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An exact solution in a gravitating fluid with a density-dependent viscosity

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Abstract. An exact nonlinear solution for a cold fluid in presence of a gravitational field and viscous dissipation is obtained using Lagrange variable. It is shown that with a density-dependent viscosity the nonlinear equation can be exactly solved. The solution indicates that in absence of viscosity and initial fluid velocity shear, density collapse occurs at time of the order of inverse Jeans frequency. The effect of viscosity is to delay the collapse but it can not halt the collapse. The initial fluid velocity shear can act in both directions: a positive one leads to delay, a negative one to a speeding up of the density collapse. This nonlinear solution may have some bearing with the structure formations in the universe.

1. Introduction

Investigations on physical processes in the interstellar gas clouds are important for a number of reasons. The star formation mechanism is believed to be due to the interaction between very high velocity interstellar gas clouds leading to the formation of a dense gaseous clump. The gaseous clump grows in mass by the gravitational Jeans instability and finally fragments into slabs. These in turn collapse further by self-gravity and evolve into stars and/or clusters of stars.

The star formation mechanisms based on the linear description of the Jeans instability have been discussed by various authors (Chandrasekhar 1961; Griv et al. 2002). Among these we mention the work of Chandrasekhar in particular, where the system, an infinite homogeneous medium, is affected by viscous effects. It has been shown that the viscosity gives rise to a stabilizing effect that may not alone overcome the Jeans instability. In this paper, we have investigated the nonlinear Jeans instability with a density-dependent viscosity and demonstrate that there is a possible delay but not a halt of the density explosion. However, in presence of fluid pressure a linear analysis indicates stabilization but unfortunately in the present nonlinear solution we have not been able to keep the fluid pressure to obtain the solution with full generality. It may, however, be possible to obtain numerical solutions in order to evaluate its effect further. In case of charged fluid (charged dust grains), a numerical proof of the stabilization of density collapse by pressure had been given by Eliasson et al. (2008).

In our work the linear instability of infinite homogeneous plasma invoking "Jeans swindle" is our starting point and then we construct a more realistic nonlinear solution taking into account the effect of zeroth-order gravitational field. Nonlinear effects are studied using the Lagrangian fluid model demonstrating the non-uniformity of density distributions and of gravitational field in an astrophysical system (Avinash and Shukla 2006; Avinash et al. 2006). We exploit the extremely beautiful and powerful Lagrangian technique to solve the nonlinear equations (Dawson 1959; Davidson and Schram 1968; Davidson 1972; Schamel 2004). Extra simplicity in this model is achieved through the assumption of one spatial dimension and by utilizing a mathematically more tractable and comprehensible case. This enables us to extend the investigation to a wider class of similar systems.

The rest of the paper is organized as follows. In Sec. 2, basic equations and linear mode are presented expressing all assumptions made. In Sec. 3, nonlinear equations are solved for the density that shows the collapse due to singularity formation in finite time. Summary is presented in Sec. 4 with a brief discussion of results.

2. Basic equations and linear mode

The fluid equations in presence of viscosity may be written as

$$\rho\left[\frac{\partial}{\partial t}+(\mathbf{u}\cdot\nabla)\right]\mathbf{u}=-\nabla p+\rho\mathbf{g}+\nabla\left(\frac{4}{3}\mu\nabla\cdot\mathbf{u}\right) \tag{2.1}$$

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\rho\mathbf{u}=0 \tag{2.2}$$

$$\nabla\cdot\mathbf{g}=-4\pi G\rho, \tag{2.3}$$

where ρ is the mass density, $\mathbf{u}$ is the fluid velocity, $\mathbf{g}$ is the gravity, and μ is the shear viscosity. Although a gravitational system, specially the Jeans instability, is usually investigated in three-dimensional fluid, for the study of nonlinear features we shall concentrate on one-dimensional analysis. This will simplify the mathematical analysis without taking the essential physics. By one-dimensional we mean all variables are functions of (x, t) only. We take gravitational field in the negative x direction and assuming $p = c_s^2 \rho$, where c_s represents sound velocity. In context of dusty, plasma both equilibrium and stability were studied in detail (Avinash 2007a,b). Also a comparative analysis on Jeans mode and dust acoustic wave had been studied in the light of dusty plasma by Pandey and his collaborators (Pandey et al. 1994).

