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Real Hypersurfaces in the Complex Quadric with Lie Invariant Structure Jacobi Operator

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Abstract. We introduce the notion of Lie invariant structure Jacobi operators for real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. The existence of invariant structure Jacobi operators implies that the unit normal vector field N becomes $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic. Then, according to each case, we give a complete classification of real hypersurfaces in $Q^m = SO_{m+2}/SO_mSO_2$ with Lie invariant structure Jacobi operators.

1 Introduction

When we consider some Hermitian symmetric spaces of rank 2, we can usually give examples of Riemannian symmetric spaces $SU_{m+2}/S(U_2U_m)$ and $SU_{2,m}/S(U_2U_m)$, which are said to be complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians, respectively (see [10, 12, 13]). These are viewed as Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure J and the quaternionic Kähler structure $\mathfrak{J}$.

The classification problems of the complex 2-plane Grassmannian with certain geometric conditions were mainly discussed in Jeong, Kim, and Suh [3], Pérez [6], and Suh [10,12,13], where the classification of *contact hypersurfaces*, *parallel Ricci tensors*, *harmonic curvatures*, and *Jacobi operators* for real hypersurfaces in $SU_{m+2}/S(U_2U_m)$ were extensively studied.

For the complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2U_m)$, Suh [14] asserted that the Reeb flow on a real hypersurface M is isometric if and only if M is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1}) \subset SU_{2,m}/S(U_2U_m)$ or a horosphere whose center at infinity is singular. More generally, this result was extended to the commuting Ricci tensor, that is, $\mathrm{Ric}\cdot\phi = \phi\cdot\mathrm{Ric}$ in Suh [11]. Here, ϕ is a tensor field of type $(1,1)$ on M defined by $\phi X = (JX)^T$, where $(JX)^T$ denotes the tangential part of JX for any vector field X on M. By virtue of this result, Suh and Kim [19] considered real hypersurfaces M in $SU_{2,m}/S(U_2U_m)$ with thenotion of Reeb invariant Ricci tensor, that is, $\mathcal{L}_\xi\mathrm{Ric} = 0$ for the Reeb vector field $\xi = -JN$, where N denotes a unit normal vector field on M, and gave a characterization of these hypersurfaces.

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Moreover, the parallel Ricci tensor for real hypersurfaces in $SU_{2,m}/S(U_2U_m)$ was investigated by Suh and Woo [21].

As another kind of Hermitian symmetric space with rank 2 of compact type different those given above, we can give an example of complex quadric $Q^m = SO_{m+2}/SO_mSO_2$, which is a complex hypersurface in complex projective space $\mathbb{C}P^{m+1}$ (see Klein [4] and Smyth [9]). The complex quadric can also be regarded as a kind of real Grassmann manifold of compact type with rank 2 (see Kobayashi and Nomizu [5]). Accordingly, the complex quadric admits two important geometric structures, a complex conjugation structure A and a Kähler structure J, which anti-commute with each other, that is, $AJ = -JA$. Then for $m \geq 2$ the triple (Q^m, J, g) is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see Klein [4] and Reckziegel [8]).

Apart from the complex structure J there is another distinguished geometric structure on Q^m, namely a parallel rank two vector bundle $\mathfrak{A}$ which contains an S^1-bundle of real structures, that is, complex conjugations A on the tangent spaces of Q^m. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$, that is, $A\mathcal{Q} \subset \mathcal{Q}$ for any complex conjugation $A \in \mathfrak{U}$, of the tangent bundle TM of a real hypersurface M in Q^m. Here the notion of parallel vector bundle $\mathfrak{A}$ means that

$$(\overline{\nabla}_X A)Y = q(X)JAY$$

for any vector fields X and Y on Q^m, where $\overline{\nabla}$ and q denote a connection and a certain 1-form defined on $T_{[z]}Q^m$ and $[z] \in Q^m$, respectively (see Smyth [9]).

For real hypersurfaces in the complex quadric Q^m, Berndt and Suh [1] have classified the problem of isometric Reeb flow, which was mainly used in [15–17]. Now we want to introduce the following theorem.

Theorem A *Let M be a real hypersurface of the complex quadric Q^m, $m \geq 3$. The Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$.*

Recall that a nonzero tangent vector $W \in T_{[z]}Q^m$ is called singular if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for the complex quadric Q^m.

1. If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-*principal*.

2. If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/\|W\| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-*isotropic*.

Here,

$$V(A) = \{X \in T_{[z]}Q^m \mid AX = X\} \quad \text{and} \quad JV(A) = \{X \in T_{[z]}Q^m \mid AX = -X\}$$

denote the $(+1)$-eigenspace and (-1)-eigenspace, respectively, for the involution $A^2 = I$ on $T_{[z]}Q^m$, $[z] \in Q^m$.

When we consider a hypersurface M in the complex quadric Q^m under the assumption of some geometric properties, the unit normal vector field N of M in Q^m can be divided into two classes: N is either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal (see [1,2,15,16]).

In the first case where N is $\mathfrak{A}$-isotropic, we have shown in [1] that M is locally congruent to a tube over a totally geodesic $\mathbb{C}P^k$ in Q^{2k}. In the second case, when the unit normal N is $\mathfrak{A}$-principal, we proved that a contact hypersurface M in Q^m is locally congruent to a tube over a totally geodesic and totally real submanifold S^m in Q^m (see [2]).

On the other hand, Jacobi fields along geodesics of a given Riemannian manifold M satisfy an well known differential equation. Naturally the classical differential equation inspires the so-called *Jacobi operator*. That is, if R denotes the curvature operator of M, then the Jacobi operator with respect to X at $x \in M$, is defined by

$$(R_X Y)(x) = (R(Y,X)X)(x)$$

for any $Y \in T_x M$. Then $R_X \in \mathrm{End}(T_x M)$ becomes a symmetric endomorphism of the tangent bundle TM of M. Clearly, each tangent vector field X to M provides a Jacobi operator with respect to X.

