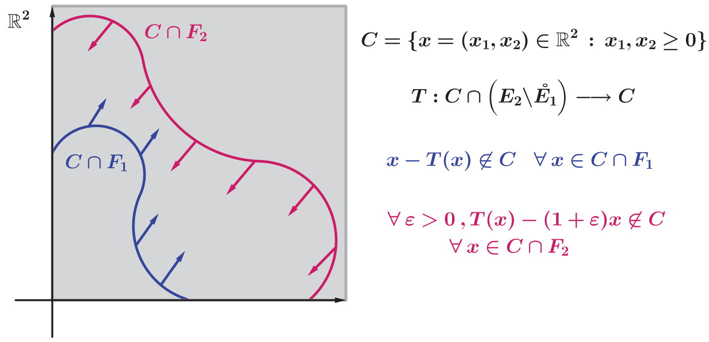

Figure 3. T is a compression of the cone C.

Definition 3.15. Let $(X, \|\cdot\|)$ be a Banach space, C a cone in X and E_1, E_2 star convex sets fulfilling condition 3.10. For $i = 1, 2$, we establish the following notation:

- F_i is the boundary of E_i, $\mathring{E}_i$ is the interior of E_i.
- $\partial_i : E_i \backslash \{0\} \longrightarrow F_i$ is the continuous mapping required by condition 3.10.
- $\beta_i : C \cap (E_i \backslash \{0\}) \longrightarrow F_i$, $x \longmapsto \beta_i(x) = ((d(0, \partial_i(x)))/(d(0, x)))$.
- $L_i = \sup\{d(0, x) \,:\, x \in F_i\}$.

Suppose that $0 \in E_1 \subset E_2$ and $F_1 \cap F_2 = \emptyset$.

A mapping $T : C \cap (E_2 \backslash \mathring{E}_1) \longrightarrow C$ is a compression of the cone C (see figure 3) if

(C_1) $x - T(x) \notin C$, for all $x \in C \cap F_1$;

(C_2) for all $\varepsilon > 0$ and $x \in C \cap F_2$, $T(x) - (1+\varepsilon)x \notin C$.

Next, some generalizations of lemma 3.6 and theorem 3.8 are proved.

Lemma 3.16. *Assume that $(X, \|\cdot\|)$ is a Banach space, C a cone in X and E a star convex set satisfying condition 3.10. Suppose that $T : C \cap F \longrightarrow C$ is a k-set contraction. We define*

$$\tilde{T} : C \cap E \longrightarrow C$$

$$x \longmapsto \tilde{T}(x) := \begin{cases} \frac{1}{\beta(x)} T(\beta(x)x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then $\tilde{T}$ is a $\tilde{k}$-set contraction for $\tilde{k} > ((L)/(d(0, F)))k$, $\tilde{k}$ as near $((L)/(d(0, F)))k$ as we please, where $L = \sup\{d(0, x) \,:\, x \in F\}$.

Proof. We prove the result in two steps:

- We first show that $\tilde{T}$ is continuous by distinguishing the cases $x_0 \neq 0$ and $x_0 = 0$.

Let $x_0 \in C \cap (E \backslash \{0\})$ and $\varepsilon > 0$ be arbitrarily fixed. As $x_0 \neq 0$, then $d(0, x_0) > 0$, therefore we can choose $\delta \in \mathbb{R}$ with $0 < \delta < d(0, x_0)$. As a consequence,

$$\tilde{T}_{|_{B(x_0,\delta)}} : B(x_0, \delta) \cap (C \cap E) \longrightarrow C$$

$$x \longmapsto \tilde{T}_{|_{B(x_0,\delta)}}(x) = \frac{1}{\beta(x)} T(\beta(x)x)$$

is continuous, because T, β and $1/\beta$ are continuous too.

Now, let $x_0 = 0$. Since X is a Banach space, in particular, a metric space, we can characterize the continuity property of $\tilde{T}$ at 0 working with convergent sequences. Suppose that $\{x_n\}_{n \in \mathbb{N}} \subset C \cap E$ converges to 0. We can assume that $x_n \neq 0$ for all $n \in \mathbb{N}$, so

$$\|\tilde{T}(x_n)\| = \left\| \frac{1}{\beta(x_n)} T(\beta(x_n)x_n) \right\| = \frac{1}{\beta_{x_n}} \|T(\beta_{x_n} x_n)\|.$$

If $n \in \mathbb{N}$, then $\beta_{x_n} x_n \in C \cap F$. Besides, as $C \cap F$ is bounded and T is a k-set contraction, then $T(F \cap C)$ is necessarily a bounded set in order to consider its measure of noncompactness. Therefore, there exists $K \in \mathbb{R}^+$ such that $\sup\{T(\beta_{x_n} x_n) \,:\, n \in \mathbb{N}\} < K$, so

$$\|\tilde{T}(x_n)\| \leqslant \frac{1}{\beta_{x_n}} K.$$

Using the properties proved in lemma 3.13, it is satisfied that $\{((1)/(\beta_{x_n}))\}_{n \in \mathbb{N}}$ converges to 0, then $\{\|\tilde{T}(x_n)\|\}_{n \in \mathbb{N}}$ also converges to 0. As $\tilde{T}(0) = 0$, the continuity of $\tilde{T}$ at $x_0 = 0$ is proved.

- Secondly, we prove that $\tilde{k}$ can be taken as near $((L)/(d(0, F)))k$ as we please, with $\tilde{T}$ being a $\tilde{k}$-set contraction. Let us consider $A \subset C \cap E$. A is a bounded set since it is a subset of E, which is bounded. Just like before, we distinguish two cases: $\alpha(A) = 0$ and $\alpha(A) \neq 0$.

  If $\alpha(A) = 0$, then $\overline{A}$ is a compact set. Besides, $\tilde{T}$ is continuous, so $\tilde{T}(\overline{A})$ is also a compact set. Using that the measure of noncompactness of a compact set is null and $\tilde{T}(A) \subseteq \tilde{T}(\overline{A})$, then we have

$$\alpha(\tilde{T}(A)) \leqslant \alpha(\tilde{T}(\overline{A})) = k\alpha(A) = 0, \quad \forall\, k \in \mathbb{R},\ k \geqslant 0.$$

  Now, assume that $\alpha(A) \neq 0$. Suppose that $k > 0$ (if this is not true, we can take $\hat{k} > 0$ instead of k). As $k, \alpha(A) > 0$, there exists $d > 0$ such that $((k\alpha(A))/(2)) > d > 0$. We fix arbitrarily some d satisfying these conditions.