For the linear analysis we recapitulate the results of Chandrasekhar (1961) in an infinite homogeneous medium. The linear part is intended to introduce the reader to the concept of Jeans instability and frequency and not to a correct treatment of the linear stability problem with proper equilibrium. Similar to the Jeans treatment we invoke Jeans swindle in the linear problem in which we assume in equilibrium, $g = 0$.

Neglecting all nonlinear terms, linearized equations in one-spatial dimension become

$$\frac{\partial u_1}{\partial t} = -c_s^2 \frac{1}{\rho_0} \frac{\partial \rho_1}{\partial x} - g_1 + \bar{\eta} \frac{\partial^2 u_1}{\partial x^2}, \tag{2.4}$$

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial u_1}{\partial x} = 0, \tag{2.5}$$

$$\frac{\partial g_1}{\partial x} = 4\pi G \rho_1, \tag{2.6}$$

where $\bar{\eta} = 4\mu/3\rho_0$. Variables with subscript 1 are the perturbed quantities and those with subscript 0 are the equilibrium quantities. A simple algebra reduces (2.4)–(2.6) as

$$\left[\frac{\partial^2}{\partial t^2} - c_s^2 \frac{\partial^2}{\partial x^2} - 4\pi G \rho_0 - \bar{\eta} \frac{\partial}{\partial t} \frac{\partial^2}{\partial x^2}\right] \left(\frac{\rho_1}{\rho_0}\right) = 0. \tag{2.7}$$

Assuming the perturbation $\rho_1 \sim \exp(ikx - i\omega t)$ we have the dispersion relation as

$$\omega^2 + i\omega\bar{\eta}k^2 + (\omega_J^2 - k^2 c_s^2) = 0, \tag{2.8}$$

where $\omega_J^2 = 4\pi G \rho_0$. In absence of pressure ($c_s = 0$) and viscosity ($\bar{\eta} = 0$), there is a purely growing instability known as the Jeans instability. Surely both pressure and viscosity act as stabilizing agents. In the next section, we shall study the nonlinear solution in a cold fluid.

3. Nonlinear analysis

The basic nonlinear equations to study this problem are

$$\rho \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x}\right) u = -c_s^2 \frac{\partial \rho}{\partial x} - \rho g + \frac{\partial}{\partial x} \left(\frac{4}{3} \mu \frac{\partial u}{\partial x}\right), \tag{3.1}$$

$$\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x}\right) \rho = -\rho \frac{\partial u}{\partial x}, \tag{3.2}$$

and Poisson's equation for the gravitational field g can be written as

$$\frac{\partial g}{\partial x} = 4\pi G \rho, \tag{3.3}$$

where G is the gravitational constant. It is to be noted here that in (3.1) and (3.2), there are same convective operators. These operators can be simplified if we transform Eulerian variables (x, t) to Lagrangian variables (ξ, τ) such that

$$\xi = x - \int_0^\tau u(\xi, \tau') d\tau', \qquad \tau = t. \tag{3.4}$$

With such transformations, the derivative operators can be written as

$$\frac{\partial}{\partial \tau} \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x}, \qquad \frac{\partial \xi}{\partial x} \equiv \left[1 + \int_0^\tau \frac{\partial u}{\partial \xi} d\tau'\right]^{-1}. \tag{3.5}$$

The continuity (3.2) now can be expressed as

$$\rho(\xi, \tau) = \frac{\rho(\xi, 0)}{\left(1 + \int_0^\tau \frac{\partial u}{\partial \xi} d\tau'\right)}. \tag{3.6}$$

