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Cyclic parallel structure Jacobi operator for real hypersurfaces in complex two-plane Grassmannians

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In this paper, from the property of Killing for structure Jacobi tensor $\mathbb{R}_\xi$, we introduce a new notion of *cyclic parallelism of structure Jacobi operator* R_ξ on real hypersurfaces in the complex two-plane Grassmannians. By virtue of geodesic curves, we can give the equivalent relation between cyclic parallelism of R_ξ and Killing property of $\mathbb{R}_\xi$. Then, we classify all Hopf real hypersurfaces with *cyclic parallel structure Jacobi operator* in complex two-plane Grassmannians.

Keywords: Hopf real hypersurface; complex two-plane Grassmannians; (quadratic) Killing tensor; cyclic parallelism; structure Jacobi operator

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1. Introduction

In the class of complex Grassmannians of rank 2, we can give the examples of Hermitian symmetric spaces $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$ and $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2U_m)$, which are said to be *complex two-plane Grassmannians* of compact type and *complex hyperbolic two-plane Grassmannians* of non-compact type, respectively. They are viewed as Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure J and the quaternionic Kähler structure $\mathcal{J} = \text{span}\{J_1, J_2, J_3\}$ (see [**6**, **11**, **15**, **31**, **33**, **38**]). Among them, in this paper we will consider our subject on complex two-plane Grassmannians and its real hypersurfaces with *cyclic parallel structure Jacobi operator*.

Now let us denote by $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$ the set of all complex 2-dimensional linear subspaces in the complex Euclidean space $\mathbb{C}^{m+2}$. If $m = 1$, then we see that $G_2(\mathbb{C}^3)$ is isometric to the 2-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature 8. And the isomorphism $\mathrm{Spin}(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented 2-dimensional linear subspaces in $\mathbb{R}^6$. So, we will consider $m \geqslant 3$ hereafter, unless otherwise stated.

Recall that a non-zero vector field X of Hermitian symmetric spaces $(\bar{M}, g)$ of rank 2 is called *singular* if it is tangent to more than one maximal flat in $\bar{M}$. In particular, there are exactly two types of singular tangent vectors X of $G_2(\mathbb{C}^{m+2})$ which are characterized by the geometric properties $JX \in \mathcal{J}X$ and $JX \perp \mathcal{J}X$ (see [3, 4]).

The Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}
&\bar{R}(X,Y)Z \\
&= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\
&\quad + \sum_{\nu=1}^{3} \{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\} \\
&\quad + \sum_{\nu=1}^{3} \{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},
\end{aligned} \tag{1.1}$$

where $\{J_1,\ J_2,\ J_3\}$ is any canonical local basis of $\mathcal{J}$ and the tensor g of type (0,2) stands for the Riemannian metric on complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ (see [3, 4, 9]).

For a real hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, we have the following two natural geometric conditions: *the* 1*-dimensional distribution* $\mathcal{C}^\perp = \mathrm{span}\{\xi\}$ *and the* 3*-dimensional distribution* $\mathcal{Q}^\perp = \mathrm{span}\{\xi_1, \xi_2, \xi_3\}$ *are invariant under the shape operator* A *of* M. Here the almost contact structure vector field ξ defined by $\xi = -JN$ is said to be a *Reeb* vector field, where N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. The *almost contact 3-structure* vector fields ξ_1, ξ_2, ξ_3 spanning the 3-dimensional distribution $\mathcal{Q}^\perp$ of M in $G_2(\mathbb{C}^{m+2})$ are defined by $\xi_\nu = -J_\nu N$ $(\nu = 1, 2, 3)$, such that $TM = \mathcal{Q} \oplus \mathcal{Q}^\perp = \mathcal{C} \oplus \mathcal{C}^\perp$. By using these invariant conditions for two kinds of distributions $\mathcal{C}^\perp$ and $\mathcal{Q}^\perp$ in $TG_2(\mathbb{C}^{m+2})$, Berndt and Suh gave a classification of real hypersurfaces in complex two-plane Grassmannians as follows:

Theorem A ([4])**.** *Let M be a connected real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Then both $\mathcal{C}^\perp$ and $\mathcal{Q}^\perp$ are invariant under the shape operator A of M if and only if*

> $(\mathcal{T}_A)$ *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*
>
> $(\mathcal{T}_B)$ *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

On the other hand, we say that a real hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ is *Hopf* if and only if the Reeb vector field ξ is Hopf, that is, $A\xi \in \mathcal{C}^{\perp}$. In addition, when the distribution $\mathcal{Q}^{\perp}$ of M in $G_2(\mathbb{C}^{m+2})$ is invariant under the shape operator, M is said to be a $\mathcal{Q}^{\perp}$*-invariant real hypersurface*.

Moreover, we say that the Reeb flow of M in $G_2(\mathbb{C}^{m+2})$ is *isometric*, when the Reeb vector field ξ of M is Killing. It implies that the metric tensor g of M is invariant under the Reeb flow of ξ, that is, $\mathcal{L}_\xi g = 0$ where $\mathcal{L}_\xi$ denotes the Lie derivative along the direction of ξ. Related to this notion, for complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, Berndt and Suh gave a remarkable characterization for real hypersurface of type $(\mathcal{T}_A)$ mentioned in theorem A (see [**5**]).

Indeed, the notion of isometric Reeb flow is regarded as a typical example of Killing vector fields which are classical objects of differential geometry. As mentioned above, Killing vector fields are defined by vanishing of the Lie derivative of metric tensor g with respect to a vector X, that is, $\mathcal{L}_X g = 0$. Recently, the notion of isometric Reeb flow is considered for real hypersurfaces in Hermitian symmetric spaces including complex Grassmannians and complex quadrics, etc. (see [**5**, **7**, **32**, **35**]). By using Lie algebraic method given in [**1**, **2**, **10**], Berndt–Suh [**8**] gave a complete classification of real hypersurfaces with isometric Reeb flow in Hermitian symmetric spaces.

Let us consider a Killing tensor field which is a generalization of a Killing vector field on $(\bar{M}, g)$. Let $\mathbb{K}$ be a tensor field of type $(0, k)$ on $(\bar{M}, g)$. Then, $\mathbb{K}$ is said to be *Killing* if the complete symmetrization of $\nabla\mathbb{K}$ vanishes. That is, it means that $\mathbb{K}$ satisfies

$$(\nabla_X \mathbb{K})(X, X, \dots, X) = 0$$

for any vector field X. It follows that for such a Killing tensor, the expression $\mathbb{K}(\dot{\gamma}, \dot{\gamma}, \cdots, \dot{\gamma})$ is constant along any geodesic γ (see [**29**]). In particular, the existing literature on symmetric Killing tensors is huge, especially coming from theoretical physics (see [**12**, **29**]). As examples of such a symmetric Killing tensor, real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with *Killing shape operator* were considered by Lee and Suh (see [**20**]). Recently, Lee, Woo and Suh [**21**] considered the notion of *Killing normal Jacobi operator* of Hopf real hypersurfaces in complex Grassmannians of rank 2. In addition, Suh gave a classification for Hopf real hypersurfaces with *Killing Ricci tensor* in complex Grassmannians of rank 2 (see [**36**, **37**]).

Now, we define a structure Jacobi tensor $\mathbb{R}_\xi$ which is a symmetric tensor field of type (0,2) on M in $G_2(\mathbb{C}^{m+2})$ given by

$$\mathbb{R}_\xi(Y, Z) = g(R_\xi Y, Z) \tag{1.2}$$

for any tangent vector fields Y and Z on M. Here, R_ξ is a symmetric tensor field of type (1,1) on M (so-called, the *structure Jacobi operator* of M). If the structure Jacobi tensor $\mathbb{R}_\xi$ satisfies

$$(\nabla_X \mathbb{R}_\xi)(X, X) = 0$$

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for any tangent vector field X on M, then $\mathbb{R}_\xi$ is said to be *Killing*. Taking the covariant derivative of (1.2), the property of Killing with respect to $\mathbb{R}_\xi$ becomes

$$(\nabla_X \mathbb{R}_\xi)(X, X) = g((\nabla_X R_\xi)X, X) = 0. \tag{1.3}$$

By virtue of the linearization, (1.3) can be rearranged as

$$g\big((\nabla_X R_\xi)Y, Z\big) + g\big((\nabla_Y R_\xi)Z, X\big) + g\big((\nabla_Z R_\xi)X, Y\big) = 0 \tag{1.4}$$

for any tangent vector fields X, Y and $Z \in TM$. If the structure Jacobi operator R_ξ of M in $G_2(\mathbb{C}^{m+2})$ satisfies (1.4), we say that R_ξ is *cyclic parallel*. Moreover, by local existence and uniqueness theorem for geodesics, (1.4) can be interpreted that the structure Jacobi curvature $\mathbb{R}_\xi(\dot\gamma, \dot\gamma) := g(R_\xi\dot\gamma, \dot\gamma)$ is constant along the geodesic γ with $\gamma(0) = p$ and $\dot\gamma(0) = X_p$ for any point $p \in M$ and any tangent vector $X(p) = X_p \in T_pM$.

From the assumption of structure Jacobi operator being cyclic parallel, first we assert that the unit normal vector field N becomes singular as follows:

Theorem 1. *Let M be a Hopf real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ for $m \geqslant 3$. If M has a cyclic parallel structure Jacobi operator, then the normal vector field N of M is singular.*

Next, by using theorem 1 we give a classification of Hopf real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$, with cyclic parallel structure Jacobi operator as follows:

Theorem 2. *Let M be a Hopf real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Then the structure Jacobi operator R_ξ of M is cyclic parallel if and only if M is locally congruent to an open part of a tube of $r = (\pi/4\sqrt{2})$ around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

2. Preliminaries

As mentioned in the introduction, the complete classifications of real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$, satisfying two invariant conditions for the distributions $\mathcal{C}^\perp = \mathrm{span}\{\xi\}$ and $\mathcal{Q}^\perp = \mathrm{span}\{\xi_1, \xi_2, \xi_3\}$ was given in [**4**].

In fact, in [**3**, **4**] Berndt and Suh gave the characterizations of the singular unit normal vector N of M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$: *There are two types of singular normal vector, those N for which $JN \perp \mathcal{J}N$, and those for which $JN \in \mathcal{J}N$*. In other words, it means that $\xi \in \mathcal{Q}$ or $\xi \in \mathcal{Q}^\perp$ because $JN = -\xi$, $\mathcal{J}N = \mathrm{span}\{\xi_1, \xi_2, \xi_3\} = \mathcal{Q}^\perp$, and $TM = \mathcal{Q} \oplus \mathcal{Q}^\perp$. The following proposition tells us that the normal vector field N on the model spaces of $(\mathcal{T}_A)$ is singular of type of $JN \in \mathcal{J}N$, that is, $\xi \in \mathcal{Q}^\perp$.