From such a viewpoint, for a real hypersurface M in the complex quadric Q^m the *structure Jacobi operator* R_ξ is defined by

$$R_\xi = R(\cdot,\xi)\xi \in End\ (T_z M), \quad z \in M,$$

where R denotes the curvature tensor of the complex quadric Q^m. Of course, the structure Jacobi operator R_ξ is a symmetric endomorphism of M in Q^m.

The structure Jacobi operator R_ξ of M in Q^m is said to be *Lie invariant* if the operator R_ξ satisfies

$$(\mathcal{L}_X R_\xi)Y = 0$$

for any $X, Y \in T_z M$, $z \in M$, where the Lie derivative $(\mathcal{L}_X R_\xi)Y$ is defined by

$$\begin{aligned}(\mathcal{L}_X R_\xi)Y &= [X, R_\xi(Y)] - R_\xi([X,Y])\\ &= \nabla_X(R_\xi(Y)) - \nabla_{R_\xi(Y)}X - R_\xi(\nabla_X Y - \nabla_Y X)\\ &= (\nabla_X R_\xi)Y - \nabla_{R_\xi(Y)}X + R_\xi(\nabla_Y X).\end{aligned}$$

Recently, for real hypersurfaces in the complex quadric Q^m we investigated the notions of parallel Ricci tensor, harmonic curvature, commuting Ricci tensor and Lie invariant normal Jacobi operator, which are respectively given by $\nabla \mathrm{Ric} = 0$, $\delta\mathrm{Ric} = 0$, $\mathrm{Ric}\cdot\phi = \phi\cdot\mathrm{Ric}$(see Suh [16], [17], Suh and Hwang [18]). Then, motivated by such facts and the classification of isometric Reeb flow due to Theorem A, Suh and Kim ([20]) gave the following theorem for real hypersurfaces in the complex quadric Q^m with Lie invariant normal Jacobi operator, that is, $\mathcal{L}_X \overline{R}_N = 0$.

Theorem B *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$, with Lie invariant normal Jacobi operator. Then M is locally congruent to a tube of radius r over a totally geodesic complex k-dimensional complex projective space $\mathbb{C}P^k$ in the complex $2k$-dimensional complex quadric Q^{2k}.*

On the other hand, from the assumption of Ricci parallel or harmonic curvature, it was difficult for us to derive the fact that the unit normal vector field N is either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal. So in [16, 17] we gave a classification with the further assumption that N is $\mathfrak{A}$-isotropic. But fortunately, when we consider Lie invariant structure Jacobi operator, that is, $\mathcal{L}_X R_\xi = 0$ for any tangent vector field X on M

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in Q^m, we can assert that the unit normal vector field N becomes either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal as follows.

Main Theorem 1 *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$, with Lie invariant structure Jacobi operator. Then the unit normal vector field N is singular, that is, N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

Then, motivated by Theorem 1 and Theorem A due to Berndt and Suh [1], we can give a classification theorem for real hypersurfaces in the complex quadric Q^m with Lie invariant structure Jacobi operator. Now we want to assert the following, which is quite different from Theorem B.

Main Theorem 2 *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$ with Lie invariant structure Jacobi operator. Then M is locally congruent to one of the following:*

(i) *a tube of radius $\frac{\pi}{4}$ over a totally geodesic complex k-dimensional complex projective space $\mathbb{C}P^k$ in Q^{2k}, $m = 2k$;*

(ii) *a hypersurface that has at most five distinct constant principal curvatures α, 0, and the solution of the cubic equation*

$$\alpha x^3 - x^2 + 2\alpha x + 1 = 0,$$

where $\alpha = g(S\xi, \xi)$ denotes the Reeb function on M,

(iii) *a hypersurface that has four distinct constant principal curvatures given α, 0, $-\frac{\alpha}{\alpha^2+2}$, and $-\frac{1}{\alpha}$ with multiplicities 1, 2, $m-2$, and $m-2$, respectively,*

(iv) *a hypersurface that has three distinct constant principal curvatures α, and two distinct roots given by*

$$\lambda = \frac{\alpha^2 - 2 \pm \sqrt{\alpha^4 + 12\alpha^2 + 4}}{2\alpha}$$

with multiplicities 1, $m-1$, and $m-1$, respectively,

(v) *a hypersurface that has three distinct constant principal curvatures α and two distinct roots given by*

$$\lambda = \frac{\alpha \pm \sqrt{\alpha^2 + 4}}{2}$$

with multiplicities 1, $m-1$, and $m-1$, respectively, provided with non-vanishing Reeb function α.

In Main Theorem 2, if the unit normal vector field N is $\mathfrak{A}$-isotropic, then M is locally congruent to a real hypersurface of type (i), (ii), or (iii). If N is $\mathfrak{A}$-principal, M is locally congruent to one of type (iv) or (v). Moreover, the case (i) in Theorem 2 is a special case of the case (ii) when the Reeb function α is vanishing.

Our paper is organized as follows. In Section 2 we present basic material about the complex quadric Q^m, including its Riemannian curvature tensor and a description of its singular tangent vectors. Apart from the complex structure J, there is another distinguished geometric structure on Q^m, namely a parallel rank two vector bundle $\mathfrak{A}$ that contains an S^1-bundle of real structures on the tangent spaces of Q^m. In Section 3 we investigate the geometry of the maximal subbundle $\mathcal{Q}$ and introduce the equation of Codazzi. In Section 4 we give a complete proof of Theorem 1, which acts as a key lemma for the proof of Theorem 2 according to the $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic unit normal vector field.

In Section 5 we give a contradiction for real hypersurfaces in Q^m with Lie invariant normal Jacobi operator if they have the $\mathfrak{A}$-principal unit normal. Finally, in Section 6, we present the proof of our Theorem 2 when M admits the $\mathfrak{A}$-isotropic unit normal. In order to do this, we introduce Lemma 6.1, saying that $SAN = 0$ and $SA\xi = 0$ for a Hopf real hypersurface with $\mathfrak{A}$-isotropic unit normal vector field N. Lemma 6.1 is crucial for the proof of Main Theorem 2. From this, together with the equation of Gauss between the curvature tensors $R(X,Y)Z$ for M and $\overline{R}(X,Y)Z$ for Q^m respectively, we give a complete proof of Main Theorem 2.