Therefore from the above analysis, we can write $\rho(\xi, \tau)/\rho(\xi, 0) = \partial \xi / \partial x$. Using these relations, the momentum and continuity equations are simplified as

$$\frac{\partial u}{\partial \tau} = -g - c_s^2 \frac{1}{\hat{\rho}(\xi, 0)} \frac{\partial \hat{\rho}}{\partial \xi} + \frac{1}{\rho_0 \hat{\rho}} \frac{\hat{\rho}}{\hat{\rho}(\xi, 0)} \frac{\partial}{\partial \xi} \left[\frac{4}{3} \mu \frac{\hat{\rho}}{\hat{\rho}(\xi, 0)} \frac{\partial u}{\partial \xi}\right] \tag{3.7}$$

$$\frac{\partial}{\partial \tau} \left(\frac{1}{\hat{\rho}}\right) = \frac{1}{\hat{\rho}(\xi, 0)} \frac{\partial u}{\partial \xi}. \tag{3.8}$$

$$\frac{\partial g}{\partial \xi} = 4\pi G \rho_0 \hat{\rho}(\xi, 0), \tag{3.9}$$

where $\hat{\rho} = \rho/\rho_0$ and ρ_0 is a constant normalizing density. These equations can be further simplified by introducing Lagrange mass variable

$$\zeta = \int_0^\xi \hat{\rho}(\xi, 0) d\xi. \tag{3.10}$$

Consequently, the above equations are simplified as

$$\frac{\partial u}{\partial \tau} = -g - c_s^2 \frac{\partial \hat{\rho}}{\partial \zeta} + \eta \frac{\partial^2 u}{\partial \zeta^2} \tag{3.11}$$

$$\frac{\partial}{\partial \tau} \left(\frac{1}{\hat{\rho}}\right) = \frac{\partial u}{\partial \zeta} \tag{3.12}$$

$$\frac{\partial g}{\partial \zeta} = 4\pi G \rho_0. \tag{3.13}$$

For an analytical progress we have defined μ as inversely proportional to the fluid mass density, i.e. $4\mu\rho/3 = \eta$, where η is constant viscosity coefficient. Substituting (3.11) in (3.13) we have

$$\frac{\partial}{\partial \zeta} \left[\frac{\partial u}{\partial \tau} + c_s^2 \frac{\partial \hat{\rho}}{\partial \zeta} - \eta \frac{\partial^2 u}{\partial \zeta^2}\right] = -4\pi G \rho_0. \tag{3.14}$$

In principle, we have to solve (3.14) in order to obtain the full generality of the solution proposed in the model equations. However, this equation is difficult to solve analytically exactly. Therefore, for analytical progress we solve this nonlinear equation using cold fluid approximation, i.e. neglecting pressure term equivalent to $c_s = 0$ in (3.14). In a way, we are investigating the effect of viscosity on Jeans instability in a cold gravitating fluid. Using (3.12) we have

$$\frac{\partial}{\partial \tau}\left[\frac{\partial}{\partial \tau} - \eta\frac{\partial^2}{\partial \zeta^2}\right]\left(\frac{1}{\hat{\rho}}\right) = -4\pi G\rho_0. \tag{3.15}$$

Integrating (3.15) with respect to time we have

$$\left[\frac{\partial}{\partial \tau} - \eta\frac{\partial^2}{\partial \zeta^2}\right]\left(\frac{1}{\hat{\rho}}\right) = -4\pi G\rho_0\tau + c_1(\zeta), \tag{3.16}$$

where $c_1(\zeta)$ is the integration constant that can be determined from initial condition. It is to be noted here that from (3.12)

$$\frac{\partial}{\partial \tau}\left(\frac{1}{\hat{\rho}}\right) = \frac{\partial u}{\partial \zeta}, \qquad \text{where from}$$

$$c_1(\zeta) = \left[\frac{\partial u}{\partial \zeta} - \eta\frac{\partial^2}{\partial \zeta^2}\left(\frac{1}{\rho}\right)\right]_{\tau=0}.$$

