J. Fluid Mech. (2012), *vol.* 707, *pp.* 111–128.
doi:10.1017/jfm.2012.266

The mean electromotive force generated by elliptic instability

K. A. Mizerski[1]†, K. Bajer[2,3] and H. K. Moffatt[4]

[1] Department of Mechanics and Physics of Fluids, Institute of Fundamental Technological Research, Polish Academy of Sciences, ul. Pawińskiego 5B, 02-106, Warsaw, Poland

[2] Faculty of Physics, University of Warsaw, ul. Pasteura 7, 02-093 Warsaw, Poland

[3] Interdisciplinary Centre for Mathematical and Computer Modelling, University of Warsaw, ul. Pawinskiego 5a, 02-106 Warsaw, Poland

[4] DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK

(Received 8 July 2011; revised 16 January 2012; accepted 4 June 2012; first published online 12 July 2012)

The mean electromotive force (EMF) associated with exponentially growing perturbations of an Euler flow with elliptic streamlines in a rotating frame of reference is studied. We are motivated by the possibility of dynamo action triggered by tidal deformation of astrophysical objects such as accretion discs, stars or planets. Ellipticity of the flow models such tidal deformations in the simplest way. Using analytical techniques developed by Lebovitz & Zweibel (*Astrophys. J.*, vol. 609, 2004, pp. 301–312) in the limit of small elliptic (tidal) deformations, we find the EMF associated with each resonant instability described by Mizerski & Bajer (*J. Fluid Mech.*, vol. 632, 2009, pp. 401–430), and for arbitrary ellipticity the EMF associated with unstable horizontal modes. Mixed resonance between unstable hydrodynamic and magnetic modes and resonance between unstable and oscillatory horizontal modes both lead to a non-vanishing mean EMF which grows exponentially in time. The essential conclusion is that interactions between unstable eigenmodes with the same wave-vector $\boldsymbol{k}$ can lead to a non-vanishing mean EMF, without any need for viscous or magnetic dissipation. This applies generally (and not only to the elliptic instabilities considered here).

Key words: dynamo theory, MHD turbulence, parametric instability

1. Introduction

In this paper we continue the earlier study of Mizerski & Bajer (2009) (hereafter MB09) of the so-called elliptic instability, i.e. the instability of an unbounded, two-dimensional, linear flow with elliptic streamlines, influenced by two physical effects: uniform axial magnetic field and uniform steady rotation of the frame of reference. The basic flow, steady in the rotating frame of reference, is characterized by three parameters: ζ (a measure of eccentricity of the streamlines), ω (the average angular velocity of recirculation), and Ω (the constant background rotation).

In MB09 (see also Bajer & Mizerski 2012) we have mapped out the regimes of the (ζ, ω, Ω) space where this configuration is prone to what we propose to call

† Email address for correspondence: krzysztof.mizerski@gmail.com

'MER instability' (i.e. Magneto-Elliptic-Rotational instability). Here we calculate the mean electromotive force (EMF) associated with the unstable modes in this type of flow, in similar manner to Moffatt's (1970, 1978) calculation of the EMF associated with inertial waves in a rotating fluid. We are motivated by the astrophysical problem of dynamo action in tidally distorted objects, with the ellipticity modelling the tidal deformation (Suess 1970; Kerswell & Malkus 1998), or by elliptical vortex patches in accretion discs. Simple global models of tidally distorted stars and planets based on the so-called Riemann ellipsoids (Chandrasekhar 1969; Aldridge, Lumb & Henderson 1989) are based on the linear elliptic flow considered here, which from the point of view of perturbations strongly localized on streamlines is effectively unbounded (see Lebovitz & Lifschitz 1996; Mizerski & Bajer 2011). Such linear basic flow can similarly model localized patches of elliptic streamlines which often appear in real systems as a result of Kelvin–Helmholtz instability. In the accretion-disc context, the flow models the cores of vortices induced by differential rotation; this idea has been extensively exploited using the local 'shearing-box' approximation of Hawley, Gammie & Balbus (1995) and Lesur & Papaloizou (2009).

The MER instability depends on the presence of the magnetic field. The question we address here is whether the same instability may contribute to the generation of a mean electromotive force (a step on the way to dynamo action). In MB09, the MER instability growth rates derived from Floquet theory were also computed numerically. By contrast, the results in the present paper are purely analytical, derived from the asymptotic theory for small ellipticity (up to $O(\zeta)$). This is not a serious limitation, because slightly non-axisymmetric vortices are abundant while strongly deformed vortices are unlikely to persist.

One type of resonance, called 'mixed' or 'HM' (i.e. hydro-magnetic), is particularly promising. This resonance produces a pair of helical modes propagating in opposite directions, and they generate non-zero electromotive forces. The unstable MER wave grows exponentially and with it grows the EMF; however, oscillations are also present (see equations (3.26*a*–*c*)). When $\zeta \to 0$ the electromotive force does not vanish (the ζ^0 term is given by (3.26*a*–*c*)), but is purely oscillatory in this limit. The two unstable MER waves produced by the HM resonance, propagating up and down the background field, generate mean electromotive forces with opposite signs. A net EMF generation occurs only if one of these two modes is 'preferred' by the system. This could be the result of either preferential excitation, or preferential suppression when, for example, one of the modes passes through a region of enhanced dissipation or is reflected or quenched by a stratified layer.

The unstable MER wave of HM type is particularly interesting because it generates a finite EMF even in the ideal fluid limit. Both in mean-field dynamo theory and in Braginsky's theory of weakly asymmetric dynamos, calculations have shown that the kinematic α-effect is diffusive in origin: see Moffatt (1974, 1976), chapters 7 and 8 of Moffatt (1978) and references therein, Moffatt (1983), Soward (1972) and Braginsky (1964*a*,*b*, 1975, 1976, 1978). More importantly, for a single helical wave, more akin to the MER waves considered here than turbulence with assumed scale separation, a simple calculation of the α-effect also shows that it vanishes in the limit $\eta \to 0$ (Moffatt 1978, chapter 7.7). Arguments have been advanced that in some circumstances a turbulent α-effect may still persist in the limit $\eta \to 0$ (Seehafer 1995; see also the discussion of the high-conductivity limit in chapter 4.2.2 of Rüdiger & Hollerbach 2004) but the issue is far from being settled.

An important feature of the analysis is that it reveals how interactions between eigenmodes can create a non-zero EMF; this mechanism is effective whether dissipation is present or not. The analysis of the unstable horizontal modes in § 3 further confirms this mechanism.

In this paper we aim to study the generation of the mean EMF in the spirit of Moffatt's (1970) calculation for inertial waves in circular flows. We consider a restricted case in which the mean magnetic field is vertical. We shall focus on the limit of small departures from axial symmetry of the basic flow, i.e. $\zeta \ll 1$, and we will build on the results of MB09, where the MER instability problem was formulated and the growth rates were obtained.

2. Mean electromotive force in an ideal fluid

It was shown by Moffatt (1978, p. 163 and pp. 248–255 for inertial waves in particular) that in the linear regime the mean electromotive force

$$\boldsymbol{\mathcal{E}} = \langle \boldsymbol{u}' \times \boldsymbol{B}' \rangle \tag{2.1}$$

(where $\langle . \rangle$ denotes a spatial average) generated by perturbation fields $\boldsymbol{u}'$ and $\boldsymbol{B}'$ corresponding to a single eigenmode can be non-zero only if dissipative effects are present in the system. This holds true as long as both fields $\boldsymbol{u}'$ and $\boldsymbol{B}'$ have the same wave-vector $\boldsymbol{k}$ and correspond to the same eigenvalue. However, in the linear regime one can also consider linear combinations of eigenmodes corresponding to the same $\boldsymbol{k}$ but different eigenvalues. The necessary phase shift between velocity and magnetic field, which for a single mode can only be created by the presence of dissipation, is no longer necessary when we consider the interactions between eigenmodes in the general form $\boldsymbol{u}'_j = \boldsymbol{v}_j \mathrm{e}^{\mathrm{i}(\varpi_j t + \boldsymbol{k}\cdot\boldsymbol{x})}$ and $\boldsymbol{B}'_j = \boldsymbol{b}_j \mathrm{e}^{\mathrm{i}(\varpi_j t + \boldsymbol{k}\cdot\boldsymbol{x})}$, i.e.

