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CLASSIFICATION OF BOTT MANIFOLDS UP TO DIMENSION 8

SUYOUNG CHOI

Department of Mathematics, Ajou University, San 5, Woncheondong, Yeongtonggu, Suwon 443-749, Republic of Korea (schoi@ajou.ac.kr)

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Abstract We show that three- and four-stage Bott manifolds are classified up to diffeomorphism by their integral cohomology rings. In addition, any cohomology ring isomorphism between two three-stage Bott manifolds can be realized by a diffeomorphism between the Bott manifolds.

Keywords: Bott tower; Bott manifold; cohomological rigidity; strong cohomological rigidity

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1. Introduction

A *Bott tower* of height n is a sequence of projective bundles

$$B_\bullet\colon B_n \xrightarrow{\pi_n} B_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\}, \tag{1.1}$$

where, for $i = 1, \dots, n$, ξ_i is the trivial complex line bundle and $\underline{\mathbb{C}}$ is a complex line bundle over B_{i-1}, and $\pi_i\colon B_i = P(\underline{\mathbb{C}} \oplus \xi_i) \to B_{i-1}$ is a projective bundle over B_{i-1}. We call B_n an n-stage Bott manifold and $B_\bullet$ a Bott tower structure of B_n. Note that an n-stage Bott manifold is of real dimension $2n$. A one-stage Bott manifold is the complex projective space $\mathbb{C}P^1$ of complex dimension 1. A two-stage Bott manifold is known as a *Hirzebruch surface*. Hirzebruch [**5**] has shown that the topological type of a Hirzebruch surface $\Sigma_a = P(\underline{\mathbb{C}} \oplus \gamma^{\otimes a})$ is completely determined by the parity of a, where γ is the tautological line bundle over $\mathbb{C}P^1$, i.e. Σ_a is homeomorphic to Σ_b if and only if $a \equiv b (\operatorname{mod} 2)$. In addition, one can easily see that $H^*(\Sigma_0)$ and $H^*(\Sigma_1)$ are not isomorphic as graded rings. Later, it is shown that this classification also holds in the smooth category (see [**7**]), and stimulates the following conjecture (see [**4**]).

Conjecture 1.1 (cohomological rigidity conjecture for Bott manifolds). *Let B_n and B'_n be n-stage Bott manifolds. Then, B_n is diffeomorphic to B'_n if and only if $H^*(B_n)$ is isomorphic to $H^*(B'_n)$ as graded rings.*

More strongly, we conjecture the following.

Conjecture 1.2 (strong cohomological rigidity conjecture for Bott manifolds). *For any cohomology ring isomorphism φ between two Bott manifolds, there is a diffeomorphism that induces φ.*

Conjecture 1.1 is known to be true for $n \leqslant 3$ (see [**3**]) and Conjecture 1.2 is known to be true for $n \leqslant 2$ (see [**2**] or Theorem 2.2). However, they have been open for the higher cases. In this paper, we shall show that Conjecture 1.2 is true for three-stage Bott manifolds, and that Conjecture 1.1 is true for four-stage Bott manifolds; namely, we have the following theorems.

Theorem A (Theorem 3.1). *For any cohomology ring isomorphism φ between two three-stage Bott manifolds there is a diffeomorphism between them that induces φ.*

Theorem B (Theorem 3.3). *Let B_4 and B_4' be four-stage Bott manifolds. Then, B_4 is diffeomorphic to B_4' if and only if $H^*(B_4)$ is isomorphic to $H^*(B_4')$ as graded rings.*

2. Cohomology rings and square vanishing elements

We recall a Bott tower in (1.1), and one can express

$$B_j = P(\underline{\mathbb{C}} \oplus \gamma^{\alpha_j}) \quad \text{with } \alpha_j \in H^2(B_{j-1}),$$

where $\underline{\mathbb{C}}$ denotes the trivial complex line bundle and γ^{α_j} denotes the complex line bundle over B_{j-1} with α_j as the first Chern class for $j = 1, \dots, n$. Using the Borel–Hirzebruch formula [**1**] for the cohomology ring of the projective bundle, we have that $H^*(B_j)$ is a free module over $H^*(B_{j-1})$ via the map π_j^* on the two generators 1 and x_j of degree 0 and 2, respectively. The ring structure is determined by the single relation

$$x_j^2 = \pi_j^*(\alpha_j)x_j,$$

where x_j is the first Chern class of the tautological line bundle over B_j.