2 The Complex Quadric

For more background to this section, we refer the reader to [4, 5, 8, 15–17]. The complex quadric Q^m is the complex hypersurface in $\mathbb{C}P^{m+1}$, which is defined by the equation $z_0^2 + \cdots + z_{m+1}^2 = 0$, where $z_0, \ldots, z_{m+1}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$. We equip Q^m with the Riemannian metric g that is induced from the Fubini–Study metric $\overline{g}$ on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The Fubini–Study metric $\overline{g}$ is defined by $\overline{g}(X,Y) = \Phi(JX,Y)$ for vector fields X and Y on $\mathbb{C}P^{m+1}$ and a globally closed $(1,1)$-form Φ given by $\Phi = -4i\partial\overline{\partial}\log f_j$ on an open set $U_j = \{[z^0, z^1, \ldots, z^{m+1}] \in \mathbb{C}P^{m+1} \mid z^j \neq 0\}$, where the function f_j denotes $f_j = \sum_{k=0}^{m+1} t_j^k \overline{t}_j^k$, and $t_j^k = \frac{z^k}{z^j}$ for $j,k = 0,\ldots,m+1$. Then, naturally, the Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure (J,g) on the complex quadric Q^m.

The complex projective space $\mathbb{C}P^{m+1}$ is a Hermitian symmetric space of the special unitary group SU_{m+2}, namely, $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$. We denote by $o = [0,\ldots,0,1] \in \mathbb{C}P^{m+1}$ the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$. The special orthogonal group $SO_{m+2} \subset SU_{m+2}$ acts on $\mathbb{C}P^{m+1}$ with cohomogeneity one. The orbit containing o is a totally geodesic real projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$. The second singular orbit of this action is the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. This homogeneous space model leads to the geometric interpretation of the complex quadric Q^m as the Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. It also gives a model of Q^m as a Hermitian symmetric space of rank 2. The complex quadric Q^1 is isometric to a sphere S^2 with constant curvature, and Q^2 is isometric to the Riemannian product of two 2-spheres with constant curvature. For this reason we will assume $m \geq 3$ from now on.

Now let us denote by $A_{\overline{z}}$ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to the unit normal $\overline{z}$. It is defined by $A_{\overline{z}}w = \overline{\nabla}_w\overline{z} = \overline{w}$ for a complex Euclidean connection $\overline{\nabla}$ induced from $\mathbb{C}^{m+2}$ and all $w \in T_{[z]}Q^m$. That is, the shape operator $A_{\overline{z}}$ is just a complex conjugation restricted to $T_{[z]}Q^m$. Moreover, it satisfies the following for any $w \in T_{[z]}Q^m$ and any $\lambda \in S^1 \subset \mathbb{C}$

$$\begin{aligned} A_{\lambda\overline{z}}^2 w &= A_{\lambda\overline{z}}A_{\lambda\overline{z}}w = A_{\lambda\overline{z}}\lambda\overline{w} \\ &= \lambda A_{\overline{z}}\lambda\overline{w} = \lambda\overline{\nabla}_{\lambda\overline{w}}\overline{z} = \lambda\overline{\lambda}\,\overline{\overline{w}} = |\lambda|^2 w = w. \end{aligned}$$

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Accordingly, $A^2_{\lambda\bar{z}} = I$ for any $\lambda \in S^1$. So the shape operator $A_{\bar{z}}$ becomes an anti-commuting involution such that $A^2_{\bar{z}} = I$ and $AJ = -JA$ on the complex vector space $T_{[z]}Q^m$ and

$$T_{[z]}Q^m = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}}),$$

where $V(A_{\bar{z}}) = \mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the $(+1)$-eigenspace and $JV(A_{\bar{z}}) = i\mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the (-1)-eigenspace of $A_{\bar{z}}$. That is, $A_{\bar{z}}X = X$ and $A_{\bar{z}}JX = -JX$, respectively, for any $X \in V(A_{\bar{z}})$.

There is a geometric interpretation of these conjugations. The complex quadric Q^m can be viewed as the complexification of the m-dimensional sphere S^m. Through each point $[z] \in Q^m$, there exists a one-parameter family of real forms of Q^m that are isometric to the sphere S^m. These real forms are congruent to each other under action of the center SO_2 of the isotropy subgroup of SO_{m+2} at $[z]$. The isometric reflection of Q^m in such a real form S^m is an isometry, and the differential at $[z]$ of such a reflection is a conjugation on $T_{[z]}Q^m$. In this way the family $\mathfrak{A}$ of conjugations on $T_{[z]}Q^m$ corresponds to the family of real forms S^m of Q^m containing $[z]$, and the subspaces $V(A) \subset T_{[z]}Q^m$ correspond to the tangent spaces $T_{[z]}S^m$ of the real forms S^m of Q^m.

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$:

$$\begin{aligned}\bar{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ - g(JX,Z)JY - 2g(JX,Y)JZ + g(AY,Z)AX \\ - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY.\end{aligned}$$

Note that J and each complex conjugation A anti-commute, that is, $AJ = -JA$ for each $A \in \mathfrak{A}$.

For every unit tangent vector $W \in T_{[z]}Q^m$ there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY$$

for some $t \in [0, \pi/4]$. Here $\mathfrak{A}$-principal W corresponds to the value $t = 0$ and $\mathfrak{A}$-isotropic W to $t = \pi/4$. Here the vector $W = X$ for $X \in V(A)$ with $t = 0$ is singular, because for any unit vector $Y \in V(A)$ orthogonal to X, $\mathbb{R}X + \mathbb{R}JY$ is a maximal flat that contains X. Also the vector $W = (X + JY)/\sqrt{2}$ for $t = \frac{\pi}{4}$ is singular, because W is $\mathfrak{A}$-isotropic, and the kernel of the Jacobi operator $\bar{R}_W$ is $\mathbb{R}W \oplus \mathbb{C}AW$. Then it follows that for any $\mu \in S^1$, $\mathfrak{a} := \mathbb{R}(X + JY) \oplus \mathbb{R}(\mu(X - JY))$ is a maximal flat that includes the vector W.