Proposition A ([**4**, **9**]). *Let $(\mathcal{T}_A)$ be the tube of radius $0 < r < \frac{\pi}{\sqrt{8}}$ around the totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. Then the following statements hold:*

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Table 1. *Principal curvatures of a model space of type* $(\mathcal{T}_A)$

Type	Eigenvalues	Eigenspace	Multiplicity
$(\mathcal{T}_A)$	$\alpha=\sqrt{8}\cot(\sqrt{8}r)$	$T_\alpha=\mathcal{C}^\perp=\text{span}\{\xi\}=\text{span}\{\xi_1\}$	1
	$\beta=\sqrt{2}\cot(\sqrt{2}r)$	$T_\beta=\mathcal{C}\ominus\mathcal{Q}=\text{span}\{\xi_2,\xi_3\}$	2
	$\lambda=-\sqrt{2}\tan(\sqrt{2}r)$	$T_\lambda=E_{-1}=\{X\in\mathcal{Q}\,\vert\,\phi X=\phi_1 X\}$	$2m-2$
	$\mu=0$	$T_\mu=E_{+1}=\{X\in\mathcal{Q}\,\vert\,\phi X=-\phi_1 X\}$	$2m-2$

1. $(\mathcal{T}_A)$ *is a Hopf hypersurface.*

2. *Every unit normal vector field* N *of* $(\mathcal{T}_A)$ *is singular and of type* $JN \in \mathcal{J}N$.

3. *The eigenvalues and their corresponding eigenspaces and multiplicities are given in Table 1.*

4. *The Reeb flow on* $(\mathcal{T}_A)$ *is isometric.*

In proposition A, the notion of isometric Reeb flow gave a kind of characterizations of real hypersurface of type $(\mathcal{T}_A)$. Like for such an investigation, many geometric conditions were considered as characterizations of the model space of $(\mathcal{T}_A)$ in complex two-plane Grassmannians (see [**14**, **22**, **23**, **25**, **26**, **28**, **39**, **40**]).

On the other hand, by using the notion of isometric Reeb flow, that is, the shape operator A of a Hopf real hypersurface M in $G_2(\mathbb{C}^{m+2})$ commutes with structure tensor ϕ, that is, $A\phi = \phi A$, Berndt and Suh gave:

$$\begin{aligned}(\nabla_X A)Y &= -\eta(Y)\phi X + (X\alpha)\eta(Y)\xi + \alpha g(A\phi X, Y)\xi - g(A^2\phi X, Y)\xi \\ &\quad - \sum_{i=1}^{3}\{\eta_\nu(Y)\phi_\nu X + g(\phi_\nu\xi, Y)\phi\phi_\nu X + 2g(\phi_\nu\xi, X)\phi\phi_\nu Y \\ &\quad + g(\phi_\nu\xi, X)\eta_\nu(Y)\xi - \eta_\nu(\xi)g(\phi_\nu X, Y)\xi + g(\phi_\nu X, Y)\xi_\nu \\ &\quad - \eta(X)\eta_\nu(Y)\phi_\nu\xi + g(\phi_\nu\phi X, Y)\phi_\nu\xi\}\end{aligned} \tag{2.1}$$

for any tangent vector fields X and Y on M (see proposition 4 in [**5**]). In fact, from (iv) in proposition A, we see that the shape operator A of $(\mathcal{T}_A)$ satisfies $A\phi = \phi A$. Thus, the above equation (2.1) holds on $(\mathcal{T}_A)$ and it can be rearranged as

$$\begin{aligned}(\nabla_X A)Y &= -\eta(Y)\phi X + \alpha g(A\phi X, Y)\xi - g(A^2\phi X, Y)\xi \\ &\quad - \sum_{i=1}^{3}\{\eta_\nu(Y)\phi_\nu X + g(\phi_\nu\xi, Y)\phi\phi_\nu X + 2g(\phi_\nu\xi, X)\phi\phi_\nu Y \\ &\quad + g(\phi_\nu\xi, X)\eta_\nu(Y)\xi - \eta_\nu(\xi)g(\phi_\nu X, Y)\xi + g(\phi_\nu X, Y)\xi_\nu \\ &\quad - \eta(X)\eta_\nu(Y)\phi_\nu\xi + g(\phi_\nu\phi X, Y)\phi_\nu\xi\}\end{aligned} \tag{2.2}$$

for any tangent vector fields X and Y on $T(\mathcal{T}_A) = T_\alpha \oplus T_\beta \oplus T_\lambda \oplus T_\mu$.

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3. Fundamental equations of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

We use some references [17, 27, 34] to recall the Riemannian geometry of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$, and some fundamental formulas including the Codazzi and Gauss equations for a real hypersurface in $G_2(\mathbb{C}^{m+2})$.

Let M be a real hypersurface of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$, that is, a submanifold of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal field of M in $G_2(\mathbb{C}^{m+2})$ and S the shape operator of M with respect to N, that is, $\bar{\nabla}_X N = -SX$. The Kähler structure J of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ, ξ, η, g). Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of the quaternionic Kähler structure $\mathcal{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M. Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{3.1}$$

for any tangent vector X on a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a normal vector of M in $G_2(\mathbb{C}^{m+2})$. Then the following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

$$\begin{aligned} &\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \quad \phi\xi_\nu = \phi_\nu\xi, \quad \eta_\nu(\phi X) = \eta(\phi_\nu X), \\ &\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \quad \phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}, \end{aligned} \tag{3.2}$$

where we have used that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$.

On the other hand, from the parallelism of J and $\mathcal{J}$ which are defined by

$$\bar{\nabla}_X J = 0 \ \text{ and } \ \bar{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \ (\nu \bmod 3),$$

together with Gauss and Weingarten formulas, it follows that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{3.3}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{3.4}$$

$$\begin{aligned} (\nabla_X\phi_\nu)Y = &-q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y \\ &+ \eta_\nu(Y)AX - g(AX, Y)\xi_\nu. \end{aligned} \tag{3.5}$$

Combining these formulas, we find the following

$$\begin{aligned} \nabla_X(\phi_\nu\xi) &= \nabla_X(\phi\xi_\nu) \\ &= (\nabla_X\phi)\xi_\nu + \phi(\nabla_X\xi_\nu) \\ &= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu\phi AX \\ &\quad - g(AX, \xi)\xi_\nu + \eta(\xi_\nu)AX. \end{aligned} \tag{3.6}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1, 2, 3$, it follows that

$$\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu. \tag{3.7}$$

Finally, using the explicit expression for the Riemannian curvature tensor $\bar{R}$ of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ in the introduction, the Codazzi and

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Gauss equations of M in $G_2(\mathbb{C}^{m+2})$ are given respectively by

$$
\begin{aligned}
(\nabla_X A)Y - (\nabla_Y A)X = {} & \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\
& + \sum_{\nu=1}^{3} \{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\} \\
& + \sum_{\nu=1}^{3} \{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\} \\
& + \sum_{\nu=1}^{3} \{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\xi_\nu
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\
& - 2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY \\
& + \sum_{\nu=1}^{3} \Big\{ g(\phi_\nu Y, Z)\phi_\nu X - g(\phi_\nu X, Z)\phi_\nu Y - 2g(\phi_\nu X, Y)\phi_\nu Z \\
& + g(\phi_\nu\phi Y, Z)\phi_\nu\phi X - g(\phi_\nu\phi X, Z)\phi_\nu\phi Y \\
& + \eta(X)\eta_\nu(Z)\phi_\nu\phi Y - \eta(Y)\eta_\nu(Z)\phi_\nu\phi X \\
& + \eta(Y)g(\phi_\nu\phi X, Z)\xi_\nu - \eta(X)g(\phi_\nu\phi Y, Z)\xi_\nu \Big\}
\end{aligned}
\tag{3.9}
$$

for any tangent vector fields X, Y and Z on M.

On the other hand, we can derive some important facts from the geometric condition of M being Hopf, that is, $A\xi = \alpha\xi$ where $\alpha = g(A\xi, \xi)$. Among them, we introduce the following formulas which are induced from the Codazzi equation:

Lemma A ([5]). *If M is a connected orientable Hopf real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$, then*

$$
\operatorname{grad} \alpha = (\xi\alpha)\xi + 4\sum_{\nu=1}^{3} \eta_\nu(\xi)\phi_\nu\xi \tag{3.10}
$$

and

$$
\begin{aligned}
& 2A\phi AX - \alpha A\phi X - \alpha\phi AX \\
& \quad = 2\phi X + 2\sum_{\nu=1}^{3} \{\eta_\nu(X)\phi_\nu\xi - g(\phi_\nu\xi, X)\xi_\nu + \eta_\nu(\xi)\phi_\nu X\} \\
& \qquad - 4\sum_{\nu=1}^{3} \{\eta(X)\eta_\nu(\xi)\phi_\nu\xi - \eta_\nu(\xi)g(\phi_\nu\xi, X)\xi\}
\end{aligned}
\tag{3.11}
$$

for any tangent vector field X on M in $G_2(\mathbb{C}^{m+2})$.

4. Proof of theorem 1

Let M be a Hopf real hypersurface with cyclic parallel structure Jacobi operator in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$.

From (3.9) the structure Jacobi operator $R_\xi \in \mathrm{End}(TM)$ is given as follows

$$\begin{aligned} R_\xi(Y) &= R(Y,\xi)\xi \\ &= Y - \eta(Y)\xi + \alpha AY - \alpha^2\eta(Y)\xi \\ &\quad - \sum_{\nu=1}^{3}\left\{\eta_\nu(Y)\xi_\nu - \eta(Y)\eta_\nu(\xi)\xi_\nu - 3g(\phi_\nu\xi, Y)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi Y\right\} \end{aligned} \tag{4.1}$$

for any tangent vector field $Y \in TM$ (see [19, 24]).

