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SUBGEOMETRIC RATES OF CONVERGENCE FOR MARKOV PROCESSES UNDER SUBORDINATION

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Abstract

We are interested in the rate of convergence of a subordinate Markov process to its invariant measure. Given a subordinator and the corresponding Bernstein function (Laplace exponent), we characterize the convergence rate of the subordinate Markov process; the key ingredients are the rate of convergence of the original process and the (inverse of the) Bernstein function. At a technical level, the crucial point is to bound three types of moment (subexponential, algebraic, and logarithmic) for subordinators as time t tends to ∞. We also discuss some concrete models and we show that subordination can dramatically change the speed of convergence to equilibrium.

Keywords: Rate of convergence; invariant measure; subordination; moment estimate; Bernstein function; Markov process

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1. Introduction and main result

The notion of stochastic stability of a Markov process is of fundamental importance both in theoretical studies and in practical applications. There are various characterizations of geometric (or exponential) ergodicity; see, e.g. [5], [6], [13], [20], [21], [22], and the references therein. In recent years, there has been considerable interest in subgeometric (or subexponential) ergodicity; see, e.g. [4], [10], [12], [11], [14], [23] for developments in this direction. In this paper we study subgeometric convergence rates of Markov processes under subordination in the sense of Bochner.

Let $X = \{X_t : t \ge 0\}$ be a Markov process with state space $(E, \mathscr{B}(E))$ and transition function $P^t(x, \mathrm{d}y)$. We assume that E is a locally compact and separable metric space and we denote by $\mathscr{B}(E)$ the corresponding Borel σ-algebra. Let $f : E \to [1, \infty)$ be a measurable control function; the *f-norm* of a signed measure μ on E is defined as $\|\mu\|_f := \sup_{|g|\le f} |\mu(g)|$. Here, the supremum ranges over all measurable g which are dominated by f and $\mu(g) := \int g \, \mathrm{d}\mu$. It is not difficult to see that $\|\cdot\|_f \ge \|\cdot\|_{\mathrm{TV}}$ always holds for the total variation norm $\|\cdot\|_{\mathrm{TV}}$; if f is bounded then the norms $\|\cdot\|_f$ and $\|\cdot\|_{\mathrm{TV}}$ are even equivalent.

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The convergence behaviour of a process X to a stationary distribution π in the f-norm can be captured by estimates of the form

$$\|P^t(x,\cdot)-\pi\|_f \le C(x)r(t), \qquad x\in E,\ t\ge 0, \tag{1.1}$$

where $C(x)\in(0,\infty)$ is a constant depending on $x\in E$ and $r\colon[0,\infty)\to(0,1]$ is the nonincreasing *rate function*. We say that X displays *subgeometric convergence in f-norm*, if the rate function r satisfies $r(t)\downarrow 0$ and $\log r(t)/t\uparrow 0$ as $t\to\infty$. Such r are called *subgeometric rates*.

In many cases, the convergence rate r can be explicitly given, and typical examples are

$$r(t)=\mathrm{e}^{-\theta t^{\delta}}, \qquad r(t)=(1+t)^{-\beta}, \qquad r(t)=[1+\log(1+t)]^{-\gamma}, \tag{1.2}$$

where $\theta>0$, $\delta\in(0,1]$, and $\beta,\gamma>0$ are some constants; see Section 4 below for specific models. Note that $r(t)=\mathrm{e}^{-\theta t}$ is the classical exponential convergence rate. Some authors refer to the above examples as subexponential, algebraic, and logarithmic rates, respectively.

Bochner's subordination is a means to obtain more general (and also interesting) jump-type Markov processes from a given Markov process through a random time change by an independent nondecreasing Lévy process (a subordinator). Among the most interesting examples are the symmetric α-stable Lévy processes, which can be viewed as subordinate to Brownian motions. It is known that many fine properties of Markov processes (and the corresponding Markov semigroups) are preserved under subordination; see [16], [9] for Harnack and shift Harnack inequalities for subordinate semigroups, [24], [15] for Nash and Poincaré inequalities under subordination, and [8] for the quasiinvariance property of subordinate Brownian motion.

Let us recall the basics of Bochner's subordination. Let $S=\{S_t\colon t\ge 0\}$ be a *subordinator* (without killing), i.e. a nondecreasing Lévy process on $[0,\infty)$ with Laplace transform

$$\mathbb{E}\mathrm{e}^{-uS_t}=\mathrm{e}^{-t\phi(u)}, \quad u>0,\ t\ge 0.$$

The characteristic (Laplace) exponent $\phi\colon(0,\infty)\to(0,\infty)$ is a *Bernstein function*, i.e. ϕ is of class C^{∞} such that $(-1)^{n-1}\phi^{(n)}\ge 0$ for all $n=1,2,\dots$; it is well known that every Bernstein function admits a unique (Lévy–Khintchine) representation

$$\phi(u)=bu+\int_{(0,\infty)}(1-\mathrm{e}^{-uy})\nu(\mathrm{d}y), \qquad u>0, \tag{1.3}$$

where $b\ge 0$ is the drift parameter and ν is a Lévy measure, i.e. a measure on $(0,\infty)$ satisfying $\int_{(0,\infty)}(y\wedge 1)\nu(\mathrm{d}y)<\infty$. Our main reference for Bernstein functions and subordination is [25]. Assume that S and X are independent processes. The *subordinate process* defined by the random time-change $X_t^{\phi}:=X_{S_t}$ is again a Markov process; if X has an invariant probability measure π then π is also invariant for the subordinate process X^{ϕ}. This follows immediately from the form of the subordinate Markov transition function which is given by

$$P_{\phi}^t(x,\mathrm{d}y)=\int_{[0,\infty)}P^s(x,\mathrm{d}y)\mu_t(\mathrm{d}s),$$

where $\mu_t:=\mathbb{P}(S_t\in\cdot)$ is the transition probability of S_t; the integral is understood in the sense of vague convergence of probability measures.

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We are interested in the following question. Assume that P^t is subgeometrically convergent to π with respect to the f-norm as $t \to \infty$; how fast will P^t_ϕ tend to π? More precisely, we need to find a suitable nonincreasing function r_ϕ on $(0, \infty)$ such that $\lim_{t\to\infty} r_\phi(t) = 0$ and

$$\|P^t_\phi(x, \cdot) - \pi\|_f \le C(x) r_\phi(t), \qquad x \in E,\ t > 0, \tag{1.4}$$

for some positive constant $C(x)$ depending only on $x \in E$. As we will see, if the convergence rates of the original process X are of the three typical forms in (1.2) then we are able to derive convergence rates for the subordinate Markov process under some reasonable assumptions on the underlying subordinator.

Note that any nontrivial Bernstein function ϕ is strictly increasing. In this paper the inverse function of ϕ will be denoted by ϕ^{-1}.

We can now state the main result of our paper.

Theorem 1.1. *Let X be a Markov process and S an independent subordinator with Bernstein function ϕ of the form (1.3).*

(i) *Assume that (1.1) holds with rate $r(t) = \mathrm{e}^{-\theta t^\delta}$ for some constants $\theta > 0$ and $\delta \in (0, 1]$. If $\nu(\mathrm{d}y) \ge c y^{-1-\alpha}\,\mathrm{d}y$ for some constants $c > 0$ and $\alpha \in (0, 1)$ then (1.4) holds with rate*

$$r_\phi(t) = \exp[-C t^{\delta/(\alpha(1-\delta)+\delta)}],$$

where $C = C(\theta, \delta, c, \alpha) > 0$.