$$\begin{aligned} \boldsymbol{\mathcal{E}} &= \mathrm{Re}[(\boldsymbol{v}_j \mathrm{e}^{\mathrm{i}\varpi_j t} + C\boldsymbol{v}_k \mathrm{e}^{\mathrm{i}\varpi_k t})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \times (\boldsymbol{b}_j^* \mathrm{e}^{-\mathrm{i}\varpi_j t} + C^*\boldsymbol{b}_k^* \mathrm{e}^{-\mathrm{i}\varpi_k t})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}] \\ &= \mathrm{Re}[C^*\boldsymbol{v}_j \times \boldsymbol{b}_k^* \mathrm{e}^{\mathrm{i}\Delta\varpi t} + C\boldsymbol{v}_k \times \boldsymbol{b}_j^* \mathrm{e}^{-\mathrm{i}\Delta\varpi t}], \end{aligned} \tag{2.2}$$

where C is a complex constant and $\Delta\varpi = \varpi_j - \varpi_k$. The expression (2.2) is in general non-zero. However, such an EMF typically oscillates in time, since we have considered two different eigenmodes which means that $\Delta\varpi$ is in general non-zero (except for the case of multiple eigenvalues, on which we comment in the concluding section). Such oscillations are superimposed on exponential growth if the interacting modes considered are unstable.

This behaviour will be exemplified below for two cases of elliptical mode instability in the absence of dissipative effects: first when the instability is via the parametric instability mechanism; and second when the wave-vector of both modes is parallel to the mean field (the 'horizontal mode' situation). A non-zero EMF is generated in both cases.

Such a rapidly oscillating EMF, i.e. oscillating on a time scale much faster then the typical time scale of the mean magnetic field, when interacting with mean shear, was found by Proctor (2007), Bushby & Proctor (2010) and Richardson & Proctor (2010) to act as an efficient dynamo mechanism even in the absence of the mean α-effect. This provides a motivation for the study of generation of rapidly oscillating electromotive forces.

3. The resonant unstable MER modes

3.1. *The stability problem*

The basic flow of the elliptic vortex can be expressed in the form

$$\boldsymbol{u}_0 = \omega[-Ey, E^{-1}x, 0] = \boldsymbol{G}\boldsymbol{x} \quad \text{with } \omega > 0,\ E \geqslant 1, \tag{3.1}$$

where $\omega(E + E^{-1}) \equiv 2\gamma$ and $(E - E^{-1})/(E + E^{-1}) \equiv \epsilon$ are its vorticity and strain respectively and $\boldsymbol{G}$ is the velocity gradient tensor. The system is rotating with angular velocity $(0, 0, \Omega)$ and is permeated by a uniform magnetic field $(0, 0, B_0)$. The fluid is assumed incompressible (of density ρ), inviscid and perfectly conducting. Perturbations

$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}', \quad \boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}' \tag{3.2}$$

are considered, and the governing magnetohydrodynamic (MHD) equations are linearized; these linearized equations admit solutions of the form

$$(\boldsymbol{u}', \boldsymbol{B}') = (\boldsymbol{v}(t), \boldsymbol{b}(t)) \exp[\mathrm{i}\boldsymbol{k}(t)\boldsymbol{\cdot}\boldsymbol{x}], \tag{3.3}$$

where

$$\boldsymbol{k}(t) = (k_x, k_y, k_z) = k_0(\sin\vartheta\cos\omega t, E\sin\vartheta\sin\omega t, \cos\vartheta), \tag{3.4}$$

a wave-vector varying periodically in time (θ is the minimal angle between the wave-vector and the z axis). With $\tau = \omega t$, the following time-periodic change of variables,

$$c_1 = E\frac{k_x}{\omega}v_y - E^{-1}\frac{k_y}{\omega}v_x, \quad c_2 = \frac{k_x}{\omega}v_x + \frac{k_y}{\omega}v_y, \tag{3.5a}$$

$$c_3 = E\frac{k_x}{\omega}b_y - E^{-1}\frac{k_y}{\omega}b_x, \quad c_4 = \frac{k_x}{\omega}b_x + \frac{k_y}{\omega}b_y, \tag{3.5b}$$

leads to a Floquet problem for the vector $\boldsymbol{c}$,

$$\frac{\mathrm{d}\boldsymbol{c}}{\mathrm{d}\tau} = \mathscr{C}(\tau)\boldsymbol{c}, \tag{3.6}$$

where $\mathscr{C}(\tau, \zeta, \cos\vartheta, H, \mathscr{R})$ is a complex 4×4 matrix, 2π-periodic in τ, and depending on the dimensionless parameters E, $\mathscr{R} = \Omega/\omega$ and $H = k_0 B_0/\sqrt{\mu_0\rho}\,\omega$, a dimensionless measure of the field strength. It possesses the property that for $\zeta = 0 (E = 1)$ it is independent of time, thus its eigenvalues, denoted by $\mathrm{i}\varpi_j$ with $j = 1, \ldots, 4$, correspond to the leading-order frequencies of oscillations of perturbations. The general solution of (3.6) is a linear superposition of Floquet modes (Bender & Orszag 1978)

$$\boldsymbol{c}(\tau) = \mathrm{e}^{\sigma\tau}\boldsymbol{f}(\tau), \quad \boldsymbol{c}(0) = \boldsymbol{f}(0), \tag{3.7}$$

where $\boldsymbol{f}(\tau)$ is 2π-periodic and σ is the (generally complex) growth rate. Writing this solution in matrix form,

$$\boldsymbol{c}(\tau) = \mathscr{M}(\tau)\boldsymbol{c}(0), \tag{3.8}$$

we have

$$\boldsymbol{c}(2\pi) = \exp(2\pi\sigma)\boldsymbol{c}(0) = \mathscr{M}(2\pi)\boldsymbol{c}(0), \tag{3.9}$$

so that $\exp 2\pi\sigma$ is an eigenvalue $\Lambda(E, \theta, \mathscr{R}, H)$ of $\mathscr{M}(2\pi)$, i.e.

$$2\pi\sigma = \ln\Lambda. \tag{3.10}$$

The matrix $\mathscr{M}(2\pi)$ has the property (first noted in the special case $\Omega = 0$ by Lebovitz & Zweibel 2004) that if Λ is an eigenvalue, then so are Λ^{-1} and the complex conjugate Λ^*. The system becomes unstable when $\mathrm{Re}\sigma > 0$, i.e. when an eigenvalue Λ of $\mathscr{M}(2\pi)$ and simultaneously $(\Lambda^*)^{-1}$ leave the unit circle in the complex Λ-plane, and hence the destabilization occurs via resonance between two eigenvalues. For the sake of brevity the matrix $\mathscr{M}(2\pi)$ will be denoted by $\boldsymbol{M}$. The eigenvalues of the matrix $\boldsymbol{M}_0 = \boldsymbol{M}(\zeta = 0)$, will be denoted by $\lambda_j = \exp \mathrm{i}\varpi_j$ with $j = 1, \ldots, 4$.