Using this formula inductively on j and regarding $H^*(B_j)$ as a graded subring of $H^*(B_n)$ through the projections in (1.1), namely, setting $x_i := \pi_n^* \circ \cdots \circ \pi_{i+1}^*(x_i)$, we see that

$$H^*(B_n) = \mathbb{Z}[x_1, \dots, x_n]/\langle x_j^2 = \alpha_j x_j \mid j = 1, \dots, n\rangle,$$

where $\alpha_1 = 0$, and $\alpha_j = \sum_{i=1}^{j-1} A_j^i x_i$ with $A_j^i \in \mathbb{Z}$ for $j = 2, \dots, n$. Since complex line bundles are classified by their first Chern classes, as is well known, a Bott tower $B_\bullet$ in (1.1) is completely determined by the list of integers A_j^i $(1 \leqslant i < j \leqslant n)$. In addition, we note that there is the natural filtration of $H^*(B_n)$

$$H^*(B_1) \xhookrightarrow{\pi_2^*} H^*(B_2) \xhookrightarrow{\pi_3^*} \cdots \xhookrightarrow{\pi_n^*} H^*(B_n).$$

Now, let us consider an element in $H^2(B_n)$ whose square vanishes. Assume that a primitive element $z = ax_j + u$ in $H^2(B_n)$ satisfies $z^2 = 0$, where a is a non-zero integer and u is a linear combination of x_is for $i < j$. Then, $z^2 = a^2x_j^2 + 2ax_ju + u^2 = 0 \in H^*(B_n)$, i.e. $2au = -a^2\alpha_j$ and $u^2 = 0$. This implies that a square vanishing element should be of the form $z = ax_j - \frac{1}{2}a\alpha_j$ with $\alpha_j^2 = 0$. Therefore, a primitive element in $H^2(B_n)$ whose square vanishes is either $x_j - \frac{1}{2}\alpha_j$ or $2x_j - \alpha_j$ up to sign for some j, where $\alpha_j^2 = 0$ in both cases. Let $X(B_n)$ be the set of all primitive square vanishing elements of $H^*(B_n)$ up to sign. Then $|X(B_n)|$ is equal to the number of js satisfying $\alpha_j^2 = 0$, and hence is less than or equal to n. Furthermore, for $n \geqslant 2$, we have $|X(B_n)| \geqslant 2$ since $\alpha_1^2 = \alpha_2^2 = 0$. We say that B_n is $\mathbb{Q}$-*trivial* if its cohomology ring is isomorphic to that of $(\mathbb{C}P^1)^n$ with $\mathbb{Q}$-coefficients as graded rings.

Proposition 2.1. *B_n is $\mathbb{Q}$-trivial if and only if $\alpha_j^2 = 0$ in $H^*(B_n)$ for all $j = 1, \dots, n$.*

Proof. If $\alpha_j^2 = 0$, then $(x_j - \alpha_j/2)^2 = 0$ in $H^*(B_n; \mathbb{Q})$ because $x_j^2 = \alpha_j x_j$. Since $x_j - \alpha_j/2$ for $j = 1, \dots, n$ generate $H^*(B_n; \mathbb{Q})$ as a graded ring, this shows that B_n is $\mathbb{Q}$-trivial. Conversely, if B_n is $\mathbb{Q}$-trivial, there are n primitive elements in $H^2(B_n)$ up to sign whose squares vanish, which implies the converse by the above discussion. □

It is known that the strong cohomological rigidity holds for the class of $\mathbb{Q}$-trivial Bott manifolds; namely, we have the following theorem.

Theorem 2.2 (Choi and Masuda [2]). *Any cohomology ring isomorphism between two $\mathbb{Q}$-trivial Bott manifolds is realizable by a diffeomorphism.*

Put $t = |X(B_n)|$. A Bott tower $B_\bullet$ is said to be *well ordered* if $\alpha_j^2 = 0$ for $j = 1, \dots, t$, and $\alpha_j^2 \neq 0$ for $j = t+1, \dots, n$.