(ii) *Assume that (1.1) holds with rate $r(t) = (1 + t)^{-\beta}$ for some constant $\beta > 0$. If*

$$\liminf_{s\to\infty} \frac{\phi(s)}{\log s} > 0 \quad \textit{and} \quad \liminf_{s\downarrow 0} \frac{\phi(\lambda s)}{\phi(s)} > 1 \quad \textit{for some (hence, all) } \lambda > 1, \tag{1.5}$$

then (1.4) holds with rate

$$r_\phi(t) = 1 \wedge \left[\phi^{-1}\left(\frac{1}{t}\right)\right]^\beta.$$

(iii) *Assume that (1.1) holds with rate $r(t) = [1 + \log(1 + t)]^{-\gamma}$ for some constant $\gamma > 0$. If $\nu(\mathrm{d}y) \ge c y^{-1-\alpha}\,\mathrm{d}y$ for some constants $c > 0$ and $\alpha \in (0, 1)$ then (1.4) holds with rate*

$$r_\phi(t) = 1 \wedge \log^{-\gamma}(1 + t).$$

Remark 1.1. Typical examples for Bernstein function ϕ satisfying (1.5) are

- $\phi(s) = \log(1 + s)$;
- $\phi(s) = s^\alpha \log^\beta(1 + s)$ with $\alpha \in (0, 1)$ and $\beta \in [0, 1 - \alpha)$;
- $\phi(s) = s^\alpha \log^{-\beta}(1 + s)$ with $0 < \beta < \alpha < 1$;
- $\phi(s) = s(1 + s)^{-\alpha}$ with $\alpha \in (0, 1)$.

We refer the reader to [25] for an extensive list of such Bernstein functions.

The paper is organized as follows. In order to prove Theorem 1.1, we establish in Section 2 three types of moment estimate for subordinators; this part is interesting in its own right. The proof of Theorem 1.1 will be addressed in Section 3. Section 4 contains several concrete models for which the corresponding convergence rates can be explicitly given. For the reader's convenience, the Appendix contains some elementary calculations, which have been used in the proof of Theorem 2.1 in Section 2.

2. Moment estimates for subordinators

In this section we prove some moment estimates for subordinators, which will be crucial for the proof of our main result, Theorem 1.1. Related moment estimates for general Lévy processes and subordinators can be found in [9, Section 3]. Recently, Kühn [18] extended our results on Lévy processes to Feller processes.

Theorem 2.1. *Let S be a subordinator with Bernstein function ϕ given by (1.3).*

(i) *Let $\theta > 0$ and $\delta \in (0, 1]$. If $\nu(\mathrm{d}y) \ge cy^{-1-\alpha}\,\mathrm{d}y$ for some constants $c > 0$ and $\alpha \in (0, 1)$ then there exists a $C = C(\theta, \delta, c, \alpha) > 0$ such that*

$$\mathbb{E}\mathrm{e}^{-\theta S_t^\delta} \le \exp[-Ct^{\delta/(\alpha(1-\delta)+\delta)}] \quad \textit{for all sufficiently large } t > 1.$$

(ii) *Let $\beta > 0$.*

(a) *We have*

$$\mathbb{E}S_t^{-\beta} \ge \frac{1}{\mathrm{e}\beta\Gamma(\beta)}\left[\phi^{-1}\left(\frac{1}{t}\right)\right]^\beta \quad \textit{for all } t > 0.$$

(b) *If the Bernstein function ϕ satisfies (1.5) then there exists a $C = C(\beta) > 0$ such that*

$$\mathbb{E}S_t^{-\beta} \le C\left[\phi^{-1}\left(\frac{1}{t}\right)\right]^\beta \quad \textit{for all sufficiently large } t > 1.$$

(c) *If the Bernstein function ϕ satisfies*

$$\liminf_{s\to\infty}\frac{\phi(\lambda s)}{\phi(s)} > 1 \quad \textit{for some (hence, all) } \lambda > 1$$

then there exists a $C = C(\beta) > 0$ such that

$$\mathbb{E}S_t^{-\beta} \le C\left[\phi^{-1}\left(\frac{1}{t}\right)\right]^\beta \quad \textit{for all } t \in (0, 1].$$

(iii) *Let $\gamma > 0$.*

(a) *If $\nu(\mathrm{d}y) \ge cy^{-1-\alpha}\,\mathrm{d}y$ for some constants $c > 0$ and $\alpha \in (0, 1)$ then there exists a $C = C(\gamma, c, \alpha) > 0$ such that*

$$\mathbb{E}\log^{-\gamma}(1 + S_t) \le C\log^{-\gamma}(1 + t^{1/\alpha}) \quad \textit{for all } t > 0.$$

(b) *If $\nu(\mathrm{d}y) = cy^{-1-\alpha}\,\mathrm{d}y$ for some constants $c > 0$ and $\alpha \in (0, 1)$ then there exists a $C = C(\gamma, c, \alpha) > 0$ such that*

$$\mathbb{E}\log^{-\gamma}(1 + S_t) \ge C\log^{-\gamma}(1 + t^{1/\alpha}) \quad \textit{for all } t > 0.$$

Remark 2.1. Theorem 2.1(ii) is motivated by an argument in [2, Proof of Theorem 2.1], where the special case $\beta = \frac{1}{2}$ was treated; see also [26, Proof of Theorem 1.3]. For the estimate of $\mathbb{E}S_t^{-1/2}$ for large t, it was assumed in [2, Theorem 2.1] that the Bernstein function ϕ satisfies

$$\liminf_{s\to\infty}\frac{\phi(s)}{\log s} > 0, \qquad \liminf_{s\downarrow 0}\phi(s)|\log s| < \infty, \qquad \limsup_{s\downarrow 0}\frac{\phi^{-1}(2s)}{\phi^{-1}(s)} < \infty. \tag{2.1}$$

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By Lemma 2.2(ii) and Lemma 2.3(ii) below, the third condition in (2.1) implies that there exist constants $c_1, c_2, \kappa > 0$ such that

$$\phi(s) \le c_1 s^{\kappa}, \qquad 0 < s \le c_2,$$

and then

$$\limsup_{s\downarrow 0} \phi(s)|\log s| \le c_1 \limsup_{t\to\infty} \frac{\log t}{t^{\kappa}} = 0.$$

This means that the third condition in (2.1) implies the second, and so (2.1) can be written as

$$\liminf_{s\to\infty} \frac{\phi(s)}{\log s} > 0, \qquad \limsup_{s\downarrow 0} \frac{\phi^{-1}(\lambda s)}{\phi^{-1}(s)} < \infty \quad \text{for some (hence, all) } \lambda > 1.$$

Before we present the proof of Theorem 2.1, we need some preparation. The following useful lemma is taken from [6, Lemma A.1, p. 193]; see also [24, Lemma 5] for a special case.

Lemma 2.1. *Let $C > 0$, $h\colon [0,\infty) \to (0,1]$ be an absolutely continuous function, and $\rho\colon (0,1] \to (0,1]$ be a nondecreasing function. If*

$$h'(t) \le C\rho(h(t)) \quad \textit{for almost all } t \ge 0$$

then

$$G(h(t)) \le G(h(0)) - Ct \quad \textit{for all } t \ge 0,$$

where

$$G(v) := -\int_v^1 \frac{\mathrm{d}u}{\rho(u)}, \qquad 0 < v \le 1.$$

For any strictly increasing function $g\colon (0,\infty) \to (0,\infty)$, we denote by g^{-1} its inverse function.

Lemma 2.2. *Let $g\colon (0,\infty) \to (0,\infty)$ be a strictly increasing function.*

(i) *The following statements are equivalent:*

(a) $\lim_{t\to\infty} g(t) = \infty$ *and* $\limsup_{t\to\infty}(g^{-1}(\lambda_0 t)/g^{-1}(t)) < \infty$ *for some* $\lambda_0 > 1$*;*

(b) $\lim_{t\to\infty} g(t) = \infty$ *and* $\limsup_{t\to\infty}(g^{-1}(\lambda t)/g^{-1}(t)) < \infty$ *for all* $\lambda > 1$*;*

(c) $\liminf_{t\to\infty}(g(\lambda_0 t)/g(t)) > 1$ *for some* $\lambda_0 > 1$.

If g is concave then (a)–(c) are also equivalent to

(d) $\liminf_{t\to\infty}(g(\lambda t)/g(t)) > 1$ *for all* $\lambda > 1$.

(ii) *The following statements are equivalent:*

(a) $\limsup_{t\downarrow 0}(g^{-1}(\lambda_0 t)/g^{-1}(t)) < \infty$ *for some* $\lambda_0 > 1$*;*

(b) $\limsup_{t\downarrow 0}(g^{-1}(\lambda t)/g^{-1}(t)) < \infty$ *for all* $\lambda > 1$*;*

(c) $\liminf_{t\downarrow 0}(g(\lambda_0 t)/g(t)) > 1$ *for some* $\lambda_0 > 1$.