  We have just proved that $\tilde{T}$ is a continuous function, so there exists $\delta_d > 0$ such that $\tilde{T}((C \cap E) \cap B(0, \delta_d)) \subset B(0, d)$. Thus, for all $A \subset C \cap E$, it is satisfied that

$$\tilde{T}(A \cap B(0, \delta_d)) \subset B(0, d). \tag{3.1}$$

For any $n \in \mathbb{N}$, define $\varepsilon_n := \delta_d/n$ and, for all $m \in \mathbb{N} \cup \{0\}$, also define

$$A_m^n := \{x \in A \ : \ \|x\| \in [m\varepsilon_n, (m+1)\varepsilon_n]\}.$$

For any $n \in \mathbb{N}$, since A is a bounded set, there exists $N_n \in \mathbb{N}$ such that $\|x\| \leqslant (N_n + 1)\varepsilon_n$ for all $x \in A$. Furthermore, $A \cap B(0, \delta_d) = \{x \in A \ : \ \|x\| < \delta_d = \varepsilon_n n\}$, then

$$A \subset (A \cap B(0, \delta_d)) \cup \left(\bigcup_{m=n}^{N_n} A_m^n \right). \tag{3.2}$$

It follows from (3.1), (3.2) and properties (i), (ii) of proposition 2.8 that

$$\begin{aligned} \alpha(\tilde{T}(A)) &= \alpha\left(\tilde{T}(A \cap B_{\delta_d}) \cup \tilde{T}\left(\bigcup_{m=n}^{N_n} A_m^n \right) \right) \\ &= \alpha(\tilde{T}(A \cap B_{\delta_d}) \cup \tilde{T}(A_n^n) \cup \ldots \cup \tilde{T}(A_{N_n}^n)) \\ &= \max\left\{ \alpha(\tilde{T}(A \cap B_{\delta_d})), \alpha(\tilde{T}(A_n^n)), \ldots, \alpha(\tilde{T}(A_{N_n}^n)) \right\} \\ &\leqslant \max\left\{ \alpha(B_d), \alpha(\tilde{T}(A_n^n)), \ldots, \alpha(\tilde{T}(A_{N_n}^n)) \right\}. \end{aligned} \tag{3.3}$$

For each $m \in \{n, \ldots, N_n\}$, we have $0 \notin A_m^n$, so

$$\begin{aligned} \tilde{T}_{|_{A_m^n}} : A_m^n &\longrightarrow C \\ x &\longmapsto \tilde{T}_{|_{A_m^n}}(x) = \frac{1}{\beta(x)} T(\beta(x)x). \end{aligned}$$

We want to prove that $\tilde{T}_{|_{A_m^n}}$ is a $(1 + 1/m)\left((L)/(d(0, F))\right)k$-set contraction for each $m \in \{n, \ldots, N_n\}$. To this purpose, we make use of some auxiliary mappings that help us to prove the result by using propositions 2.12 and 2.15. Let us fix $m \in \{n, \ldots, N_n\}$ arbitrarily and define:

* $1/\beta_{|_{A_m^n}} : A_m^n \longrightarrow C$, $x \longmapsto 1/\beta_{|_{A_m^n}}(x) := 1/\beta(x)$.
* $\hat{T}_{|_{A_m^n}} : A_m^n \longrightarrow C$, $x \longmapsto \hat{T}(x) = T(\beta(x)x)$, which can be expressed as the composition $T \circ S_m^n$, where $S_m^n : A_m^n \longrightarrow F$, $x \longmapsto S_m^n(x) := \beta(x)x$.

Firstly, since β is continuous and its image is a subset of $[1, +\infty)$, we deduce that $1/\beta_{|_{A_m^n}}$ is a continuous function. Besides, for all $x \in A_m^n$, it is satisfied that $1/\beta_{|_{A_m^n}}(x) = ((d(0, x))/(d(0, \partial x))) \leqslant (((m + 1)\varepsilon_n)/(d(0, \partial x)))$, hence

$$\sup\{1/\beta(x) \ : \ x \in A_m^n\} \leqslant \sup\left\{ \frac{(m+1)\varepsilon_n}{d(0, \partial x)} \ : \ x \in A_m^n \right\} \leqslant \frac{(m+1)\varepsilon_n}{d(0, F)}.$$

Secondly, as T is a k-set contraction, we will conclude a similar behaviour for S_m^n. Let $B \subset A_m^n$, then B is bounded since $A_m^n \subset A$ is also bounded. For all $x \in B \subset A_m^n$, it is satisfied that $\|x\| \in [m\varepsilon_n, (m+1)\varepsilon_n]$, therefore $1/\|x\| \in [((1)/((m+1)\varepsilon_n)), ((1)/(m\varepsilon_n))]$. Consequently, by remark 3.14, for all $x \in B$, $((d(0, \partial(x)))/(\|x\|)) \leqslant ((L)/(m\varepsilon_n))$, then

$$S_m^n(B) = \left\{ \frac{d(0,\partial(x))}{\|x\|} x \,:\, x \in B \right\} \subset co \left\{ \{0\} \cup \frac{L}{m\varepsilon_n} B \right\}. \qquad (3.4)$$

Using (3.4) and the properties (i), (ii), (v), (vi) and (vii) of proposition 2.8, we get

$$\begin{aligned} \alpha(S_m^n(B)) &\leqslant \alpha\left(\{0\} \cup \frac{L}{m\varepsilon_n} B\right) = \max\left\{\alpha(\{0\}), \alpha\left(\frac{L}{m\varepsilon_n} B\right)\right\} \\ &= \alpha\left(\frac{L}{m\varepsilon_n} B\right) = \frac{L}{m\varepsilon_n}\alpha(B). \end{aligned}$$

Besides, S_m^n is a continuous function since β has also this property, hence S_m^n is an $L/(m\varepsilon_n)$-set contraction.

We conclude that $\hat{T}_{|A_m^n}$ is an $L/m\varepsilon_n k$-set contraction by using (i) of proposition 2.12.

Applying proposition 2.15 to the mappings $1/\beta_{|A_m^n}$ and $\hat{T}_{|A_m^n}$, we have that $\tilde{T}_{|A_m^n}$ is a $(1 + 1/m)((L)/(d(0, F)))k$-set contraction.

Then, taking into account (3.3) and (i) of proposition 2.8, we are able to establish the inequalities

$$\begin{aligned} \alpha(\tilde{T}(A)) &\leqslant \max\left\{2d, \left(1+\frac{1}{n}\right)\frac{L}{d(0,F)}k\alpha(A_n^n), \ldots, \left(1+\frac{1}{N_n}\right)\frac{L}{d(0,F)}k\alpha(A_{N_n}^n)\right\} \\ &\leqslant \max\left\{2d, \left(1+\frac{1}{n}\right)\frac{L}{d(0,F)}k\alpha(A), \ldots, \left(1+\frac{1}{N_n}\right)\frac{L}{d(0,F)}k\alpha(A)\right\} \\ &\leqslant \left(1+\frac{1}{n}\right)\frac{L}{d(0,F)}k\alpha(A), \end{aligned}$$

where we have used $2d < k\alpha(A)$ and $L \geqslant d(0, F)$. Since $n \in \mathbb{N}$ was arbitrarily fixed, we have proved that $\tilde{T}$ is a $\tilde{k}$-set contraction, with $\tilde{k} > ((L)/(d(0, F)))k$, $\tilde{k}$ as near $((L)/(d(0, F)))k$ as we please. □

Remark 3.17. It is possible to consider $\tilde{k} = ((L)/(d(0, F)))k$ in lemma 3.16.

Remark 3.18. If we want $\tilde{T}$ to be a $\tilde{k}$-set contraction with $\tilde{k} < 1$, it is needed that

$$\frac{L}{d(0,F)}k < 1,$$

that is, $k < ((d(0, F))/(L))$.