With this integration constant (3.16) in normalized form can be written as

$$\left[\frac{\partial}{\partial \hat{\tau}} - \hat{\eta}\frac{\partial^2}{\partial \hat{\zeta}^2}\right]\left(\frac{1}{\hat{\rho}}\right) = -\hat{\tau} + \left[\frac{\partial \hat{u}}{\partial \hat{\zeta}} - \hat{\eta}\frac{\partial^2}{\partial \hat{\zeta}^2}\left(\frac{1}{\hat{\rho}}\right)\right]_{\tau=0}, \tag{3.17}$$

where all the normalized variables are defined as $\hat{\tau} = \omega_J\tau$, $\hat{\zeta} = \zeta/L$, $\hat{\eta} = \eta/L^2\omega_J$, $\hat{u} = u/L\omega_J$. Hereafter we will remove all hats for simplicity of notation but keep in our mind that we are working on normalized variables. Also to simplify notation we have defined normalized specific volume $V(\zeta,\tau) = 1/\rho(\zeta,\tau)$ so that $V_0(\zeta) = 1/\rho(\zeta,0)$. Proposing the solution of (3.17), which now reads

$$\left[\frac{\partial V}{\partial \hat{\tau}} - \hat{\eta}\frac{\partial^2 V}{\partial \hat{\zeta}^2}\right] = -\tau + \left[\frac{\partial u}{\partial \zeta} - \eta\frac{\partial^2 V}{\partial \hat{\zeta}^2}\right]_{\tau=0}, \tag{3.18}$$

to be of the form

$$V(\zeta,\tau) = V_0(\zeta) - \frac{\tau^2}{2} + \phi(\zeta,\tau), \tag{3.19}$$

and substituting (3.19) in (3.18), it can be shown that ϕ satisfies diffusion equation that is given by

$$\frac{\partial \phi}{\partial \tau} = \eta\frac{\partial^2 \phi}{\partial \zeta^2} + \left(\frac{\partial u}{\partial \zeta}\right)_0, \tag{3.20}$$

which is forced diffusion equation subject to the initial condition $\phi(\zeta,0) = 0$ and subscript zero in the right-hand side signifies value of velocity shear at $\tau = 0$. Hereafter we denote $u_0'(\zeta)$ for $(\partial u/\partial\zeta)_0$. In absence of viscous dissipation, $\eta = 0$, the solution of ϕ is easily found and is given by $\phi(\zeta,\tau) = u_0'(\zeta)\tau$, which after insertion in (3.19) tells us that there is a specific point in the (ζ,τ) plane:(ζ_c,τ_c), where $V(\zeta_c,\tau_c) = 0$. Therefore (ζ_c,τ_c) is a point, where the specific volume (inverse of density) becomes zero and hence density collapses.

In presence of viscous dissipation where $\phi(\zeta,\tau)$ solves the forced diffusion (3.20) with the driving term $u_0'(\zeta)$, the specific solution may be written as

$$\phi(\zeta,\tau) = \int_0^\tau ds \int_{-\infty}^{+\infty} d\zeta' u_0'(\zeta')\frac{1}{\sqrt{4\pi\eta(\tau-s)}} \times \exp\left[-\frac{(\zeta-\zeta')^2}{4\pi\eta(\tau-s)}\right], \tag{3.21}$$

which is valid for arbitrary $u_0'(\zeta)$. It should be emphasized here that for non-viscous case, i.e. $\eta \to 0$, the above solution reduces to the known solution as obtained before. In the limit of $\eta \to 0$, the inner integrand Green's function in (3.21) becomes a delta function

$$\phi(\zeta,\tau) = \int_0^\tau ds \int_{-\infty}^{+\infty} d\zeta' u_0'(\zeta')\delta(\zeta-\zeta') = u_0'(\zeta)\int_0^\tau ds = u_0'(\zeta)\tau, \tag{3.22}$$

which gives back our old result. Moreover, the tendency of delay (or speeding up) of the collapse process is essentially maintained by the action of the viscous dissipation.

For a specific example, we now assume $u_0(\zeta) = \tanh\zeta$ and defining $\Phi(\zeta,\tau) = \phi(\zeta,\tau) + (1/\eta)\ln\cosh\zeta$ we can show that the new dependent variable Φ satisfies the standard diffusion equation, which is given by

$$\frac{\partial \Phi}{\partial \tau} = \eta\frac{\partial^2 \Phi}{\partial \zeta^2}, \tag{3.23}$$

with the initial condition $\Phi(\zeta,0) = (1/\eta)\ln\cosh\zeta$, where the equilibrium velocity amplitude is taken to be unity in normalized variable. The solution of (3.23) can be found in a standard textbook and is given by

$$\Phi(\zeta,\tau) = \frac{1}{\eta\sqrt{4\pi\eta\tau}}\int_{-\infty}^{+\infty} d\zeta' \ \ln\cosh (\zeta')\exp\left[-\frac{(\zeta-\zeta')^2}{4\eta\tau}\right]. \tag{3.24}$$