If g is concave then (a)–(c) are also equivalent to

(d) $\liminf_{t\downarrow 0}(g(\lambda t)/g(t)) > 1$ *for all* $\lambda > 1$.

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Proof. We will only show (i), since the proof of (ii) is completely analogous.

(a) $\iff$ (b) The direction (b) $\implies$ (a) is trivial. Conversely, suppose that (a) holds for some $\lambda_0 > 1$. By the monotonicity of g, (b) holds for all $\lambda \in (1, \lambda_0]$. Now assume that $\lambda > \lambda_0$ and let $k := \lfloor \log_{\lambda_0} \lambda \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of a nonnegative real number $x \geq 0$. Since (a) implies that there exist $c_1 > 1$ and $c_2 > 0$ such that

$$g^{-1}(\lambda_0 t) \leq c_1 g^{-1}(t), \qquad t \geq c_2, \tag{2.2}$$

we find that

$$g^{-1}(\lambda t) \leq g^{-1}(\lambda_0^{k+1} t) \leq c_1^{k+1} g^{-1}(t), \qquad t \geq c_2 \lambda_0^{-k},$$

and so

$$\limsup_{t\to\infty} (g^{-1}(\lambda t)/g^{-1}(t)) \leq c_1^{k+1} < \infty.$$

(a) $\iff$ (c) If (a) holds, we can apply g to both sides of (2.2) and substitute $g^{-1}(t) = s$ to obtain

$$\lambda_0 g(s) \leq g(c_1 s), \qquad s \geq g^{-1}(c_2).$$

This implies that

$$\liminf_{s\to\infty} \frac{g(c_1 s)}{g(s)} \geq \lambda_0 > 1,$$

and then (c).

Conversely, if (c) is satisfied for some $\lambda_0 > 1$ then it is clear that $\lim_{t\to\infty} g(t) = \infty$, and, moreover, we can easily reverse the above argument to deduce (a).

(c) $\iff$ (d) Assume that g is concave. If (c) holds for some $\lambda_0 > 1$ then (d) holds for all $\lambda \geq \lambda_0$. It remains to consider the $\lambda \in (1, \lambda_0)$ case. By (c), there exist $c_3 > 1$ and $c_4 > 0$ such that

$$g(\lambda_0 t) > c_3 g(t), \qquad t \geq c_4.$$

Using the concavity of g, we obtain, for any $t \geq c_4$ and $\lambda \in (1, \lambda_0)$,

$$\begin{aligned} g(\lambda t) &= g\left(\frac{\lambda_0 - \lambda}{\lambda_0 - 1} t + \frac{\lambda - 1}{\lambda_0 - 1} \lambda_0 t\right) \\ &\geq \frac{\lambda_0 - \lambda}{\lambda_0 - 1} g(t) + \frac{\lambda - 1}{\lambda_0 - 1} g(\lambda_0 t) \\ &> \frac{\lambda_0 - \lambda}{\lambda_0 - 1} g(t) + \frac{\lambda - 1}{\lambda_0 - 1} c_3 g(t), \end{aligned}$$

which yields

$$\liminf_{t\to\infty} \frac{g(\lambda t)}{g(t)} \geq \frac{\lambda_0 - \lambda}{\lambda_0 - 1} + \frac{\lambda - 1}{\lambda_0 - 1} c_3 > \frac{\lambda_0 - \lambda}{\lambda_0 - 1} + \frac{\lambda - 1}{\lambda_0 - 1} = 1.$$

This completes the proof. □

Below we extend a lemma which was originally proved in [17].

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Lemma 2.3. *Let* $g\colon (0,\infty)\to(0,\infty)$ *be a nondecreasing function.*

(i) *If*

$$\liminf_{t\to\infty}\frac{g(\lambda t)}{g(t)}>1\quad \textit{for some } \lambda>1$$

then there exist positive constants c_1, κ_1, M *such that*

$$g(t)\ge c_1 t^{\kappa_1}\quad \textit{for all } t\in[M,\infty).$$

(ii) *If*

$$\liminf_{t\downarrow 0}\frac{g(\lambda t)}{g(t)}>1\quad \textit{for some } \lambda>1$$

then there exist positive constants c_2, κ_2, m *such that*

$$g(t)\le c_2 t^{\kappa_2}\quad \textit{for all } t\in(0,m].$$

Proof. The first assertion can be found in [17, Lemma 3.8]. Part (ii) can be shown in a similar way. By our assumption, there exist $c_3>1$ and $m>0$ such that

$$g(\lambda t)\ge c_3 g(t),\qquad 0<t\le m.$$

This implies that, for any $n\in\mathbb{N}$,

$$g(t)\le c_3^{-n}g(\lambda^n t),\qquad 0<t\le\frac{m}{\lambda^{n-1}}.$$

Let $t\in(0,m]$ and set

$$n_t:=\left\lfloor \log_\lambda\frac{m}{t}\right\rfloor+1.$$

From this, we obtain

$$g(t)\le g\bigg(\frac{m}{\lambda^{n_t-1}}\bigg)\le c_3^{-n_t}g\bigg(\lambda^{n_t}\frac{m}{\lambda^{n_t-1}}\bigg)\le c_3^{-\log_\lambda(m/t)}g(\lambda m)=\frac{g(\lambda m)}{m^{\log_\lambda c_3}}t^{\log_\lambda m},$$

thus completing the proof. □

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. (i) We split the proof of this part into four steps.

Step 1. Without loss of generality, we may assume that S has no drift part, i.e. the infinitesimal generator of S is given by

$$\mathcal{L}g(x)=\int_{(0,\infty)}(g(x+y)-g(x))\nu(\mathrm{d}y),\qquad g\in C_b^1(\mathbb{R}).$$

For $\delta\in(0,1]$, set

$$g(x):=\mathrm{e}^{-\theta x^\delta},\qquad x\ge 0.$$

By Dynkin's formula, we have

$$\mathbb{E}g(S_t)=\mathbb{E}g(S_s)+\mathbb{E}\bigg\{\int_s^t\mathcal{L}g(S_u)\,\mathrm{d}u\bigg\},\qquad 0\le s\le t. \tag{2.3}$$

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Step 2. We will now estimate $\mathcal{L}g(x)$ for $x > 0$. Since $\nu(\mathrm{d}y) \geq cy^{-1-\alpha}\,\mathrm{d}y$, we have

$$\begin{aligned}\mathcal{L}g(x) &= \int_{(0,\infty)} (\mathrm{e}^{-\theta(x+y)^\delta} - \mathrm{e}^{-\theta x^\delta})\nu(\mathrm{d}y)\\ &\leq c\mathrm{e}^{-\theta x^\delta}\int_0^\infty (\mathrm{e}^{-\theta x^\delta((1+yx^{-1})^\delta-1)} - 1)\frac{\mathrm{d}y}{y^{1+\alpha}}\\ &= cx^{-\alpha}\mathrm{e}^{-\theta x^\delta}\int_0^\infty (\mathrm{e}^{-\theta x^\delta((1+z)^\delta-1)} - 1)\frac{\mathrm{d}z}{z^{1+\alpha}}.\end{aligned}$$