Lemma 2.3. *Every Bott manifold B_n admits a well-ordered Bott tower structure.*

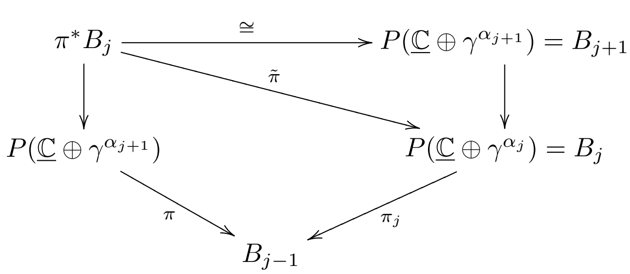
Proof. Consider any Bott tower structure of B_n that is not well ordered. In other words, there exists at least one j such that $\alpha_j^2 \neq 0$ but $\alpha_{j+1}^2 = 0$. Remember that

$$\alpha_{j+1} = \sum_{i=1}^{j-1} A_{j+1}^i x_i + A_{j+1}^j x_j.$$

If $A_{j+1}^j \neq 0$ and, as assumed, $\alpha_{j+1}^2 = 0$, then we get

$$\alpha_{j+1} = A_{j+1}^j x_j - (A_{j+1}^j/2)\alpha_j$$

with $\alpha_j^2 = 0$, which is a contradiction. Hence, $A_{j+1}^j = 0$. We can interchange the labels j and $j+1$, which proves the lemma by the following procedure: since $A_{j+1}^j = 0$, $\gamma^{\alpha_{j+1}}$ can be regarded as a complex bundle over B_{j-1}. Let $\pi\colon P(\mathbb{C} \oplus \gamma^{\alpha_{j+1}}) \to B_{j-1}$ be the corresponding projection. Then,



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where $\pi^* B_j$ is the pullback of $B_j \to B_{j-1}$ by π. One can then see that $\pi^* B_j$ is diffeomorphic to B_{j+1}, and it gives another Bott tower structure of B_n that is obtained from $B_\bullet$ by interchanging the jth and $j+1$th stages. $\square$

From now on, we only consider Bott manifolds with well-ordered Bott tower structure; namely, we assume that any Bott tower that appears in this paper is well ordered.

Let B'_n be another Bott manifold. Suppose that $H^*(B_n)$ and $H^*(B'_n)$ are isomorphic as graded rings. A graded ring isomorphism $\varphi\colon H^*(B_n) \to H^*(B'_n)$ is said to be k-stable if there is a graded ring isomorphism $h_k\colon H^*(B_k) \to H^*(B'_k)$ that makes the diagram

$$\begin{array}{ccc} H^*(B_k) & \xhookrightarrow{\pi_n^* \circ \cdots \circ \pi_{k+1}^*} & H^*(B_n) \\ \Big\downarrow{\scriptstyle h_k} & & \Big\downarrow{\scriptstyle \varphi} \\ H^*(B'_k) & \xhookrightarrow{\pi_n'^* \circ \cdots \circ \pi_{k+1}'^*} & H^*(B_n) \end{array}$$

commute. We note that φ should send elements in $X(B_n)$ to elements in $X(B'_n)$ up to sign, and $X(B_n)$ forms a basis of $\pi_n^* \circ \cdots \circ \pi_{t+1}^*(H^2(B_t))$, where $t = |X(B_n)|$. It implies that $|X(B_n)| = |X(B'_n)|$ and φ is t-stable.

Theorem 2.4 (Ishida [6]). *Let B_n and B'_n be two Bott manifolds. If there is an isomorphism $\varphi\colon H^*(B_n) \to H^*(B'_n)$ that is $(n-1)$-stable, and if h_{n-1} is realizable by a diffeomorphism between B_{n-1} and B'_{n-1}, then so is φ by a diffeomorphism between B_n and B'_n.*

3. Classification of low-stage Bott manifolds

Note that there is only one one-stage Bott manifold $\mathbb{C}P^1$, and every two-stage Bott manifold is $\mathbb{Q}$-trivial. Hence, by Theorem 2.2, the strong cohomological rigidity holds for one- and two-stage Bott manifolds.

Theorem 3.1. *For any cohomology ring isomorphism φ between two three-stage Bott manifolds, there is a diffeomorphism between them that induces φ.*

Proof. If three-stage Bott manifolds are $\mathbb{Q}$-trivial, then, by Theorem 2.2, φ can be realized by diffeomorphism. Otherwise, namely, if they are not $\mathbb{Q}$-trivial, then φ should be 2-stable. Since the strong cohomological rigidity holds for two-stage Bott manifolds, by Theorem 2.4, φ is realizable. $\square$

Now we prepare one lemma for proving the cohomological rigidity for four-stage Bott manifolds.