Theorem 3.19. *Let $(X, \|\cdot\|)$ be a Banach space, C a cone in X and E_1, E_2 star convex sets satisfying condition 3.10. Suppose that $T : C \cap (E_2 \backslash \mathring{E}_1) \longrightarrow C$ is a k-set contraction, with $k < ((d(0, F_1))/(L_1))$, and a compression of the cone C according to definition 3.15. Then T has at least one fixed point in $C \cap (E_2 \backslash \mathring{E}_1)$.*

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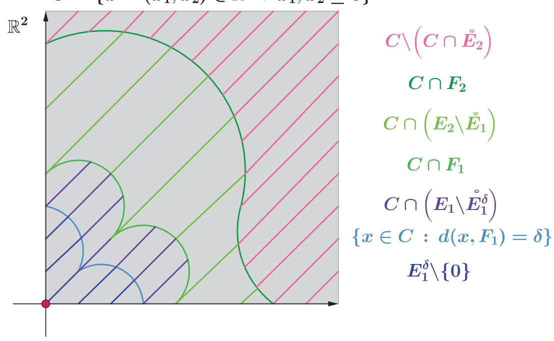

Figure 4. Example of sets used in the definition of $\overline{T}$.

Proof. First, we consider a mapping $\overline{T}$ that is an extension of $T : C \cap (E_2 \backslash \mathring{E}_1) \longrightarrow C$. To that purpose, we should take into account that, since $0 \in E_1 \backslash F_1$, then $d(0, F_1) > 0$. Hence, it is possible to choose $\delta \in \mathbb{R}$ such that $0 < \delta < d(0, F_1)$. Fix arbitrarily some $\delta \in \mathbb{R}$ satisfying such conditions and define the set

$$E_1^\delta := \{x \in E_1 \,:\, d(x, F_1) \geqslant \delta\}.$$

We consider $\overline{T}$ defined by (see figure 4)

$$\overline{T} : C \longrightarrow C$$

$$x \longmapsto \overline{T}(x) := \begin{cases} \delta h, & \|x\| = 0, \\ \dfrac{1}{\beta_1(x)} T(\beta_1(x)x) + \delta h, & x \in C \cap (E_1^\delta \backslash \{0\}), \\ \dfrac{1}{\beta_1(x)} T(\beta_1(x)x) + d(x, F_1)h, & x \in C \cap (E_1 \backslash \mathring{E}_1^\delta), \\ T(x), & x \in C \cap (E_2 \backslash \mathring{E}_1), \\ T(\partial_2^C(x)), & x \in C \backslash (C \cap \mathring{E}_2); \end{cases}$$

where $h \in C$ and

$$\|h\| > (1/\delta)[\sup\{d(0, x) \,:\, x \in C \cap E_1\} + \sup\{\|T(\beta_1(x)x)\| \,:\, x \in C \cap E_1\}].$$

We show that there exists such an h:

- Since $C \cap E_1$ is a bounded set, there exists $\sup\{d(0, x) \,:\, x \in C \cap E_1\} < +\infty$.
- For each $x \in C \cap E_1$, $\beta_1(x)x \in C \cap F_1$ and, as T is a k-set contraction, then the image of the bounded set $C \cap F_1$ is also bounded, hence there exists

$$\sup\{\|T(\beta_1(x)x)\| \,:\, x \in C \cap E_1\} < +\infty.$$

As C is a cone in X, it is possible to choose such an h.

The different expressions of $\overline{T}$ coincide in the intersection of the sets $C \cap (E_1^\delta \backslash \{0\})$, $C \cap (E_1 \backslash \mathring{E}_1^\delta)$, $C \cap (E_2 \backslash \mathring{E}_1)$ and $C \backslash (C \cap \mathring{E}_2)$, so $\overline{T}$ is well defined. Moreover, $\overline{T}$ is continuous in $C \backslash \{0\}$ and the proof of the continuity of this mapping at $x = 0$ is quite similar to the one given in the lemma 3.16.

Our purpose is to apply proposition 3.1, so we need to select a bounded, closed and convex set such that the restriction of $\overline{T}$ to this set is a $\overline{k}$-set contraction with $\overline{k} < 1$. In this way, we are going to prove some properties:

(a) There exists $R_1 \in \mathbb{R}^+$ such that $R_1 = \sup\{\|\overline{T}(x)\| \,:\, x \in C \cap E_2\}$.
We complete the proof by distinguishing four cases, because the definition of $\overline{T}$ depends on the subset of $C \cap E_2$ which is considered:
- If $x = 0$, $\|\overline{T}(x)\| = \delta \|h\|$.
- If $x \in C \cap (E_1^\delta \backslash \{0\})$, $\|\overline{T}(x)\| = \|((1)/(\beta_1(x)))T(\beta_1(x)x) + \delta h\|$. Using the triangular inequality and that, if $\beta_1(x) \in [1, +\infty)$, then $((1)/(\beta_1(x))) \in (0, 1]$, we obtain $\|\overline{T}(x)\| \leqslant \|T(\beta_1(x)x)\| + \delta\|h\|$. As $C \cap F_1 \subset C \cap (E_2 \backslash \mathring{E}_1)$ is a bounded set and $T : C \cap (E_2 \backslash \mathring{E}_1) \longrightarrow C$ is a k-set contraction, then $T(C \cap F_1)$ is also bounded, therefore there exists $M_1 \in \mathbb{R}^+$ such that $\|T(x)\| \leqslant M_1$ for all $x \in C \cap F_1$. Besides, for all $x \in C \cap (E_1^\delta \backslash \{0\})$, $\beta_1(x)x \in C \cap F_1$, so we conclude

$$\|\overline{T}(x)\| \leqslant M_1 + \delta\|h\|, \quad \forall\, x \in C \cap (E_1^\delta \backslash \{0\}).$$

- If $x \in C \cap (E_1 \backslash \mathring{E}_1^\delta)$, then

$$\|\overline{T}(x)\| \leqslant M_1 + \delta\|h\|$$

by the existence of M_1 proved before.
- If $x \in C \cap (E_2 \backslash \mathring{E}_1)$, as this is a bounded set and T is a k-set contraction, there exists $M_2 \in \mathbb{R}^+$ such that $\|\overline{T}(x)\| = \|T(x)\| \leqslant M_2$, for all $x \in C \cap (E_2 \backslash \mathring{E}_1)$.

Thus, we conclude that there exists $R_1 \in \mathbb{R}^+$ with the above property.

(b) There exists $R_2 \in \mathbb{R}^+$ such that $R_2 = \sup\{d(0, x) \,:\, x \in C \cap E_2\}$, since $C \cap E_2$ is a bounded set.

Now, let $R = \max\{R_1, R_2\} \in \mathbb{R}^+$ and $B_R = \{x \in C \,:\, d(0, x) \leqslant R\}$, then it is obtained that $\overline{T}(B_R) \subseteq B_R$.

We want to prove that $\overline{T}_{|B_R} : B_R \longrightarrow B_R$ satisfies the hypotheses of proposition 3.1. Indeed:

- B_R is a bounded, closed and convex set, because it is the intersection of the closed and convex set C with the bounded, closed and convex set $\overline{B}(0, R) = \{x \in X \,:\, d(0, x) \leqslant R\}$.
- $\overline{T}_{|B_R}$ is clearly continuous.