3 Some General Equations

Let M be a real hypersurface in Q^m and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M and η the corresponding 1-form defined by $\eta(X) = g(\xi, X)$ for any tangent vector field X on M. The tangent bundle TM of M splits orthogonally into

$TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

At each point $z \in M$, we define a maximal $\mathfrak{A}$-invariant subspace of T_zM, $z \in M$ as

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}.$$

Then we want to introduce an important lemma that will be used in the proof of our main theorem.

Lemma 3.1 ([15]) *For each $z \in M$, we have the following.*

(i) *If N_z is $\mathfrak{A}$-principal, then $\mathcal{Q}_z = \mathcal{C}_z$.*
(ii) *If N_z is not $\mathfrak{A}$-principal, there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $N_z = \cos(t)X + \sin(t)JY$ for some $t \in (0, \pi/4]$. Then we have $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$.*

We now assume that M is a Hopf hypersurface. Then the Reeb vector field $\xi = -JN$ satisfies $S\xi = \alpha\xi$, where S denotes the shape operator of the real hypersurfaces M with the smooth function $\alpha = g(S\xi, \xi)$ on M. When we consider the transform JX by the Kähler structure J on Q^m for any vector field X on M in Q^m, we can put $JX = \phi X + \eta(X)N$ for a unit normal N to M. We now consider the Codazzi equation,

$$\begin{aligned} g\big((\nabla_X S)Y - (\nabla_Y S)X, Z\big) &= \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) \\ &\quad + g(X, AN)g(AY, Z) - g(Y, AN)g(AX, Z) \\ &\quad + g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z). \end{aligned} \tag{3.1}$$

Putting $Z = \xi$ in (3.1), we get

$$\begin{aligned} g\big((\nabla_X S)Y - (\nabla_Y S)X, \xi\big) = &-2g(\phi X, Y) \\ &+ g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ &- g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi). \end{aligned} \tag{3.2}$$

On the other hand, we have

$$\begin{aligned} g\big((\nabla_X S)Y - (\nabla_Y S)X, \xi\big) &= g\big((\nabla_X S)\xi, Y\big) - g\big((\nabla_Y S)\xi, X\big) \\ &= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g\big((S\phi + \phi S)X, Y\big) - 2g(S\phi SX, Y). \end{aligned} \tag{3.3}$$

Comparing (3.2) and (3.3) and putting $X = \xi$ yields

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi). \tag{3.4}$$

Reinserting this into (3.3) yields

$$\begin{aligned} g\big((\nabla_X S)Y - (\nabla_Y S)X, \xi\big) &= -2g(\xi, AN)g(X, A\xi)\eta(Y) + 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ &\quad + 2g(\xi, AN)g(Y, A\xi)\eta(X) - 2g(Y, AN)g(\xi, A\xi)\eta(X) \\ &\quad + \alpha g\big((\phi S + S\phi)X, Y\big) - 2g(S\phi SX, Y). \end{aligned}$$

Altogether this implies

$$\begin{aligned}0 = 2g(S\phi SX, Y) - \alpha g\big((\phi S + S\phi)X, Y\big) - 2g(\phi X, Y)\\ + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi)\\ - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi)\\ + 2g(\xi, AN)g(X, A\xi)\eta(Y) - 2g(X, AN)g(\xi, A\xi)\eta(Y)\\ - 2g(\xi, AN)g(Y, A\xi)\eta(X) + 2g(Y, AN)g(\xi, A\xi)\eta(X).\end{aligned}$$

At each point $z \in M$ we can choose $A \in \mathfrak{A}_z$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \le t \le \frac{\pi}{4}$ (see [8, Proposition 3]). Note that t is a function on M. First of all, since $\xi = -JN$, we have

$$\begin{aligned}AN &= \cos(t)Z_1 - \sin(t)JZ_2,\\ \xi &= \sin(t)Z_2 - \cos(t)JZ_1,\\ A\xi &= \sin(t)Z_2 + \cos(t)JZ_1.\end{aligned}$$

This implies $g(\xi, AN) = 0$, and hence

$$\begin{aligned}0 = 2g(S\phi SX, Y) - \alpha g\big((\phi S + S\phi)X, Y\big) - 2g(\phi X, Y)\\ + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi)\\ - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi)\\ - 2g(X, AN)g(\xi, A\xi)\eta(Y) + 2g(Y, AN)g(\xi, A\xi)\eta(X).\end{aligned}$$

We now apply this result to get more information for the Reeb function α on Hopf hypersurfaces in Q^m.

Lemma 3.2 ([15]) *Let M be a Hopf hypersurface in Q^m such that the normal vector field N is $\mathfrak{A}$-principal everywhere. Then α is constant. Moreover, if $X \in \mathcal{C}$ is a principal curvature vector of M with principal curvature λ, then $2\lambda \ne \alpha$ and ϕX is a principal curvature vector of M with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.*

Lemma 3.3 ([1]) *Let M be a Hopf hypersurface in Q^m, $m \ge 3$, such that the normal vector field N is $\mathfrak{A}$-isotropic everywhere. Then α is constant.*

4 Invariant Structure Jacobi Operator and a Key Lemma

By the Gauss equation, the curvature tensor $R(X, Y)Z$ for a real hypersurface M in Q^m induced from the curvature tensor $\overline{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugation $A \in \mathfrak{A}$ as follows:

$$\begin{aligned}R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y\\ - 2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY\\ + g(JAY, Z)JAX - g(JAX, Z)JAY + g(SY, Z)SX\\ - g(SX, Z)SY\end{aligned}$$

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for any $X, Y, Z \in T_zM$, $z \in M$. Then the structure Jacobi operator $\overline{R}_N$ is defined in such a way that

$$R_\xi(X) = R(X, \xi)\xi$$

for any tangent vector field X in T_zM and the unit normal N of M in T_zQ^m, $z \in Q^m$. Then the structure Jacobi operator R_ξ becomes a symmetric operator on the tangent space T_zM, $z \in M$, of Q^m. From this, by the complex structure J and the complex conjugations $A \in \mathfrak{A}$, together with the fact that $g(A\xi, N) = 0$ and $\xi = -JN$ in Section 3, the structure Jacobi operator R_ξ is given by

$$\begin{aligned}(4.1)\qquad R_\xi(X) = X - \eta(X)\xi + \beta(AX)^T - g(AX,\xi)A\xi - g(AX,N)(AN)^T \\ + \alpha SX - g(SX,\xi)S\xi\end{aligned}$$

for any $Y \in T_zM$, $z \in M$, where the function β is defined by $\beta = g(A\xi, \xi) = -g(AN, N)$.