Taking the covariant derivative of (4.1) along the direction of X implies

$$\begin{aligned} (\nabla_X R_\xi)Y &= \nabla_X(R_\xi Y) - R_\xi(\nabla_X Y) \\ &= -g(\phi AX, Y)\xi - \eta(Y)\phi AX \\ &\quad - \sum_{\nu=1}^{3}\Big[g(\phi_\nu AX, Y)\xi_\nu + 2\eta(Y)g(\phi_\nu\xi, AX)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \\ &\quad + 3g(\phi_\nu AX, \phi Y)\phi_\nu\xi + 3\eta(Y)\eta_\nu(AX)\phi_\nu\xi \\ &\quad - 3g(\phi_\nu\xi, Y)\phi_\nu\phi AX + 3\alpha\eta(X)g(\phi_\nu\xi, Y)\xi_\nu \\ &\quad - 4\eta_\nu(\xi)g(\phi_\nu\xi, Y)AX - 4\eta_\nu(\xi)g(AX, Y)\phi_\nu\xi \\ &\quad - 2g(\phi_\nu\xi, AX)\phi_\nu\phi Y\Big] \\ &\quad + g((\nabla_X A)\xi, \xi)AY + \alpha(\nabla_X A)Y - \alpha g((\nabla_X A)Y, \xi)\xi \\ &\quad - \alpha g(AY, \phi AX)\xi - \alpha\eta(Y)(\nabla_X A)\xi - \alpha\eta(Y)A\phi AX \end{aligned} \tag{4.2}$$

for any tangent vector fields X and Y on M (see [19]). From this and using symmetric property of the structure Jacobi operator R_ξ in $G_2(\mathbb{C}^{m+2})$, the cyclic parallelism of the structure Jacobi operator (1.4) can be rearranged as follows:

$$\begin{aligned} 0 &= g\big((\nabla_X R_\xi)Y, Z\big) + g\big((\nabla_Y R_\xi)Z, X\big) + g\big((\nabla_Z R_\xi)X, Y\big) \\ &= g\big((\nabla_X R_\xi)Y, Z\big) + g\big((\nabla_Y R_\xi)X, Z\big) \\ &\quad + g(A\phi X, Z)\eta(Y) + \eta(X)g(A\phi Y, Z) + (\xi\alpha)g(AX, Y)\eta(Z) \\ &\quad - \alpha(\xi\alpha)\eta(X)\eta(Y)\eta(Z) + \alpha^2\eta(Y)g(A\phi X, Z) - \alpha\eta(Y)g(A\phi AX, Z) \\ &\quad + \alpha\eta(Y)g(A\phi AX, Z) + \alpha\eta(X)g(A\phi AY, Z) - \alpha(\xi\alpha)\eta(X)\eta(Y)\eta(Z) \\ &\quad + \alpha^2\eta(X)g(A\phi Y, Z) - \alpha\eta(X)g(A\phi AY, Z) + \alpha g((\nabla_X A)Y, Z) \\ &\quad + \alpha g(\phi X, Y)\eta(Z) + \alpha\eta(X)g(\phi Y, Z) + 2\alpha\eta(Y)g(\phi X, Z) \end{aligned}$$

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$$
\begin{aligned}
&+\sum_{\nu=1}^{3}\Big[\eta_\nu(Y)g(A\phi_\nu X,Z)-2\eta(X)\eta_\nu(Y)g(A\phi_\nu\xi,Z)+\eta_\nu(X)g(A\phi_\nu Y,Z)\\
&+3g(\phi_\nu\xi,Y)g(A\phi_\nu\phi X,Z)-3\eta(X)g(\phi_\nu\xi,Y)g(A\xi_\nu,Z)\\
&+3g(\phi_\nu\xi,X)g(A\phi\phi_\nu Y,Z)-3\alpha g(\phi_\nu\xi,X)\eta_\nu(Y)\eta(Z)\\
&+4\eta_\nu(\xi)g(\phi_\nu\xi,X)g(AY,Z)+4\eta_\nu(\xi)g(\phi_\nu\xi,Y)g(AX,Z)\\
&+2g(\phi_\nu\phi X,Y)g(A\phi_\nu\xi,Z)+4g(AX,Y)\eta_\nu(\xi)g(\phi_\nu\xi,Z)\\
&-4\alpha\eta(X)\eta(Y)\eta_\nu(\xi)g(\phi_\nu\xi,Z)-4\alpha\eta(X)\eta(Y)\eta_\nu(\xi)g(\phi_\nu\xi,Z)\Big]\\
&+\alpha\sum_{\nu=1}^{3}\Big[g(\phi_\nu X,Y)\eta_\nu(Z)+\eta_\nu(X)g(\phi_\nu Y,Z)+2\eta_\nu(Y)g(\phi_\nu X,Z)\\
&-g(\phi_\nu\phi X,Y)g(\phi_\nu\xi,Z)+g(\phi_\nu\xi,X)g(\phi\phi_\nu Y,Z)\\
&+\eta_\nu(\phi X)\eta_\nu(Y)\eta(Z)+\eta(X)\eta_\nu(Y)g(\phi_\nu\xi,Z)\Big],
\end{aligned}
\tag{4.3}
$$

where we have used

$$
\begin{aligned}
g((\nabla_Z A)\xi,X)&=(Z\alpha)\eta(X)-\alpha g(A\phi X,Z)+g(A\phi AX,Z)\\
&=(\xi\alpha)\eta(Z)\eta(X)+4\sum_{\nu=1}^{3}\eta_\nu(\xi)g(\phi_\nu\xi,Z)\eta(X)\\
&\quad-\alpha g(A\phi X,Z)+g(A\phi AX,Z)
\end{aligned}
$$

and

$$
\begin{aligned}
&g((\nabla_Z A)X,Y)\\
&=g((\nabla_X A)Z,Y)+\eta(Z)g(\phi X,Y)-\eta(X)g(\phi Z,Y)-2g(\phi Z,X)\eta(Y)\\
&\quad+\sum_{\nu=1}^{3}\{\eta_\nu(Z)g(\phi_\nu X,Y)-\eta_\nu(X)g(\phi_\nu Z,Y)-2g(\phi_\nu Z,X)\eta_\nu(Y)\}\\
&\quad+\sum_{\nu=1}^{3}\{\eta_\nu(\phi Z)g(\phi_\nu\phi X,Y)-\eta_\nu(\phi X)g(\phi_\nu\phi Z,Y)\}\\
&\quad+\sum_{\nu=1}^{3}\{\eta(Z)\eta_\nu(\phi X)-\eta(X)\eta_\nu(\phi Z)\}\eta_\nu(Y)
\end{aligned}
$$

for any tangent vector fields X, Y and Z on M. Deleting Z from (4.3) and using (4.2) gives

$$
\begin{aligned}
&-g(\phi AX,Y)\xi-\eta(Y)\phi AX-g(\phi AY,X)\xi-\eta(X)\phi AY+\eta(Y)A\phi X\\
&+\eta(X)A\phi Y+(\xi\alpha)g(AX,Y)\xi-2\alpha(\xi\alpha)\eta(X)\eta(Y)\xi+\alpha^2\eta(Y)A\phi X\\
&+\alpha^2\eta(X)A\phi Y+\alpha(\nabla_X A)Y+\alpha g(\phi X,Y)\xi+\alpha\eta(X)\phi Y+2\alpha\eta(Y)\phi X
\end{aligned}
$$

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$$
\begin{aligned}
&- \sum_{\nu=1}^{3} \Big[g(\phi_\nu AX, Y)\xi_\nu + 2\eta(Y) g(\phi_\nu \xi, AX)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \\
&+ 3g(\phi_\nu AX, \phi Y)\phi_\nu \xi + 3\eta(Y)\eta_\nu(AX)\phi_\nu\xi - 3g(\phi_\nu\xi, Y)\phi_\nu\phi AX \\
&+ 3\alpha\eta(X) g(\phi_\nu\xi, Y)\xi_\nu - 4\eta_\nu(\xi) g(\phi_\nu\xi, Y)AX \\
&\qquad - 4\eta_\nu(\xi) g(AX, Y)\phi_\nu\xi - 2g(\phi_\nu\xi, AX)\phi_\nu\phi Y \\
&+ g(\phi_\nu AY, X)\xi_\nu + 2\eta(X) g(\phi_\nu\xi, AY)\xi_\nu + \eta_\nu(X)\phi_\nu AY \\
&+ 3g(\phi_\nu AY, \phi X)\phi_\nu\xi + 3\eta(X)\eta_\nu(AY)\phi_\nu\xi - 3g(\phi_\nu\xi, X)\phi_\nu\phi AY \\
&+ 3\alpha\eta(Y) g(\phi_\nu\xi, X)\xi_\nu - 4\eta_\nu(\xi) g(\phi_\nu\xi, X)AY \\
&- 4\eta_\nu(\xi) g(AY, X)\phi_\nu\xi - 2g(\phi_\nu\xi, AY)\phi_\nu\phi X \Big] \\
&+ \sum_{\nu=1}^{3} \Big[\eta_\nu(Y) A\phi_\nu X - 2\eta(X)\eta_\nu(Y) A\phi_\nu\xi + \eta_\nu(X) A\phi_\nu Y \\
&+ 3g(\phi_\nu\xi, Y) A\phi_\nu\phi X - 3\eta(X) g(\phi_\nu\xi, Y) A\xi_\nu \\
&+ 3g(\phi_\nu\xi, X) A\phi\phi_\nu Y - 3\alpha g(\phi_\nu\xi, X)\eta_\nu(Y)\xi \\
&+ 4\eta_\nu(\xi) g(\phi_\nu\xi, X)AY + 4\eta_\nu(\xi) g(\phi_\nu\xi, Y)AX \\
&+ 2g(\phi_\nu\phi X, Y) A\phi_\nu\xi + 4g(AX, Y)\eta_\nu(\xi)\phi_\nu\xi \\
&- 4\alpha\eta(X)\eta(Y)\eta_\nu(\xi)\phi_\nu\xi - 4\alpha\eta(X)\eta(Y)\eta_\nu(\xi)\phi_\nu\xi \Big] \\
&+ \alpha \sum_{\nu=1}^{3} \Big[g(\phi_\nu X, Y)\xi_\nu + \eta_\nu(X)\phi_\nu Y + 2\eta_\nu(Y)\phi_\nu X - g(\phi_\nu\phi X, Y)\phi_\nu\xi \\
&+ g(\phi_\nu\xi, X)\phi\phi_\nu Y + \eta_\nu(\phi X)\eta_\nu(Y)\xi + \eta(X)\eta_\nu(Y)\phi_\nu\xi \Big] \\
&+ g((\nabla_X A)\xi, \xi)AY - \alpha g((\nabla_X A)Y, \xi)\xi - \alpha g(AY, \phi AX)\xi \\
&- \alpha\eta(Y) A\phi AX + g((\nabla_Y A)\xi, \xi)AX - \alpha g((\nabla_Y A)X, \xi)\xi \\
&- \alpha g(AX, \phi AY)\xi - \alpha\eta(X) A\phi AY + \alpha(\nabla_X A)Y - \alpha\eta(Y)(\nabla_X A)\xi \\
&+ \alpha(\nabla_Y A)X - \alpha\eta(X)(\nabla_Y A)\xi = 0. \qquad (4.4)
\end{aligned}
$$

On the other hand, by using the Codazzi equation (3.8) and (3.10) in the latter part of (4.4), we obtain

$$
\begin{aligned}
&g((\nabla_X A)\xi, \xi)AY - \alpha g((\nabla_X A)Y, \xi)\xi - \alpha g(AY, \phi AX)\xi - \alpha\eta(Y) A\phi AX \\
&+ g((\nabla_Y A)\xi, \xi)AX - \alpha g((\nabla_Y A)X, \xi)\xi - \alpha g(AX, \phi AY)\xi - \alpha\eta(X) A\phi AY \\
&+ \alpha(\nabla_X A)Y + \alpha(\nabla_Y A)X - \alpha\eta(Y)(\nabla_X A)\xi - \alpha\eta(X)(\nabla_Y A)\xi \\
&= (\xi\alpha)\eta(X)AY + 4\sum_{\nu=1}^{3} \eta_\nu(\xi) g(\phi_\nu\xi, X)AY - \alpha g(A\phi AX, Y)\xi
\end{aligned}
$$