The initial condition is readily satisfied since in the limit $\tau \to 0$ the Green's function in the above equation becomes a delta function $\delta(\zeta-\zeta')$. Therefore complete solution for $\rho(\zeta,\tau)$ can be written as

$$\rho(\zeta,\tau) = \frac{\rho(\zeta,0)}{1-\rho(\zeta,0)\left[\tau^2/2 + (1/\eta)\ln\cosh\zeta - \Phi(\zeta,\tau)\right]}. \tag{3.25}$$

The obtained solution is complete if we can express the Lagrangian variable ζ in terms of Eulerian variable x. To do this we consider the relation $\rho(\xi,\tau)/\rho(\xi,0) = \partial\xi/\partial x$ as obtained before and we can write

$$\frac{\partial \zeta}{\partial x} = \frac{\partial \zeta}{\partial \xi}\frac{\partial \xi}{\partial x} = \rho(\xi,0)\frac{\rho(\xi,\tau)}{\rho(\xi,0)} = \rho(\zeta,\tau).$$

Using (3.25) for $\rho(\zeta,\tau)$ we have got a relation between Eulerian and Lagrangian variables as

$$x = \int_0^\zeta \frac{1}{\rho(\zeta',0)}d\zeta' - \frac{\tau^2}{2}\int_0^\zeta d\zeta' - \frac{1}{\eta}\int_0^\zeta \ln\cosh$$

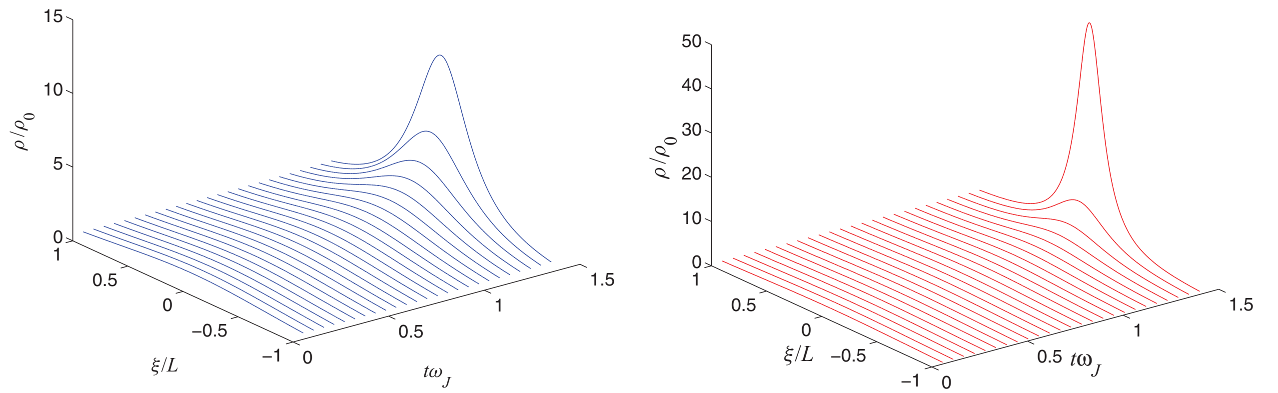

Figure 1. (Color online) Fluid density in Lagrangian variables $(\xi - \tau)$(left) shows large amplitude behavior with time. In this plot, we have taken $\hat{\eta} = 0.001$ and $\omega_J \tau = 1.35$. In the right side, same plot has been shown with same viscosity parameter but $\omega_J \tau = 1.4$. Figure indicates a density collapse in finite time.