On the other hand, the definition of the Lie derivative of the structure Jacobi operator R_ξ gives

$$(4.2)\qquad (\mathcal{L}_X R_\xi)(Y) = \mathcal{L}_X(R_\xi(Y)) - R_\xi(\mathcal{L}_X Y) = [X, R_\xi(Y)] - R_\xi([X,Y])$$

for any tangent vector fields X and Y on M in Q^m. Moreover, the derivative of R_ξ is given by

$$(4.3)\qquad (\nabla_X R_\xi)Y = \nabla_X(R_\xi Y) - R_\xi(\nabla_X Y).$$

Now let us suppose that the structure Jacobi operator R_ξ is Lie parallel; that is, $\mathcal{L}_X R_\xi = 0$. Then (4.2) gives

$$[X, R_\xi(Y)] - R_\xi\big([X,Y]\big) = 0.$$

From this, together with (4.3), it follows that

$$(4.4)\qquad (\nabla_X R_\xi)Y = \nabla_{R_\xi(Y)}X - R_\xi(\nabla_Y X).$$

Then, putting $X = \xi$ in the above equation and using (4.1), we have

$$\begin{aligned}(\nabla_\xi R_\xi)Y &= \nabla_{R_\xi Y}\xi - R_\xi\nabla_Y\xi = \phi S R_\xi Y - R_\xi\phi S Y \\ &= \phi SY + \beta\phi S(AY)^T - g(AT,\xi)\phi SA\xi - g(AY,N)\phi S(AN)^T \\ &\quad + \alpha\phi S^2 Y - \big\{\phi SY + \beta(A\phi SY)^T - g(A\phi SY,\xi)A\xi \\ &\qquad - g(A\phi SY, N)(AN)^T + \alpha S\phi SY\big\}.\end{aligned}$$

From this, taking the inner product with the unit normal vector field N, we have

$$\begin{aligned}(4.5)\qquad 0 &= -\big\{\beta g(A\phi SY, N) - g(A\phi SY, N)g(AN,N)\big\} \\ &= -\big\{\beta - g(AN,N)\big\}\, g\big(A\phi SY, N\big) \\ &= 2g(AN,N)g(A\phi SY,N).\end{aligned}$$

Then at some points $x \in M$, the unit normal vector N_x is $\mathfrak{A}$-isotropic, that is, $\beta(x) = 0$ holds, whereas at other points $y \in M$, we have $\beta(y) \neq 0$. Then by (4.5), on such points we know that $(S\phi AN)_y = 0$. This gives us a motivation to consider the open subset $\mathfrak{U} = \{x \in M \mid \beta(x) \neq 0\}$. Then we assert the following lemma.

Lemma 4.1 *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$, with Lie invariant structure Jacobi operator. Then on the open subset $\mathfrak{U}$ of M, we have $S\phi AN = 0$.*

On the other hand, putting $Y = \xi$ in (4.4), we have

$$\begin{aligned}
0 &= (\nabla_X R_\xi)\xi - \nabla_{R_\xi \xi} X + R_\xi(\nabla_\xi X) \\
&= \nabla_X(R_\xi(\xi)) - R_\xi(\nabla_X \xi) + R_\xi(\nabla_\xi X) \\
&= -R_\xi(\phi SX) + R_\xi(\nabla_\xi X) \\
&= -\big\{\phi SX + \beta(A\phi SX)^T - g(A\phi SX, \xi)A\xi - g(A\phi SX, N)(AN)^T \\
&\qquad\qquad + \alpha S\phi SX - g(S\phi SX, \xi)S\xi\big\} \\
&\quad + \big\{\nabla_\xi X - \eta(\nabla_\xi X)\xi + \beta(A\nabla_\xi X)^T - g(A\nabla_\xi X, \xi)A\xi \\
&\qquad\qquad - g(A\nabla_\xi X, N)(AN)^T + \alpha S\nabla_\xi X - g(S\nabla_\xi X, \xi)S\xi\big\}.
\end{aligned}$$

From this, taking the inner product with the unit normal vector field N and using $S\phi AN = 0$, we have

$$0 = -2g(AN, N)g(\nabla_\xi X, AN).$$

Then we assert the following lemma.

Lemma 4.2 *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$, with Lie invariant structure Jacobi operator. Then on the open subset $\mathfrak{U}$ of M, we have $g(\nabla_\xi X, AN) = 0$ for any vector field X on M.*

Then on the open subset $\mathfrak{U}$, by Lemma 4.2, we have, for any tangent vector field X on M,

$$\begin{aligned}
0 &= g(\nabla_\xi(\phi X), AN) = g(\phi\nabla_\xi X, AN) \\
&= g(J\nabla_\xi X - \eta(\nabla_\xi X)N, AN) = -g(\nabla_\xi X, A\xi).
\end{aligned} \tag{4.6}$$

Then putting $X = A\xi$ in (4.6), we have naturally

$$\begin{aligned}
0 &= g(\nabla_\xi(A\xi), AN) \\
&= -q(\xi)g(AN, AN) + \alpha g(AN, AN) - \alpha\beta g(N, AN) \\
&= -\{q(\xi) - \alpha\} + \alpha\beta^2,
\end{aligned} \tag{4.7}$$

because we have used

$$\nabla_\xi(A\xi) = q(\xi)JA\xi + \alpha AN - \alpha\beta N,$$

where q denotes a certain 1-form defined on $T_{[z]}Q^m$, $[z] \in Q^m$ as in the introduction.