3.2. *The unstable modes*

In this subsection we shall find the unstable eigenmodes for the case of mixed hydro-magnetic resonance described in MB09, which is in fact the only resonant case leading to non-vanishing electromotive force. (In appendix B we demonstrate that the other cases of hydro–hydro and magnetic–magnetic resonances do not generate an EMF.) Following MB09, we assume from here on that the ellipticity of the basic flow is small:

$$\zeta = \tfrac{1}{2}(E - E^{-1}) \ll 1. \tag{3.11}$$

Let Γ be defined so that to first order, each eigenvalue of the matrix $\boldsymbol{M}$ takes the form

$$\Lambda = \mathrm{e}^{2\pi\sigma} = \mathrm{e}^{\mathrm{i}2\pi\varpi}(1 + \zeta\Gamma), \tag{3.12}$$

so that the growth rate is given by

$$\sigma = \zeta\frac{\mathrm{Re}\Gamma}{2\pi} + \mathrm{i}\left(\varpi + \zeta\frac{\mathrm{Im}\Gamma}{2\pi}\right) + O(\zeta^2). \tag{3.13}$$

The following results will be needed throughout the calculations:

$$\cos\vartheta = [(1+\mathscr{R})^2 + H^2]^{-1/2}, \tag{3.14}$$

$$\varpi_1 = (1+\mathscr{R})\cos\vartheta + 1, \quad \varpi_3 = (1+\mathscr{R})\cos\vartheta - 1, \tag{3.15a}$$

$$\varpi_2 = -(1+\mathscr{R})\cos\vartheta - 1, \quad \varpi_4 = -(1+\mathscr{R})\cos\vartheta + 1, \tag{3.15b}$$

$$\varpi_1 - \varpi_3 = \varpi_4 - \varpi_2 = 2. \tag{3.15c}$$

The four eigenvalues of $\boldsymbol{M}$ are listed in (A 2) in appendix A (note the dependence of $\boldsymbol{M}$ on ζ). The parameter Γ takes different values for each of these four eigenvalues, and for $\zeta \neq 0$ depends on the angle ϑ. For the most unstable mode we have

$$\Gamma_\pm = \pm\frac{\pi}{2}(1-\cos^2\vartheta)(1-(1+\mathscr{R})^2\cos^2\vartheta) - \mathrm{i}\pi(1-\cos^2\vartheta)(1+\mathscr{R})\cos\vartheta(1-(1+\mathscr{R})^2\cos^2\vartheta) \tag{3.16}$$

(a corresponding calculation for any unstable mode leads to the same qualitative conclusions). The eigenvalues associated with instability are Λ_1 and Λ_4, since the real parts of the growth rates in (3.13),

$$\mathrm{Re}\sigma_1 = \mathrm{Re}\sigma_4 = \zeta\,\mathrm{Re}\Gamma_+/2\pi + O(\zeta^2), \tag{3.17}$$

are positive only for these eigenvalues.

The eigenproblem is solved in appendix A, and the eigenmodes at $\tau = 2\pi$ are obtained. For finite $\zeta \ll 1$ we present the results only for $\tau = 2\pi$, since it is enough to demonstrate the generation of exponentially growing EMFs, and calculations for arbitrary time τ are far more involved. Hence, now using (A 4) and the transformation (A 1) together with the solenoidal conditions $\boldsymbol{k}\cdot\boldsymbol{v} = 0$ and $\boldsymbol{k}\cdot\boldsymbol{b} = 0$ taken at $\tau = 2\pi$, we obtain the following form of the eigenmode associated with Λ_1:

$$\left.\begin{aligned} \boldsymbol{v}_1 &= \mathrm{e}^{2\pi\sigma_1}\frac{\omega}{k_0\sin\vartheta}\begin{bmatrix}\cos\vartheta\,(\varpi_1+\mathrm{i}\varpi_3)\\ -\varpi_3+\mathrm{i}\varpi_1\\ -\sin\vartheta\,(\varpi_1+\mathrm{i}\varpi_3)\end{bmatrix},\\ \boldsymbol{b}_1 &= \mathrm{e}^{2\pi\sigma_1}\frac{\omega H\cos\vartheta}{k_0\sin\vartheta}\begin{bmatrix}\cos\vartheta\,(1+\mathrm{i})\\ -1+\mathrm{i}\\ -\sin\vartheta\,(1+\mathrm{i})\end{bmatrix}. \end{aligned}\right\} \tag{3.18}$$

The exponential growth in time is apparent through the term $\mathrm{e}^{2\pi\mathrm{Re}\sigma_1}$; however, at leading order ($\zeta=0$), we have $2\pi\sigma_1=\mathrm{i}2\pi(1+\mathscr{R})\cos\vartheta$, and the modes are stable. The relations

$$-\mathrm{i}\boldsymbol{k}\times\boldsymbol{v}_1^*=k_0\boldsymbol{v}_1^* \quad\text{and}\quad \mathrm{i}\boldsymbol{k}\times\boldsymbol{b}_1=k_0\boldsymbol{b}_1 \tag{3.19}$$

are satisfied, and for the mode associated with $\Lambda_4=\Lambda_1^*$ we find that

$$\boldsymbol{v}_4=\mathrm{i}\boldsymbol{v}_1^*,\quad \boldsymbol{b}_4=-\mathrm{i}\boldsymbol{b}_1^* \quad\text{and}\quad \mathrm{i}\boldsymbol{k}\times\boldsymbol{v}_4^*=k_0\boldsymbol{v}_4^*,\quad \mathrm{i}\boldsymbol{k}\times\boldsymbol{b}_4=-k_0\boldsymbol{b}_4. \tag{3.20}$$

Equation (3.19) ensures that these modes are helical and of maximal helicity (see appendix C for further details concerning the helicity and group velocity of the unstable modes).

It is worth mentioning that, for all three cases considered, only the leading-order expressions in $\zeta\ll 1$ for the eigenmodes can be obtained without calculation of higher-order corrections to the eigenvalues and the matrix $\boldsymbol{M}$. This is because for $\zeta=0$ in the resonant cases there are always two-dimensional eigenspaces associated with the resonant eigenvalue, and the first-order correction in ζ only specifies one vector from such an eigenspace. To obtain full expressions, in all three cases, for the first-order corrections to the eigenmodes, one would have to go to order ζ^2 in the calculation of the eigenvalues and $\boldsymbol{M}$. Therefore the eigenmodes (3.18) and (3.20) do not depend on the ellipticity; however, for finite $\zeta\ll 1$ they grow exponentially in time with the growth rate $\zeta\,\mathrm{Re}\Gamma_+/2\pi$.

In the following section, we use our leading-order results to calculate the mean electromotive force $\boldsymbol{\mathcal{E}}$ for rotating elliptical flow, which turns out to be non-zero only in the mixed-resonance case.

3.3. *The EMF generated by the resonant unstable modes*

For the space average, we may simply average over a wavelength $2\pi/k_z$ in the z-direction:

$$\boldsymbol{\mathcal{E}}=\sqrt{\mu_0\rho}\,\langle\mathrm{Re}\,\boldsymbol{u}'\times\mathrm{Re}\,\boldsymbol{B}'\rangle_{2\pi/k_z}=\tfrac{1}{2}\sqrt{\mu_0\rho}\,\mathrm{Re}\,(\boldsymbol{v}\times\boldsymbol{b}^*), \tag{3.21}$$

where the factor $\sqrt{\mu_0\rho}$ appears because the perturbation field was scaled with $1/\sqrt{\mu_0\rho}$. We stress here that, since we are dealing with unstable modes ($\boldsymbol{v}\sim\mathrm{e}^{\mathrm{Re}\sigma_1\tau}$ and $\boldsymbol{b}\sim\mathrm{e}^{\mathrm{Re}\sigma_1\tau}$), the EMF grows exponentially in time with the growth rate $2\mathrm{Re}\sigma_1$. In the mixed-resonance case, $\boldsymbol{\mathcal{E}}$ can be evaluated at $\tau=2\pi$ for both modes propagating upwards and downwards, associated with eigenvalues Λ_1, Λ_4, respectively:

$$\boldsymbol{\mathcal{E}}^{(1)}(\tau=2\pi)=\frac{\omega B_0}{k_0}\left[-\frac{2\cos\vartheta}{\sin\vartheta},0,-\frac{2\cos^2\vartheta}{\sin^2\vartheta}\right]\mathrm{e}^{4\pi\mathrm{Re}\sigma_1}+O(\zeta), \tag{3.22}$$

$$\boldsymbol{\mathcal{E}}^{(4)}(\tau=2\pi)=\frac{\omega B_0}{k_0}\left[\frac{2\cos\vartheta}{\sin\vartheta},0,\frac{2\cos^2\vartheta}{\sin^2\vartheta}\right]\mathrm{e}^{4\pi\mathrm{Re}\sigma_1}+O(\zeta)=-\boldsymbol{\mathcal{E}}^{(1)}. \tag{3.23}$$

For this mixed-resonance case, $\cos\vartheta$ is given by (4.9) in MB09, so that explicitly

$$\begin{aligned}\mathscr{E}_x^{(4)}(\tau=2\pi) &= -\mathscr{E}_x^{(1)}(\tau=2\pi)\\ &= \frac{\omega B_0}{k_0}\frac{2}{\left[(1+\mathscr{R})^2-1+4B_0^2/\mathscr{B}_0^2\right]^{1/2}}\mathrm{e}^{4\pi\mathrm{Re}\sigma_1}+O(\zeta),\end{aligned} \tag{3.24}$$

$$\begin{aligned}\mathscr{E}_z^{(4)}(\tau=2\pi) &= -\mathscr{E}_z^{(1)}(\tau=2\pi)\\ &= \frac{\omega B_0}{k_0}\frac{2}{(1+\mathscr{R})^2-1+4B_0^2/\mathscr{B}_0^2}\mathrm{e}^{4\pi\mathrm{Re}\sigma_1}+O(\zeta),\end{aligned} \tag{3.25}$$

where $\mathscr{B}_0=2\sqrt{\mu_0\rho}\,\omega/k_0$ is a constant with the dimension of magnetic field. As in Moffatt (1978), it is also the case here that if there exists a mechanism of preferential excitation of upward or downward propagating waves, the total EMF may be expected to be non-zero.