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$$
\begin{aligned}
&-\alpha\eta(Y)A\phi AX-\alpha(\xi\alpha)\eta(X)\eta(Y)\xi-4\alpha\sum_{\nu=1}^{3}\eta_{\nu}(\xi)g(\phi_{\nu}\xi,X)\eta(Y)\xi\\
&-\alpha^{2}g(\phi AX,Y)\xi+\alpha g(A\phi AX,Y)\xi+(\xi\alpha)\eta(Y)AX\\
&+4\sum_{\nu=1}^{3}\eta_{\nu}(\xi)g(\phi_{\nu}\xi,Y)AX+\alpha g(A\phi AX,Y)\xi\\
&-\alpha\eta(X)A\phi AY-\alpha(\xi\alpha)\eta(X)\eta(Y)\xi\\
&-4\alpha\sum_{\nu=1}^{3}\eta_{\nu}(\xi)\eta(X)g(\phi_{\nu}\xi,Y)\xi-\alpha^{2}g(\phi AY,X)\xi+\alpha g(A\phi AY,X)\xi\\
&+2\alpha(\nabla_{X}A)Y+\alpha\eta(Y)\phi X-\alpha\eta(X)\phi Y-2\alpha g(\phi Y,X)\xi\\
&+\alpha\sum_{\nu=1}^{3}\big\{\eta_{\nu}(Y)\phi_{\nu}X-\eta_{\nu}(X)\phi_{\nu}Y-2g(\phi_{\nu}Y,X)\xi_{\nu}\eta_{\nu}(\phi Y)\phi_{\nu}\phi X\big\}\\
&+\alpha\sum_{\nu=1}^{3}\big\{-\eta_{\nu}(\phi X)\phi_{\nu}\phi Y+\eta(Y)\eta_{\nu}(\phi X)\xi_{\nu}-\eta(X)\eta_{\nu}(\phi Y)\xi_{\nu}\big\}\\
&-\alpha\eta(Y)\Big\{(\xi\alpha)\eta(X)\xi+4\sum_{\nu=1}^{3}\eta_{\nu}(\xi)g(\phi_{\nu}\xi,X)\xi+\alpha\phi AX-A\phi AX\Big\}\\
&-\alpha\eta(X)\Big\{(\xi\alpha)\eta(Y)\xi+4\sum_{\nu=1}^{3}\eta_{\nu}(\xi)g(\phi_{\nu}\xi,Y)\xi+\alpha\phi AY-A\phi AY\Big\}. \qquad (4.5)
\end{aligned}
$$

From now on, we want to prove that the normal vector field N of a Hopf real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is singular. Then by the meaning of singularity mentioned in the introduction, we see that either $\xi \in \mathcal{Q}$ or $\xi \in \mathcal{Q}^{\perp}$ where $\mathcal{Q}$ is the maximal quaternionic subbundle of $TM = \mathcal{Q} \oplus \mathcal{Q}^{\perp}$. In order to do this, we may put the Reeb vector field ξ as follows:

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \qquad (*)$$

for unit vector fields $X_0 \in \mathcal{Q}$ and $\xi_1 \in \mathcal{Q}^{\perp}$ with $\eta(X_0)\eta(\xi_1) \neq 0$. By using the notation (*) we obtain that *the Reeb function α is constant along the direction of ξ if and only if the distribution $\mathcal{Q}$- or the $\mathcal{Q}^{\perp}$-component of the structure vector field ξ is invariant by the shape operator, that is $AX_0 = \alpha X_0$ and $A\xi_1 = \alpha\xi_1$* (see [**13**, **18**]). From this fact, we obtain the following useful formulas for Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$.

Lemma 4.1. *Let M be a Hopf real hypersurface with non-vanishing geodesic Reeb flow in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. If the distribution $\mathcal{Q}$ or $\mathcal{Q}^{\perp}$ component of the structure vector field ξ is invariant by the shape operator, then the following formulas hold:*

1. $A\phi X_0 = \mu\phi X_0$,
2. $A\phi\xi_1 = \mu\phi\xi_1$,
3. $A\phi_1 X_0 = \mu\phi_1 X_0$

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where the function μ *is given by* $\mu = (\alpha^2 + 4\eta^2(X_0)/\alpha)$.

Proof. Putting $X = X_0$ in (3.11) and using $AX_0 = \alpha X_0$, it yields

$$\alpha A\phi X_0 = \alpha^2\phi X_0 + 2\phi X_0 + 2\eta(\xi_1)\phi_1 X_0 - 4\eta(X_0)\eta(\xi_1)\phi_1\xi, \tag{4.6}$$

where we have used $g(\phi_\nu\xi, X_0) = 0$ for $\nu = 1, 2, 3$ and $\eta_2(\xi) = \eta_3(\xi) = 0$.

On the other hand, by (*) we obtain

$$\phi_1\xi = \eta(X_0)\phi_1 X_0 + \eta(\xi_1)\phi_1\xi_1 = \eta(X_0)\phi_1 X_0. \tag{4.7}$$

In addition, from (*) and $\phi_1\xi = \phi_1\xi$ we have

$$\begin{aligned} 0 = \phi\xi &= \eta(X_0)\phi X_0 + \eta(\xi_1)\phi\xi_1 \\ &= \eta(X_0)\phi X_0 + \eta(\xi_1)\phi_1\xi \\ &= \eta(X_0)\phi X_0 + \eta(\xi_1)\eta(X_0)\phi_1 X_0, \end{aligned}$$

which means

$$\phi X_0 = -\eta(\xi_1)\phi_1 X_0 \tag{4.8}$$

because of $\eta(X_0)\eta(\xi_1) \neq 0$. Substituting (4.7) and (4.8) to (4.6), we get

$$\alpha A\phi X_0 = \alpha^2\phi X_0 + 4\eta^2(X_0)\phi X_0 = (\alpha^2 + 4\eta^2(X_0))\phi X_0.$$

Since M has non-vanishing geodesic Reeb flow, we see that the vector field ϕX_0 is principal with corresponding principal curvature $\mu = (\alpha^2 + 4\eta^2(X_0)/\alpha)$.

Similarly, using (4.7) and (4.8), together with $\eta(X_0)\eta(\xi_1) \neq 0$, the formula (4.6) gives (b) and (c). □

When the Reeb function α is vanishing, Pérez and Suh gave the following

Lemma B ([27]). *Let* M *be a Hopf real hypersurface in* $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. *If* M *has vanishing geodesic Reeb flow, then the unit normal vector field* N *of* M *is singular, that is, either* $\xi \in \mathcal{Q}$ *or* $\xi \in \mathcal{Q}^\perp$.

REMARK 4.2. By using the method in the proof of lemma B, we can assert that *if* M *is a Hopf real hypersurface with constant Reeb curvature, then the unit normal vector field* N *of* M *is singular*. In fact, since M has constant Reeb function, (3.10) becomes

$$4\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\xi = 0$$

By using (*), this equation yields $\eta(\xi_1)\phi_1\xi = 0$. From our assumption of $\eta(X)\eta(\xi_1) \neq 0$ and (4.7), it leads to $\phi_1 X_0 = 0$. Taking the inner product with $\phi_1 X_0$, it implies

$$g(\phi_1 X_0, \phi_1 X_0) = -g(\phi_1^2 X_0, X_0) = g(X_0, X_0) - \big(\eta_1(X_0)\big)^2 = 1,$$

which gives us a contradiction.

By using lemma B, in the latter part of this section, we prove that the normal vector field N of M is singular, when a Hopf real hypersurface M in $G_2(\mathbb{C}^{m+2})$ has non-vanishing geodesic Reeb flow $\alpha = g(A\xi, \xi)$.

LEMMA 4.3. *Let M be a Hopf real hypersurface with non-vanishing geodesic Reeb flow in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. If the structure Jacobi operator R_ξ of M is cyclic parallel, then the unit normal vector field N of M is singular.*

Proof. In [**16**], Lee and Loo show that *if M is Hopf, then the Reeb function α is constant along the direction of structure vector field ξ*, that is, $\xi\alpha = 0$. Then we see that the distribution $\mathcal{Q}$- and the $\mathcal{Q}^\perp$-component of ξ are invariant by the shape operator A, that is $AX_0 = \alpha X_0$ and $A\xi_1 = \alpha\xi_1$.

Bearing in mind of these facts, putting $X = X_0$ and $Y = \xi_1$ in (4.4) and using (4.5), we obtain

$$\begin{aligned}
&-\alpha\eta(X_0)\phi\xi_1 + \mu\eta(\xi_1)\phi X_0 + \mu\eta(X_0)\phi\xi_1 + 3\alpha(\nabla_{X_0}A)\xi_1 + 2\alpha\eta(\xi_1)\phi X_0\\
&-\alpha^3\eta(\xi_1)\phi X_0 + \mu\alpha^2\eta(\xi_1)\phi X_0 - \alpha^3\eta(X_0)\phi\xi_1 + \mu\alpha^2\eta(X_0)\phi\xi_1\\
&+\sum_{\nu=1}^{3}\Big[\alpha\eta_\nu(\xi_1)\phi_\nu X_0 - 3\alpha g(\phi_\nu X_0, \phi\xi_1)\phi_\nu\xi - 2\alpha\eta(X_0)\eta_\nu(\xi_1)\phi_\nu\xi + \eta_\nu(\xi_1)A\phi_\nu X_0\\
&\qquad - 2\eta(X_0)\eta_\nu(\xi_1)A\phi_\nu\xi - 8\alpha\eta(X_0)\eta(\xi_1)\eta_\nu(\xi)\phi_\nu\xi + \alpha\eta_\nu(\xi_1)\phi_\nu X_0\Big] = 0,
\end{aligned}$$

where we have used $g(\phi\xi_1, X_0) = -g(\phi X_0, \xi_1) = 0$ and

$$g(\phi_\nu X_0, \xi_1) = g(\phi_\nu\xi, X_0) = g(\phi_\nu\xi, \xi_1) = g(\phi_\nu\phi X_0, \xi_1) = 0$$

for all $\nu = 1, 2, 3$. Since $\eta_2(\xi) = \eta_3(\xi) = 0$, together with $g(\phi_1 X_0, \phi_1 X_0) = 1$, this equation can be rearranged as

$$\begin{aligned}
&-\alpha\eta(X_0)\phi\xi_1 + \mu\eta(\xi_1)\phi X_0 + \mu\eta(X_0)\phi\xi_1 + 3\alpha(\nabla_{X_0}A)\xi_1\\
&+2\alpha\eta(\xi_1)\phi X_0 - \alpha^3\eta(\xi_1)\phi X_0 + \mu\alpha^2\eta(\xi_1)\phi X_0\\
&-\alpha^3\eta(X_0)\phi\xi_1 + \mu\alpha^2\eta(X_0)\phi\xi_1 + 2\alpha\phi_1 X_0 - 5\alpha\eta(X_0)\phi\xi_1\\
&+\mu\phi_1 X_0 - 2\mu\eta(X_0)\phi_1\xi - 8\alpha\eta(X_0)\big(\eta(\xi_1)\big)^2\phi_1\xi = 0.
\end{aligned} \tag{4.9}$$