On the other hand, for $X \perp A\xi$, $X \in T_z M$, and $X \perp \xi$, we know that

$$\begin{aligned}
0 &= g(\nabla_\xi X, A\xi) = -g\big(X, \nabla_\xi(A\xi)\big) \\
&= -g\big(X, q(\xi)JA\xi + \alpha AN - \alpha\beta N\big) \\
&= -g\big(X, -q(\xi)AN + \alpha AN\big) = \big(q(\xi) - \alpha\big)g(X, AN).
\end{aligned}$$

From this, putting $X = AN^T$, we have $q(\xi) = \alpha$. Substituting this into (4.7), the Reeb function α is then vanishing on the open subset $\mathfrak{U}$. Then (3.4) gives that $g(Y, AN)$

$g(\xi, A\xi) = 0$ for any vector field Y on $\mathfrak{U}$ on M. This means that $AN = N$; that is, the unit normal vector field N is $\mathfrak{A}$-principal. Here the $\mathfrak{A}$-principalness of the unit vector N has been shown only at the points of the open set $\mathfrak{U}$. However, this implies that $\mathfrak{U} = \{x \in M \mid AN_x = N_x\}$ is closed. Because M is connected, it follows that either $\mathfrak{U} = \varnothing$ holds, meaning that N is $\mathfrak{A}$-isotropic on M, or else $\mathfrak{U} = M$ holds, meaning that N is $\mathfrak{A}$-principal on M. That is, we assert the following lemma.

Lemma 4.3 *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$, with Lie invariant structure Jacobi operator. Then the unit normal vector field N is $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.*

5 Invariant Normal Jacobi Operator with $\mathfrak{A}$-principal Normal Vector Field

In this section let us consider a real hypersurface M in a complex quadric with $\mathfrak{A}$-principal unit normal vector field. Then the unit normal vector field N satisfies $AN = N$ for a complex conjuagation $A \in \mathfrak{A}$. This also implies that $A\xi = -\xi$ for the Reeb vector field $\xi = -JN$.

Then the structure Jacobi operator R_ξ in Section 4 becomes

$$R_\xi(X) = X - \eta(X)\xi + \beta(AX)^T - \eta(X)\xi - g(AX, N)(AN)^T \\ + \alpha SX - g(SX, \xi)S\xi$$

for any $Y \in T_zM$, $z \in M$, where the function β denotes $\beta = g(A\xi, \xi)$. Moreover, the formula (4.2) for the $\mathfrak{A}$-principal unit normal vector field, that is, $A\xi = -\xi$, becomes

$$(\nabla_\xi R_\xi)Y = \beta\phi SAY + \alpha\phi S^2 Y - \{\beta A\phi SY + \alpha S\phi SY\}, \tag{5.1}$$

and, using the constancy of the Reeb function α in Lemma 3.2 and the function $\beta = -1$, we have the formula

$$(\nabla_\xi R_\xi)Y = \nabla_\xi(R_\xi(Y)) - R_\xi(\nabla_\xi Y) = \beta(\nabla_\xi A)Y + \alpha(\nabla_\xi S)Y. \tag{5.2}$$

Then (5.1) and (5.2), together with the function $\beta = -1$, give

$$-\phi SAY + \alpha\phi S^2 Y + \{A\phi SY - \alpha S\phi SY\} = -(\nabla_\xi A)Y + \alpha(\nabla_\xi S)Y. \tag{5.3}$$

On the other hand, the Codazzi equation gives

$$\begin{aligned}(\nabla_\xi S)Y &= (\nabla_Y S)\xi + \phi Y - \phi AY \\ &= \nabla_Y(S\xi) - S\nabla_Y\xi + \phi Y - \phi AY \\ &= (Y\alpha)\xi + \alpha\phi SY - S\phi SY + \phi Y - \phi AY.\end{aligned}$$

From this, (5.3) becomes

$$-\phi SAY + \alpha\phi S^2 Y + A\phi SY = -2\alpha\phi AY + \alpha^2\phi SY + \alpha\phi Y - \alpha\phi AY, \tag{5.4}$$

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where we have used the derivative formula

$$
\begin{aligned}
(\nabla_\xi A)X &= \nabla_\xi(AX) - A\nabla_\xi X \\
&= \overline{\nabla}_\xi(AX) - \sigma(\xi, AX) - A\nabla_\xi X \\
&= (\overline{\nabla}_\xi A)X + A\sigma(\xi, X) - \alpha g(\xi, AX)N \\
&= q(\xi)JAX + 2\alpha\eta(X)N.
\end{aligned} \tag{5.5}
$$

Taking the inner product of (5.5) with the unit normal vector field N, we have

$$q(\xi) = 2\alpha. \tag{5.6}$$

Remark 5.1 When the Reeb function α is vanishing, by (5.2), (5.5), and (5.6), the structure Jacobi operator R_ξ is parallel along the Reeb direction.

From (5.6), together with (5.4), we have

$$-\phi SAY + \alpha\phi S^2Y + A\phi SY = -3\alpha\phi AY + \alpha^2\phi SY + \alpha\phi Y. \tag{5.7}$$

Now let us consider the following three cases.

Case 1. $Y \in V(A) \cap T_zM$, $z \in M$. Since M is Hopf, that is, $S\xi = \alpha\xi$, we can put $SY = \lambda Y$ for $Y \in \mathcal{C} = \xi^\perp$, and use $AY = Y$ and $A\phi Y = -\phi Y$. Then (5.7) gives

$$-\lambda\phi Y + \alpha\lambda^2\phi Y - \lambda\phi Y = \alpha^2\lambda\phi Y - 2\alpha\phi Y.$$

From this, if the Reeb function α vanishes, then all $\lambda = 0$. This means M is totally geodesic, which gives a contradiction to the Codazzi equation (see Suh [15]). So in Case 1, the Reeb function α cannot be vanishing. Then we have $(\alpha\lambda - 2)(\lambda - \alpha) = 0$. This gives

$$\lambda = \alpha \qquad \text{or} \qquad \lambda = \frac{2}{\alpha}. \tag{5.8}$$

Moreover, by Lemma 3.2, we know that $S\phi X = \mu\phi X$, $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$.

