Bull. Aust. Math. Soc. **100** (2019), 5–12
doi:10.1017/S0004972719000091

ON THE CROSSING NUMBER OF THE CARTESIAN PRODUCT OF A SUNLET GRAPH AND A STAR GRAPH

MICHAEL HAYTHORPE✉ and ALEX NEWCOMBE

(Received 31 October 2018; accepted 20 December 2018; first published online 8 February 2019)

Abstract

The exact crossing number is only known for a small number of families of graphs. Many of the families for which crossing numbers have been determined correspond to cartesian products of two graphs. Here, the cartesian product of the sunlet graph, denoted $\mathcal{S}_n$, and the star graph, denoted $K_{1,m}$, is considered for the first time. It is proved that the crossing number of $\mathcal{S}_n \square K_{1,2}$ is n, and the crossing number of $\mathcal{S}_n \square K_{1,3}$ is $3n$. An upper bound for the crossing number of $\mathcal{S}_n \square K_{1,m}$ is also given.

2010 *Mathematics subject classification*: primary 05C10; secondary 68R10.

Keywords and phrases: crossing number, cartesian product, star, sunlet graph.

1. Introduction

Consider a simple graph G with vertex set $V(G)$ and edge set $E(G)$. A *drawing* is an embedding of the graph in the plane, in the sense that each vertex $v \in V(G)$ is assigned coordinates in the plane, and each edge $e \in E(G)$ is drawn as a curve starting and ending at the coordinates of its end points. A *good drawing* is one in which edges have at most one point in common, no more than two edges cross at a single point and edges which share an end point do not cross. For a given drawing D of the graph G, the *crossings* in the drawing, denoted $cr_D(G)$, can then be computed as the number of times two edges intersect at points other than at their end points. The *crossing number* $cr(G)$ of a graph G is the smallest number of crossings over all possible drawings of G. It is well known that any drawing of G which contains $cr(G)$ crossings is a good drawing.

The *crossing number problem*, being the problem of determining the crossing number of a graph, is known to be NP-hard [11]; this is true even for graphs constructed by adding a single edge to a planar graph [7]. Indeed, the crossing number problem is known to be notoriously difficult and is still unsolved even for very small instances. For example, the crossing number of K_{13} has still not been determined, although it is known to be either 223 or 225 [1, 18]. However, the crossing number has been determined for some infinite families of graphs. In many such cases, the family is

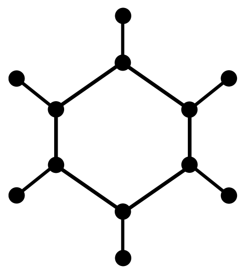

Figure 1. The sunlet graph $\mathcal{S}_6$.

created by taking the cartesian product of members of two smaller graph families. To the best of the authors' knowledge, the first such result published was due to Harary *et al.* [13] in 1973, who conjectured that the crossing number of $C_m \square C_n$, that is, the cartesian product of two arbitrarily large cycles, would be $n(m-2)$ for $n \geq m \geq 3$. To date, this conjecture remains unproven, although a number of partial results have been determined. Specifically, the conjecture is known to be true for $m \leq 7$ and also whenever $n \geq m(m-1)$ [2, 3, 5, 10, 12, 19, 20]. Other infinite graph families, for which the crossing numbers of their cartesian products have been studied, include paths and stars [6, 14, 15], complete graphs and cycles [21], cycles and stars [14, 15], wheels and trees [17] and cycles with the 2-power of paths [16].

In the present work, we expand this growing literature by considering the cartesian product of a sunlet graph and a star graph. The sunlet graph on $2n$ vertices, denoted $\mathcal{S}_n$ for $n \geq 3$, is constructed by attaching n pendant edges to an n-cycle C_n; see Figure 1 for an example of $\mathcal{S}_6$. The star graph on $m+1$ vertices consists of one vertex of degree m attached to m vertices of degree 1. It is usually denoted S_m; however, to avoid confusion with the notation for the sunlet graph, we note that the star graph is equivalent to the complete bipartite graph $K_{1,m}$ and use that notation instead. We will show that $cr(\mathcal{S}_n \square K_{1,m}) \leq nm(m-1)/2$ for $n \geq 3$ and $m \geq 1$. We will also prove that the crossing number meets this bound precisely for $m = \{1, 2, 3\}$ and conjecture that it does so for all $m \in \mathbb{Z}+$.