From (4.7) and (4.8), (4.9) becomes

$$\begin{aligned}
&\eta^2(X_0)\{-6\alpha - \mu - \alpha^3 + \mu\alpha^2 - 8\alpha\eta^2(\xi_1)\}\phi_1 X_0\\
&\quad - \eta^2(\xi_1)\{\mu + 2\alpha - \alpha^3 + \mu\alpha^2\}\phi_1 X_0\\
&\quad + (2\alpha + \mu)\phi_1 X_0 + 3\alpha(\nabla_{X_0}A)\xi_1 = 0.
\end{aligned} \tag{4.10}$$

On the other hand, from (3.4) and (3.10), the assumption $A\xi_1 = \alpha\xi_1$ yields

$$\begin{aligned}
(\nabla_X A)\xi_1 &= (X\alpha)\xi_1 + \alpha\nabla_X\xi_1 - A(\nabla_X\xi_1)\\
&= 4\eta(\xi_1)g(\phi_1\xi, X)\xi_1 + \alpha\{q_3(X)\xi_2 - q_2(X)\xi_3 + \phi_1 AX\}\\
&\quad - q_3(X)A\xi_2 + q_2(X)A\xi_3 - A\phi_1 AX
\end{aligned}$$

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for any tangent vector field X on M. From this, taking the inner product with $\phi_1 X_0$ to (4.10) and (3.4), together with $\alpha\mu = \alpha^2 + 4\eta^2(X_0)$, we get

$$\begin{aligned} &\eta^2(X_0)\{-14\alpha - \mu + 12\alpha\eta^2(X_0)\} - \eta^2(\xi_1)\{\mu + 2\alpha + 4\alpha\eta^2(X_0)\} \\ &\quad + 2\alpha + \mu - 12\alpha\eta^2(X_0) = 0, \end{aligned} \tag{4.11}$$

where we have used $g(\phi_1 X_0, \phi_1 X_0) = 1$, $\eta^2(X_0) + \eta^2(\xi_1) = 1$, and

$$\begin{aligned} g((\nabla_{X_0} A)\xi_1, \phi_1 X_0) &= \alpha g(\phi_1 A X_0, \phi_1 X_0) - g(A\phi_1 A X_0, \phi_1 X_0) \\ &= \alpha^2 - \alpha\mu = -4\eta^2(X_0). \end{aligned}$$

By using non-vanishing Reeb function $\alpha \neq 0$ and $\alpha\mu = \alpha^2 + 4\eta^2(X_0)$, together with $\eta^2(\xi_1) = 1 - \eta^2(X_0)$, (4.11) becomes

$$\eta^2(X_0)\{-28\alpha^2 + 16\alpha^2\eta^2(X_0)\} = 0. \tag{4.12}$$

By virtue of $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ in (*) for $\eta(X_0)\eta(\xi_1) \neq 0$, and our assumption of non-vanishing geodesic Reeb flow, that is, $\alpha \neq 0$, (4.12) implies that $\eta^2(X_0) = \frac{7}{4}$. Since the structure vector field ξ is unit, we should have $\eta^2(X_0) + \eta^2(\xi_1) = 1$. From these facts, we obtain $\eta^2(\xi_1) = -\frac{3}{4}$. It makes a contradiction. This means that either $\xi = \eta(X_0)X_0 = \pm X_0 \in \mathcal{Q}$ or $\xi = \eta(\xi_1)\xi_1 = \pm\xi_1 \in \mathcal{Q}^\perp$, which gives the unit normal vector field N is singular. □

Summing up lemmas B and **4.3**, we assert that our theorem 1 in the introduction.

5. Cyclic parallel structure Jacobi operator for $JN \in \mathcal{J}N$

Hereafter, let M be a Hopf real hypersurface with cyclic parallel structure Jacobi operator in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ for $m \geqslant 3$. Then by theorem 1, our discussions can be divided into two cases accordingly as the Reeb vector field $\xi \in \mathcal{Q}^\perp$ or $\xi \in \mathcal{Q}$.

In this section, we consider the case of $\xi \in \mathcal{Q}^\perp$ (i.e. $JN \in \mathcal{J}N$ where N is a unit normal vector field on M in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$). Since $\mathcal{Q}^\perp$ is 3-dimensional distribution defined by $\mathcal{Q}^\perp = \text{span}\{\xi_1, \xi_2, \xi_3\}$, we may put $\xi = \xi_1$. From this, we give an important lemma as follows.

Lemma 5.1. *Let M be a real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Let $J_1 \in \mathcal{J}$ be the almost Hermitian structure such that $JN = J_1 N$ (or $\xi = \xi_1$). Then we obtain*

$$\phi A X = 2g(AX, \xi_3)\xi_2 - 2g(AX, \xi_2)\xi_3 + \phi_1 A X$$

for any tangent vector field X on M.

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Proof. Differentiating $\xi = \xi_1$ along any vector field $X \in TM$ and using (3.4), we obtain

$$\begin{aligned} \phi AX &= \nabla_X \xi \\ &= \nabla_X \xi_1 = q_3(X)\xi_2 - q_2(X)\xi_3 + \phi_1 AX. \end{aligned} \tag{5.1}$$

Taking the inner product of (5.1) with ξ_2 and ξ_3, we obtain

$$g(\phi AX, \xi_2) = q_3(X) + g(\phi_1 A\xi, \xi_2)$$

and

$$g(\phi AX, \xi_3) = -q_2(X) + g(\phi_1 A\xi, \xi_3)$$

respectively. It follows that

$$q_3(X) = 2g(AX, \xi_3) \quad \text{and} \quad q_2(X) = 2g(AX, \xi_2).$$

From this, (5.1) becomes

$$\phi AX = 2g(AX, \xi_3)\xi_2 - 2g(AX, \xi_2)\xi_3 + \phi_1 AX \tag{5.2}$$

for any tangent vector field X on M. Moreover, taking the symmetric part of (5.2) we obtain

$$A\phi X = 2\eta_3(X)A\xi_2 - 2\eta_2(X)A\xi_3 + A\phi_1 X. \tag{5.3}$$

□

Then, by virtue of lemma 5.1, we prove the following

LEMMA 5.2. *Let M be a Hopf hypersurface with cyclic parallel structure Jacobi operator in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. If the Reeb vector field ξ belongs to $\mathcal{Q}^\perp$ (i.e. $\xi = \xi_1$), then the distribution $\mathcal{Q}^\perp$ is invariant by the shape operator A of M, that is, $g(A\mathcal{Q}, \mathcal{Q}^\perp) = 0$.*

Proof. By (3.10) we obtain $X\alpha = (\xi\alpha)\eta(X)$ for any $X \in TM$, when the Reeb vector field ξ belongs to the distribution $\mathcal{Q}$. From this and taking the inner product of (4.4) with ξ, we have

$$\begin{aligned} &- g(\phi AX, Y) + g(A\phi X, Y) + (\xi\alpha)g(AX, Y) - \alpha(\xi\alpha)\eta(X)\eta(Y) + 3\alpha^2 g(\phi AX, Y) \\ &- \alpha g(A\phi AX, Y) + 3\alpha g(\phi X, Y) + \alpha^2 g(A\phi X, Y) - \alpha^2 g(\phi AX, Y) \\ &+ \sum_{\nu=1}^{3} \Big[- \eta_\nu(\xi)g(\phi_\nu AX, Y) - g(\phi_\nu\xi, AX)\eta_\nu(Y) - 3g(AX, \xi_\nu)g(\phi_\nu\xi, Y) \\ &\qquad + 4\alpha\eta_\nu(\xi)\eta(X)g(\phi_\nu\xi, Y) + \eta_\nu(\xi)g(A\phi_\nu X, Y) - \eta_\nu(X)g(\phi_\nu\xi, AY) \\ &\qquad - 3g(\phi_\nu\xi, X)g(\xi_\nu, AY) + 4\alpha\eta_\nu(\xi)g(\phi_\nu\xi, X)\eta(Y) \\ &\qquad - 9\alpha g(\phi_\nu\xi, X)\eta_\nu(Y) - 3\alpha\eta_\nu(X)g(\phi_\nu\xi, Y) + 3\alpha\eta_\nu(\xi)g(\phi_\nu X, Y) \Big] = 0, \end{aligned}$$

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where we have used

$$g((\nabla_X A)Y,\xi) = g((\nabla_X A)\xi, Y) = (X\alpha)\eta(Y) + \alpha g(\phi AX, Y) - g(A\phi AX, Y)$$
$$= (\xi\alpha)\eta(X)\eta(Y) + \alpha g(\phi AX, Y) - g(A\phi AX, Y),$$

$$g(\phi_\nu \phi AX, \xi) = g(\phi\phi_\nu \xi, AX) = g(\phi^2\xi_\nu, AX) = -g(\xi_\nu, AX) + \alpha\eta(\xi_\nu)\eta(X)$$

and

$$g(\phi_\nu\phi X, \xi) = g(\phi^2\xi_\nu, X) = -\eta_\nu(X) + \eta_\nu(\xi)\eta(X)$$

for any tangent vector fields X and Y on M.

On the other hand, from the assumption $\xi = \xi_1 \in \mathcal{Q}^\perp$ we get $\phi_2\xi = \phi_2\xi_1 = -\xi_3$ and $\phi_3\xi = \phi_3\xi_1 = \xi_2$. By using these formulas into the preceding equation, we get

$$\begin{aligned} &- g(\phi AX, Y) + g(A\phi X, Y) + (\xi\alpha)g(AX, Y) - \alpha(\xi\alpha)\eta(X)\eta(Y) \\ &+ 2\alpha^2 g(\phi AX, Y) - \alpha g(A\phi AX, Y) + 3\alpha g(\phi X, Y) + \alpha^2 g(A\phi X, Y) \\ &- g(\phi_1 AX, Y) - 2\eta_3(AX)\eta_2(Y) + 2\eta_2(AX)\eta_3(Y) \\ &+ g(A\phi_1 X, Y) - 2\eta_2(X)g(A\xi_3, Y) + 2\eta_3(X)g(A\xi_2, Y) \\ &+ 6\alpha\eta_3(X)\eta_2(Y) - 6\alpha\eta_2(X)\eta_3(Y) + 3\alpha g(\phi_1 X, Y) = 0. \end{aligned} \tag{5.4}$$

Deleting Y from (5.4), we get

$$\begin{aligned} &- \phi AX + A\phi X + (\xi\alpha)AX - \alpha(\xi\alpha)\eta(X)\xi + 2\alpha^2\phi AX - \alpha A\phi AX + \alpha^2 A\phi X \\ &- \phi_1 AX - 2\eta_3(AX)\xi_2 + 2\eta_2(AX)\xi_3 + A\phi_1 X - 2\eta_2(X)A\xi_3 + 2\eta_3(X)A\xi_2 \\ &+ 3\alpha\big\{2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3 + \phi X + \phi_1 X\big\} = 0 \end{aligned} \tag{5.5}$$

for any tangent vector field X on M.