Now from (5.8), we consider the first case $\lambda = \alpha$. Then from also $(\alpha\mu - 2)(\mu - \alpha) = 0$ and $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$, naturally we can consider two subcases:

$$\mu = \frac{\alpha^2+2}{\alpha} = \alpha \quad \text{or} \quad \mu = \frac{\alpha^2+2}{\alpha} = \frac{2}{\alpha}.$$

The above two subcases can be valid only for a non-vanishing Reeb function α. The first subcase gives us a contradiction. From the second subcase we get $\alpha = 0$, which gives a contradiction for non-vanishing α. So we cannot consider the first case.

Next it remains only to consider the second case of (5.8), that is, $\lambda = \frac{2}{\alpha}$. This case can be also considered for a non-vanishing Reeb function α. In this case, by (5.8), the function μ becomes $\mu = \frac{4\alpha}{4-\alpha^2} = \alpha$ or $\mu = \frac{4\alpha}{4-\alpha^2} = \frac{2}{\alpha}$. Then the first subcase implies $\alpha = 0$, which also gives us a contradiction. The second subcase $\mu = \frac{4\alpha}{4-\alpha^2} = \frac{2}{\alpha}$ is valid only for $\alpha^2 = \frac{4}{3}$, but $\lambda = \mu = \frac{2}{\alpha}$ implies $S\phi = \phi S$, which means that the Reeb flow is isometric. But Berndt and Suh [1] proved that the unit normal vector field N is $\mathfrak{A}$-isotropic if $S\phi = \phi S$. Accordingly, we conclude that Case 1 cannot be considered.

Case 2. $Y \in JV(A) \cap T_z M$, $z \in M$. In this case, $AY = -Y$, $A\phi Y = -\phi AY = \phi Y$. Then (5.7) gives

$$\phi SY + \alpha\phi S^2 Y + A\phi SY = 3\alpha\phi Y + \alpha^2\phi SY + \alpha\phi Y.$$

From this, putting $SY = \lambda Y$ for $Y \in \mathcal{C}$, and using $A\phi Y = \phi Y$, we have

$$\alpha\lambda^2 - (\alpha^2 - 2)\lambda - 4\alpha = 0.$$

So M has three distinct constant principal curvatures α with multiplicities 1, $m-1$, and $m-1$, and two distinct roots given by

$$\lambda = \frac{\alpha^2 - 2 \pm \sqrt{\alpha^4 + 12\alpha^2 + 4}}{2\alpha}.$$

Case 3. $Y \in \mathcal{C}_z \setminus (V(A) \cup JV(A))$, $z \in M$. Then we can put $Y = Z + W$ for some non-vanishing two unit vector fields $Z \in V(A)$ and $W \in JV(A)$. From this, it follows that

$$AY = A(Z + W) = Z - W.$$

So for $SY = \lambda Y$ for $Y \in \mathcal{C} = [\xi]^\perp$, where $[\xi]^\perp$ denotes the orthogonal complement of the Reeb vector field ξ in $T_z M$, $z \in M$, we have

$$A\phi SY = \lambda A\phi Y = -\lambda\phi AY = -\lambda(\phi Z - \phi W).$$

From this, (5.7) implies that

$$(5.9) \quad -\phi S(Z - W) + \alpha\lambda^2(\phi Z + \phi W) - \lambda(\phi Z - \phi W) = \\ -3\alpha(\phi Z - \phi W) + \alpha^2\lambda(\phi Z + \phi W) + \alpha(\phi Z + \phi W).$$

Then taking the inner product of (5.9) with the vector fields ϕZ and ϕW respectively, we get

$$(5.10) \quad -g(SZ, Z) + g(SW, Z) + \alpha\lambda^2 - \lambda = -3\alpha + \alpha^2\lambda + \alpha,$$

$$(5.11) \quad -g(SZ, W) + g(SW, W) + \alpha\lambda^2 + \lambda = 3\alpha + \alpha^2\lambda + \alpha.$$

On the other hand, $SY = \lambda Y$ gives $SZ + SW = \lambda Z + \lambda W$. Then, taking the inner products with two unit vector fields Z and W, we get $g(SW, Z) = -g(SZ, Z) + \lambda$ and $g(SW, W) + g(SZ, W) = \lambda$, respectively. Subtracting these two equations, we have

$$g(SZ, Z) = g(SW, W).$$

Now adding equations (5.10) and (5.11) and using the above formula, we have

$$\lambda^2 - \alpha\lambda - 1 = 0,$$

provided that the Reeb function α is non-vanishing. When the Reeb function α vanishes, we get no information; only identity holds. So by Lemma 3.2, M has three distinct constant principal curvatures α and

$$\lambda = \frac{\alpha \pm \sqrt{\alpha^2 + 4}}{2}$$

with multiplicities 1, $m-1$, and $m-1$, respectively.

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6 Invariant Structure Jacobi Operator with $\mathfrak{A}$-isotropic Normal Vector Field

Under the assumption of $\mathfrak{A}$-isotropic unit normal, the structure Jacobi operator R_ξ in Section 4 becomes

$$R_\xi(X) = X - \eta(X)\xi - g(AX,\xi)A\xi - g(AX,N)(AN)^T + \alpha SX - \alpha^2\eta(X)\xi$$

for any $Y \in T_zM$, $z \in M$. Under the assumption of $\mathfrak{A}$-isotropic and $\mathcal{L}_X R_\xi = 0$, we have

$$0 = (\mathcal{L}_X R_\xi)Y = (\nabla_X R_\xi)Y - \nabla_{R_\xi(Y)}X + R_\xi(\nabla_Y X).$$