2. Upper bound

We begin by providing an upper bound for $cr(\mathcal{S}_n \square K_{1,m})$. In what follows, let the vertex labels of $K_{1,m}$ be v_0 for the vertex of degree m and $v_1, v_2, \ldots, v_m$ for the vertices of degree 1. Let the vertex labels of $\mathcal{S}_n$ be $u_0, u_1, u_2, \ldots, u_{n-1}$ for the vertices on the cycle and let u'_i denote the pendant vertex attached to u_i.

Theorem 2.1. *The crossing number of $\mathcal{S}_n \square K_{1,m}$ is no larger than $nm(m-1)/2$ for $n \geq 3$, $m \geq 1$.*

Proof. It is easy to check that $\mathcal{S}_n \square K_{1,1}$ is planar; for instance, a planar drawing of $\mathcal{S}_6 \square K_{1,1}$ is illustrated in Figure 2, which can obviously be extended for any n. It then

https://doi.org/10.1017/S0004972719000091 Published online by Cambridge University Press

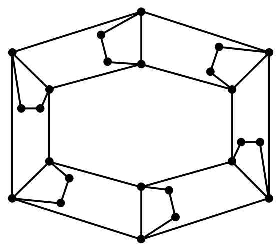

FIGURE 2. Planar drawing of $\mathcal{S}_6 \square K_{1,1}$.

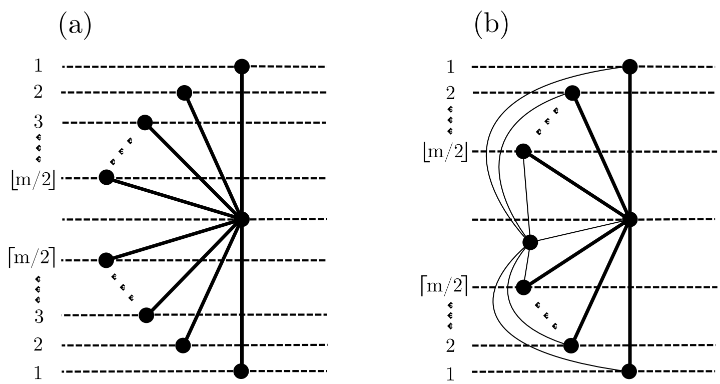

FIGURE 3. In (a), the construction of a drawing of the subgraph $K_{1,m} \square C_n$. In (b), the extension which will be subdivided to produce a drawing of $K_{1,m} \square \mathcal{S}_n$.

suffices to give a procedure for drawing the graph $\mathcal{S}_n \square K_{1,m}$, $m \geq 2$, so that the number of crossings meets the proposed upper bound.

First, note that $\mathcal{S}_n \square K_{1,m}$ contains $C_n \square K_{1,m}$ as a subgraph. Begin by drawing the subgraph $C_n \square K_{1,m}$ in the manner illustrated in Figure 3(a). For a given $i = 0, 1, \ldots, n-1$, the thick edges represent $((v_0, u_i), (v_j, u_i))$ for $j = 0, 1, \ldots, m$. The dashed edges represent $((v_j, u_i), (v_j, u_{i+1}))$ and $((v_j, u_i), (v_j, u_{i-1}))$ for $j = 0, 1, \ldots, m$. Then it is easy to see that the dashed edges can be joined to the corresponding sections for $i+1$ and $i-1$ to complete a drawing of $K_{1,m} \square C_n$ without introducing any additional crossings. Hence, the number of crossings in this drawing of the subgraph $C_n \square K_{1,m}$ is

$$n\left(\sum_{k=1}^{\lfloor \frac{m}{2} \rfloor - 1} k + \sum_{k=1}^{\lceil \frac{m}{2} \rceil - 1} k\right) = n\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor.$$

Next, we extend this drawing to a drawing of $\mathcal{S}_n \square K_{1,m}$ in the following way. For each $i = 0, 1, \ldots, n-1$, place a vertex in the region between the centre horizontal (dashed) edge $((v_0, u_i), (v_0, u_{i+1}))$ and the first thick edge on the side which possesses $\lceil m/2 \rceil$ vertices and join this new vertex to each of the vertices (v_j, u_i) for $j = 0, 1, \ldots, m$

https://doi.org/10.1017/S0004972719000091 Published online by Cambridge University Press

as in Figure 3(b). Then the number of crossings in this graph is equal to

$$n\left(\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor+\sum_{k=1}^{\lfloor\frac{m}{2}\rfloor}k+\sum_{k=1}^{\lceil\frac{m}{2}\rceil-1}k\right)=n\left\lfloor\frac{m}{2}\right\rfloor\left(\left\lfloor\frac{m-1}{2}\right\rfloor+\left\lceil\frac{m}{2}\right\rceil\right)=\frac{nm(m-1)}{2}.$$