- $\overline{T}_{|B_R}$ is a $\overline{k}$-set contraction with $\overline{k} < 1$. To prove it, we define another helpful auxiliary mappings:

$$T_1 : B_R \cap E_1 \longrightarrow B_R \quad \text{is given by}$$

$$T_1(x) := \begin{cases} \delta h, & x = 0; \\ \dfrac{1}{\beta_1(x)} T(\beta_1(x)x) + \delta h, & x \in B_R \cap \left(E_1^\delta \backslash \{0\}\right); \\ \dfrac{1}{\beta_1(x)} T(\beta_1(x)x) + d(x, F_1)h, & x \in B_R \cap \left(E_1 \backslash \mathring{E}_1^\delta\right). \end{cases}$$

$$T_2 : B_R \backslash (B_R \cap \mathring{E}_1) \longrightarrow B_R \quad \text{is given by}$$

$$T_2(x) := \begin{cases} T(x), & x \in B_R \cap (E_2 \backslash \mathring{E}_1), \\ T(\partial_2^C(x)), & x \in B_R \backslash (B_R \cap \mathring{E}_2). \end{cases}$$

It is possible to express T_1 as the sum of two mappings T_1^1 and T_1^2. The mapping

$$T_1^1 : B_R \cap E_1 \longrightarrow B_R$$
$$x \longmapsto T_1^1(x) := \begin{cases} \delta h, & x \in B_R \cap E_1^\delta, \\ d(x, F_1)h, & x \in B_R \cap (E_1 \backslash \mathring{E}_1^\delta), \end{cases}$$

is a 0-set contraction. In fact, let $A \subset B_R \cap E_1$, then A is bounded and $T_1^1(A) = T_1^1(A \cap (B_R \cap E_1^\delta)) \cup T_1^1(A \cap (B_R \cap (E_1 \backslash \mathring{E}_1^\delta)))$. As

$$T_1^1(A \cap (B_R \cap (E_1 \backslash E_1^\delta))) \subset \overline{\text{co}}\{\{0\} \cup \{\delta h\}\},$$

by using properties (i), (ii), (iii), (vi) and (vii) of proposition 2.8, we can conclude

$$\alpha(T_1^1(A)) \leqslant \max\{\alpha(\{\delta h\}), \alpha(\text{co}\{\{0\} \cup \{\delta h\}\})\} = 0.$$

Furthermore, the mapping

$$T_1^2 : B_R \cap E_1 \longrightarrow B_R$$
$$x \longmapsto T_1^2(x) := \begin{cases} 0, & x = 0, \\ \dfrac{1}{\beta_1(x)} T(\beta_1(x)x), & x \neq 0, \end{cases}$$

is a k_1^2-set contraction with $k_1^2 < 1$, because it is the restriction to $B_R \cap E_1$ of $\tilde{T}$ in lemma 3.16 with $E = E_1$ and $F = F_1$.

Using (ii) of proposition 2.12, then $T_1 = T_1^1 + T_1^2$ is a $k_1 = 0 + k_1^2$-set contraction with $k_1 < 1$.

Besides, T_2 is a k_2-set contraction with $k_2 = k$, because it can be written as the composition $T \circ S$, where $S : B_R \backslash (B_R \cap \mathring{E}_1) \longrightarrow B_R \cap (E_2 \backslash \mathring{E}_1)$ is given by

$$S(x) := \begin{cases} x, & x \in B_R \cap (E_2 \backslash \mathring{E}_1), \\ \partial_2^C(x) = \beta_2 \left(\dfrac{d(0, F_2)}{||x||} x \right) \dfrac{d(0, F_2)}{||x||} x, & x \in B_R \backslash (B_R \cap \mathring{E}_2), \end{cases}$$

is a 1-set contraction. Therefore, hypothesis (i) of proposition 2.12 is satisfied and, as a consequence, T_2 is a k-set contraction. We now prove that S is a 1-set contraction. For this, let us consider $\lambda : B_R \backslash (B_R \cap \mathring{E}_1) \longrightarrow \mathbb{R}^+$ given by

$$\lambda(x) := \begin{cases} 1, & x \in B_R \cap (E_2 \backslash \mathring{E}_1), \\ \beta_2 \left(\dfrac{d(0, F_2)}{||x||} x \right) \dfrac{d(0, F_2)}{||x||}, & x \in B_R \backslash (B_R \cap \mathring{E}_2), \end{cases}$$

which is a continuous function and satisfies

$$\sup\{\lambda(x) \; : \; x \in B_R \cap (B_R \backslash \mathring{E}_1)\} \leqslant 1.$$

Hence, by using proposition 2.15, we conclude that S is a 1-set contraction since the identity also fulfills this property.

Applying corollary 2.14 to T_1 and T_2, we get that $\overline{T}_{|B_R}$ is a $\overline{k}$-set contraction with $\overline{k} = \max\{k_1, k\} < 1$.

Therefore, the hypotheses of proposition 3.1 are satisfied and $\overline{T}_{|B_R}$ has at least one fixed point $\overline{x}$. We only have to prove that $\overline{x} \in C \cap (E_2 \backslash \mathring{E}_1)$. To that purpose, we consider four cases:

Case 1: Suppose that $\overline{x} = 0$.
So $\overline{T}(0) = 0$, then $\delta\|h\| = 0$ and this is not possible since $\delta > 0, \|h\| > 0$.

Case 2: Assume that $\overline{x} \in B_R \cap (E_1^\delta \backslash \{0\})$.
Consequently, $\overline{T}(\overline{x}) = ((1)/(\beta_1(\overline{x})))T(\beta_1(\overline{x})\overline{x}) + \delta h = \overline{x}$, so

$$\|h\| \leqslant \frac{1}{\delta}\|\overline{x}\| + \frac{1}{\delta\beta_1(\overline{x})}\|T(\beta_1(\overline{x})\overline{x})\|$$

and it is a contradiction with the selection of h in the definition of the mapping $\overline{T} : C \longrightarrow C$.

Case 3: Let $\overline{x} \in B_R \cap (E_1 \backslash E_1^\delta)$.
Since $\overline{x}$ is a fixed point, then $\overline{T}(\overline{x}) = ((1)/(\beta_1(\overline{x})))T(\beta_1(\overline{x})\overline{x}) + d(\overline{x}, F_1)h = \overline{x}$, so

$$\overline{x} - \frac{1}{\beta_1(\overline{x})} T(\beta_1(\overline{x})\overline{x}) = d(\overline{x}, F_1)h \in C,$$

due to $d(\overline{x}, F_1) \geqslant 0$. Moreover, $\beta_1(\overline{x}) \in [1, +\infty)$, thus

$$\beta_1(\overline{x})\overline{x} - T(\beta_1(\overline{x})\overline{x}) \in C, \quad \text{where } \beta_1(\overline{x})\overline{x} \in F_1 \cap C,$$

which contradicts the hypothesis $(C1)$ for T a compression of the cone C.

Case 4: Suppose that $\overline{x} \in B_R \backslash (B_R \cap E_2)$.
Let us define $y_{\overline{x}} = ((d(0, F_2))/(||\overline{x}||))\overline{x}$, then

$$\begin{aligned}\overline{T}(\overline{x}) &= T(\partial_2^C(\overline{x})) = T(\partial_2(y_{\overline{x}})) \\ &= T(\beta_2(y_{\overline{x}})y_{\overline{x}}) = T\left(\frac{d(0, \partial_2(y_{\overline{x}}))}{\|y_{\overline{x}}\|}\frac{d(0, F_2)}{\|\overline{x}\|}\overline{x}\right).\end{aligned}$$