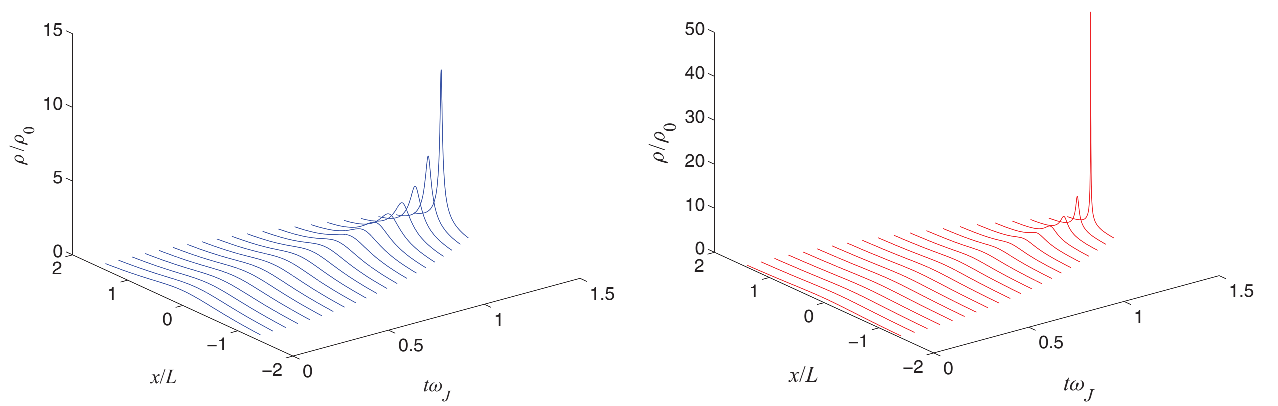

Figure 2. (Color online) Fluid density in Eulerian variables $(x - t)$(left) shows large amplitude behavior with time. In this plot, we have taken $\hat{\eta} = 0.001$ and $\omega_J \tau = 1.35$. In the right side, same plot has been shown with same viscosity parameter but $\omega_J \tau = 1.4$. Figure indicates a density collapse in finite time.

$$\zeta' d\zeta' + \int_0^{\zeta} \Phi(\zeta', \tau) d\zeta', \tag{3.26}$$

which clearly indicates that this relation crucially depends on the initial density distribution. For a specific initial density profile $\rho(\zeta, 0) = \text{sech}^2 \zeta$ we have obtained density in Eulerian variables (x, t). Since in (3.26) the last two integration is difficult to obtain analytically we have calculated x, ζ relation numerically and density is plotted with Lagrangian (Fig. 1) and Eulerian variables (Fig. 2). Both plots show collapse behavior with increasing time, which indicates that viscosity cannot arrest the density collapse. In Eulerian variables, density contours are more stretched as shown in Fig. 2. To find the condition in which the solutions are valid is $\rho \geqslant 0$. This implies that

$$1 - \rho(\zeta, 0) \left[\frac{\tau^2}{2} + (1/\eta) \ln \cosh(\zeta) - \Phi \right] \geqslant 0. \tag{3.27}$$

From the density solution (3.25) it is clear that in absence of velocity shear ($1/\eta$ term in normalized unit) and viscosity, density collapse occur at time $\tau = \sqrt{2}/\omega_J(x)$. We have not observed that due to viscosity density solution splits that is reported in electron plasma wave dynamics (Infeld et al. 2009). Finally, we mentioned that a new mechanism for matter clumping prior to the onset of gravitational contraction, being based on nonlinear magnetosonic wave steepening and collapse and treated similarly by Lagrangian variables, has been proposed recently (Chakrabarti et al. 2011, 2013).

4. Summary

In a short summary, we like to point out that the present analysis of density collapse due to Jeans instability is obtained through a very simplified model. In this model, nonlinear time-dependent processes are involved in a cold fluid. Non-relativistic gravitohydrodynamic equations are solved and the solutions are physically interesting in the sense that the density bursts are found indicating the signature of nonlinear instability even in presence of viscosity. From mathematical point of view, it is interesting to note that the Lagrangian mass variable transformation (Sack and Schamel 1987) is introduced to reduce the spatial partial derivative in some analytically tractable form. Perhaps calculations with these transformation in the nonlinear states are very useful and powerful. We have not introduced spatial variations in more than one dimension to avoid complexity in calculations. The type of solution considered represents a class of nonlinear solutions that may arise in many such similar physical situations. Full numerical solution of basic equations is of general interest including pressure term and we hope to report this in future.

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