Noting that

$$z \geq (\theta^{-1}x^{-\delta}+1)^{1/\delta} - 1 \quad\Longrightarrow\quad \mathrm{e}^{-\theta x^\delta((1+z)^\delta-1)} - 1 \leq -(1-\mathrm{e}^{-1}),$$

we conclude that

$$\begin{aligned}\mathcal{L}g(x) &\leq cx^{-\alpha}\mathrm{e}^{-\theta x^\delta}\int_{(\theta^{-1}x^{-\delta}+1)^{1/\delta}-1}^\infty (\mathrm{e}^{-\theta x^\delta((1+z)^\delta-1)} - 1)\frac{\mathrm{d}z}{z^{1+\alpha}}\\ &\leq -(1-\mathrm{e}^{-1})cx^{-\alpha}\mathrm{e}^{-\theta x^\delta}\int_{(\theta^{-1}x^{-\delta}+1)^{1/\delta}-1}^\infty \frac{\mathrm{d}z}{z^{1+\alpha}}\\ &= -(1-\mathrm{e}^{-1})c\alpha^{-1}\mathrm{e}^{-\theta x^\delta}[(\theta^{-1}+x^\delta)^{1/\delta} - x]^{-\alpha}\\ &= -C_1\rho(g(x)),\end{aligned}\tag{2.4}$$

where $C_1 := (1-\mathrm{e}^{-1})c\alpha^{-1}\theta^{\alpha/\delta}$ and

$$\rho(u) := u[(1-\log u)^{1/\delta} - (-\log u)^{1/\delta}]^{-\alpha}, \qquad 0 < u \leq 1.$$

Step 3. Some lengthy, but otherwise elementary, calculations (see Lemma A.1 in the Appendix) yield that ρ is convex and strictly increasing on $(0, 1]$. Therefore, (2.3) and (2.4) together with Tonelli's theorem and Jensen's inequality yield, for $0 \leq s \leq t$,

$$\begin{aligned}\mathbb{E}g(S_t) - \mathbb{E}g(S_s) &\leq -C_1\mathbb{E}\bigg\{\int_s^t \rho(g(S_u))\,\mathrm{d}u\bigg\}\\ &= -C_1\int_s^t \mathbb{E}\rho(g(S_u))\,\mathrm{d}u\\ &\leq -C_1\int_s^t \rho(\mathbb{E}g(S_u))\,\mathrm{d}u.\end{aligned}$$

Setting

$$h(t) := \mathbb{E}g(S_t), \qquad t \geq 0,$$

we find that

$$\frac{h(t)-h(s)}{t-s} \leq -C_1\frac{1}{t-s}\int_s^t \rho(h(u))\,\mathrm{d}u, \qquad 0 \leq s < t.$$

Due to (2.3), h is absolutely continuous on $[0, \infty)$, and so we can let $t \downarrow s$ to obtain

$$h'(s) \leq -C_1\rho(h(s)) \quad \text{for almost all } s \geq 0.$$

According to Lemma 2.1, we have

$$G(h(t)) \leq G(1) - C_1 t, \qquad t \geq 0,$$

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where

$$G(v) := -\int_v^1 \frac{\mathrm{d}u}{\rho(u)}, \qquad 0 < v \le 1.$$

Clearly, G is strictly increasing with $\lim_{r\downarrow 0} G(r) = -\infty$ and $G(1) = 0$. Thus, we obtain

$$h(t) \le G^{-1}(G(1) - C_1 t) = G^{-1}(-C_1 t), \qquad t \ge 0, \tag{2.5}$$

where G^{-1} is the inverse function of G.

Step 4. In order to find a lower bound for $G(v)$, we first observe that, for $v \in (0, 1]$,

$$\begin{aligned} G(v) &= -\int_v^1 u^{-1}[(1-\log u)^{1/\delta} - (-\log u)^{1/\delta}]^\alpha \,\mathrm{d}u \\ &= -\int_0^{-\log v} [(1+s)^{1/\delta} - s^{1/\delta}]^\alpha \,\mathrm{d}s. \end{aligned}$$

It is easy to see that, for $s \ge 0$,

$$(1+s)^{1/\delta} - s^{1/\delta} = \frac{1}{\delta}\int_s^{1+s} u^{(1-\delta)/\delta}\,\mathrm{d}u \le \frac{1}{\delta}(1+s)^{(1-\delta)/\delta} \le \frac{1}{\delta}2^{(1-\delta)/\delta}\max\{s^{(1-\delta)/\delta}, 1\}.$$

Thus, for $v \in (0, \mathrm{e}^{-1})$, we have

$$\begin{aligned} \int_0^{-\log v} [(1+s)^{1/\delta} - s^{1/\delta}]^\alpha \,\mathrm{d}s &\le \frac{1}{\delta^\alpha}2^{\alpha(1-\delta)/\delta} + \frac{1}{\delta^\alpha}2^{\alpha(1-\delta)/\delta}\int_1^{-\log v} s^{\alpha(1-\delta)/\delta}\,\mathrm{d}s \\ &\le C_2 + C_2(-\log v)^{(\alpha(1-\delta)+\delta)/\delta} \end{aligned}$$

for some $C_2 = C_2(\delta, \alpha) > 0$. This means that, for all $v \in (0, \mathrm{e}^{-1})$,

$$G(v) \ge -C_2 - C_2(-\log v)^{(\alpha(1-\delta)+\delta)/\delta},$$

from which we can easily deduce that

$$G^{-1}(-C_1 t) \le \exp[-(C_1 C_2^{-1} t - 1)^{\delta/(\alpha(1-\delta)+\delta)}], \qquad t > 2C_1^{-1}C_2.$$

Combining this with (2.5), the assertion follows.

(ii) We prove these three assertions separately.

- Using the identity

$$x^{-\beta} = \frac{1}{\Gamma(\beta)}\int_0^\infty \mathrm{e}^{-ux}u^{\beta-1}\,\mathrm{d}u, \qquad x \ge 0, \tag{2.6}$$

  and Tonelli's theorem, we obtain, for $t > 0$,

$$\mathbb{E}S_t^{-\beta} = \frac{1}{\Gamma(\beta)}\mathbb{E}\bigg\{\int_0^\infty \mathrm{e}^{-uS_t}u^{\beta-1}\,\mathrm{d}u\bigg\} = \frac{1}{\Gamma(\beta)}\int_0^\infty \mathrm{e}^{-t\phi(u)}u^{\beta-1}\,\mathrm{d}u.$$

  Changing variables according to $v = \phi(u)$, we obtain

$$\begin{aligned} \Gamma(\beta)\mathbb{E}S_t^{-\beta} &= \int_0^\infty \mathrm{e}^{-tv}[\phi^{-1}(v)]^{\beta-1}\frac{\mathrm{d}v}{\phi'(\phi^{-1}(v))} \\ &= \frac{1}{\beta}\int_0^\infty \mathrm{e}^{-tv}\,\mathrm{d}\{[\phi^{-1}(v)]^\beta\}. \end{aligned} \tag{2.7}$$

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Thus, we obtain the lower bound, since, for any $t > 0$,

$$\beta\Gamma(\beta)\mathbb{E}S_t^{-\beta} \geq \int_0^{1/t} \mathrm{e}^{-tv}\,\mathrm{d}\{[\phi^{-1}(v)]^\beta\} \geq \mathrm{e}^{-1}\bigg[\phi^{-1}\bigg(\frac{1}{t}\bigg)\bigg]^\beta.$$

- By (1.5), and Lemmas 2.2(ii) and 2.3(ii), there exist constants $c_1, c_2 > 1$ and $c_3, c_4, \kappa > 0$ such that

$$\phi^{-1}(2s) \leq c_1\phi^{-1}(s), \qquad 0 < s \leq c_2, \tag{2.8}$$

$$\phi(s)| \leq c_3 s^\kappa, \qquad 0 < s \leq c_2, \tag{2.9}$$

$$\phi^{-1}(s) \leq \mathrm{e}^{c_4 s}, \qquad s \geq c_2. \tag{2.10}$$

By (2.7), the integration by parts formula, and (2.10), we find that, for $t > \beta c_4$,

$$\beta\Gamma(\beta)\mathbb{E}S_t^{-\beta} = \int_0^\infty [\phi^{-1}(v)]^\beta\,\mathrm{d}\{\mathrm{e}^{-tv}\} = \int_0^\infty \bigg[\phi^{-1}\bigg(\frac{s}{t}\bigg)\bigg]^\beta \mathrm{e}^{-s}\,\mathrm{d}s.$$