As $\|y_{\overline{x}}\| = ((d(0, F_2))/(\|\overline{x}\|))\|\overline{x}\| = d(0, F_2)$, then $\overline{T}(\overline{x}) = T(((d(0, \partial_2 (y_{\overline{x}})))/(\|\overline{x}\|))\overline{x})$. Take $\varepsilon = ((\|\overline{x}\|)/(d(0, \partial_2(y_{\overline{x}})))) - 1$, then $\varepsilon > 0$ since $((\|\overline{x}\|)/(d(0, \partial_2(y_{\overline{x}})))) > 1$. We can express $\overline{x}$ as $(1 + \varepsilon)((d(0, \partial_2(y_{\overline{x}})))/ (\|\overline{x}\|))\overline{x}$, then

$$\overline{T}(\overline{x}) = T\left(\frac{d(0, \partial_2(y_{\overline{x}}))}{\|\overline{x}\|}\overline{x}\right) = (1 + \varepsilon)\frac{d(0, \partial_2(y_{\overline{x}}))}{\|\overline{x}\|}\overline{x}.$$

Due to $((d(0, \partial_2(y_{\overline{x}})))/(\|\overline{x}\|))\overline{x} \in C \cap F_2$ and $T(((d(0, \partial_2(y_{\overline{x}})))/(\|\overline{x}\|))\overline{x}) - (1 + \varepsilon)((d(0, \partial_2(y_{\overline{x}})))/(\|\overline{x}\|))\overline{x} = 0 \in C$, we obtain a contradiction with the hypothesis $(C2)$ for T a compression of the cone C.

Finally, the fixed point of $\overline{T}$ belongs to $C \cap (E_2 \backslash \mathring{E}_1)$, since $\overline{T}$ and T coincide on this set, then we conclude that T has a fixed point in the mentioned set. □

Potter, in [**9**], asserts that, using similar techniques as the ones in the proof of theorem 3.8, it is possible to generalize the fixed point result due to Krasnosel'skii by considering hypothesis (1.2) instead of (1.1), but the proof is not given. On the other hand, Các and Gatica prove it in [**3**] following the steps of Krasnosel'skii to prove the mentioned classical result with hypothesis (1.2). In the future, we will consider the case of expansive mappings working with sets which satisfy condition 3.10.

3.1. Admissible sets defined by functionals

The above-mentioned fixed point theorem due to Potter provides the existence of fixed points in certain subsets of a Banach space $(X, ||\cdot||)$. These subsets are determined by the norm $||\cdot||$, for instance, given $r \in \mathbb{R}$, $r > 0$ and C a cone in X, we define

$$B_r := C \cap \{x \in X \,:\, ||x|| \leqslant r\}, \quad S_r := C \cap \{x \in X \,:\, ||x|| = r\}.$$

This result guarantees, for some $r, R \in \mathbb{R}$, $0 < r < R$, and some particular mapping T, that there exists at least a fixed point of T in $B_R \backslash \mathring{B}_r = C \cap \{x \in X \,:\, r \leqslant ||x|| \leqslant R\}$.

We have just proved a generalization of this result working with some particular star convex sets. Now, our aim is to determine conditions over a functional $\varphi : X \longrightarrow [0, +\infty)$ such that, for $r \in \mathbb{R}$, $r > 0$, the sets

$$E_r := \{x \in X \,:\, \varphi(x) \leqslant r\}, \quad F_r := \{x \in X \,:\, \varphi(x) = r\}$$

satisfy the condition 3.10, and derive some consequences as corollary of theorem 3.19.

REMARK 3.20. If $\varphi = ||\cdot||$, then $B_r = C \cap E_r$ and $S_r = C \cap F_r$. This allows to consider functionals with weaker properties in comparison with the norm.

THEOREM 3.21. *Let* $(X, \|\cdot\|)$ *be a Banach space,* $r \in \mathbb{R}, r > 0$ *and* $\varphi : X \longrightarrow [0, +\infty)$ *satisfying:*

(F1) *If* $x \in X$ *is such that* $\varphi(x) \leqslant r$*, then* $\varphi(\lambda x) \leqslant r$ *for all* $\lambda \in [0, 1]$.

(F2) φ *is a continuous functional.*

(F3) *For all* $x \in E_r$, $\varphi(x) = 0 \Leftrightarrow x = 0$.

(F4) $\varphi(\lambda x) = \lambda\varphi(x)$ *for all* $\lambda \in (0, +\infty), x \in E_r$.

(F5) *There exists* $m \in \mathbb{R}, m > 0$ *such that* $m\|x\| \leqslant \varphi(x)$ *for all* $x \in X$ *with* $\|x\| > \varphi(x)$*, or* $\liminf_{\|x\|\to+\infty} \varphi(x) > r$.

Under these assumptions, E_r *and* F_r *satisfy condition* 3.10.

Proof. We proceed step by step, that is, we prove each one of the properties required to E_r and F_r by using the appropriate hypotheses of φ. Indeed:

- $(F1)$ is equivalent to E_r being a star convex set.
- $(F2)$ implies that E_r is closed and $\varphi^{-1}([0, r))$ is open in X. Indeed, we can write $E_r = \varphi^{-1}([0, r])$. As $[0, r]$ is a closed subset of $([0, +\infty), |\cdot|)$, where $|\cdot|$ is the absolute value for real numbers, then E_r is closed since it is the preimage of a closed set by a continuous function. Besides, we can assert that $\varphi^{-1}([0, r))$ is open in X since $[0, r)$ is an open subset of $[0, +\infty)$.
- (F_3) implies that $0 \notin F_r$, because $\varphi(x) = r > 0$ for all $x \in F_r$.
- By hypothesis (F_4), we prove two conditions over the sets E_r and F_r. First, we show that F_r is the boundary of E_r. We have just proved that E_r is closed and $E_r \backslash F_r = \varphi^{-1}([0, r))$ is open, so the boundary of E_r is a subset of F_r. Let us consider $y \in F_r$ arbitrarily, we want to prove that y is a boundary point of E_r. As $y \in F_r \subset E_r$, we must prove that, for all $\varepsilon > 0$, $B(y, \varepsilon) \cap (X \backslash E_r) \neq \emptyset$. We take $z = (1 + ((\varepsilon)/(2\|y\|)))y \in B(y, \varepsilon)$, then it is satisfied

  $$\varphi(z) = \left(1 + \frac{\varepsilon}{2\|y\|}\right)\varphi(y),$$

  and, as $(1 + ((\varepsilon)/(2\|y\|))) > 1$, we can conclude that $\varphi(z) > \varphi(y) = r$. Therefore, $z \in X \backslash E_r$ and y is a boundary point of E_r. Secondly, there exists a mapping $\partial : E_r \backslash \{0\} \longrightarrow F_r$, $x \longmapsto \partial(x) := ((r)/(\varphi(x)))x$, which satisfies the desired conditions for mapping ∂. The function ∂ is clearly well-defined and continuous, by using $(F2)$ and $(F3)$. Let $x \in E_r \backslash \{0\}$, $\varphi(\partial(x)) = \varphi(((r)/(\varphi(x)))x) = r$, then $\partial(x) \in F_r$. Moreover, if $x \in F_r$, then $\varphi(x) = r$ and,

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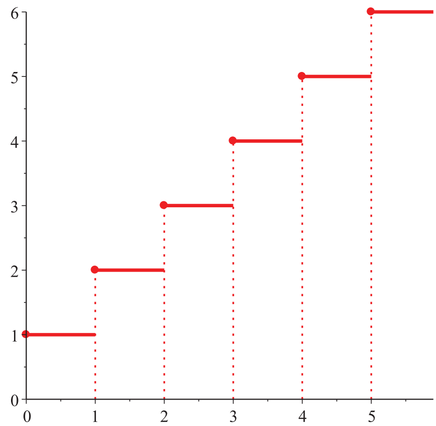

Figure 5. Graph of function φ, which is upper semicontinuous.

therefore, $\partial(x) = x$. Furthermore, by $(F4)$, $\partial(\lambda x) = \partial(x)$ for all $x \in E_r$ and $\lambda \in (0, 1]$.