The term $\mathrm{e}^{4\pi\mathrm{Re}\sigma_1}$ indicates exponential growth of $\boldsymbol{\mathcal{E}}^{(1)}$ and $\boldsymbol{\mathcal{E}}^{(4)}$. Moreover, since $\boldsymbol{k}$ is 2π-time-periodic, Floquet theory implies that $\boldsymbol{v}$ and $\boldsymbol{b}$ share this property, and so the components of the EMFs also have additional oscillatory time-dependence of general type $(C_1\cos 2\tau+C_2\sin 2\tau)(C_3\cos\tau+C_4\sin\tau+C_5)+O(\zeta)$, where the C_j $(j=1\ldots 5)$ are constants.

Since the growth rate of the modes is proportional to the ellipticity $\zeta\ll 1$, it follows that at any finite τ, and in particular at $\tau=2\pi$, the exponential growth can be expanded in powers of $\zeta\tau$ and at leading order $(\zeta=0)$ we get $\mathrm{e}^{4\pi\mathrm{Re}\sigma_1}=1+O(\zeta)$. The exact form of the oscillatory time-dependence of the EMFs at leading order can therefore be easily obtained by considering the limit $\zeta=0$. In this limit, in which the basic flow has circular streamlines, these EMFs are non-zero even though magnetic diffusivity and viscosity have been neglected. This is a consequence of the general feature of ideal MHD problems described in § 2, namely that in any linear problem we may always take a linear combination of eigenmodes with the same $\boldsymbol{k}$ to calculate $\boldsymbol{\mathcal{E}}$. Such modes, in the case of circular flow, are eigenvectors of the matrix $\mathscr{S}(\zeta=0)$ defined in MB09, which is the analogue of the matrix $\mathscr{C}(\zeta=0)$, but before the transformation (3.5), rather than $\mathscr{M}(\zeta=0,\tau=2\pi)$. In the analysed case of the mixed-type resonance, for $j=1$, $k=3$ in (2.2), the expression $\mathrm{Re}(\boldsymbol{v}\times\boldsymbol{b}^*)$ is non-zero only if $\Delta\varpi\neq 0$, because otherwise the matrix $\mathscr{S}(\zeta=0)$ is degenerate and there is only one eigenvector associated with its double eigenvalue $\varpi_1=\varpi_3$. The case at hand is defined by $\Delta\varpi=2$, and from the asymptotic analysis for $\zeta\ll 1$ in § 3.2 and appendix A we get $C=\mathrm{i}$, which picks up a certain eigenvector from the two-dimensional eigenspace at $\zeta=0$. Therefore, taking the eigenvectors from the first and third columns of the matrix (A 1) in appendix A, and remembering that the wave-vector $\boldsymbol{k}$ in the transformation (3.5) depends on time, for the case $\zeta=0$, i.e. the basic flow with circular streamlines, we obtain the following form of the electromotive force:

$$\mathscr{E}_x^{(1)}=-\frac{\omega B_0}{k_0}\frac{2}{\left[(1+\mathscr{R})^2-1+4B_0^2/\mathscr{B}_0^2\right]^{1/2}}\cos 2\tau\cos\tau, \tag{3.26a}$$

$$\mathscr{E}_y^{(1)}=-\frac{\omega B_0}{k_0}\frac{2}{\left[(1+\mathscr{R})^2-1+4B_0^2/\mathscr{B}_0^2\right]^{1/2}}\cos 2\tau\sin\tau, \tag{3.26b}$$

$$\mathscr{E}_z^{(1)}=-\frac{\omega B_0}{k_0}\frac{2}{(1+\mathscr{R})^2-1+4B_0^2/\mathscr{B}_0^2}\cos 2\tau. \tag{3.26c}$$

As explained above, for finite ellipticity $\zeta \ll 1$ the components of the EMF grow exponentially in time with the growth rate $2\mathrm{Re}\sigma_1$, and $\mathrm{Re}\sigma_1$ is the growth rate for mixed-resonance modes given in (3.17). It is worth pointing out that since the growth rate is proportional to ζ, the time scale associated with the growth of the components of $\boldsymbol{\mathcal{E}}$ is much larger than the period of oscillations determined by the basic frequency ω. When ζ is finite, the oscillatory time-dependence of the EMF is also modified, and the departure from axial symmetry introduces additional terms of type $\zeta f(B_0)\cos n\tau \cos\tau$ (or, in general, with any of the cosines replaced by sine) where n is an integer, but the function $f(B_0)$ is not yet known. In general the dependence of the EMF on B_0 is quite complex, since $\boldsymbol{\mathcal{E}}$ depends on B_0 also through the growth rate σ_1. However, it is important to realize that in the limit of weak magnetic field, typically taken in dynamo considerations for the period of initial growth, the growth rate of the mixed mode depends on the magnetic field as $\mathrm{Re}\sigma_1 \sim H^2$. This implies very weak amplification of the only unstable resonant modes which produce non-zero EMF. Note that the weak growth does not apply to unstable horizontal modes considered in the following section, which may also produce non-zero EMF.

From (3.26*a–c*) we can also observe that the effect of background rotation is quite significant, since finite $\mathscr{R}$ substantially modifies the form of the B_0-dependence. Moreover, the precise structure of the EMF depends also on the amplitude of the perturbations, which cannot be established within the framework of linear analysis and could only be obtained through nonlinear analysis. Nevertheless, for some initial period of time during which nonlinear terms can be neglected, the small amplitude is determined by the initial conditions and hence is arbitrary.

4. The EMF generated by unstable horizontal modes

We now turn to calculation of the electromotive force generated by purely horizontal modes, i.e. those having only horizontal components and for which $k_x = k_y = 0$. In this case, the MHD evolution equations reduce to a system of four linear ODEs, which can be written in the vector form analogous to (3.6),

$$\mathrm{d}\boldsymbol{s}/\mathrm{d}\tau = \mathscr{S}\boldsymbol{s} \quad \text{and} \quad \boldsymbol{s} = [v_x, v_y, b_x, b_y], \tag{4.1}$$

where the matrix

$$\mathscr{S} = \begin{bmatrix} 0 & \epsilon + \mathfrak{w} & \mathrm{i}h & 0 \\ \epsilon - \mathfrak{w} & 0 & 0 & \mathrm{i}h \\ \mathrm{i}h & 0 & 0 & -(1+\epsilon) \\ 0 & \mathrm{i}h & 1-\epsilon & 0 \end{bmatrix} \tag{4.2}$$

is independent of time. Here $h = k_z B_0/\sqrt{\mu_0\rho}\gamma$, $\mathfrak{w} = 1 + 2Ro^{-1}$ is the so-called 'tilting vorticity' (Cambon *et al.* 1994; Leblanc 1997; Leblanc & Cambon 1997), $Ro^{-1} = \Omega/\gamma = \mathscr{R}\sqrt{1-\epsilon^2}$ is the inverse of the Rossby number, and $\epsilon = (E - E^{-1})/(E + E^{-1})$, as previously defined, is the strain in the basic flow. Time is scaled with the basic vorticity, thus $t = \tau/\gamma$. It is straightforward to calculate the eigenvalues of the matrix $\mathscr{S}$:

$$\sigma_1 = \sqrt{\epsilon^2 - \chi_+^2}, \quad \sigma_2 = -\sqrt{\epsilon^2 - \chi_+^2}, \tag{4.3a}$$

$$\sigma_3 = \sqrt{\epsilon^2 - \chi_-^2}, \quad \sigma_4 = -\sqrt{\epsilon^2 - \chi_-^2}, \tag{4.3b}$$

where

$$\chi_{\pm}=\sqrt{\tfrac{1}{4}\left(\mathfrak{w}+1\right)^{2}+h^{2}}\pm\tfrac{1}{2}(\mathfrak{w}-1). \tag{4.4}$$