Lemma 3.2. *Let $B_n = P(\underline{\mathbb{C}} \oplus \gamma^\alpha)$ and $B'_n = P(\underline{\mathbb{C}} \oplus \gamma^\beta)$ be two projective bundles over an $(n-1)$-stage Bott manifold B_{n-1}. If there exists $u \in H^2(B_{n-1})$ such that $\alpha = \beta - 2u$ and $u(u-\beta) = 0$, then B_n is isomorphic to B'_n as bundles. In particular, $P(\underline{\mathbb{C}} \oplus \gamma^\alpha)$ is isomorphic to $P(\underline{\mathbb{C}} \oplus \gamma^{-\alpha})$.*

Proof. Note that $P(\underline{\mathbb{C}} \oplus \gamma^{\beta})$ is isomorphic to $P(\gamma^u \oplus \gamma^{\beta+u})$. The total Chern class of $\gamma^{-u} \oplus \gamma^{\beta-u}$ is $(1-u)(1+\beta-u) = 1+\beta-2u+u(u-\beta) = 1+\alpha$. Hence, $\gamma^{-u} \oplus \gamma^{\beta-u}$ and $\underline{\mathbb{C}} \oplus \gamma^{\alpha}$ are isomorphic by [**6**, Theorem 3.1], as are $P(\underline{\mathbb{C}} \oplus \gamma^{\beta})$ and $P(\underline{\mathbb{C}} \oplus \gamma^{\alpha})$. □

Theorem 3.3. *Let B_4 and B_4' be four-stage Bott manifolds. Then B_4 is diffeomorphic to B_4' if and only if $H^*(B_4)$ is isomorphic to $H^*(B_4')$ as graded rings.*

Proof. Let $\varphi\colon H^*(B_4) \to H^*(B_4')$ be a graded ring isomorphism. If both B_4 and B_4' are $\mathbb{Q}$-trivial, then, by Theorem 2.2, φ can be realized by a diffeomorphism. If $|X(B_4)| = 3$, then, combining Theorem 3.1 and Theorem 2.4, φ can also be realized. Hence, for the above two cases, B_4 and B_4' are diffeomorphic.

Assume that $|X(B_4)| = 2$. We denote by y_j, β_j and B_j^i those elements in $H^*(B_4')$ that correspond to x_j, α_j and A_j^i in $H^*(B_4)$ for $j = 1, \dots, 4$. Since φ is 2-stable, φ induces a ring isomorphism

$$\begin{array}{ccc} H^*(B_4)/\pi_4^* \circ \pi_3^*(H^*(B_2)) & \longrightarrow & H^*(B_4')/\pi_4'^* \circ \pi_3'^*(H^*(B_2')) \\ \| & & \| \\ \mathbb{Z}[x_3, x_4]/\langle x_3^2 = 0, x_4^2 = A_4^3 x_3 x_4 \rangle & & \mathbb{Z}[y_3, y_4]/\langle y_3^2 = 0, y_4^2 = B_4^3 y_3 y_4 \rangle \end{array}$$

Hence, since it preserves the set of primitive square vanishing elements, we conclude that A_4^3 and B_4^3 have the same parity, and $\varphi(x_3)$ is either $\epsilon y_3 + w$, $\epsilon(y_4 - (B_4^3/2)y_3) + w$ (if B_4^3 is even) or $\epsilon(2y_4 - B_4^3 y_3) + w$ (if B_4^3 is odd), where $\epsilon = \pm 1$ and w is a linear combination of y_1 and y_2.

Case 1 ($\boldsymbol{\varphi(x_3) = \epsilon y_3 + w}$). Note that φ is 3-stable. Hence, φ can be realized by a diffeomorphism.

Case 2 ($\boldsymbol{\varphi(x_3) = \epsilon(y_4 - (B_4^3/2)y_3) + w}$). Note that B_4^3 (say, b) is even. If $b = 0$, then we may interchange the third and fourth stages of its Bott tower structure as in Lemma 2.3. Hence, $\varphi(x_3)$ would be 3-stable, and hence it can be realized. Suppose that $b \neq 0$. Since $x_3(x_3 - \alpha_3) = 0$,

$$\begin{aligned} 0 = \varphi(x_3(x_3 - \alpha_3)) &= \left(\epsilon y_4 - \frac{\epsilon b}{2} y_3 + w\right)\left(\epsilon y_4 - \frac{\epsilon b}{2} y_3 + w - \varphi(\alpha_3)\right) \\ &= y_4(y_4 - b y_3 + 2\epsilon w - \epsilon\varphi(\alpha_3)) \\ &\qquad + \tfrac{1}{4} b y_3 (b y_3 - 4\epsilon w + 2\epsilon\varphi(\alpha_3)) + w^2 - w\varphi(\alpha_3). \end{aligned}$$