- Finally, $(F5)$ implies that E_r is a bounded set.

□

Remark 3.22. We know that the norm is a continuous function, so we think if it is possible to relax this condition required to the functional. Is it enough to work with an upper or lower semicontinuous function φ? Next, we prove that this assumption is not enough.

Assume that φ is upper semicontinuous. We justify that $E_r \backslash F_r$ is open and E_r is not necessarily closed:

- Let $r \in \mathbb{R}$, $r > 0$, then $\varphi^{-1}([0, r))$ is open. We prove it by contradiction, so suppose that $\varphi^{-1}([0, r))$ is not open, then there exists $y \in \varphi^{-1}([0, r))$ such that y is not an interior point. Therefore, for all $\delta \in \mathbb{R}$, $\delta > 0$, it holds that

$$B(y, \delta) \nsubseteq \varphi^{-1}([0, r)). \tag{3.5}$$

  As $\varphi(y) < r$, we can take $\varepsilon > 0$ such that $\varphi(y) + \varepsilon < r$. Besides, by using that φ is upper semicontinuous, there exists δ_ε^y such that, for all $x \in B(y, \delta_\varepsilon^y)$, $0 \leqslant \varphi(x) < \varphi(y) + \varepsilon < r$. As a consequence, $B(y, \delta_\varepsilon^y) \subset \varphi^{-1}([0, r))$ and it contradicts (3.5). Therefore, we conclude that $\varphi^{-1}([0, r))$ is an open set.

- However, $\varphi^{-1}([0, r])$ is not always closed. As an example, we work with the function

$$\begin{aligned} \varphi : [0, +\infty) &\longrightarrow [0, +\infty) \\ x &\longmapsto \varphi(x) = \lfloor x \rfloor + 1, \end{aligned}$$

  where $\lfloor \cdot \rfloor$ denotes the floor function (see figure 5).

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First of all, we prove that φ is an upper semicontinuous function. Let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$ and $y \in [0, +\infty)$ be arbitrarily fixed. We take $\delta_y = ((d(y, \varphi(y)))/(2)) > 0$, so

$$\varphi(B_{[0,+\infty)}(y, \delta_y)) \subset [0, \varphi(y + \delta_y)) \subset [0, \varphi(y)) \subset [0, \varphi(y) + \varepsilon).$$

Then, φ is upper semicontinuous.

Secondly, for each $r \in \mathbb{R}$, $r > 0$, it is satisfied that: If $r \in (0, 1)$, then $\varphi^{-1}([0, r]) = \emptyset$. If $r \in [1, +\infty)$, then $\varphi^{-1}([0, r]) = [0, \lfloor r \rfloor)$, which is an open set of $([0, +\infty), |\cdot|)$.

Assume that φ is lower semicontinuous. We justify that E_r is closed and $E_r \backslash F_r$ is not necessarily open.

- Let $r \in \mathbb{R}$, $r > 0$, then $\varphi^{-1}([0, r])$ is closed. As $\varphi^{-1}([0, r]) = X \backslash \varphi^{-1}((r, +\infty))$, we prove that $\varphi^{-1}((r, +\infty))$ is an open subset of $[0, +\infty)$ by contradiction. Let us assume that $\varphi^{-1}((r, +\infty))$ is not open, so there exists at least a point $y \in \varphi^{-1}(r, +\infty)$ such that for each $\delta \in \mathbb{R}$, $\delta > 0$, it is fulfilled

$$B(y, \delta) \nsubseteq \varphi^{-1}((r, +\infty)). \tag{3.6}$$

  Now, as $\varphi(y) > r$, we can take $\varepsilon > 0$ such that $\varphi(y) - \varepsilon > r$. Besides, by using that φ is lower semicontinuous, there exists $\delta_\varepsilon^y > 0$ such that, for all $x \in B(y, \delta_\varepsilon^y)$, $r < \varphi(y) - \varepsilon < \varphi(x) < +\infty$. As a consequence, $B(y, \delta_\varepsilon^y) \subset \varphi^{-1}((r, +\infty))$, which contradicts (3.6).

- Nevertheless, $\varphi^{-1}([0, r))$ is not always open. For example, we consider the function

$$\varphi : [0, +\infty) \longrightarrow [0, +\infty)$$
$$x \longmapsto \varphi(x) := \begin{cases} \lfloor x \rfloor, & x \notin \mathbb{N}; \\ x - 1, & x \in \mathbb{N}; \end{cases}$$

  where $\lfloor \cdot \rfloor$ is the floor function (see figure 6).

  First, we show that φ is lower semicontinuous. Let $y = 0$ and $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, since

$$\varphi([0, \delta)) \subset [0, +\infty) = [\varphi(0), +\infty) \subset (\varphi(0) - \varepsilon, +\infty), \quad \forall\, \delta \in \mathbb{R},\ \delta > 0,$$

  φ is clearly lower semicontinuous at 0. Let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, and $y \in (0, +\infty)$ be arbitrarily fixed. We take $\delta_y = ((d(y, \varphi(y)))/(2)) > 0$ and it is possible to prove that

$$\varphi(B_{[0,+\infty)}(y, \delta_y)) \subset (\varphi(y) - \varepsilon, +\infty).$$

  As a consequence, we can assert that φ is lower semicontinuous.

  Next, let $r \in \mathbb{R}$, $r > 0$, it is satisfied that: If $r \in \mathbb{N}$, then $\varphi^{-1}([0, r)) = [0, r]$, which is a closed subset of $[0, +\infty)$. If $r \notin \mathbb{N}$, then $\varphi^{-1}([0, r)) = [0, \lfloor r \rfloor + 1]$, which is a closed subset of $[0, +\infty)$.

To sum up, it is not enough to work with an upper or lower semicontinuous function.

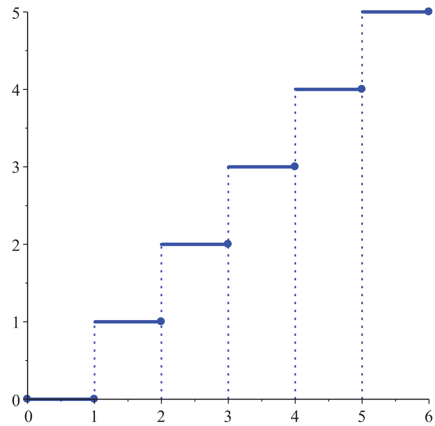

Figure 6. Graph of function φ, which is lower semicontinuous.

Now, we rewrite theorem 3.19 by using functionals.