Since for all values h and $\mathfrak{w}$, $\chi_{-}\geqslant 1\geqslant\epsilon$, the only unstable mode is associated with the eigenvalue σ_1, which is real and positive if $\chi_{+}^{2}<1$, i.e. $\mathfrak{w}<(2-h^2)/2$, and $\epsilon\geqslant|\chi_{+}|$, i.e. if the absolute vorticity $\mathfrak{w}+1$ satisfies

$$\frac{(1-\epsilon)^{2}-h^{2}}{1-\epsilon}<\mathfrak{w}+1<\frac{(1+\epsilon)^{2}-h^{2}}{1+\epsilon}. \tag{4.5}$$

The eigenvalues $\sigma_{3,4}$ are always purely imaginary, whereas $\sigma_{1,2}$ are either real (and opposite in sign with $\sigma_1=-\sigma_2>0$) if (4.5) is satisfied, or purely imaginary (or zero) if (4.5) is not satisfied.

In the following we assume that the condition (4.5) is satisfied and hence the eigenmode 1 is unstable. We now calculate the general form of the eigenmodes of the matrix $\mathscr{S}$ associated with eigenvalue σ_j ($j=1,2,3,4$):

$$v_x=\frac{\omega}{k_z}h\sigma_j(1-\mathfrak{w}), \tag{4.6a}$$

$$v_y=\frac{\omega}{k_z}h[(\epsilon+1)(\epsilon-\mathfrak{w})-h^2-\sigma_j^2], \tag{4.6b}$$

$$b_x=-\mathrm{i}\frac{\omega}{k_z}[\sigma_j^2(1+\epsilon)+h^2(\epsilon+\mathfrak{w})-(\epsilon+1)(\epsilon^2-\mathfrak{w}^2)], \tag{4.6c}$$

$$b_y=-\mathrm{i}\frac{\omega}{k_z}\sigma_j(\epsilon^2-\mathfrak{w}^2-h^2-\sigma_j^2). \tag{4.6d}$$

The electromotive force defined in (2.1) can be calculated via

$$\mathscr{E}_z=\tfrac{1}{2}\sqrt{\mu_0\rho}\,\mathrm{Re}\,(\boldsymbol{v}\times\boldsymbol{b}^*)_z, \tag{4.7}$$

and since $\boldsymbol{v}\times\boldsymbol{b}^*$ has only a z-component, the only non-zero component of $\boldsymbol{\mathscr{E}}$ is indeed $\mathscr{E}_z$, which for simplicity will be denoted by $\mathscr{E}$. However, although we consider only the purely horizontal modes here, the actual band of directions around the 'z' axis of wave-vectors of perturbations destabilized via the 'horizontal instability' mechanism is rather large (MB09 and Bajer & Mizerski 2012). In reality, therefore, the structure of the problem is more complex, and components other than $\mathscr{E}_z$ are also non-zero. The vector quantities $\boldsymbol{v}$ and $\boldsymbol{b}$ in (4.7) may represent only one of the four eigenvectors of the matrix $\mathscr{S}$ (associated with only one of the four eigenvalues), or may be a linear combination of these eigenmodes.

4.1. *Electromotive force for a single eigenmode*

For a single eigenmode, either stable or unstable, associated with eigenvalue σ_j, the z-component of $\boldsymbol{v}\times\boldsymbol{b}^*$ can be easily calculated:

$$(\boldsymbol{v}\times\boldsymbol{b}^*)_z=\mathrm{i}\frac{\omega^2}{k_z^2}h\{|\sigma_j|^2(1-\mathfrak{w})(\epsilon^2-\mathfrak{w}^2-h^2-\sigma_j^2)-[(\epsilon+1)(\epsilon-\mathfrak{w})-h^2-\sigma_j^2]$$
$$\times[(\sigma_j^2+\mathfrak{w}^2-\epsilon^2)(1+\epsilon)+h^2(\epsilon+\mathfrak{w})]\}. \tag{4.8}$$

This quantity is always purely imaginary and we conclude that a single horizontal eigenmode, like a single inertial wave in the circular non-dissipative vortex case (Moffatt 1978), cannot generate a non-zero electromotive force as in (4.7). However,

https://doi.org/10.1017/jfm.2012.266 Published online by Cambridge University Press

as we will now demonstrate, an exponentially growing electromotive force can be generated by interaction between eigenmodes.

4.2. *Electromotive force generated by the interaction of eigenmodes*

For a linear combination of eigenmodes 1 and 3, the electromotive force takes the form

$$\begin{aligned}\mathrm{Re}(\boldsymbol{v}\times\boldsymbol{b}^*) &= \mathrm{Re}[(\boldsymbol{v}^{(1)}\mathrm{e}^{\sigma_1 t}+C\boldsymbol{v}^{(3)}\mathrm{e}^{\mathrm{i}\omega_3 t})\mathrm{e}^{\mathrm{i}k_z z}\times(\boldsymbol{b}^{(1)*}\mathrm{e}^{\sigma_1 t}+C^*\boldsymbol{b}^{(3)*}\mathrm{e}^{-\mathrm{i}\omega_3 t})\mathrm{e}^{-\mathrm{i}k_z z}]\\ &= \mathrm{e}^{\sigma_1 t}\mathrm{Re}[C^*(v_x^{(1)}b_y^{(3)*}-v_y^{(1)}b_x^{(3)*})\mathrm{e}^{-\mathrm{i}\omega_3 t}\\ &\quad +C(v_x^{(3)}b_y^{(1)*}-v_y^{(3)}b_x^{(1)*})\mathrm{e}^{\mathrm{i}\omega_3 t}]\hat{\boldsymbol{e}}_z,\end{aligned} \tag{4.9}$$

where C is a complex constant, $\sigma_3=\mathrm{i}\omega_3=\mathrm{i}\,(\chi_-^2-\epsilon^2)^{1/2}$ and $\omega_3\in\boldsymbol{R}$. This linear combination of modes because if condition (4.5) is satisfied, then mode 1 is unstable and mode 3 is purely oscillatory. This guarantees that the real part of the final term in (2.2) is generally non-zero, and that $\mathscr{E}$ is therefore also non-zero. Furthermore we note that such an $\mathscr{E}$ generated by an unstable mode interacting with an oscillatory mode grows exponentially in time, but with superposed oscillations.

Let us now calculate the explicit form of $\mathscr{E}$ in (4.7) from (4.9) and (4.6*a–d*) choosing C real, and $h\ll 1$ (weak magnetic field):

$$\begin{aligned}\mathscr{E} &= \frac{\omega^2}{2k_z^2}Ch\,(1-\varpi)^2(1+\varpi)\sqrt{1+\epsilon}\sqrt{\epsilon-\varpi}\mathrm{e}^{\sigma_1\tau}(\sqrt{\epsilon+\varpi}\sqrt{1-\epsilon}\cos\omega_3\tau\\ &\quad +\sqrt{\epsilon-\varpi}\sqrt{1+\epsilon}\sin\omega_3\tau).\end{aligned} \tag{4.10}$$

In this case $|\varpi|<\epsilon$, $\chi_+=\varpi$ and $\chi_-=1$, so that $\sigma_1=\sqrt{\epsilon^2-\varpi^2}$, and $\omega_3=\sqrt{1-\epsilon^2}$ is equal to the frequency of circulation in the basic elliptic flow. Note that the growth rate, and thus the instability, is strong even in the limit of a weak magnetic field, $h\ll 1$. In fact, as explained by Bajer & Mizerski (2012), even an arbitrarily weak magnetic field typically leads to a powerful horizontal instability. When the tilting vorticity vanishes $\varpi=0$ (or $Ro=-2$), the growth rate is maximal, $\sigma_1=\epsilon$, and the EMF takes the form

$$\mathscr{E}=\frac{\omega^2}{2k_z^2}Ch\epsilon\sqrt{1+\epsilon}\mathrm{e}^{\sigma_1\tau}(\sqrt{1-\epsilon}\cos\omega_3\tau+\sqrt{1+\epsilon}\sin\omega_3\tau). \tag{4.11}$$

This shows that if $\varpi=0$, ellipticity of the vortex ($\epsilon\neq 0$) is necessary for the EMF to be non-zero. On the other hand (4.10) shows the importance of background rotation, since in non-rotating systems, when $\varpi=1$ ($Ro^{-1}=0$), the EMF again vanishes.