If $t > c_2^{-1}$ and $s \in (1, c_2 t)$, we set $k_s := \lfloor \log_2 s \rfloor$ and use (2.8) $1 + k_s$ times to obtain

$$\begin{aligned}
\phi^{-1}\bigg(\frac{s}{t}\bigg) &\leq \phi^{-1}\bigg(2^{1+k_s}\frac{1}{t}\bigg)\\
&\leq c_1^{1+k_s}\phi^{-1}\bigg(\frac{1}{t}\bigg)\\
&\leq c_1^{1+\log_2 s}\phi^{-1}\bigg(\frac{1}{t}\bigg)\\
&= c_1\phi^{-1}\bigg(\frac{1}{t}\bigg)s^{\log_2 c_1}.
\end{aligned}$$

Thus, for $t > \max\{c_2^{-1}, 2\beta c_4\}$,

$$\begin{aligned}
\beta\Gamma(\beta)\mathbb{E}S_t^{-\beta} &= \bigg(\int_0^1 + \int_1^{c_2 t} + \int_{c_2 t}^\infty\bigg)\bigg[\phi^{-1}\bigg(\frac{s}{t}\bigg)\bigg]^\beta \mathrm{e}^{-s}\,\mathrm{d}s\\
&\leq \bigg[\phi^{-1}\bigg(\frac{1}{t}\bigg)\bigg]^\beta + c_1^\beta\bigg[\phi^{-1}\bigg(\frac{1}{t}\bigg)\bigg]^\beta \int_1^{c_2 t} s^{\beta\log_2 c_1}\mathrm{e}^{-s}\,\mathrm{d}s\\
&\quad + \int_{c_2 t}^\infty \mathrm{e}^{-(1-\beta c_4/t)s}\,\mathrm{d}s\\
&\leq \bigg[\phi^{-1}\bigg(\frac{1}{t}\bigg)\bigg]^\beta + c_1^\beta\Gamma(1+\beta\log_2 c_1)\bigg[\phi^{-1}\bigg(\frac{1}{t}\bigg)\bigg]^\beta + 2\mathrm{e}^{-c_2 t/2}.
\end{aligned}$$

Since, from (2.9), it follows that

$$\limsup_{t\to\infty}\frac{\mathrm{e}^{-c_2 t/(2\beta)}}{\phi^{-1}(1/t)} = \limsup_{s\downarrow 0}\frac{1}{s}\exp\bigg[-\frac{c_2}{2\beta\phi(s)}\bigg] \leq \limsup_{s\downarrow 0}\frac{1}{s}\exp\bigg[-\frac{c_2}{2\beta c_3 s^\kappa}\bigg] = 0,$$

we can find some $c_5 = c_5(\beta) > 0$ such that

$$\mathrm{e}^{-c_2 t/2} \leq \bigg[\phi^{-1}\bigg(\frac{1}{t}\bigg)\bigg]^\beta, \qquad t > c_5.$$

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Therefore, we conclude that

$$\beta\Gamma(\beta)\mathbb{E}S_t^{-\beta} \le (3 + c_1^{\beta}\Gamma(1+\beta\log_2 c_1))\bigg[\phi^{-1}\bigg(\frac{1}{t}\bigg)\bigg]^{\beta}, \qquad t > \max\{c_2^{-1}, 2\beta c_4, c_5\},$$

which proves our claim.

- From our assumption and Lemma 2.2(i) we know that

$$\lim_{s\to\infty}\phi(s) = \infty \quad \text{and} \quad \limsup_{s\to\infty}\frac{\phi^{-1}(2s)}{\phi^{-1}(s)} < \infty.$$

Thus, there is a constant $c_6 > 1$ such that

$$\phi^{-1}(2s) \le c_6\phi^{-1}(s) \quad \text{for all } s \ge 1,$$

from which we obtain

$$\phi^{-1}(2^n s) \le c_6^n\phi^{-1}(s) \quad \text{for all } s \ge 1 \text{ and } n \in \mathbb{N}.$$

This, together with (2.7), yields that, for any $t \in (0, 1]$,

$$\begin{aligned}\beta\Gamma(\beta)\mathbb{E}S_t^{-\beta} &= \bigg(\int_0^{1/t} + \sum_{k=0}^{\infty}\int_{2^k/t}^{2^{k+1}/t}\bigg)\mathrm{e}^{-tv}\,\mathrm{d}\{[\phi^{-1}(v)]^{\beta}\} \\ &\le \bigg[\phi^{-1}\bigg(\frac{1}{t}\bigg)\bigg]^{\beta} + \sum_{k=0}^{\infty}\mathrm{e}^{-2^k}\bigg[\phi^{-1}\bigg(2^{k+1}\frac{1}{t}\bigg)\bigg]^{\beta} \\ &\le \bigg(1 + \sum_{k=0}^{\infty}\mathrm{e}^{-2^k}c_6^{(k+1)\beta}\bigg)\bigg[\phi^{-1}\bigg(\frac{1}{t}\bigg)\bigg]^{\beta}.\end{aligned}$$

(iii) Assume that $\nu(\mathrm{d}y) \ge cy^{-1-\alpha}\,\mathrm{d}y$. Using the Lévy–Itô decomposition of the Lévy process S_t, we denote by $\tilde{S}_t$ that part of S_t whose jumps are governed by the Lévy measure $cy^{-1-\alpha}\,\mathrm{d}y$. Clearly, $\tilde{S}_t$ has, up to a constant, the same jump behaviour as an α-stable subordinator; moreover,

$$S_t \ge \tilde{S}_t, \qquad t \ge 0.$$

By (2.6) and Tonelli's theorem, we find that, for $t > 0$,

$$\begin{aligned}\mathbb{E}\log^{-\gamma}(1+S_t) &\le \mathbb{E}\log^{-\gamma}(1+\tilde{S}_t) \\ &= \frac{1}{\Gamma(\gamma)}\mathbb{E}\bigg\{\int_0^{\infty}\mathrm{e}^{-u\log(1+\tilde{S}_t)}u^{\gamma-1}\,\mathrm{d}u\bigg\} \\ &= \frac{1}{\Gamma(\gamma)}\int_0^{\infty}\mathbb{E}\{(1+\tilde{S}_t)^{-u}\}u^{\gamma-1}\,\mathrm{d}u.\end{aligned}$$

Since $\tilde{S}_t$ behaves like an α-stable subordinator, we obtain, from [1, Equation (14)],

$$\mathbb{P}(\tilde{S}_t \in \mathrm{d}s) \le C_{\alpha,c}ts^{-1-\alpha}\mathrm{e}^{-ts^{-\alpha}}\,\mathrm{d}s,$$

where $C_{\alpha,c} > 0$ is a constant depending only on α and c; using the change of variable $v = ts^{-\alpha}$ and Tonelli's theorem, we obtain

$$\begin{aligned}\mathbb{E}\log^{-\gamma}(1+S_t) &\le \frac{C_{\alpha,c}}{\Gamma(\gamma)}t\int_0^\infty\biggl(\int_0^\infty (1+s)^{-u}s^{-1-\alpha}\mathrm{e}^{-ts^{-\alpha}}\,\mathrm{d}s\biggr)u^{\gamma-1}\,\mathrm{d}u\\ &= \frac{C_{\alpha,c}}{\alpha\Gamma(\gamma)}\int_0^\infty\biggl(\int_0^\infty\biggl(1+\biggl(\frac{t}{v}\biggr)^{1/\alpha}\biggr)^{-u}\mathrm{e}^{-v}\,\mathrm{d}v\biggr)u^{\gamma-1}\,\mathrm{d}u\\ &= \frac{C_{\alpha,c}}{\alpha\Gamma(\gamma)}\int_0^\infty\biggl(\int_0^\infty\biggl(1+\biggl(\frac{t}{v}\biggr)^{1/\alpha}\biggr)^{-u}u^{\gamma-1}\,\mathrm{d}u\biggr)\mathrm{e}^{-v}\,\mathrm{d}v\\ &= \frac{C_{\alpha,c}}{\alpha}\int_0^\infty\log^{-\gamma}\biggl(1+\biggl(\frac{t}{v}\biggr)^{1/\alpha}\biggr)\mathrm{e}^{-v}\,\mathrm{d}v\\ &= \frac{C_{\alpha,c}}{\alpha}\biggl(\int_0^1+\int_1^\infty\biggr)\log^{-\gamma}\biggl(1+\biggl(\frac{t}{v}\biggr)^{1/\alpha}\biggr)\mathrm{e}^{-v}\,\mathrm{d}v\\ &=: \frac{C_{\alpha,c}}{\alpha}(I_1+I_2).\end{aligned}$$