THEOREM 3.23. *Let* $(X, \|\cdot\|)$ *be a Banach space,* C *a cone in* X *and* $r, R \in \mathbb{R}$*, with* $0 < r < R$*. Assume that* $\varphi, \psi : X \longrightarrow [0, +\infty)$ *are functionals satisfying the hypotheses* $(F1) - (F5)$ *of theorem 3.21, for* r, R *respectively, and* $\psi(x) < (R/r)\varphi(x)$ *for all* $x \in X$ *such that* $\varphi(x) \leqslant r$*. We consider* $E_1 := \{x \in X : \varphi(x) \leqslant r\}$ *and* $E_2 := \{x \in X : \psi(x) \leqslant R\}$*. Then* E_1 *and* E_2 *fulfill condition 3.10,* $0 \in E_1 \subset E_2$ *and* $F_1 \cap F_2 = \emptyset$*. Moreover, suppose that* $T : C \cap (E_2 \backslash \mathring{E}_1) \longrightarrow C$ *is a* k*-set contraction, with* $k < ((d(0, F_1))/(L_1))$ *and a compression of the cone* C *according to definition 3.15. Then* T *has at least one fixed point in* $C \cap (E_2 \backslash \mathring{E}_1)$*.*

Proof. First of all, by theorem 3.21, since φ, ψ satisfy the hypotheses $(F1) - (F5)$, then E_i, F_i fulfill the condition 3.10, for $i = 1, 2$.

Secondly, we check that $0 \in E_1 \subset E_2$ and $F_1 \cap F_2 = \emptyset$ in order to deal with the concept of a compression of the cone C. Since $\psi(x) < (R/r)\varphi(x)$ for all $x \in E_1$:

- Let $y \in E_1$, $\psi(y) < (R/r)\varphi(y) \leqslant R$, then $y \in E_2$.
- Let $y \in F_1$, $\psi(y) < (R/r)\varphi(y) = R$, then $y \notin F_2$.

Therefore, the hypotheses of theorem 3.19 are satisfied and, as a consequence, T has at least a fixed point in $C \cap (E_2 \backslash \mathring{E}_1)$. □

REMARK 3.24. The hypothesis $\psi(x) < (R/r)\varphi(x)$, for all $x \in X$ such that $\varphi(x) \leqslant r$, of theorem 3.23, can be replaced by

$$\psi(x) \leqslant \frac{R}{r}\varphi(x), \quad \text{for all } x \in X \text{ such that } \varphi(x) < r,$$
$$\psi(x) < \frac{R}{r}\varphi(x), \quad \text{for all } x \in X \text{ such that } \varphi(x) = r.$$

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4. Application

In this last section, we consider an ordinary differential equation subject to boundary conditions and apply theorems 3.19 and 3.21 to study the existence of solutions localized in a region determined by two star convex sets defined by functionals. The same problem was considered by Avery, Henderson and O'Regan in [**2**], where the authors applied other different fixed point theorems.

To proceed, let us consider the second-order nonlinear boundary value problem

$$\begin{aligned} &x''(t) + f(x(t)) = 0, \quad t \in [0,1], \\ &x(0) = 0 = x(1), \end{aligned} \tag{4.1}$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function such that $f(z) \geqslant 0$ for all $z \geqslant 0$. We seek some solution $x \in C^2([0,1],\mathbb{R})$ to (4.1). We prove that there exists at least one solution that satisfies some properties as concavity and symmetry with respect to $t = 1/2$.

First of all, we obtain a mapping T whose fixed points correspond to the solutions of the boundary value problem. This mapping is given as follows:

$$\begin{aligned} T : \mathcal{C}([0,1],\mathbb{R}) &\longrightarrow \mathcal{C}([0,1],\mathbb{R}) \\ x &\longmapsto T(x) := Tx : [0,1] \longrightarrow \mathbb{R} \\ & \qquad t \longmapsto [Tx](t) := \int_0^1 G(t,s) f(x(s)) \mathrm{d}s, \end{aligned}$$

where G is the function:

$$\begin{aligned} G : [0,1] \times [0,1] &\longrightarrow [0,1] \\ (t,s) &\longmapsto G(t,s) := \begin{cases} t(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \\ s(1-t), & 0 \leqslant s \leqslant t \leqslant 1. \end{cases} \end{aligned}$$

Now, we apply theorem 3.19 to the mapping T. We consider the Banach space $(\mathcal{C}([0,1],\mathbb{R}), ||\cdot||)$, where $||\cdot||$ is the norm defined in example 3.4. However, we take a cone in $\mathcal{C}([0,1],\mathbb{R})$ different from the one in this example, as follows:

$$C := \left\{ x \in \mathcal{C}([0,1],\mathbb{R}) \ : \ x \geqslant 0, \ x \text{ is concave and symmetric with respect to } \frac{1}{2} \right\}.$$

Besides, we require that the function f which defines the second-order differential equation in (4.1) satisfies the following conditions.

CONDITION 4.1. *There exist real numbers* $0 < r < R$ *and* $0 < \alpha < ((R)/(r)) - 1$ *such that:*

(H$_1$) $f(z) > 8r$, *for all* $z \in [0, r]$.

(H$_2$) $f(z) \leqslant ((8R)/((1 + \alpha)))$, *for all* $z \in [0, R/\alpha]$.

Given real numbers $0 < r < R$ and $0 < \alpha < ((R)/(r)) - 1$ such that the hypotheses (H_1) and (H_2) of condition 4.1 are fulfilled, and $0 \leqslant \delta < 1/2$ fixed, we consider the following sets

$$E_1 := \left\{ x \in \mathcal{C}([0,1],\mathbb{R}) \,:\, ||x|| = \max_{t\in[0,1]} |x(t)| \leqslant r \right\},$$
$$E_2 := \left\{ x \in \mathcal{C}([0,1],\mathbb{R}) \,:\, \min_{t\in[1/2-\delta,1/2+\delta]} |x(t)| + \alpha\|x\| \leqslant R \right\}.$$

We assert that E_1 and E_2 are star convex sets fulfilling condition 3.10. E_1 is clearly under the required hypotheses. On the other hand, E_2 is a set defined by the functional

$$\varphi : \mathcal{C}([0,1],\mathbb{R}) \longrightarrow [0,+\infty)$$
$$x \longmapsto \varphi(x) := \min_{t\in[1/2-\delta,1/2+\delta]} |x(t)| + \alpha\|x\|.$$

This functional satisfies the hypotheses required in theorem 3.21. Therefore, E_2 is a star convex set fulfilling condition 3.10. Besides, the following properties are satisfied:

- ⋆ Since $\alpha < R/r - 1$, then $E_1 \subset E_2$ and $F_1 \cap F_2 = \emptyset$.
- ⋆ If $x \in C \cap F_1$, then $\|x\| = \max_{t\in[0,1]} |x(t)| = x(1/2) = r$.
- ⋆ If $x \in C \cap F_2$, then $\min_{t\in[1/2-\delta,1/2+\delta]} |x(t)| + \alpha\|x\| = x(1/2-\delta) + \alpha x(1/2) = R$.

The next result proves the existence of a nonzero, concave and symmetric with respect to $1/2$ solution to the boundary value problem (4.1).

THEOREM 4.2. *For all continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(z) \geqslant 0$ for all $z \geqslant 0$ and fulfilling condition 4.1, the boundary value problem (4.1) has at least a solution in $C \cap (E_2 \backslash \mathring{E}_1)$.*

Proof. We prove the result by checking the hypotheses of theorem 3.19.

First, $(\mathcal{C}([0,1],\mathbb{R}), ||\cdot||)$ is a Banach space, C a cone in $\mathcal{C}([0,1],\mathbb{R})$ and E_1, E_2 star convex sets satisfying condition 3.10.

Secondly, it is possible to prove that $T : \mathcal{C}([0,1],\mathbb{R}) \longrightarrow \mathcal{C}([0,1],\mathbb{R})$ is a continuous and compact mapping.

Then, we have to prove the following properties:

(i) $T(C \cap (E_2 \backslash \mathring{E}_1)) \subset C$.