On the other hand, when $\xi = \xi_1 \in \mathcal{Q}$, (3.11) gives us

$$\phi X + \phi_1 X - 2\eta_2(X)\xi_3 + 2\eta_3(X)\xi_2 = A\phi AX - \frac{\alpha}{2}A\phi X - \frac{\alpha}{2}\phi AX \tag{5.6}$$

for any tangent vector field X on M. Substituting (5.6) into (5.5), it follows that

$$\begin{aligned} &- \phi AX + A\phi X + (\xi\alpha)AX - \alpha(\xi\alpha)\eta(X)\xi + 2\alpha^2\phi AX - \alpha A\phi AX + \alpha^2 A\phi X \\ &- \phi_1 AX - 2\eta_3(AX)\xi_2 + 2\eta_2(AX)\xi_3 + A\phi_1 X - 2\eta_2(X)A\xi_3 + 2\eta_3(X)A\xi_2 \\ &+ 3\alpha\Big\{A\phi AX - \frac{\alpha}{2}A\phi X - \frac{\alpha}{2}\phi AX\Big\} = 0, \end{aligned}$$

which implies

$$\begin{aligned} &(-2 + 7\alpha^2)\phi AX + (2 - \alpha^2)A\phi X + 2(\xi\alpha)AX - 2\alpha(\xi\alpha)\eta(X)\xi \\ &\quad + 4\alpha A\phi AX - 2\big\{\phi_1 AX + 2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3\big\} \\ &\quad + 2\big\{A\phi_1 X - 2\eta_2(X)A\xi_3 + 2\eta_3(X)A\xi_2\big\} = 0 \end{aligned} \tag{5.7}$$

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for any $X \in TM$. Bearing in mind of (5.2) and (5.3), the above equation reduces to

$$\begin{aligned}(-4+7\alpha^2)\phi AX+(4-\alpha^2)A\phi X+2(\xi\alpha)AX\\ -2\alpha(\xi\alpha)\eta(X)\xi+4\alpha A\phi AX=0.\end{aligned} \tag{5.8}$$

From (5.2) and (5.3), we get

$$2\eta_3(AX)\xi_2-2\eta_2(AX)=\phi AX-\phi_1 AX \tag{5.9}$$

and

$$2\eta_3(X)A\xi_2-2\eta_2(X)A\xi_3=A\phi X-A\phi_1 X, \tag{5.10}$$

respectively. Substituting (5.9) and (5.10) into (5.7), it becomes

$$\begin{aligned}&(-2+7\alpha^2)\phi AX+(2-\alpha^2)A\phi X+2(\xi\alpha)AX-2\alpha(\xi\alpha)\eta(X)\xi+4\alpha A\phi AX\\ &-2\big\{\phi_1 AX+\phi AX-\phi_1 AX\big\}+2\big\{A\phi_1 X-A\phi X+A\phi_1 X\big\}=0,\end{aligned}$$

which yields

$$\begin{aligned}(-4+7\alpha^2)\phi AX-\alpha^2 A\phi X+2(\xi\alpha)AX-2\alpha(\xi\alpha)\eta(X)\xi\\ +4\alpha A\phi AX+4A\phi_1 X=0.\end{aligned} \tag{5.11}$$

Subtracting (5.11) from (5.8), we have $A\phi X=A\phi_1 X$, which means that $\phi AX=\phi_1 AX$ for any tangent vector field X on M. From this, (5.2) becomes

$$g(A\xi_2,X)\xi_2-g(A\xi_2,X)\xi_3=0 \tag{5.12}$$

for any tangent vector field X on M. Taking the inner product of (5.12) with ξ_2 (resp. ξ_3), we get the following for any tangent vector field X on M

$$g(A\xi_2,X)=g(AX,\xi_2)=0 \quad (\text{resp. } \ g(A\xi_3,X)=g(AX,\xi_3)=0), \tag{5.13}$$

which means that $g(A\mathcal{Q},\mathcal{Q}^\perp)=0$. It gives a complete proof of lemma 5.2. □

By theorem A and proposition A, lemma 5.2 assures that *if a Hopf real hypersurface satisfies all of geometric conditions mentioned in lemma 5.2, then M is locally congruent to an open part of the model spaces of type $(\mathcal{T}_A)$.*

From now on, we will check whether a real hypersurface of type $(\mathcal{T}_A)$ satisfies our hypothesis given in lemma 5.2. By proposition A mentioned in §2, we see that such real hypersurface is Hopf and its normal vector field satisfies $JN \in \mathcal{J}N$.

In the remaining part of this section, we want to check if the structure Jacobi operator R_ξ for a model space of type $(\mathcal{T}_A)$ satisfies the cyclic parallelism. In order to do this, we want to find some necessary and sufficient conditions for structure Jacobi operator R_ξ of a real hypersurface $(\mathcal{T}_A)$ to be cyclic parallel according to each eigenspace including the vector Y.

From such a view point, first, we consider the following case.

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Case A. $Y \in T_\lambda$

In other words, from (4.4) and (4.5), together with (2.2), the structure Jacobi operator R_ξ of a real hypersurface of type $(\mathcal{T}_A)$ satisfies the following for any tangent vector field $X \in T(\mathcal{T}_A)$

$$\begin{aligned} &3\alpha(\lambda^2 - \alpha\lambda - 2)g(\phi Y, X)\xi - 2(2\alpha - \beta - \lambda)g(\phi_2 Y, X)\xi_2 \\ &\quad - 2(2\alpha - \beta - \lambda)g(\phi_3 Y, X)\xi_3 - 2(2\alpha - \beta - \lambda)\eta_2(X)\phi_2 Y \\ &\quad - 2(2\alpha - \beta - \lambda)\eta_3(X)\phi_3 Y = 0, \end{aligned} \tag{5.14}$$

where $T(\mathcal{T}_A)$ denotes a tangent space of type $(\mathcal{T}_A)$ and we have used $\phi\phi_2 Y = \phi_2\phi Y = -\phi_3 Y \in T_\mu$ and $\phi\phi_3 Y = \phi_3\phi Y = \phi_2 Y \in T_\mu$ for any $Y \in T_\lambda$.

From now on, we want to check a solution of the equation (5.14) to be satisfied for type $(\mathcal{T}_A)$. In fact, the left side of (5.14) depends on the eigenspaces of $(\mathcal{T}_A)$ and is given as

$$\text{Left side of (5.14)} = \begin{cases} 0 & \text{for } X \in T_\alpha, \\ -2(2\alpha - \beta - \lambda)\phi_2 Y & \text{for } X = \xi_2 \in T_\beta, \\ -2(2\alpha - \beta - \lambda)\phi_3 Y & \text{for } X = \xi_3 \in T_\beta, \\ 3\alpha(\lambda^2 - \alpha\lambda - 2)g(\phi Y, X)\xi & \text{for } X \in T_\lambda, \\ -2(2\alpha - \beta - \lambda)g(\phi_2 Y, X)(\xi_2 + \xi_3) & \text{for } X \in T_\mu \end{cases}$$

for $Y \in T_\lambda$. By using $\alpha = 2\sqrt{2}\cot(2\sqrt{2}r) = \sqrt{2}(\cot(\sqrt{2}r) - \tan(\sqrt{2}r))$ and $\lambda = -\sqrt{2}\tan(\sqrt{2}r)$ with $r \in (0, (\pi/2\sqrt{2}))$, we get $\lambda^2 - \alpha\lambda - 2 = 0$. From this, the previous formula follows

$$\begin{aligned} &\text{Left side of (5.14)} \\ &= \begin{cases} 0 & \text{for } X \in T_\alpha, \\ -2(2\alpha - \beta - \lambda)\phi_2 Y & \text{for } X = \xi_2 \in T_\beta, \\ -2(2\alpha - \beta - \lambda)\phi_3 Y & \text{for } X = \xi_3 \in T_\beta, \\ 0 & \text{for } X \in T_\lambda, \\ -2(2\alpha - \beta - \lambda)g(\phi_2 Y, X)(\xi_2 + \xi_3) & \text{for } X \in T_\mu \end{cases} \end{aligned} \tag{5.15}$$

for $Y \in T_\lambda$.

Bearing in mind of proposition A, if $r = (\pi/4\sqrt{2})$, then $2\alpha - \beta - \lambda = 0$. Hence, when $Y \in T_\lambda$, the structure Jacobi operator R_ξ is cyclic parallel if and only if the radius r of the tube $(\mathcal{T}_A)$ is $(\pi/4\sqrt{2})$.

Under these situations, we consider our problem for the other cases $Y \in T_\alpha \oplus T_\beta \oplus T_\mu$ as follows.

Case B. $Y \in T_\alpha \oplus T_\beta \oplus T_\mu$ where $\alpha = \mu = 0$, $\beta = \sqrt{2}$, and $\lambda = -\sqrt{2}$

By the affect of case A in $(\mathcal{T}_A)$, we have seen that in order to be cyclic parallel for the structure Jacobi operator R_ξ of $(\mathcal{T}_A)$, the radius r of $(\mathcal{T}_A)$ should satisfy $r = (\pi/4\sqrt{2})$. From this fact, we obtain $\alpha = \mu = 0$, $\beta = \sqrt{2}$, and $\lambda = -\sqrt{2}$. Then, the left side of (4.4) becomes

$$
\begin{aligned}
&\text{Left side of (4.4)}\\
&= -g(\phi AX, Y)\xi - \eta(Y)\phi AX - g(\phi AY, X)\xi\\
&\quad - \eta(X)\phi AY + \eta(Y)A\phi X + \eta(X)A\phi Y\\
&\quad - \sum_{\nu=1}^{3}\Big[g(\phi_\nu AX, Y)\xi_\nu + 2\eta(Y)g(\phi_\nu\xi, AX)\xi_\nu + \eta_\nu(Y)\phi_\nu AX\\
&\quad + 3g(\phi_\nu AX, \phi Y)\phi_\nu\xi + 3\eta(Y)\eta_\nu(AX)\phi_\nu\xi\\
&\quad - 3g(\phi_\nu\xi, Y)\phi_\nu\phi AX - 2g(\phi_\nu\xi, AX)\phi_\nu\phi Y + \eta_\nu(X)\phi_\nu AY\\
&\quad - 2g(\phi_\nu\xi, AY)\phi_\nu\phi X + g(\phi_\nu AY, X)\xi_\nu + 2\eta(X)g(\phi_\nu\xi, AY)\xi_\nu\\
&\quad + 3g(\phi_\nu AY, \phi X)\phi_\nu\xi + 3\eta(X)\eta_\nu(AY)\phi_\nu\xi - 3g(\phi_\nu\xi, X)\phi_\nu\phi AY\\
&\quad - \eta_\nu(Y)A\phi_\nu X + 2\eta(X)\eta_\nu(Y)A\phi_\nu\xi - \eta_\nu(X)A\phi_\nu Y\\
&\quad - 3g(\phi_\nu\xi, Y)A\phi_\nu\phi X + 3\eta(X)g(\phi_\nu\xi, Y)A\xi_\nu\\
&\quad - 3g(\phi_\nu\xi, X)A\phi\phi_\nu Y - 2g(\phi_\nu\phi X, Y)A\phi_\nu\xi\Big]
\end{aligned}
\tag{5.16}
$$

for any $X \in T(\mathcal{T}_A)$ and $Y \in T_\alpha \oplus T_\beta \oplus T_\mu$.