Because $\varphi(\alpha_3)$ is a linear combination of y_1 and y_2, and $b \neq 0$, we have that

$$y_4(y_4 - b y_3 + 2\epsilon w - \epsilon\varphi(\alpha_3)) = 0 \in H^*(B_4') \tag{3.1}$$

and

$$\tfrac{1}{2} b y_3 (\tfrac{1}{2} b y_3 - 2\epsilon w + \epsilon\varphi(\alpha_3)) = 0 \in H^*(B_4'). \tag{3.2}$$

Hence, by (3.1), $\beta_4 = b y_3 - 2\epsilon w + \epsilon\varphi(\alpha_3)$. Let $u = b y_3/2$. Then, by (3.2), $u(\beta_4 - u) = 0$. Hence, by Lemma 3.2, we have an isomorphism $f\colon B_4' \to P(\underline{\mathbb{C}} \oplus \gamma^{\beta_4 - 2u})$ as bundles over

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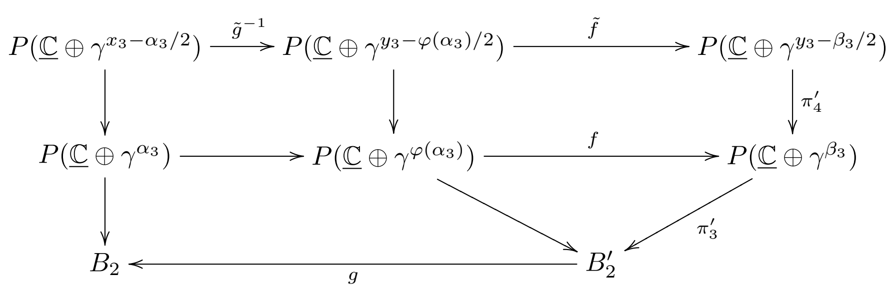
B'_3. This isomorphism gives a new Bott tower structure of B'_4 whose third and fourth stages are interchangeable. The interchange map is denoted by g. The new Bott tower structure obtained by $g \circ f(B'_4)$ is denoted by $B''_\bullet$. Note that f and g are diffeomorphisms and $B''_\bullet$ is well ordered. Hence, one can easily check that $(g^{-1})^* \circ (f^{-1})^* \circ \varphi\colon H^*(B_4) \to H^*(B''_4)$ is 3-stable. Therefore, $(g^{-1})^* \circ (f^{-1})^* \circ \varphi$ is realizable and, hence, so is φ.

Case 3 ($\varphi(x_3) = \epsilon(2y_4 - B^3_4 y_3) + w$). Note that A^3_4 (say, a) and B^3_4 (say, b) are both odd. We may also assume that $\varphi^{-1}(y_3) = \varepsilon(2x_4 - ax_3) + z$, where $\varepsilon = \pm 1$ and z is a linear combination of x_1 and x_2. Since $x_3(x_3 - \alpha_3) = 0$,

$$\begin{aligned} 0 &= \varphi(x_3(x_3 - \alpha_3)) \\ &= (2\epsilon y_4 - b\epsilon y_3 + w)(2\epsilon y_4 - b\epsilon y_3 + w - \varphi(\alpha_3)) \\ &= 4y_4\left(y_4 - by_3 + \epsilon w - \epsilon\frac{\varphi(\alpha_3)}{2}\right) + b^2 y_3\left(y_3 - \frac{2}{b}\epsilon w + \frac{1}{b}\epsilon\varphi(\alpha_3)\right) + w^2 - w\varphi(\alpha_3) \end{aligned}$$