(ii) $x - Tx \notin C$, for all $x \in C \cap F_1$.

(iii) For all $\varepsilon > 0$ and $x \in C \cap F_2$, $Tx - (1+\varepsilon)x \notin C$.

We begin by proving (i). Let $x \in C \cap (E_2 \backslash \mathring{E}_1)$, we show that $T(x) \in C$. To that purpose, we check the following items:

- For each $t \in [0,1]$, since $G(t,s) \in [0,1]$ for all $(t,s) \in [0,1] \times [0,1]$ and $f(x(s)) \geqslant 0$ for all $s \in [0,1]$, then

$$[Tx](t) = \int_0^1 G(t,s) f(x(s)) \mathrm{d}s \geqslant 0.$$

- By construction of $T(x)$, for all $t \in (0,1)$, it is satisfied that $[Tx]''(t) = -f(x(t)) \leqslant 0$, so Tx is a concave function.

- Tx is symmetric with respect to $1/2$, because

$$[Tx](t) = [Tx](1-t), \quad \text{for all } t \in [0,1].$$

  In fact, for all $t \in [0,1]$, it holds that

$$[Tx](t) = \int_0^1 G(t,s) f(x(s)) \mathrm{d}s = \int_0^t s(1-t) f(x(s)) \mathrm{d}s + \int_t^1 t(1-s) f(x(s)) \mathrm{d}s.$$

  Making the change of variable $s = 1-u$, we obtain

$$[Tx](t) = \int_{1-t}^1 (1-u)(1-t) f(x(1-u)) \mathrm{d}u + \int_0^{1-t} tu f(x(1-u)) \mathrm{d}u.$$

  Since $x \in C$, x is symmetric with respect to $1/2$, then

$$\begin{aligned} [Tx](t) &= \int_0^{1-t} u[1-(1-t)] f(x(u)) \mathrm{d}u + \int_{1-t}^1 (1-t)(1-u) f(x(u)) \mathrm{d}u \\ &= \int_0^1 G(1-t,u) f(x(u)) \mathrm{d}u = [Tx](1-t). \end{aligned}$$

Therefore, property (i) has been proved. In fact, we have just proved $Tx \in C$ for all $x \in C$.

By hypothesis (H_1), we prove that condition (ii) is fulfilled. In fact, let $x \in C \cap F_1$, as $x \geqslant 0$ and $||x|| = r$, then $0 \leqslant x(s) \leqslant r$ for all $s \in [0,1]$. Let us consider $t = 1/2$, then

$$x\left(\frac{1}{2}\right) - [Tx]\left(\frac{1}{2}\right) = r - \int_0^1 G\left(\frac{1}{2}, s\right) f(x(s)) \mathrm{d}s.$$

Now, by using (H_1), we obtain that $f(x(s)) > 8r$ for all $s \in [0,1]$, then

$$x\left(\frac{1}{2}\right) - [Tx]\left(\frac{1}{2}\right) < r - 8r \int_0^1 G\left(\frac{1}{2}, s\right) \mathrm{d}s = 0,$$

and, therefore, we conclude that $x - Tx \notin C$, because $x - Tx \geqslant 0$ is not satisfied. As $x \in C \cap F_1$ was arbitrarily fixed, then (ii) has been proved.

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By hypothesis (H_2), we prove that property (iii) is satisfied. Indeed, let $\varepsilon > 0$ and $x \in C \cap F_2$ be arbitrarily fixed. Since $x \in C \cap F_2$, then:

- The property $R = \min_{t\in[1/2-\delta,1/2+\delta]} |x(t)| + \alpha\|x\| = x(1/2-\delta) + \alpha x(1/2)$ implies that $x(1/2) = ((R - x(1/2-\delta))/(\alpha)) \leqslant R/\alpha$.
- Since $\|x\| = \max_{t\in[0,1]} x(t) = x(1/2)$, then $0 \leqslant x(s) \leqslant x(1/2) \leqslant R/\alpha$, $\forall\ s \in [0,1]$.
- As $x(1/2-\delta) \leqslant x(1/2)$, then $R = x(1/2-\delta) + \alpha x(1/2) \leqslant (1+\alpha)x(1/2)$ and, hence, $x(1/2) \geqslant ((R)/(1+\alpha))$ and, finally $(1+\varepsilon)x(1/2) > ((R)/(1+\alpha))$.

Therefore, by using hypothesis (H_2),

$$\begin{aligned}[][Tx]\left(\frac{1}{2}\right) - (1+\varepsilon)x\left(\frac{1}{2}\right) &= \int_0^1 G\left(\frac{1}{2},s\right) f(x(s))\mathrm{d}s - (1+\varepsilon)x\left(\frac{1}{2}\right)\\ &\leqslant \frac{8R}{1+\alpha}\int_0^1 G\left(\frac{1}{2},s\right)\mathrm{d}s - (1+\varepsilon)x\left(\frac{1}{2}\right) < 0.\end{aligned}$$

As a consequence, we conclude that $Tx - (1+\varepsilon)x \notin C$, because $Tx - (1+\varepsilon)x \geqslant 0$ is not satisfied. Since $\varepsilon > 0$ and $x \in C \cap F_2$ have been arbitrarily fixed, the property (iii) has been proved.

From all this, theorem 3.19 guarantees that there exists at least one fixed point x of T in $C \cap (E_2 \backslash \mathring{E}_1)$. Therefore, there exists at least one solution to the boundary value problem (4.1) in $C \cap (E_2 \backslash \mathring{E}_1)$. □

We have shown an application to boundary value problems of a generalized version of a Krasnosel'skii fixed point result. We would like to emphasize that we work with star convex sets that generalize the ones considered in the results by Potter [**9**]. Besides, with the application given we have illustrated that E_2 is an admissible set by using that this is defined by a suitable functional.

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References

1 D. R. Anderson, R. I. Avery and J. Henderson. Functional expansion-compression fixed point theorem of Leggett-Williams type. *Electron. J. Differ. Eq.* **2010** (2010), 1–9.

2 R. I. Avery, J. Henderson and D. O'Regan. Functional compression-expansion fixed point theorem. *Electron. J. Differ. Eq.* **2008** (2008), 1–12.

3 N. P. Các and J. A. Gatica. Fixed point theorems for mappings in ordered Banach spaces. *J. Math. Anal. Appl.* **71** (1979), 547–557.

4 G. Darbo. Punti uniti transformazioni a condominio non-compactto. *Rend. Semin. Mat. U. Pad.* **24** (1955), 84–92.

5 D. Güo and V. Lakshmikhantam. *Nonlinear Problems in Abstract Cones* (California: Notes and Reports in Mathematics in Science and Engineering, 1998).

6 M. A. Krasnosel'skii. Fixed points of cone-compressing or cone-expanding operators. *Soviet Math. Dokl.* **1** (1960), 1285–1288.

7 M. K. Kwong. On Krasnoselskii's cone fixed point theorem. *Fixed Point Theory A.* **2008** (2008), 18, article n 164537.

8 J. R. Munkres. *Topology* (USA: Prentice Hall, 2000).

9 A. J. B. Potter. A fixed point theorem for positive k-set contractions. *P. Edinburgh Math. Soc.* **19** (1974), 93–102.

10 W. Rudin. *Functional analysis* (New York: McGraw-Hill, 1991).

11 J. R. L. Webb. New fixed point index results and nonlinear boundary value problems. *Bull. Lond. Math. Soc.* **49** (2017), 534–547.