Subcase B-1. $Y = \xi \in T_\alpha$ where $\alpha = 0$

From this assumption, we get $AY = A\xi = \alpha\xi = 0$. Then, (5.16) becomes

$$
\begin{aligned}
&-\phi AX + A\phi X - \sum_{\nu=1}^{3}\{g(A\phi_\nu\xi, X)\xi_\nu + \eta_\nu(\xi)\phi_\nu AX + 3g(A\xi_\nu, X)\phi_\nu\xi\}\\
&\quad + \sum_{\nu=1}^{3}\{\eta_\nu(X)A\phi_\nu\xi - 3g(\phi_\nu\xi, X)A\xi_\nu + \eta_\nu(\xi)A\phi_\nu X\}\\
&= -\phi AX + A\phi X - \phi_1 AX + A\phi_1 X,
\end{aligned}
\tag{5.17}
$$

where we have used $\phi_2\xi = -\xi_3$, $\phi_3\xi = \xi_2$, and $\phi\phi_\nu\xi = \phi^2\xi_\nu = -\xi_\nu + \eta(\xi_\nu)\xi$. According to the composition of the eigenspaces for $(\mathcal{T}_A)$, we see that each eigenspace T_σ of $(\mathcal{T}_A)$ is ϕ-(or ϕ_1-)invariant, that is, $\phi T_\sigma = \phi_1 T_\sigma = T_\sigma$. From this, (5.17) vanishes on all eigenspaces of $(\mathcal{T}_A)$. So, this means that the structure Jacobi operator R_ξ is cyclic parallel when $Y \in T_\alpha$.

Subcase B-2. $Y \in T_\beta$ where $\beta = \sqrt{2}$

Since $T_\beta = \text{span}\{\xi_2, \xi_3\}$, we have the following two subcases.

- $Y = \xi_2 \in T_\beta$

  Using $\alpha = 0$, (5.16) can be rearranged as

$$
\begin{aligned}
&6\beta\eta_3(X)\xi + \beta\eta(X)\xi_3 - \phi_2 AX + 3\phi_3\phi AX\\
&\quad + 2\beta\phi_3\phi X + A\phi_2 X + 3A\phi_3\phi X
\end{aligned}
\tag{5.18}
$$

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for any eigenvector X on $(\mathcal{T}_A)$. It is well-known that for $X \in T_\lambda$ (resp. $X \in T_\mu$), by the straightforward calculation with (3.2), we obtain

$$\phi_2\phi X \underset{X\in T_\lambda}{=} \phi_2\phi_1 X \underset{3.2}{=} -\phi_3 X \in T_\mu$$
$$(\text{resp. } \phi_2\phi X \underset{X\in T_\mu}{=} -\phi_2\phi_1 X = \phi_3 X \in T_\lambda),$$
$$\phi_3\phi X \underset{X\in T_\lambda}{=} \phi_3\phi_1 X \underset{3.2}{=} \phi_2 X \in T_\mu$$
$$(\text{resp. } \phi_3\phi X \underset{X\in T_\mu}{=} -\phi_3\phi_1 X = -\phi_2 X \in T_\lambda),$$

and

$$\phi X = \phi_1 X \in T_\lambda \quad (\text{resp. } \phi X = \phi_1 X \in T_\mu).$$

Bearing in mind such properties, together with $\beta = \sqrt{2}$ and $\lambda = -\sqrt{2}$, (5.18) is identically vanishing for any tangent vector field X on $(\mathcal{T}_A)$.

- $Y = \xi_3 \in T_\beta$
  Similarly, from (5.16) we obtain

$$\begin{aligned} &-6\beta\eta_2(X)\xi - \beta\eta(X)\xi_2 - \phi_3 AX - 3\phi_2\phi AX \\ &\quad - 2\beta\phi_2\phi X + A\phi_3 X - 3A\phi_2\phi X \end{aligned} \tag{5.19}$$

for any eigenvector X on $(\mathcal{T}_A)$. More specifically, according to each eigenspace T_α, T_β, T_λ and T_μ, it follows that

$$(5.19) = \begin{cases} -\beta\xi_2 + A\phi_3\xi = -\beta\xi_2 + A\xi_2 = 0 & \text{for } X \in T_\alpha, \\ -6\beta\xi - \phi_3 A\xi_2 - 3\phi_2\phi A\xi_2 - 2\beta\phi_2\phi\xi_2 = 0 & \text{for } X = \xi_2 \in T_\beta, \\ -3\phi_2\phi A\xi_3 - 2\beta\phi_2\phi\xi_3 - 3A\phi_2\phi\xi_3 = 0 & \text{for } X = \xi_3 \in T_\beta, \\ -\lambda\phi_3 X - 3\lambda\phi_2\phi X - 2\beta\phi_2\phi X = 2(\lambda+\beta)\phi_3 X = 0 & \text{for } X \in T_\lambda, \\ -2\beta\phi_2\phi X + \lambda\phi_3 X - 3A\phi_3 X = -2\beta(\beta+\lambda)\phi_3 X = 0 & \text{for } X \in T_\mu, \end{cases}$$

where we have used $\phi_2\phi\xi_2 = -\phi_2\xi_3 = -\xi$, $\phi_2\phi\xi_2 = \phi_2\xi_2 = 0$, $\beta = \sqrt{2}$ and $\lambda = -\sqrt{2}$.

Subcase B-3. $Y \in T_\mu$ where $\mu = 0$

Since a real hypersurface of type $(\mathcal{T}_A)$ has isometric Reeb flow, we obtain $\phi Y \in T_{\mu=0}$, that is, $A\phi Y = \phi AY = \mu\phi Y = 0$ for any $Y \in T_{\mu=0}$. From this and the construction of $T_\mu = \{Y \in \mathcal{Q} \,|\, \phi Y = -\phi_1 Y\}$, we also obtain $A\phi_1 Y = -A\phi Y = 0$ for $Y \in T_{\mu=0}$. From these properties, (5.16) becomes

$$\begin{aligned} &-\sum_{\nu=1}^{3}\big\{g(\phi_\nu AX, Y)\xi_\nu + 3g(\phi_\nu AX, \phi Y)\phi_\nu\xi - 2g(\phi_\nu\xi, AX)\phi_\nu\phi Y \\ &\qquad - \eta_\nu(X)A\phi_\nu Y - 3g(\phi_\nu\xi, X)A\phi\phi_\nu Y - 2g(\phi_\nu\phi X, Y)A\phi_\nu\xi\big\} \\ &= -2(\beta+\lambda)\big\{g(\phi_2 X, Y)\xi_2 + g(\phi_3 X, Y)\xi_3 + \eta_2(X)\phi_2 Y + \eta_3(X)\phi_3 Y\big\}, \end{aligned} \tag{5.20}$$

where we have used $\phi_2\phi Y = \phi_3 Y \in T_\lambda$ and $\phi_3\phi Y = -\phi_2 Y \in T_\lambda$ for any $Y \in T_{\mu=0}$. Since $\beta = \sqrt{2}$ and $\lambda = -\sqrt{2}$, (5.20) is identically vanishing for any tangent vector field X on $(\mathcal{T}_A)$.

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Summing up these discussions, we assert that *the structure Jacobi operator* R_ξ *of a real hypersurface of type* $(\mathcal{T}_A)$ *is cyclic parallel if and only if the radius* r *of the tube around of type* $(\mathcal{T}_A)$ *is* $(\pi/4\sqrt{2})$.

6. Cyclic parallel structure Jacobi operator for $JN \perp \mathcal{J}N$

Let M be a Hopf real hypersurface with cyclic parallel structure Jacobi operator R_ξ in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Assume that the unit normal vector field N of M satisfies $JN \perp \mathcal{J}N$ (i.e. $\xi \in \mathcal{Q}$). Related to the Reeb vector field ξ of M in $G_2(\mathbb{C}^{m+2})$, Lee and Suh gave:

Theorem B ([17]). *Let* M *be a connected orientable Hopf real hypersurface in complex two-plane Grassmannians of compact type* $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. *Then the Reeb vector* ξ *belongs to the distribution* $\mathcal{Q}$ *if and only if* M *is locally congruent to an open part of* $(\mathcal{T}_B)$: *a tube around a totally geodesic* $\mathbb{H}P^n$ *in* $G_2(\mathbb{C}^{m+2})$, *where* $m = 2n$.

By virtue of theorem 1 and theorem B, we assert that *a Hopf real hypersurface* M *in complex two-plane Grassmannians* $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$, *satisfying the hypothesis in our theorem* 2 *is locally congruent to an open part of the model space mentioned in theorem* B. Hereafter, conversely, let us check whether the structure Jacobi operator R_ξ of the model space of type $(\mathcal{T}_B)$ satisfies our assumption of cyclic parallel structure Jacobi operator.

In order to do this, we introduce a proposition given in [30] as follows:

Proposition B. *Let* M *be a connected real hypersurface in complex two-plane Grassmannians* $G_2(\mathbb{C}^{m+2})$. *Suppose that* $A\mathcal{Q} \subset \mathcal{Q}$, $A\xi = \alpha\xi$, *and* ξ *is tangent to* $\mathcal{Q}$. *Then the quaternionic dimension* m *of* $G_2(\mathbb{C}^{m+2})$ *is even, say* $m = 2n$, *and* M *has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r),\ \beta = 2\cot(2r),\ \gamma = 0,\ \lambda = \cot(r),\ \mu = -\tan(r)$$

with some $r \in (0, \frac{\pi}{4})$. *The corresponding multiplicities are*

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$\begin{aligned} &T_\alpha = \mathbb{R}\xi = \mathcal{C}^\perp = \operatorname{span}\{\xi\}, \\ &T_\beta = \mathcal{J}J\xi = \operatorname{span}\{\xi_1, \xi_2, \xi_3\}, \\ &T_\gamma = \mathcal{J}\xi = \operatorname{span}\{\phi\xi_1, \phi\xi_2, \phi\xi_3\}, \\ &T_\lambda, \quad T_\mu, \end{aligned}$$

where

$$T_\lambda \oplus T_\mu = TM \ominus (\mathbb{R}\xi \oplus \mathcal{J}J\xi), \quad \mathcal{J}T_\lambda = T_\lambda, \quad \mathcal{J}T_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