The intuition developed from mean-field dynamo theory for inertial waves in rotating systems (see Moffatt 1978) suggests that dissipative effects should play a role in the process of generation of the mean electromotive force. However, this is not the case for horizontal elliptical modes. Dissipative effects are briefly considered in appendix D.

5. Conclusions

We have shown on the example of elliptical instability that the mean electromotive force in MHD systems can be generated by interactions between linear eigenmodes with the same wave-vector $\boldsymbol{k}$, and this mechanism of generation of the EMF *does not require diffusivity to work*. When at least one of the interacting modes is unstable, the generated EMF grows exponentially in time. Moreover, since interactions between two different eigenmodes are required, the electromotive force generated oscillates in time if the modes correspond to two different eigenvalues. The amplitude of the oscillating mean electromotive force then grows exponentially. (However, we note that, if two linearly stable modes correspond to the same (double) eigenvalue, then their mutual interaction as in (2.2) may introduce linear growth in time of the mean electromotive force (without oscillations).)

We have limited our attention to the case in which the mean magnetic field is vertical. This means that we have in effect been able to calculate only the components $\alpha_{xz}, \alpha_{yz}, \alpha_{zz}$ of an α-effect tensor, and this is not sufficient to demonstrate dynamo action. If the mean field has horizontal components, then these are necessarily time-dependent, being convected by the elliptical flow. The eigenvalue problem then becomes much more complex, although still amenable to numerical investigation. This is the subject of future work.

In the limit of small elliptic deformation, we calculated the EMFs for unstable modes in all three resonant cases: hydrodynamic, magnetic and mixed hydro-magnetic. Only the mixed-resonance case leads to a non-zero EMF; in this case, as the modes are unstable, the EMF grows exponentially in time and depends strongly on the magnetic field B_0. Interestingly, the calculation shows that in an ideal fluid $\boldsymbol{\mathcal{E}}$ can be non-zero even in the case of circular flow, i.e. when the ellipticity ζ is zero. However, the modes generating this $\boldsymbol{\mathcal{E}}$ are stable and oscillatory, and therefore $\boldsymbol{\mathcal{E}}$ does not grow exponentially in time, but is purely time-periodic.

We have further shown that a non-zero EMF can be generated by horizontal mode interactions, specifically interactions between an unstable and an oscillatory eigenmode. The z-component of the $\boldsymbol{\mathcal{E}}$ then grows exponentially in time. Background rotation and ellipticity of the basic flow are crucial to this behaviour.

Weak dissipation (viscous and/or magnetic) has a weak influence (as shown in appendix D), but does not change the qualitative behaviour.

The main conclusion is that for the elliptic flow considered, interaction of eigenmodes, at least one of which is unstable, can lead to sustained growth of a mean EMF even in a perfectly conducting inviscid fluid.

Acknowledgements

We wish to thank the anonymous referee for his constructive comments. This work originated when K.M. held a David Crighton Fellowship at the Department of Applied Mathematics and Theoretical Physics, University of Cambridge. The support and hospitality of DAMTP and the Trinity College are gratefully acknowledged. The final stage of this work was funded by grant IP2011 036671 (Iuventus Plus) of the Polish Ministry of Science and Higher Education. K.B. acknowledges the support of the COST Action MP0806.

Appendix A

Since all the calculations in the asymptotic regime are performed with the use of variables c_j, $j = 1, 2, 3, 4$ (see (3.5)) and in the base of eigenvectors of the matrix $\mathscr{C}_0 = \mathscr{C}(\zeta = 0)$, we start by introducing a transformation matrix that allows us to get back to original variables $\boldsymbol{v}$ and $\boldsymbol{b}$:

$$\boldsymbol{\Pi} = \frac{\omega}{Ek_0 \sin\vartheta}\begin{bmatrix} -\mathrm{i}\sigma_1 E\cos\vartheta & \mathrm{i}\sigma_2 E\cos\vartheta & -\mathrm{i}\sigma_3 E\cos\vartheta & \mathrm{i}\sigma_4 E\cos\vartheta \\ \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ HE\cos^2\vartheta & -HE\cos^2\vartheta & HE\cos^2\vartheta & -HE\cos^2\vartheta \\ \mathrm{i}H\cos\vartheta & \mathrm{i}H\cos\vartheta & \mathrm{i}H\cos\vartheta & \mathrm{i}H\cos\vartheta \end{bmatrix}. \qquad \text{(A 1)}$$

Generally the transformation matrix $\boldsymbol{\Pi} = \boldsymbol{\Pi}(\tau)$ depends on time τ but here we only need its form at $\tau = 2\pi$.

The eigenvalues of the matrix $\boldsymbol{M}$ in the case of the mixed hydro-magnetic resonance are

$$\Lambda_1 = \Lambda_4^* = \lambda(1 + \zeta\Gamma_+), \quad \Lambda_2 = \Lambda_3^* = \lambda^*(1 + \zeta\Gamma_-^*), \qquad \text{(A 2)}$$

where $\lambda = \exp[\mathrm{i}2\pi(1 + \mathscr{R})\cos\vartheta]$ and λ^* are the two double eigenvalues of the matrix $\boldsymbol{M}_0 = \boldsymbol{M}(\zeta = 0)$ (in this case there is a double resonance between the first and third eigenvalues and another between the second and fourth eigenvalues, whereas in the hydro–hydro and magnetic–magnetic resonances there is only one double eigenvalue).

The eigenproblem is defined by the matrix $\boldsymbol{M}' - \Lambda_1\boldsymbol{I}$, where $\boldsymbol{I}$ is the identity operator, which has the form

$$\boldsymbol{M}' - \Lambda_1\boldsymbol{I} = \begin{bmatrix} -\zeta\lambda\,\mathrm{Re}\Gamma_+ & \zeta\lambda\mathscr{J}'_{12} & -\zeta\mathrm{i}\lambda\,\mathrm{Re}\Gamma_+ & \zeta\lambda\mathscr{J}'_{14} \\ \zeta\lambda^*\mathscr{J}'_{21} & \lambda^* - \lambda + O(\zeta) & \zeta\lambda^*\mathscr{J}'_{23} & \zeta\mathrm{i}\lambda^*\,\mathrm{Re}\Gamma_+ \\ \zeta\mathrm{i}\lambda\,\mathrm{Re}\Gamma_+ & \zeta\lambda\mathscr{J}'_{32} & -\zeta\lambda\,\mathrm{Re}\Gamma_+ & \zeta\lambda\mathscr{J}'_{34} \\ \zeta\lambda^*\mathscr{J}'_{41} & -\zeta\mathrm{i}\lambda^*\,\mathrm{Re}\Gamma_+ & \zeta\lambda^*\mathscr{J}'_{43} & \lambda^* - \lambda + O(\zeta) \end{bmatrix}, \qquad \text{(A 3)}$$

and so the leading-order form of the eigenvector associated with Λ_1 is

$$\boldsymbol{c}'_1(\tau = 0) = [1, 0, \mathrm{i}, 0]^{\mathrm{T}} \Rightarrow \boldsymbol{c}'_1(\tau = 2\pi) = \Lambda_1\,[1, 0, \mathrm{i}, 0]^{\mathrm{T}}, \qquad \text{(A 4)}$$

where we use the notation introduced in MB09 and the primed variables are expressed in the base of eigenvectors of the matrix $\mathscr{C}_0$. Of course, in an unbounded domain each Floquet mode is also a solution of the full nonlinear Navier–Stokes and induction equations (because the divergence-free conditions imply vanishing of the nonlinear terms), and hence the amplitude of the eigenvector in (A 4) is arbitrary.