Finally, if every new edge is subdivided, except for the ones emanating from (v_0, u_i) for $i = 0, 1, \ldots, n-1$, the resulting graph is isomorphic to $\mathcal{S}_n \square K_{1,m}$. Since subdividing edges does not alter the number of crossings, we conclude that it is possible to draw $\mathcal{S}_n \square K_{1,m}$ with $nm(m-1)/2$ crossings. □

3. Exact results

We now consider $\mathcal{S}_n \square K_{1,m}$ for some small values of m and show that the crossing number coincides precisely with the upper bound from Section 2. Denote that upper bound by $U(n, m) := nm(m-1)/2$. As noted previously, $\mathcal{S}_n \square K_{1,1}$ is planar; see Figure 2. This agrees with $U(n, 1) = 0$. Next, we will consider the cases when $m = 2$ and $m = 3$.

In what follows, we will utilise some properties of subgraphs of $\mathcal{S}_n \square K_{1,m}$, which we denote by H_i for each $i = 0, 1, 2, \ldots, n-1$. In particular, H_i is defined as the subgraph induced by the union of the following, disjoint, sets of edges:

$$\begin{aligned}
a_i &:= \{((v_j, u_i), (v_j, u_{i+1})) \mid j = 0, 1, \ldots, m\};\\
b_i &:= \{((v_j, u_i), (v_j, u_i')) \mid j = 0, 1, \ldots, m\};\\
b_i' &:= \{((v_0, u_i'), (v_j, u_i')) \mid j = 1, \ldots, m\};\\
c_i &:= \{((v_j, u_i), (v_j, u_{i-1})) \mid j = 0, 1, \ldots, m\};\\
t_i &:= \{((v_0, u_i), (v_j, u_i)) \mid j = 1, \ldots, m\};\\
t_{i+1} &:= \{((v_0, u_{i+1}), (v_j, u_{i+1})) \mid j = 1, \ldots, m\};\\
t_{i-1} &:= \{((v_0, u_{i-1}), (v_j, u_{i-1})) \mid j = 1, \ldots, m\}.
\end{aligned}$$

A detailed illustration of H_i, for the case $m = 3$, is displayed in Figure 4. For each $i = 0, 1, 2, \ldots, n-1$, there is a corresponding H_i in $\mathcal{S}_n \square K_{1,m}$ and H_i and H_j contain common edges when $j = i+1$ or $j = i-1$.

We now consider the case when $m = 2$. Note that $U(n, 2) = n$. In what follows, we use the following notation: consider a drawing D of a graph which contains two edge sets a and b. Then $cr_D(a)$ is equal to the number of crossings on the edges of a in D. Similarly, $cr_D(a, b)$ is equal to the number of crossings in D between edge pairs, such that one edge is contained in a and the other is contained in b.

Lemma 3.1. *The crossing number of $\mathcal{S}_n \square K_{1,2}$ is equal to n.*

Proof. From Theorem 2.1, we know that $cr(\mathcal{S}_n \square K_{1,2}) \le n$. Hence, the task now is to show that the reverse inequality holds. Let H_i' be the subgraph H_i without the edges t_i. An illustration of H_i' is displayed in Figure 5.

It is clear that H_i' is homeomorphic to $K_{3,3}$ and so there exists at least one crossing in the subdrawing D' of H_i'. Furthermore, at least one crossing in D' involves two edges

https://doi.org/10.1017/S0004972719000091 Published online by Cambridge University Press

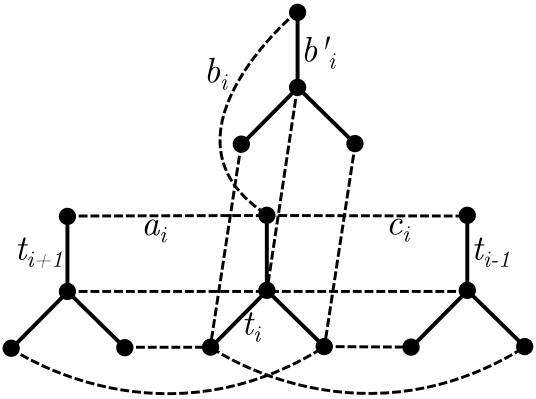

FIGURE 4. The subgraph H_i of $\mathcal{S}_n \square K_{1,3}$. The labels for each set of edges lie next to one edge belonging to that set. In this drawing, the thick lines correspond to the sets t_{i-1}, t_i, t_{i+1} and b'_i.