Hence, $\beta_4 = by_3 - \epsilon w + \epsilon(\varphi(\alpha_3)/2)$, $\beta_3 = 2\epsilon w/b - \epsilon\varphi(\alpha_3)/b$ and $w^2 = w\varphi(\alpha_3)$. Note that $\beta_3^2 = (1/b^2)\varphi(\alpha_3^2) \neq 0 \in H^4(B'_4)$. Similarly, we also have $\alpha_3^2 = (1/a^2)\varphi^{-1}(\beta_3^2)$. Thus, $\alpha_3^2 = (1/a^2b^2)\alpha_3^2$. Since α_3^2 does not vanish, $a^2b^2 = 1$. Hence, $|a| = |b| = 1$. We may assume that $a = b = 1$ by Lemma 3.2. Then $\beta_3 = 2\epsilon w - \epsilon\varphi(\alpha_3)$ and $\beta_4 = y_3 - \epsilon w + \epsilon\varphi(\alpha_3)/2 = y_3 - \beta_3/2$. Similarly, we have $\alpha_4 = x_3 - \alpha_3/2$. By Lemma 3.2, we have a bundle isomorphism $f\colon P(\underline{\mathbb{C}} \oplus \gamma^{\varphi(\alpha_3)}) \to B'_3$ over B'_2. Then we obtain the pullback $f^*B'_4 = P(\underline{\mathbb{C}} \oplus \gamma^{y_3 - \varphi(\alpha_3)/2})$ of B'_4 by f, and we obtain the induced diffeomorphism $\tilde{f}\colon P(\underline{\mathbb{C}} \oplus \gamma^{y_3 - \varphi(\alpha_3)/2}) \to B'_4$. On the other hand, since any cohomology ring isomorphism between two Hirzebruch surfaces is realizable, we consider a diffeomorphism $g\colon B'_2 \to B_2$ that induces φ restricted by $H^*(B_2)$. We also obtain, then, the pullback ${g^{-1}}^*(f^*B'_4) = P(\underline{\mathbb{C}} \oplus \gamma^{x_3 - \alpha_3/2})$ of $f^*B'_4$ by g^{-1}, and we also have the induced diffeomorphism $\tilde{g}^{-1}\colon P(\underline{\mathbb{C}} \oplus \gamma^{x_3 - \alpha_3/2}) \to f^*B'_4$; see the following diagram



Note that $P(\underline{\mathbb{C}} \oplus \gamma^{x_3 - \alpha_3/2}) = P(\underline{\mathbb{C}} \oplus \gamma^{\alpha_4})$, and, hence, $P(\underline{\mathbb{C}} \oplus \gamma^{\alpha_4}) \to P(\underline{\mathbb{C}} \oplus \gamma^{\alpha_3}) \to B_2$ is a Bott tower structure of B_4. Hence, $\tilde{f} \circ \tilde{g}^{-1}$ is a diffeomorphism between B_4 and B'_4.

In the three cases above we have shown that B_4 and B'_4 are diffeomorphic, which proves the theorem. □

Example 3.4. Let B_4 be a four-stage Bott manifold with the Bott tower structure $P(\underline{\mathbb{C}} \oplus \gamma^{x_3 - \alpha_3/2}) \to P(\underline{\mathbb{C}} \oplus \gamma^{\alpha_3}) \to B_2$. Consider four homomorphisms $\varphi_k\colon H^*(B_4) \to H^*(B_4)$ $(k = 1, \ldots, 4)$ defined by

(1) $\varphi_1(x_1) = x_1$, $\varphi_1(x_2) = x_2$, $\varphi_1(x_3) = 2x_4 - x_3 + \alpha_3$ and $\varphi_1(x_4) = x_4$;

(2) $\varphi_2(x_1) = x_1$, $\varphi_2(x_2) = x_2$, $\varphi_2(x_3) = 2x_4 - x_3 + \alpha_3$ and $\varphi_2(x_4) = x_4 - x_3 + \frac{1}{2}\alpha_3$;

(3) $\varphi_3(x_1) = x_1$, $\varphi_3(x_2) = x_2$, $\varphi_3(x_3) = -2x_4 + x_3$ and $\varphi_3(x_4) = -x_4$;

(4) $\varphi_4(x_1) = x_1$, $\varphi_4(x_2) = x_2$, $\varphi_4(x_3) = -2x_4 + x_3$ and $\varphi_4(x_4) = -x_4 + x_3 - \frac{1}{2}\alpha_3$.

Then they are all well defined and are graded ring isomorphisms. Moreover, they are all under the third case of the proof of Theorem 3.3.

We remark that a cohomology ring isomorphism φ is realizable unless it is under the last case of the proof of Theorem 3.3. However, we do not know whether φ of the last case is realizable or not. In order to prove the strong cohomological rigidity for four-stage Bott manifolds, we require that any automorphism of the cohomology ring of B_4 with the Bott tower structure $P(\underline{\mathbb{C}} \oplus \gamma^{x_3-\alpha_3/2}) \to P(\underline{\mathbb{C}} \oplus \gamma^{\alpha_3}) \to B_2$ under the last case is realizable. We note that there are only finitely many such automorphisms. Since we may assume that $\varphi(x_1) = x_1$ and $\varphi(x_2) = x_2$, there are only four essential automorphisms φ_k ($k = 1, \dots, 4$).

Problem 3.5. Are the automorphisms φ_k ($k = 1, \dots, 4$) realizable?

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