First,

$$I_1 \le \log^{-\gamma}(1+t^{1/\alpha})\int_0^1 \mathrm{e}^{-v}\,\mathrm{d}v = (1-\mathrm{e}^{-1})\log^{-\gamma}(1+t^{1/\alpha}).$$

Since the function $x \mapsto x^{-1}\log(1+x)$ is strictly decreasing for $x > 0$, we have

$$\frac{\log(1+x)}{x} > \frac{\log(1+\lambda)}{\lambda}, \qquad 0 < x < \lambda. \tag{2.11}$$

For $v > 1$, using this inequality with $x = (tv^{-1})^{1/\alpha}$ and $\lambda = t^{1/\alpha}$, we obtain

$$\log\biggl(1+\biggl(\frac{t}{v}\biggr)^{1/\alpha}\biggr) > \frac{\log(1+t^{1/\alpha})}{t^{1/\alpha}}\biggl(\frac{t}{v}\biggr)^{1/\alpha} = \frac{\log(1+t^{1/\alpha})}{v^{1/\alpha}},$$

and so

$$I_2 \le \log^{-\gamma}(1+t^{1/\alpha})\int_1^\infty v^{\gamma/\alpha}\mathrm{e}^{-v}\,\mathrm{d}v \le \Gamma\biggl(\frac{\gamma}{\alpha}+1\biggr)\log^{-\gamma}(1+t^{1/\alpha}).$$

These estimates give the upper bound in Theorem 2.1(iii)(a).

If $\nu(\mathrm{d}y) = cy^{-1-\alpha}\,\mathrm{d}y$ then, for any $t > 0$, the distribution of S_t coincides with that of $t^{1/\alpha}S_1$, and it is easy to see that $\mathbb{E}\log(1+S_t) < \infty$ for all $t > 0$. Since the function $x \mapsto x^{-\gamma}$ is convex for $x \in (0,\infty)$, it follows from Jensen's inequality that

$$\mathbb{E}\log^{-\gamma}(1+S_t) \ge [\mathbb{E}\log(1+S_t)]^{-\gamma} = [\mathbb{E}\log(1+t^{1/\alpha}S_1)]^{-\gamma}.$$

For $t \ge 1$, since

$$\begin{aligned}\log(1+t^{1/\alpha}S_1) &\le \log[(1+t)^{1/\alpha}+(1+t)^{1/\alpha}S_1]\\ &= \log(1+S_1)+\frac{1}{\alpha}\log(1+t)\\ &\le \biggl(\frac{\log(1+S_1)}{\log 2}+\frac{1}{\alpha}\biggr)\log(1+t)\\ &\le \biggl(\frac{\log(1+S_1)}{\log 2}+\frac{1}{\alpha}\biggr)\log(1+t^{1/\alpha}),\end{aligned}$$

it holds that

$$\mathbb{E}\log^{-\gamma}(1+S_t) \geq \left(\frac{\mathbb{E}\log(1+S_1)}{\log 2}+\frac{1}{\alpha}\right)^{-\gamma}\log^{-\gamma}(1+t^{1/\alpha}). \tag{2.12}$$

For $t\in(0,1)$, from Lemma A.2 with $\tau=t^{1/\alpha}\in(0,1)$ and $x=S_1$, we obtain

$$\begin{aligned}\mathbb{E}\log^{-\gamma}(1+S_t) &\geq \left[\mathbb{E}\left\{\frac{1}{\log 2}\log(1+t^{1/\alpha})\log(1+S_1)\right\}\right]^{-\gamma}\\ &= \left(\frac{\mathbb{E}\log(1+S_1)}{\log 2}\right)^{-\gamma}\log^{-\gamma}(1+t^{1/\alpha}). \end{aligned}\tag{2.13}$$

Therefore, from (2.12) and (2.13), we obtain the desired lower bound in Theorem 2.1(iii)(b). □

3. Proof of Theorem 1.1

Theorem 1.1 follows at once from Lemma 3.1 below together with the corresponding upper moment bounds for subordinators derived in Theorem 2.1.

Lemma 3.1. *If (1.1) holds with some rate function r then so does (1.4) with rate function $r_\phi(t)=\mathbb{E}r(S_t)$.*

Proof. By the definition of $P_\phi^t(x,\mathrm{d}y)$ and (1.1), we find that, for all $t>0$ and $x\in E$,

$$\begin{aligned}\|P_\phi^t(x,\cdot)-\pi\|_f &= \left\|\int_{[0,\infty)}(P^s(x,\cdot)-\pi)\mu_t(\mathrm{d}s)\right\|_f\\ &\leq \int_{[0,\infty)}\|P^s(x,\cdot)-\pi\|_f\,\mu_t(\mathrm{d}s)\\ &\leq C(x)\int_{[0,\infty)}r(s)\mu_t(\mathrm{d}s)\\ &= C(x)\mathbb{E}r(S_t),\end{aligned}$$

and the claim follows. □

4. Examples

In this section we discuss three models for which we are able to obtain explicit convergence rates, so that our main result Theorem 1.1 can be applied.

4.1. Q-processes on $\mathbb{Z}_+$

Let $Q=(q_{ij})_{i,j\in\mathbb{Z}_+}$ be a Q-matrix with

$$q_{00}=-\lambda_0,\qquad q_{0i}=\lambda_0 p_i,\qquad q_{i0}=-q_{ii}=\lambda_i,\qquad i\geq 1,$$

and $q_{ij}=0$ otherwise, where $(p_i)_{i\geq1}$ and $(\lambda_i)_{i\geq 0}$ are two sequences of positive numbers with $\sum_{i=1}^\infty p_i=1$ and $\sup_{i\geq0}\lambda_i<\infty$. It is well known that there exists a unique Q-process with transition semigroup $P^t=\sum_{n=0}^\infty (tQ)^n/n!$; see [5, Corollary 2.24]. Moreover, we assume that

$$\liminf_{i\to\infty}\lambda_i=0 \quad\text{and}\quad \sum_{i=1}^\infty p_i\lambda_i^{-1}<\infty.$$

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Under these assumptions, it is easy to see that the process admits an invariant distribution π given by

$$\pi_0 = \left(1 + \lambda_0 \sum_{j=1}^{\infty} p_j \lambda_j^{-1}\right)^{-1}, \qquad \pi_i = \pi_0 \lambda_0 p_i \lambda_i^{-1},\ i \geq 1.$$

For this toy model, it is known, see [14, (the proofs of) Propositions 12 and 14], that:

- if

$$\sum_{i=1}^{\infty} p_i (1 \vee \lambda_i^{-1}) \lambda_i^{-1/2} \mathrm{e}^{\theta^2 \lambda_i^{-1}} < \infty \quad \text{for some } \theta > 0$$

then (1.1) holds for any $q \in [0, 1]$ with

$$r(t) = \mathrm{e}^{-2\theta q \sqrt{t}}, \qquad f(i) = (1 + \lambda_i^{-1/2} \mathrm{e}^{\theta^2 \lambda_i^{-1}})^{1-q};$$

- if

$$\sum_{i=1}^{\infty} p_i \lambda_i^{-1-\theta} < \infty \quad \text{for some } \theta \geq 0$$

then (1.1) holds for any $\beta \in [0, \theta]$ with

$$r(t) = (1+t)^{-\beta}, \qquad f(i) = 1 + \lambda_i^{\beta - \theta};$$

- if

$$\sum_{i=1}^{\infty} p_i (1 \vee \lambda_i^{-1}) \log^{\theta}(1 \vee \lambda_i^{-1}) < \infty \quad \text{for some } \theta \geq 0$$

then (1.1) holds for any $\gamma \in [0, \theta]$ with

$$r(t) = [1 + \log(1+t)]^{-\gamma}, \qquad f(i) = [1 + \log(1 \vee \lambda_i^{-1})]^{\theta - \gamma}.$$

4.2. Diffusion processes on $\mathbb{R}^d$

Consider the stochastic differential equation (SDE)

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t, \qquad X_0 = x, \tag{4.1}$$

where $\{B_t : t \geq 0\}$ is a standard d-dimensional Brownian motion, $b\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ are locally Lipschitz, σ is bounded, and the smallest eigenvalue of $a(x) := \sigma(x)\sigma^{\top}(x)$ is bounded away from 0 in every bounded domain. If there exist constants $p \in (0, 1)$, $C > 0$, and $M > 0$ such that

$$\langle b(x), x \rangle \leq -C|x|^{1-p}, \qquad |x| \geq M,$$

then (4.1) has a unique solution with infinite lifetime and an invariant probability measure π; moreover, (1.1) holds for any $q \in (0, 1)$ with

$$r(t) = \exp[-C_1 q t^{(1-p)/(1+p)}], \qquad f(x) = 1 + (1 + |x|)^{-2p(1-q)} \exp[C_2(1-q)|x|^{1-p}]$$

for some positive constants C_1 and C_2; see [10, Theorem 5.4].