Appendix B

Here we calculate the eigenmodes associated with the other two unstable cases, namely the hydro–hydro (case 1) and magnetic–magnetic (case 3) destabilizing resonances.

Case 1: Resonance between two hydrodynamic modes

For this case we have

$$\cos\vartheta = \left[(1 + \mathscr{R}) + \sqrt{(1 + \mathscr{R})^2 + H^2}\right]^{-1}, \qquad \text{(B 1)}$$

$$\varpi_1 = -\varpi_2 = 1, \quad \varpi_3 = -\varpi_4 = \frac{(1 + \mathscr{R}) - \sqrt{(1 + \mathscr{R})^2 + H^2}}{(1 + \mathscr{R}) + \sqrt{(1 + \mathscr{R})^2 + H^2}}, \quad \lambda = 1, \qquad \text{(B 2)}$$

and so the eigenvalue associated with the unstable mode is

$$\Lambda_1 = 1 + \zeta\Gamma \quad \text{where } \Gamma^2 = \frac{\pi^2}{4}(1+\cos\vartheta)^4 - \pi^2\left(\frac{2a}{\cos\vartheta} - \frac{(1+\mathscr{R})\cos\vartheta\,(1-\cos^2\vartheta)}{1-(1+\mathscr{R})\cos\vartheta}\right)^2. \tag{B 3}$$

The matrix $\boldsymbol{M}' - \Lambda_1 \boldsymbol{I}$ in this case takes the form (with primed variables expressed in the base of eigenvectors of the matrix $\mathscr{C}_0$):

$$\boldsymbol{M}' - \Lambda_1\boldsymbol{I} = \begin{bmatrix} \zeta\,(\mathrm{i}\kappa - \Gamma) & -\mathrm{i}\zeta\,\Gamma_m & \zeta\,\mathscr{J}'_{13} & \zeta\,\mathscr{J}'_{14} \\ \mathrm{i}\zeta\,\Gamma_m & \zeta\,(-\mathrm{i}\kappa - \Gamma) & \zeta\,\mathscr{J}'_{23} & \zeta\,\mathscr{J}'_{24} \\ \zeta \mathrm{e}^{\mathrm{i}2\pi\varpi_3}\,\mathscr{J}'_{31} & \zeta \mathrm{e}^{\mathrm{i}2\pi\varpi_3}\,\mathscr{J}'_{32} & \mathrm{e}^{\mathrm{i}2\pi\varpi_3} - 1 + O(\zeta) & \zeta \mathrm{e}^{\mathrm{i}2\pi\varpi_3}\,\mathscr{J}'_{34} \\ \zeta \mathrm{e}^{-\mathrm{i}2\pi\varpi_3}\,\mathscr{J}'_{41} & \zeta \mathrm{e}^{-\mathrm{i}2\pi\varpi_3}\,\mathscr{J}'_{42} & \zeta \mathrm{e}^{-\mathrm{i}2\pi\varpi_3}\,\mathscr{J}'_{43} & \mathrm{e}^{-\mathrm{i}2\pi\varpi_3} - 1 + O(\zeta) \end{bmatrix}, \tag{B 4}$$

where $\kappa = \mathscr{J}'_{11} + \mathrm{i}2\pi a/\cos\vartheta$, $\Gamma_m = \pi(1+\cos\vartheta)^2/2$ is the maximal value of Γ achieved for the value of a given below (4.7) in MB09 and $\Gamma^2 = \Gamma_m^2 - \kappa^2$. This results in the following leading-order expression for the components $[c'_1, c'_2, c'_3, c'_4]^{\mathrm{T}}$ of the eigenvector $\boldsymbol{c}'\,(\tau = 0)$ of the propagator matrix $\boldsymbol{M}'$:

$$\left\{\begin{aligned} &(\mathrm{i}\kappa - \Gamma)\,c'_1 - \mathrm{i}\Gamma_m c'_2 = 0 + O(\zeta), \\ &\mathrm{i}\Gamma_m c'_1 - (\mathrm{i}\kappa + \Gamma)\,c'_2 = 0 + O(\zeta), \\ &\quad c'_3 = O(\zeta), \quad c'_4 = O(\zeta). \end{aligned}\right. \tag{B 5}$$

Since $\boldsymbol{c}'(\tau = 2\pi) = \Lambda_1 \boldsymbol{c}'(\tau = 0)$ at leading order, we obtain

$$\boldsymbol{c}'\,(\tau = 2\pi) = \left[1, \frac{\kappa + \mathrm{i}\Gamma}{\Gamma_m}, 0, 0\right]^{\mathrm{T}}. \tag{B 6}$$

Using the transformation (A 1) together with the solenoidal conditions $\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v} = 0$ and $\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{b} = 0$ taken at $\tau = 2\pi$, we may write

$$\left.\begin{aligned} \boldsymbol{v} &= \frac{\omega}{k_0 \sin\vartheta}\begin{bmatrix} \cos\vartheta\left(1 + \dfrac{\kappa + \mathrm{i}\Gamma}{\Gamma_m}\right) \\ \mathrm{i} + \dfrac{\Gamma - \mathrm{i}\kappa}{\Gamma_m} \\ -\sin\vartheta\left(1 + \dfrac{\kappa + \mathrm{i}\Gamma}{\Gamma_m}\right) \end{bmatrix}, \\ \boldsymbol{b} &= \frac{\omega H\cos\vartheta}{k_0 \sin\vartheta}\begin{bmatrix} \cos\vartheta\left(1 - \dfrac{\kappa + \mathrm{i}\Gamma}{\Gamma_m}\right) \\ \mathrm{i} - \dfrac{\Gamma - \mathrm{i}\kappa}{\Gamma_m} \\ -\sin\vartheta\left(1 - \dfrac{\kappa + \mathrm{i}\Gamma}{\Gamma_m}\right) \end{bmatrix}. \end{aligned}\right\} \tag{B 7}$$

Case 3: Resonance between two magnetic modes

Now the parameters take new values as follows:

$$\cos\vartheta = \left[\sqrt{(1+\mathscr{R})^2 + H^2} - (1+\mathscr{R})\right]^{-1}, \tag{B 8}$$

$$\varpi_4 = -\varpi_3 = 1, \quad \varpi_2 = -\varpi_1 = \frac{(1+\mathscr{R}) + \sqrt{(1+\mathscr{R})^2 + H^2}}{(1+\mathscr{R}) - \sqrt{(1+\mathscr{R})^2 + H^2}}, \quad \lambda = 1, \tag{B 9}$$

and so for this case the eigenvalue associated with the unstable mode is

$$\Lambda_1 = 1 + \zeta\Gamma \quad \text{where } \Gamma^2 = \frac{\pi^2}{4}(1-\cos\vartheta)^4 - \pi^2\left(\frac{2a}{\cos\vartheta} + \frac{(1+\mathscr{R})\cos\vartheta\left(1-\cos^2\vartheta\right)}{1+(1+\mathscr{R})\cos\vartheta}\right)^2. \tag{B 10}$$

Now the matrix $\boldsymbol{M}' - \Lambda_1\boldsymbol{I}$ has the form

$$\boldsymbol{M}' - \Lambda_1\boldsymbol{I} = \begin{bmatrix} e^{i2\pi\varpi_1} - 1 + O(\zeta) & \zeta e^{i2\pi\varpi_1}\mathscr{J}'_{12} & \zeta e^{i2\pi\varpi_1}\mathscr{J}'_{13} & \zeta e^{i2\pi\varpi_1}\mathscr{J}'_{14} \\ \zeta e^{-i2\pi\varpi_1}\mathscr{J}'_{21} & e^{-i2\pi\varpi_1} - 1 + O(\zeta) & \zeta e^{-i2\pi\varpi_1}\mathscr{J}'_{23} & \zeta e^{-i2\pi\varpi_1}\mathscr{J}'_{24} \\ \zeta\mathscr{J}'_{31} & \zeta\mathscr{J}'_{32} & \zeta(-i\kappa - \Gamma) & -i\zeta\Gamma_m \\ \zeta\mathscr{J}'_{41} & \zeta\mathscr{J}'_{42} & i\zeta\Gamma_m & \zeta(i\kappa - \Gamma) \end{bmatrix}, \tag{B 11}$$