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In order to check the converse part, we assume that the structure Jacobi operator R_ξ of our model space of type $(\mathcal{T}_B)$ satisfies the property of cyclic parallelism. Accordingly, by $A\phi\xi_\nu = 0$ for $\nu = 1, 2, 3$, the property (1.4) can be rearranged as

$$\begin{aligned}
&g(X, A\phi Y)\xi - \eta(Y)\phi AX - g(X, \phi AY)\xi - \eta(X)\phi AY + \eta(Y)A\phi X \\
&+ \eta(X)A\phi Y + \alpha^2\eta(Y)A\phi X + \alpha^2\eta(X)A\phi Y + 3\alpha(\nabla_X A)Y \\
&+ \alpha g(\phi X, Y)\xi + 3\alpha\eta(Y)\phi X + \alpha^2 g(X, A\phi Y)\xi - \alpha^2 g(\phi AY, X)\xi \\
&- 2\alpha g(\phi Y, X)\xi - \alpha^2\eta(Y)\phi AX - \alpha^2\eta(X)\phi AY \\
&+ \sum_{\nu=1}^{3}\Big[-g(\phi_\nu AX, Y)\xi_\nu - \eta_\nu(Y)\phi_\nu AX - 3g(\phi_\nu AX, \phi Y)\phi_\nu\xi \\
&- 3\eta(Y)\eta_\nu(AX)\phi_\nu\xi + 3g(\phi_\nu\xi, Y)\phi_\nu\phi AX \\
&- 3\alpha\eta(X)g(\phi_\nu\xi, Y)\xi_\nu + 2g(\phi_\nu\xi, AX)\phi_\nu\phi Y \\
&- g(\phi_\nu AY, X)\xi_\nu - 2\eta(X)g(\phi_\nu\xi, AY)\xi_\nu - \eta_\nu(X)\phi_\nu AY \\
&- 3g(\phi_\nu AY, \phi X)\phi_\nu\xi - 3\eta(X)\eta_\nu(AY)\phi_\nu\xi + 3g(\phi_\nu\xi, X)\phi_\nu\phi AY \\
&- 3\alpha\eta(Y)g(\phi_\nu\xi, X)\xi_\nu + 2g(\phi_\nu\xi, AY)\phi_\nu\phi X + \eta_\nu(Y)A\phi_\nu X \\
&+ \eta_\nu(X)A\phi_\nu Y + 3g(\phi_\nu\xi, Y)A\phi_\nu\phi X - 3\eta(X)g(\phi_\nu\xi, Y)A\xi_\nu \\
&+ 3g(\phi_\nu\xi, X)A\phi\phi_\nu Y - 3\alpha g(\phi_\nu\xi, X)\eta_\nu(Y)\xi + \alpha g(\phi_\nu X, Y)\xi_\nu \\
&+ 2\alpha\eta_\nu(Y)\phi_\nu X - \alpha g(\phi_\nu\phi X, Y)\phi_\nu\xi + \alpha g(\phi_\nu\xi, X)\phi\phi_\nu Y \\
&+ \alpha\eta_\nu(\phi X)\eta_\nu(Y)\xi + \alpha\eta(X)\eta_\nu(Y)\phi_\nu\xi + \alpha\eta_\nu(Y)\phi_\nu X \\
&- 2\alpha g(\phi_\nu Y, X)\xi_\nu + \alpha\eta_\nu(\phi Y)\phi_\nu\phi X - \alpha\eta_\nu(\phi X)\phi_\nu\phi Y \\
&+ \alpha\eta(Y)\eta_\nu(\phi X)\xi_\nu - \alpha\eta(X)\eta_\nu(\phi Y)\xi_\nu\Big] = 0
\end{aligned} \tag{6.1}$$

for any tangent vector field X on type $(\mathcal{T}_B)$.

Bearing in mind of our assumption, the structure Jacobi operator R_ξ for the tube of type $(\mathcal{T}_B)$ is cyclic parallel, taking $Y \in T_\alpha$ in (6.1) yields

$$\begin{aligned}
&- \phi AX + A\phi X + \alpha^2 A\phi X + 2\alpha^2\phi AX - 3\alpha A\phi AX + 3\alpha\phi X \\
&- 3\sum_{\nu=1}^{3}\Big[\beta\eta_\nu(X)\phi_\nu\xi + 3\alpha g(\phi_\nu\xi, X)\xi_\nu + \alpha\eta_\nu(X)\phi_\nu\xi + \beta g(\phi_\nu\xi, X)\xi_\nu\Big] = 0,
\end{aligned} \tag{6.2}$$

where we have used $(\nabla_X A)\xi = \alpha\phi AX - A\phi AX$ and $\phi\phi_\nu\xi = \phi^2\xi_\nu = -\xi_\nu$. Furthermore, taking $X = \xi_\mu \in T_\beta$ in (6.2) follows

$$\begin{aligned}
0 &= -\phi A\xi_\mu + A\phi\xi_\mu + \alpha^2 A\phi\xi_\mu + 2\alpha^2\phi A\xi_\mu - 3\alpha A\phi A\xi_\mu + 3\alpha\phi\xi_\mu \\
&\quad - 3\sum_{\nu=1}^{3}\Big[\beta\eta_\nu(\xi_\mu)\phi_\nu\xi + 3\alpha g(\phi_\nu\xi, \xi_\mu)\xi_\nu + \alpha\eta_\nu(\xi_\mu)\phi_\nu\xi + \beta g(\phi_\nu\xi, \xi_\mu)\xi_\nu\Big] \\
&= 2\beta(\alpha^2 - 2)\phi_\mu\xi,
\end{aligned}$$

which implies $\beta(\alpha^2-2)=0$. Since $\beta = 2\cot(2r)$ for $r \in (0,(\pi/4))$, we obtain $\alpha^2 = 2$.

On the other hand, taking $X \in T_\lambda$ in (6.2), together with $\phi T_\lambda = T_\mu$, provides

$$\begin{aligned} 0 &= -\lambda\phi X + \mu\phi X + \alpha^2\mu\phi X + 2\alpha^2\lambda\phi X - 3\alpha\lambda\mu\phi X + 3\alpha\phi X \\ &= 3(\beta+2\alpha)\phi X, \end{aligned}$$

where we have used $\alpha^2=2$, $\lambda\mu = (\cot r)\cdot(-\tan r) = -1$, and $\lambda+\mu = 2\cot(2t) = \beta$.

Applying a method to (6.2) that is done above, the left side of (6.2) according to each eigenspace of type $(\mathcal{T}_\beta)$ is given as

$$\text{Left side of (6.2)} = \begin{cases} 0 & \text{for } X \in T_\alpha, \\ 2\beta(\alpha^2-2)\phi_\mu\xi & \text{for } X = \xi_\mu \in T_\beta, \\ -6(\beta+2\alpha)\xi_\mu & \text{for } X = \phi_\mu\xi \in T_\gamma, \\ 3(\beta+2\alpha)\phi X & \text{for } X \in T_\lambda, \\ 3(\beta+2\alpha)\phi X & \text{for } X \in T_\mu. \end{cases}$$

Now, as the other case we consider the case $Y \in T_\lambda$. Then, by using $JT_\lambda = T_\mu$ and $\mathcal{J}T_\lambda = T_\lambda$, equation (6.1) is rearranged as

$$\begin{aligned} & g(X,A\phi Y)\xi - g(X,\phi AY)\xi - \eta(X)\phi AY + \eta(X)A\phi Y + \alpha^2\eta(X)A\phi Y \\ & + 3\alpha(\nabla_X A)Y + \alpha g(\phi X,Y)\xi + \alpha^2 g(X,A\phi Y)\xi \\ & - \alpha^2 g(\phi AY,X)\xi - 2\alpha g(\phi Y,X)\xi - \alpha^2\eta(X)\phi AY \\ & + \sum_{\nu=1}^{3}\Big[-g(\phi_\nu AX,Y)\xi_\nu - 3g(\phi_\nu AX,\phi Y)\phi_\nu\xi - g(\phi_\nu AY,X)\xi_\nu \\ & - \eta_\nu(X)\phi_\nu AY - 3g(\phi_\nu AY,\phi X)\phi_\nu\xi + 3g(\phi_\nu\xi,X)\phi_\nu\phi AY \\ & + \eta_\nu(X)A\phi_\nu Y + \alpha g(\phi_\nu X,Y)\xi_\nu + 3g(\phi_\nu\xi,X)A\phi\phi_\nu Y \\ & - \alpha g(\phi_\nu\phi X,Y)\phi_\nu\xi + \alpha g(\phi_\nu\xi,X)\phi\phi_\nu Y \\ & - 2\alpha g(\phi_\nu Y,X)\xi_\nu + \alpha g(\phi_\nu\xi,X)\phi_\nu\phi Y\Big] \\ & = (\mu-\lambda-3\alpha+\alpha^2\mu-\alpha^2\lambda)g(X,\phi Y)\xi + (\lambda+\mu+\alpha^2\mu-\alpha^2\lambda)\eta(X)\phi Y \\ & + 3\alpha(\nabla_X A)Y + \sum_{\nu=1}^{3}\Big[-3\alpha g(\phi_\nu Y,X)\xi_\nu + (3\mu+3\lambda-\alpha)g(\phi\phi_\nu Y,X)\phi_\nu\xi\Big] \\ & + \sum_{\nu=1}^{3}(3\lambda+3\mu+2\alpha)g(\phi_\nu\xi,X)\phi\phi_\nu Y = 0 \end{aligned} \tag{6.3}$$

for any tangent vector field X on type $(\mathcal{T}_B)$. Restricting $X \in T_\alpha$ in (6.3) provides

$$(\lambda+\mu+\alpha^2\mu-\alpha^2\lambda)\phi Y + 3\alpha(\nabla_\xi A)Y = 0 \tag{6.4}$$

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for any $Y \in T_\lambda$. By the equation of Codazzi (3.8), we get

$$\begin{aligned}(\nabla_\xi A)Y &= (\nabla_Y A)\xi + \phi Y + \sum_{\nu=1}^{3}\{-\eta_\nu(Y)\phi_\nu\xi - 3g(\phi_\nu\xi, Y)\xi_\nu\}\\ &= \alpha\phi AY - A\phi AY + \phi Y = (\alpha\lambda - \lambda\mu + 1)\phi Y\end{aligned}$$

for any $Y \in T_\lambda$. From this, (6.4) becomes

$$(\lambda + \mu + \alpha^2\mu - \alpha^2\lambda + 3\alpha^2\lambda - 3\alpha\lambda\mu + 3\alpha)\phi Y = 0.$$

Since $\alpha^2 = 2$, $\beta + 2\alpha = 0$, $\lambda + \mu = \beta$ and $\lambda\mu = -1$, the previous equation gives

$$\beta + 2\mu + 4\lambda + 6\alpha = -2(\beta - \mu - 2\lambda) = 0, \tag{6.5}$$

which gives us a contradiction. In fact, by proposition B we see that $\beta = 2\cot(2r)$, $\lambda = \cot(r)$ and $\mu = -\tan(r)$ where $r \in (0, \frac{\pi}{4})$. From this, we get

$$\beta - \mu - 2\lambda = -\frac{1}{\tan r},$$

which means that the function $\beta - \mu - 2\lambda$ is non-vanishing for any $r \in (0, (\pi/4))$.

Summing up those documents in this section, we can assert that *there does not exist a Hopf real hypersurface in complex two-plane Grassmannians* $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$, *with cyclic parallel structure Jacobi operator when the normal vector field of* M *is of type* $JN \perp \mathcal{J}N$.

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