Set

$$K := \frac{\lambda_+ - \lambda_- + \Lambda}{2\lambda_+},$$

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where

$$\lambda_+ := \sup_{x\neq 0} \frac{\langle a(x)x, x\rangle}{|x|^2}, \qquad \lambda_- := \inf_{x\neq 0} \frac{\langle a(x)x, x\rangle}{|x|^2}, \qquad \Lambda := \sup_{x\in\mathbb{R}^d} \operatorname{Tr}(a(x)).$$

In order to obtain algebraic rates of convergence, we need the following condition.

(A) There exist constants $C > K$ and $M > 0$ such that $\langle b(x), x\rangle \le -C\lambda_+$, $|x| \ge M$.

It is not difficult to see that under (A) there is a unique nonexplosive solution to the SDE (4.1).

Proposition 4.1. *Assume that (A) holds. Then (1.1) holds for any* $\beta \in (0, C-K)$ *and any* $m \in (0, C-K-\beta)$ *with*

$$r(t) = (1+t)^{-\beta}, \qquad f(x) = 1 + |x|^{2m}.$$

4.3. SDEs driven by α-stable processes

Consider the SDE

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \mathrm{d}Z_t, \qquad X_0 = x, \tag{4.2}$$

where $\{Z_t : t \ge 0\}$ is an α-stable ($0 < \alpha < 2$) Lévy process on $\mathbb{R}^d$, and $b\colon \mathbb{R}^d \to \mathbb{R}^d$ is continuous such that

$$\langle b(x) - b(y), x - y\rangle \le L|x-y|^2, \qquad x, y \in \mathbb{R}^d \quad \text{for some } L \in \mathbb{R}.$$

Under these assumptions there exists a unique nonexplosive solution to the SDE (4.2), which is (strong) Feller by the dimension-free Harnack inequality, see [7], [28], and Lebesgue irreducible; see, e.g. [19].

Proposition 4.2. *(i) If there exist constants* $p \ge 0$, $C > 0$, *and* $M > 0$ *such that*

$$\langle b(x), x\rangle \le -C|x|^2 \log^{-p}(1+|x|), \qquad |x| \ge M, \tag{4.3}$$

then (1.1) holds for any $q \in (0,1)$ *and any* $m \in (0, 1 \wedge (d-1+\alpha))$ *with*

$$r(t) = \exp[-\tilde{C} q t^{1/(1+p)}], \qquad f(x) = 1 + |x|^{m(1-q)}$$

for some constant $\tilde{C} = \tilde{C}(C, p, M, m) > 0$.

(ii) If there exist $p \in (0, 1 \wedge (d-1+\alpha))$, $C > 0$, *and* $M > 0$ *such that*

$$\langle b(x), x\rangle \le -C|x|^{p+2-1\wedge(d-1+\alpha)}, \qquad |x| \ge M, \tag{4.4}$$

then (1.1) holds for any $\beta \in (0, p/(1\wedge(d-1+\alpha)-p))$ *and any* $0 < m < p - \beta(1 \wedge (d-1+\alpha) - p)$ *with*

$$r(t) = (1+t)^{-\beta}, \qquad f(x) = 1 + |x|^m.$$

Remark 4.1. If (4.3) holds with $p = 0$ then we obtain exponential rates of convergence in Proposition 4.2(i); see, e.g. [19, Lemma 2.4].

4.4. Proofs of Propositions 4.1 and 4.2

Our proofs of Propositions 4.1 and 4.2 are based on the following Foster–Lyapunov criterion, which is a special case of [10, Theorem 3.2]; see also [12, Theorem 2.8] for the corresponding result for discrete-time Markov chains.

Let X be a Markov process with generator $(A, \mathcal{D})$. It is well known that, for $g := Af$ and $f \in \mathcal{D}$, the process $M_t^f := f(X_t) - f(X_0) - \int_0^t g(X_s)\,\mathrm{d}s$ is a martingale. Recall that the extended generator consists of all pairs (f, g) of measurable functions such that M_t^f is a local martingale for some unique g; note that we do not require $f \in \mathcal{D}$ nor $g = Af$; see [3, pp. 25, 26] for details. We denote the extended generator by $(\mathcal{A}, D(\mathcal{A}))$.

Proposition 4.3. *Let X be a Markov process on the state space $(E, \mathcal{B}(E))$ with extended generator $(\mathcal{A}, D(\mathcal{A}))$. Assume that*

(i) *some skeleton chain (i.e. a Markov chain with transition kernel P^T for some $T > 0$) is ψ-irreducible for some σ-finite measure ψ;*

(ii) *there exist a closed petite set B, a constant $b > 0$, a continuous function $V \in D(\mathcal{A})$, $V\colon E \to [1, \infty)$ with $\sup_B V < \infty$, and a nondecreasing differentiable concave function $\varphi\colon [1, \infty) \to (0, \infty)$ satisfying $\lim_{x\to\infty} \varphi'(x) = 0$ and*

$$\mathcal{A}V(x) \le -\varphi \circ V(x) + b\,\mathbf{1}_B(x), \qquad x \in E, \tag{4.5}$$

where $\mathbf{1}$ is the indicator function. Then there exists an invariant probability measure π such that $\pi(\varphi \circ V) < \infty$, and (1.1) holds for any $q \in (0, 1)$ with

$$r(t) = 1 \wedge (\varphi \circ H_\varphi^{-1}(t))^{-q} \quad \textit{and} \quad f = 1 \vee (\varphi \circ V)^{1-q},$$

where H_φ^{-1} is the inverse of the function

$$H_\varphi(u) = \int_1^u \frac{\mathrm{d}x}{\varphi(x)}, \qquad u \ge 1.$$

Proof of Proposition 4.1. As was pointed out in [10, p. 908], it is a standard argument that (A) ensures the existence of a unique invariant probability measure π, and that any skeleton chain is π-irreducible. Thus, we know that every compact set is a closed petite set; see [27, Theorems 5.1 and 7.1].