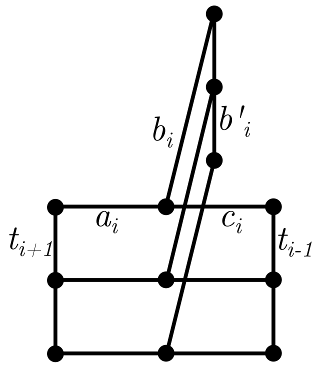

FIGURE 5. The subgraph H'_i of $\mathcal{S}_n \square K_{1,2}$. The labels for each set of edges lie next to one edge belonging to that set.

which come from the edge sets $(a_i \cup t_{i+1}), (b_i \cup b'_i)$ and $(c_i \cup t_{i-i})$, but do not both come from the same edge set. That is,

$$cr_{D'}((a_i \cup t_{i+1}), (b_i \cup b'_i)) + cr_{D'}((a_i \cup t_{i+1}), (c_i \cup t_{i-1})) + cr_{D'}((b_i \cup b'_i), (c_i \cup t_{i+1})) \geq 1.$$

Hence, it is clear that there is at least one crossing in each H'_i which does not occur in any other H'_j for $i \neq j$, which leads immediately to the result. □

Next, we consider the case when $m = 3$. Note that $U(n, 3) = 3n$. In order to handle this case, we first need to prove two intermediate results, Lemmas 3.2 and 3.3.

LEMMA 3.2. *For $m = 3$, consider the following four edge sets: $(a_i \cup t_{i+1})$, $(b_i \cup b'_i)$, $(c_i \cup t_{i-1})$ and t_i. Then, in any good drawing of the subgraph H_i, there are at least three crossings for which the two edges involved in the crossing are not in the same set.*

https://doi.org/10.1017/S0004972719000091 Published online by Cambridge University Press

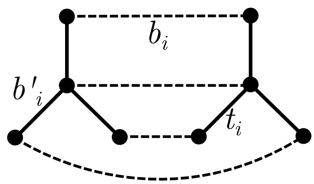

FIGURE 6. The drawing of the subgraph induced by F_i if F_i is not crossed by itself.

PROOF. The subgraph H_i is homeomorphic to $K_{1,3,3}$ and Asano [4] proved that $\text{cr}(K_{1,3,3}) = 3$. Any drawing of H_i corresponds to some drawing of $K_{1,3,3}$. Any drawing of $K_{1,3,3}$ has at least three crossings between pairs of edges which are not incident. These crossings correspond precisely to crossings in the drawing of H_i which satisfy the lemma. □

LEMMA 3.3. *For $n \geq 3$, let D be a drawing of $S_n \square K_{1,3}$. If, for each $i = 0, 1, 2, \ldots, n-1$, the edges $t_i \cup b_i \cup b'_i$ are crossed two or fewer times in D, then D has at least $3n$ crossings.*

PROOF. Let F_i denote the edge set $t_i \cup b_i \cup b'_i$. Note that F_i is a subgraph of H_i. Then, from Lemma 3.2,

$$cr_D(a_i \cup t_{i+1}, F_i) + cr_D(c_i \cup t_{i-1}, F_i) + cr_D((a_i \cup t_{i+1}), (c_i \cup t_{i-1})) + cr_D(F_i, F_i) \geq 3. \tag{3.1}$$

Assume that $cr_D(F_i) \leq 2$ for all $i = 0, 1, 2, \ldots, n-1$. It will be shown that if $cr_D(t_{i+1}, F_i) \neq 0$, or if $cr_D(t_{i-1}, F_i) \neq 0$, then a contradiction arises.

Suppose that $cr_D(t_{i+1}, F_i) = 1$. Note that the edges of b_{i+1} link to all of the end points of t_{i+1}. Since the subgraph induced by F_i is 2-connected, it is clear that it is impossible to draw $(b_{i+1} \cup b'_{i+1})$ without creating an additional crossing on the edges of F_i. Since the subgraph induced by $F_i \cup c_i \cup t_{i-1}$ is isomorphic to $P_2 \square K_{1,3}$, where P_2 denotes the path graph on three vertices, and $cr(P_2 \square K_{1,3}) = 1$ [14], it follows that

$$cr_D(c_i \cup t_{i-1}, F_i) + cr_D(F_i, F_i) \geq 1.$$

This would imply that F_i is crossed at least three times, but, by assumption, $cr_D(F_i) \leq 2$. Hence, $cr_D(t_{i+1}, F_i) \neq 1$. An analogous argument can be made for t_{i-1}, which, similarly, implies that $cr_D(t_{i-1}, F_i) \neq 1$ as well.