where $\kappa = -i\mathscr{J}'_{44} + 2\pi a/\cos\vartheta$, $\Gamma_m = \pi(1-\cos\vartheta)^2/2$ is the maximal value of Γ achieved for the value of a given below (4.18) in MB09 and $\Gamma^2 = \Gamma_m^2 - \kappa^2$, and the leading-order expression for the eigenvector $\boldsymbol{c}'$ $(\tau = 0)$ of the propagator matrix $\boldsymbol{M}'$ is

$$\left\{\begin{aligned} &c'_1 = O(\zeta), \quad c'_2 = O(\zeta), \\ &-(i\kappa + \Gamma)\,c'_3 - i\Gamma_m c'_4 = 0 + O(\zeta), \\ &i\Gamma_m c'_3 + (i\kappa - \Gamma)\,c'_4 = 0 + O(\zeta), \end{aligned}\right. \tag{B 12}$$

and so at leading order (because $\boldsymbol{c}'(\tau = 2\pi) = \Lambda_1\boldsymbol{c}'(\tau = 0)$),

$$\boldsymbol{c}'(\tau = 2\pi) = \left[0, 0, -\frac{\kappa + i\Gamma}{\Gamma_m}, 1\right]^{\mathrm{T}}. \tag{B 13}$$

As for case 1, we obtain

$$\left.\begin{aligned} \boldsymbol{v} &= \frac{\omega}{k_0\sin\vartheta}\begin{bmatrix} \cos\vartheta\left(\dfrac{\kappa + i\Gamma}{\Gamma_m} - 1\right) \\ i - \dfrac{\Gamma - i\kappa}{\Gamma_m} \\ -\sin\vartheta\left(\dfrac{\kappa + i\Gamma}{\Gamma_m} - 1\right) \end{bmatrix}, \\ \boldsymbol{b} &= \frac{\omega H\cos\vartheta}{k_0\sin\vartheta}\begin{bmatrix} -\cos\vartheta\left(\dfrac{\kappa + i\Gamma}{\Gamma_m} + 1\right) \\ \dfrac{\Gamma - i\kappa}{\Gamma_m} + i \\ \sin\vartheta\left(\dfrac{\kappa + i\Gamma}{\Gamma_m} + 1\right) \end{bmatrix}. \end{aligned}\right\} \tag{B 14}$$

We can easily infer from the general considerations of appendix C below and (3.21) that the mean electromotive force is zero for all times in both cases of resonances between modes of the same type, 1 and 3.

Appendix C. Properties of the resonant modes

Here we comment on some particular properties of the analysed system. With the unstable modes calculated in § 3.2 and appendix B ((B 7) for case 1, (B 14) for case 3, and (3.18) and (3.20) for case 2) we can calculate the helicity densities at $\tau = 2\pi$ for the kinetic helicity $\mathscr{H}_k = \mathrm{Re}(\boldsymbol{u}') \cdot \mathrm{Re}(\boldsymbol{w}')$, the magnetic helicity $\mathscr{H}_m = \mathrm{Re}(\boldsymbol{A}') \cdot \mathrm{Re}(\boldsymbol{B}')$ and the cross-helicity $\mathscr{H}_c = \mathrm{Re}(\boldsymbol{u}') \cdot \mathrm{Re}(\boldsymbol{B}')$, where $\boldsymbol{w}' = \nabla \times \boldsymbol{u}'$ is the vorticity of the perturbation velocity field and $\boldsymbol{A} = \boldsymbol{A}_0 + \boldsymbol{A}'$ is a vector potential of the magnetic field $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}' = \nabla \times \boldsymbol{A} = \nabla \times \boldsymbol{A}_0 + \nabla \times \boldsymbol{A}'$ with $\boldsymbol{A}_0 = [-(1/2)B_0 y, (1/2)B_0 x, 0]$. A straightforward calculation shows that all the helicity densities at $\tau = 0$ are zero in cases 1 and 3, and only case 2 of the mixed resonance between the hydrodynamic and magnetic modes leads to helical unstable perturbations (with all three helicity densities non-zero). It is easily demonstrated that for all three cases, the helicity density at time τ is proportional to its initial value; hence, these helicity densities vanish in cases 1 and 3 and are non-zero in case 2 for all times. Helicity provides the simplest indicator of lack of reflectional symmetry in the system, although it is known that this is no guarantee of dynamo action.

We may also show that the unstable modes belonging to the class of mixed resonance are the only ones that propagate energy. The group velocity of the unstable modes is defined as $\boldsymbol{c}_g = \omega \nabla_{\boldsymbol{k}} (\mathrm{Im}\, \sigma)$, where $\nabla_{\boldsymbol{k}}$ is the gradient operator with respect to $\boldsymbol{k}$ and, as we know from MB09, $\mathrm{Im}\sigma$ depends on $\cos\vartheta$ and k_0. Again, a straightforward calculation shows that in cases 1 and 3 the group velocity vanishes,

$$c_g = 0 + O(\zeta) \quad \text{(cases 1 and 3)}, \tag{C 1}$$

so to leading order in ζ these modes do not propagate energy. However, in case 2 the group velocity at leading order is not zero; averaged over time, it has only a z-component given by

$$\langle \boldsymbol{c}_g \rangle = \left[-v_A \frac{H\,(1+\mathscr{R})}{\left[(1+\mathscr{R})^2 + H^2\right]^2} + \frac{\omega}{k_0}(1+\mathscr{R}) \frac{(1+\mathscr{R})^2 + H^2 - 1}{(1+\mathscr{R})^2 + H^2} \right] \hat{\boldsymbol{e}}_z, \tag{C 2}$$

where $v_A = B_0/\sqrt{\mu\rho}$, so that energy is transported in the z-direction with this velocity. Because the resonant case 2 is the case of double resonance, there also exists an unstable mode associated with the eigenvalue Λ^*, for which the energy propagates in the opposite direction and the group velocity $\boldsymbol{c}_g' = -\boldsymbol{c}_g$. The helicities are also of opposite sign for the two unstable modes (one associated with Λ and the other with Λ^*). Nevertheless, as pointed out by Moffatt (1978), in a system in which preferential excitation of upward or downward propagating waves is present, the net helicity will be negative or positive respectively.

Appendix D

In the presence of magnetic (η) and viscous (ν) dissipation, the matrix $\mathscr{S}$ (equations (4.1) and (4.2)) becomes

$$\mathscr{S}_0 = \begin{bmatrix} -Re^{-1} & 1+\epsilon+2Ro^{-1} & \mathrm{i}h & 0 \\ -(1-\epsilon)-2Ro^{-1} & -Re^{-1} & 0 & \mathrm{i}h \\ \mathrm{i}h & 0 & -Rm^{-1} & -(1+\epsilon) \\ 0 & \mathrm{i}h & 1-\epsilon & -Rm^{-1} \end{bmatrix}, \tag{D 1}$$

where $Re = \gamma/(\nu k_0^2)$ and $Rm = \gamma/(\eta k_0^2)$ are the Reynolds and magnetic Reynolds numbers respectively. By defining

$$\mathfrak{D}_\pm = Rm^{-1} \pm Re^{-1}, \quad \Lambda = -\tfrac{1}{2}\mathfrak{D}_+ + \lambda, \tag{D 2}$$

where Λ is an eigenvalue of the matrix $\mathscr{S}$, the eigenproblem for the matrix $\mathscr{S}$ becomes independent of $\mathfrak{D}_+$ and depends only on $\mathfrak{D}_-$. This shows first that if the diffusivities are equal, $\eta = \nu$, i.e. $\mathfrak{D}_- = 0$, then the only effect of dissipation is reduction of the growth rate to $\sigma_1 - \mathfrak{D}_+$. Second, if we assume $\mathfrak{D}_- \ll 1$, then a straightforward perturbative analysis shows that, just as for an ideal fluid, a single unstable horizontal mode cannot generate the electromotive force, and interactions with oscillatory modes are necessary. Moreover, the effect of dissipation is mostly apparent through decrease of the growth rate. As in the ideal-fluid situation, the EMF cannot be generated in a non-rotating system, i.e. for $\mathfrak{w} = 1$ we get $\mathscr{E} = 0$, and the ellipticity is also crucial for generation of a strong electromotive force.

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