Fix $\beta \in (0, C - K)$ and $m \in (0, C - K - \beta)$. Observe that $C^2(\mathbb{R}^d) \subset D(\mathcal{A})$. Choose a test function $V \in C^2(\mathbb{R}^d)$ such that $V(x) = (1 + |x|)^{2m+2\beta+2}$ for $|x| > M$. By (A), we obtain, for all $|x| > M$,

$$\begin{aligned}
\mathcal{A}V(x) \le{}& -C\lambda_+(2m + 2\beta + 2)(1 + |x|)^{2m+2\beta+1}|x|^{-1}\\
&+ \lambda_+(m + \beta + 1)(2m + 2\beta + 1)(1 + |x|)^{2m+2\beta}\\
&- \lambda_-(m + \beta + 1)(1 + |x|)^{2m+2\beta+1}|x|^{-1}\\
&+ \Lambda(m + \beta + 1)(1 + |x|)^{2m+2\beta+1}|x|^{-1}\\
\le{}& -2\lambda_+(m + \beta + 1)(C - K - \beta - m)(1 + |x|)^{2m+2\beta}\\
=:{}& -C_1(1 + |x|)^{2m+2\beta}.
\end{aligned}$$

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This implies that the Foster–Lyapunov condition (4.5) holds with $\varphi(x) = C_1 x^{(m+\beta)/(m+\beta+1)}$, $B = \{x \in \mathbb{R}^d : |x| \le M\}$, and some constant $b > 0$. Using Proposition 4.3 with $q = \beta/(m+\beta)$, we know that (1.1) holds with

$$r(t) = 1 \wedge \bigg[C_1^{-\beta/(m+\beta)}\bigg(1 + \frac{C_1}{m+\beta+1}t\bigg)^{-\beta}\bigg], \qquad f(x) = 1 \vee [C_1^{m/(m+\beta)}(1+|x|)^{2m}].$$

This completes the proof. □

Proof of Proposition 4.2. (i) First of all, it is clear that

$$\bigg\{V \in C^2(\mathbb{R}^d) : \bigg|\int_{|y|>1} (V(x+y) - V(x)) \frac{\mathrm{d}y}{|y|^{d+\alpha}}\bigg| < \infty \text{ for all } x \in \mathbb{R}^d\bigg\} \subset D(\mathcal{A}).$$

Fix $m \in (0, 1 \wedge (d-1+\alpha))$ and choose a test function $V \in C^2(\mathbb{R}^d)$ such that $V(x) = (1+|x|)^m$ for $|x| > M$. Then $V \in D(\mathcal{A})$ and

$$\mathcal{A}V(x) = \langle b(x), \nabla V(x)\rangle + \mathcal{A}_1 V(x) + \mathcal{A}_2 V(x),$$

where

$$\mathcal{A}_1 V(x) := \int_{|y|>1} (V(x+y) - V(x)) \frac{c_{d,\alpha}}{|y|^{d+\alpha}}\,\mathrm{d}y,$$

$$\mathcal{A}_2 V(x) := \int_{0<|y|\le 1} (V(x+y) - V(x) - \langle \nabla V(x), y\rangle) \frac{c_{d,\alpha}}{|y|^{d+\alpha}}\,\mathrm{d}y.$$

Since $|\mathcal{A}_1 V(x)| = o(1)$ and $|\mathcal{A}_2 V(x)| = o(1)$ as $|x| \to \infty$, see the proof of [19, Lemma 2.4], we obtain from (4.3) that, for $|x| > M$,

$$\begin{aligned}\mathcal{A}V(x) &\le -Cm(1+|x|)^{m-1}|x|\log^{-p}(1+|x|) + o(1)\\ &\le -C_1 V(x)(1 + p + \log V(x))^{-p}\end{aligned}$$

for some $C_1 = C_1(C, p, M, m) > 0$. Hence, (4.5) is satisfied with $\varphi(x) = C_1 x(1 + p + \log x)^{-p}$, and the claim follows from Proposition 4.3 and some straightforward calculations.

(ii) Set $\varrho := 1 \wedge (d-1+\alpha) - p$ and fix $\beta \in (0, p/\varrho)$ and $m \in (0, p - \beta\varrho)$. Choose $V \in C^2(\mathbb{R}^d)$ such that $V(x) = (1+|x|)^{m+\varrho(\beta+1)}$ for $|x| > M$. As in part (i), it is not difficult to obtain from (4.4) that (4.5) holds with $\varphi(x) = C_2 x^{(m+\varrho\beta)/(m+\varrho(\beta+1))}$ for some $C_2 = C_2(C, p, M, m) > 0$. Thus, the assertion follows from Proposition 4.3 with $q = \varrho\beta/(m+\varrho\beta)$. □

Appendix A

Lemma A.1. *Let* $\tau \ge 1$, $\alpha \in (0, 1)$, *and*

$$g(x) = x[(1-\log x)^\tau - (-\log x)^\tau]^{-\alpha}, \qquad 0 < x \le 1.$$

Then the function g *is convex and strictly increasing on* $(0, 1]$.

Proof. Obviously, we need to prove the statement only for $x \in (0, 1)$. By a direct calculation, we find that, for $x \in (0, 1)$,

$$\begin{aligned}g'(x) &= \frac{g(x)}{x} + \tau\alpha\frac{g(x)}{x}[(1-\log x)^\tau - (-\log x)^\tau]^{-1}[(1-\log x)^{\tau-1} - (-\log x)^{\tau-1}]\\ &= \xi(x)\eta(x),\end{aligned}$$

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where

$$\xi(x) := \frac{g(x)}{x} = [(1-\log x)^\tau - (-\log x)^\tau]^{-\alpha},$$
$$\eta(x) := 1 + \tau\alpha[(1-\log x)^\tau - (-\log x)^\tau]^{-1}[(1-\log x)^{\tau-1} - (-\log x)^{\tau-1}].$$

Since $\xi(x) > 0$ and $\eta(x) > 0$ for all $x \in (0,1)$, g is strictly increasing on $(0,1)$. Noting that $g''(x) = \xi'(x)\eta(x) + \xi(x)\eta'(x)$, it suffices to prove that $\xi'(x) \geq 0$ and $\eta'(x) \geq 0$ for all $x \in (0,1)$. For $x \in (0,1)$, we have

$$\xi'(x) = \frac{\tau\alpha}{x}[(1-\log x)^\tau - (-\log x)^\tau]^{-\alpha-1}[(1-\log x)^{\tau-1} - (-\log x)^{\tau-1}] \geq 0,$$

and

$$\begin{aligned}\frac{1}{\tau\alpha}\eta'(x) &= \frac{\tau}{x}[(1-\log x)^\tau - (-\log x)^\tau]^{-2}[(1-\log x)^{\tau-1} - (-\log x)^{\tau-1}]^2 \\ &\quad - \frac{\tau-1}{x}[(1-\log x)^\tau - (-\log x)^\tau]^{-1}[(1-\log x)^{\tau-2} - (-\log x)^{\tau-2}] \\ &= \frac{1}{x}[(1-\log x)^\tau - (-\log x)^\tau]^{-2}\zeta(x),\end{aligned}$$

where

$$\begin{aligned}\zeta(x) &:= \tau[(1-\log x)^{\tau-1} - (-\log x)^{\tau-1}]^2 \\ &\quad - (\tau-1)[(1-\log x)^\tau - (-\log x)^\tau][(1-\log x)^{\tau-2} - (-\log x)^{\tau-2}] \\ &= (1-\log x)^{2\tau-2} + (-\log x)^{2\tau-2} - 2\tau(1-\log x)^{\tau-1}(-\log x)^{\tau-1} \\ &\quad + (\tau-1)(1-\log x)^\tau(-\log x)^{\tau-2} + (\tau-1)(1-\log x)^{\tau-2}(-\log x)^\tau.\end{aligned}$$

It remains to check that $\zeta(x) \geq 0$ for all $x \in (0,1)$. From the elementary inequality

$$y^2 + z^2 \geq 2yz, \qquad y, z \geq 0,$$

it follows that, for $x \in (0,1)$,

$$\begin{aligned}\zeta(x) &\geq 2(1-\log x)^{\tau-1}(-\log x)^{\tau-1} - 2\tau(1-\log x)^{\tau-1}(-\log x)^{\tau-1} \\ &\quad + (\tau-1)(1-\log x)^\tau(-\log x)^{\tau-2} + (\tau-1)(1-\log x)^{\tau-2}(-\log x)^\tau \\ &= (\tau-1)(1-\log x)^{\tau-2}(-\log x)^{\tau-2} \\ &\geq 0,\end{aligned}$$

and the proof is complete. □

Lemma A.2. *Let* $\tau \in (0,1)$ *and* $x \geq 0$. *Then*

$$\log(1+\tau x) \leq \frac{1}{\log 2}\log(1+\tau)\log(1+x).$$

Proof. From (2.11) we infer that

$$\frac{\log(1+\tau)}{\tau} > \log 2.$$

This, together with Bernoulli's inequality, yields

$$1 + \tau x \leq (1+x)^\tau \leq (1+x)^{\log(1+\tau)/\log 2},$$

and the claim follows by taking the logarithm on both sides. □

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