Suppose that $cr_D(t_{i+1}, F_i) = 2$. Then, since $cr_D(F_i) \leq 2$, it must be the case that $cr_D(F_i, F_i) = 0$ and hence, without loss of generality, the subdrawing of the subgraph induced by F_i is equivalent to the drawing displayed in Figure 6.

Now consider the rest of the subgraph H_i, which includes edge sets $(a_i \cup t_{i+1})$ and $(c_i \cup t_{i-1})$. Note that the edges c_i link to all of the end points of t_i and these do not lie on a common face of D, so $(c_i \cup t_{i-1})$ cannot be drawn without crossing F_i at least once. This would imply that F_i is crossed at least three times, but, by assumption, $cr_D(F_i) \leq 2$. Hence, $cr_D(t_{i+1}, F_i) \neq 2$. An analogous argument can be made for t_{i-1}, which, similarly, implies that $cr_D(t_{i-1}, F_i) \neq 2$ as well.

https://doi.org/10.1017/S0004972719000091 Published online by Cambridge University Press

Since $cr_D(F_i) \le 2$, the only possibility is that $cr_D(t_{i+1}, F_i) = cr_D(t_{i-1}, F_i) = 0$ and so (3.1) simplifies to

$$cr_D(a_i, F_i) + cr_D(c_i, F_i) + cr_D((a_i \cup t_{i+1}), (c_i \cup t_{i-1})) + cr_D(F_i, F_i) \ge 3. \qquad (3.2)$$

It is easy to see that any crossing counted by the left-hand side of (3.2) is not counted for any other $j \neq i$. Hence, summing (3.2) over $i = 0, 1, 2, \ldots, n-1$ provides the result. □

Finally, we are ready to propose the theorem for $m = 3$.

THEOREM 3.4. *For $n \ge 3$, the crossing number of $\mathcal{S}_n \Box K_{1,3}$ is equal to $3n$.*

PROOF. We will prove the result by induction. The base case where $n = 3$, corresponding to a graph on 24 vertices, was proved computationally, utilising the exact crossing minimisation methods of Chimani and Wiedera [8], which are available at http://crossings.uos.de. The proof comes from a solution to an appropriately constructed integer linear program and shows that $cr(\mathcal{S}_3 \Box K_{1,3}) = 9$. The proof file is available and can be provided by the corresponding author if desired.

Now assume that $cr(\mathcal{S}_n \Box K_{1,3}) = 3n$ for each $n = 3, \ldots, k-1$, but that for $n = k$ there exists a drawing with strictly fewer than $3k$ crossings. Let D denote such a drawing. By Lemma 3.3, there must be at least one i such that the edges of F_i are crossed at least three times in D. Hence, the edges F_i could be deleted and the number of crossings remaining would be strictly less than $3(k-1)$. However, once F_i is deleted, the resulting graph is homeomorphic to $\mathcal{S}_{k-1} \Box K_{1,3}$, which, by the inductive assumption, has crossing number equal to $3(k-1)$. This is a contradiction and hence any drawing for $n = k$ must have at least $3k$ crossings. This, combined with Theorem 2.1, implies that $cr(\mathcal{S}_k \Box K_{1,3}) = 3k$ and inductively we obtain the result. □

We conclude by conjecturing that the upper bound described in Theorem 2.1 coincides precisely with the crossing number in all cases. To provide evidence supporting this conjecture, we used QuickCross [9], a recently developed crossing minimisation heuristic, to find good drawings of $\mathcal{S}_n \Box K_{1,m}$ for $n, m \le 20$. In all cases, QuickCross was able to find an embedding that agrees with the conjecture but was never able to find an embedding with fewer crossings.

CONJECTURE 3.5. For $n \ge 3$, $m \ge 1$,

$$cr(\mathcal{S}_n \Box K_{1,m}) = \frac{nm(m-1)}{2}.$$

References

[1] B. M. Ábrego, O. Aichholzer, S. Fernández-Merchant, T. Hackl, J. Pammer, A. Pilz, P. Ramos, G. Salazar and B. Vogtenhuber, 'All good drawings of small complete graphs', in: *Proc. 31st European Workshop on Computational Geometry (EuroCG) 2015*, Book of Abstracts, eurocg15.fri.uni-lj.si/pub/eurocg15-book-of-abstracts.pdf, 57–60.

[2] M. Anderson, R. B. Richter and P. Rodney, 'The crossing number of $C_6 \times C_6$', *Congr. Numer.* **117** (1996), 97–107.

[3] M. Anderson, R. B. Richter and P. Rodney, 'The crossing number of $C_7 \times C_7$', *Congr. Numer.* **125** (1996), 97–117.

[4] K. Asano, 'The crossing number of $K_{1,3,n}$ and $K_{2,3,n}$', *J. Graph Theory* **10** (1986), 1–8.

[5] L. W. Beineke and R. D. Ringeisen, 'On the crossing numbers of products of cycles and graphs of order four', *J. Graph Theory* **4**(2) (1980), 145–155.

[6] D. Bokal, 'On the crossing number of Cartesian products with paths', *J. Combin. Theory Ser. B* **97**(3) (2007), 381–384.

[7] S. Cabello and B. Mohar, 'Adding one edge to planar graphs makes crossing number and 1-planarity hard', *SIAM J. Comput.* **42**(5) (2013), 1803–1829.

[8] M. Chimani and T. Wiedera, 'An ILP-based proof system for the crossing number problem', in: *24th European Symposium on Algorithms (ESA 2016), Aarhus, Denmark*, Leibniz International Proceedings in Informatics, 56 (Schloss Dagstuhl, Dagstuhl, Germany, 2016), 29.1–29.13.

[9] K. Clancy, M. Haythorpe and A. Newcombe, 'An effective crossing minimisation heuristic based on star insertion', preprint available at arXiv:abs/1804.09900.

[10] A. M. Dean and R. B. Richter, 'The crossing number of $C_4 \times C_4$', *J. Graph Theory* **19**(1) (1995), 125–129.

[11] M. R. Garey and D. S. Johnson, 'Crossing number is NP-complete', *SIAM J. Algebr. Discrete Methods* **4**(3) (1983), 312–316.

[12] L. Y. Glebsky and G. Salazar, 'The crossing number of $C_m \times C_n$ is as conjectured for $n \geq m(m+1)$', *J. Graph Theory* **47**(1) (2004), 53–72.

[13] F. Harary, P. C. Kainen and A. J. Schwenk, 'Toroidal graphs with arbitrarily high crossing numbers', *Nanta Math.* **6**(1) (1973), 58–67.

[14] S. Jendrol and M. Ščerbová, 'On the crossing numbers of $S_m \times P_n$ and $S_m \times C_n$', *Casopis pro Pestováni Mat.* **107** (1982), 225–230.

[15] M. Klešč, 'On the crossing numbers of Cartesian products of stars and paths or cycles', *Math. Slovaca* **41**(2) (1991), 113–120.

[16] M. Klešč and D. Kravecová, 'The crossing number of $P_n^2 \square C_3$', *Discrete Math.* **312** (2012), 2096–2101.

[17] M. Klešč, J. Petrillová and M. Valo, 'On the crossing numbers of cartesian products of wheels and trees', *Discrete Math. Graph Theory* **37**(2) (2017), 399–413.

[18] D. McQuillan, S. Pan and R. B. Richter, 'On the crossing number of K_{13}', *J. Combin. Theory Ser. B* **115** (2015), 224–235.

[19] R. B. Richter and C. Thomassen, 'Intersections of curve systems and the crossing number of $C_5 \times C_5$', *Discrete Comput. Geom.* **13**(1) (1995), 149–159.

[20] R. D. Ringeisen and L. W. Beineke, 'The crossing number of $C_3 \times C_n$', *J. Combin. Theory Ser. B* **24**(2) (1978), 134–136.

[21] W. Zheng, X. Lin, Y. Yang and C. Deng, 'On the crossing number of $K_m \square C_n$ and $K_{m,l} \square P_n$', *Discrete Appl. Math.* **156** (2008), 1892–1907.

MICHAEL HAYTHORPE, 1284 South Road, Clovelly Park, 5042,
College of Science and Engineering, Flinders University, Australia
e-mail: michael.haythorpe@flinders.edu.au

ALEX NEWCOMBE, 1284 South Road, Clovelly Park, 5042,
College of Science and Engineering, Flinders University, Australia
e-mail: alex.newcombe@flinders.edu.au

https://doi.org/10.1017/S0004972719000091 Published online by Cambridge University Press