From this, putting $Y = \xi$ and using $R_\xi(\xi) = 0$, we have

$$(6.1)\qquad \begin{aligned}(\nabla_X R_\xi)\xi &= \nabla_{R_\xi(\xi)}X - R_\xi\nabla_\xi X\\ &= -\{\nabla_\xi X - \eta(\nabla_\xi X)\xi - g(A\nabla_\xi X,\xi)A\xi - g(A\nabla_\xi X,N)AN\\ &\quad + \alpha S\nabla_\xi X - \alpha^2\eta(\nabla_\xi X)\xi\}.\end{aligned}$$

Moreover, differentiating the structure Jacobi operator R_ξ gives

$$(6.2)\qquad \begin{aligned}(\nabla_X R_\xi)\xi &= \nabla_X(R_\xi(\xi)) - R_\xi(\nabla_X\xi)\\ &= -\left\{\phi SX - g(A\phi SX,\xi)A\xi - g(A\phi SX,N)AN + \alpha S\phi SX\right\}.\end{aligned}$$

Then from (6.1) and (6.2),

$$(6.3)\qquad \begin{aligned}&\nabla_\xi X - \eta(\nabla_\xi X)\xi - g(A\nabla_\xi X,\xi)A\xi - g(A\nabla_\xi X,N)AN\\ &\quad + \alpha S\nabla_\xi X - \alpha^2\eta(\nabla_\xi X)\xi\\ &= \phi SX - g(A\phi SX,\xi)A\xi - g(A\phi SX,N)AN + \alpha S\phi SX.\end{aligned}$$

Then we can prove the following lemma for a Hopf hypersurface in Q^m with $\mathfrak{A}$-isotropic unit normal.

Lemma 6.1 *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$, with $\mathfrak{A}$-isotropic unit normal. Then we have*

$$SAN = 0 \quad \textit{and} \quad SA\xi = 0.$$

Proof Let us denote by $\mathcal{C} - \mathcal{Q} = \mathrm{Span}[A\xi, AN]$. Since N is isotropic, $g(AN,N) = 0$ and $g(A\xi,\xi) = 0$. Differentiating $g(AN,N) = 0$ and using $(\overline{\nabla}_X A)Y = q(X)JAY$ and the equation of Weingarten, we know that

$$\begin{aligned}0 &= g(\overline{\nabla}_X(AN),N) + g(AN,\overline{\nabla}_X N)\\ &= g(q(X)JAN - ASX,N) - g(AN,SX)\\ &= -2g(ASX,N).\end{aligned}$$

Then $SAN = 0$. Moreover, by differentiating $g(A\xi,N) = 0$ and using $g(AN,N) = 0$, we have

$$\begin{aligned}0 &= g(\overline{\nabla}_X(A\xi),N) + g(A\xi,\overline{\nabla}_X N)\\ &= g(q(X)JA\xi + A(\phi SX + g(SX,\xi)N),N) - g(SA\xi,X)\\ &= -2g(SA\xi,X)\end{aligned}$$

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for any $X \in T_zM$, $z \in M$, where in the third equality we have used $\phi AN = JAN = -AJN = A\xi$. Then it follows that $SA\xi = 0$, which completes the proof of our assertion. ■

By Lemma 3.3, it is known that the Reeb function α is constant. So we can consider two cases: $\alpha = 0$ and $\alpha \neq 0$.

Case 1: $\alpha = 0$ By Lemma 6.1, for any $X \in \mathcal{C}$ the formula (6.3) with $\alpha = 0$ gives

$$\phi SX = \nabla_\xi X - g(A\nabla_\xi X, \xi)A\xi - g(A\nabla_\xi X, N)AN. \tag{6.4}$$

Now let us consider that $SX = \lambda X$, $X \in \mathcal{Q}$. Then it follows that $S\phi X = \mu\phi X$, $\mu = \frac{1}{\lambda}$. Then (6.4) gives that

$$\lambda\phi X = \nabla_\xi X - g(A\nabla_\xi X, \xi)A\xi - g(A\nabla_\xi X, N)AN. \tag{6.5}$$

Moreover, if we consider ϕX such that $S\phi X = \mu\phi X$ in (6.4), it follows that

$$\begin{aligned} -\mu X &= \nabla_\xi(\phi X) - g(A\nabla_\xi(\phi X), \xi)A\xi - g(A\nabla_\xi(\phi X), N)AN \\ &= \phi\nabla_\xi X - g(A\phi\nabla_\xi X, \xi)A\xi - g(A\phi\nabla_\xi X, N)AN. \end{aligned} \tag{6.6}$$

Then, by transforming the structure tensor ϕ to (6.6), we have

$$\begin{aligned} \mu\phi X &= \nabla_\xi X - g(A\phi\nabla_\xi X, \xi)\phi A\xi - g(A\phi\nabla_\xi X, N)\phi AN \\ &= \nabla_\xi X - g(\nabla_\xi X, AN)AN - g(\nabla_\xi X, A\xi)A\xi. \end{aligned}$$

Comparing this with (6.5), it follows that $(\lambda - \mu)\phi X = 0$. Then $\lambda = \mu = \frac{1}{\lambda}$, so $\lambda = \pm 1$. In such a case, the expression of the shape operator becomes

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & -1 \end{bmatrix},$$

where the multiplicities of the principal curvatures 1 and −1 are respectively $2p$ and $2q$, where $p + q = m - 2$. So, by virtue of a theorem due to Berndt and Suh [1] and Suh [16], M is locally congruent to a tube of radius $r = \frac{\pi}{4}$ over a totally geodesic $\mathbb{C}P^k$ in Q^{2k}.

Case 2: $\alpha \neq 0$. In this case, also by Lemma 6.1, we have

$$S\xi = \alpha\xi, \quad SAN = 0, \quad \text{and} \quad SA\xi = 0.$$

Then for $X \in \mathcal{Q}$, (6.3) gives

$$\begin{aligned} \nabla_\xi X - \eta(\nabla_\xi X)\xi - g(A\nabla_\xi X, \xi)A\xi - g(A\nabla_\xi X, N)AN& \\ + \alpha S\nabla_\xi X - \alpha^2\eta(\nabla_\xi X)\xi& \\ = \phi SX + \alpha S\phi SX.& \end{aligned} \tag{6.7}$$

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Then for $SX = \lambda X$, $X \in \mathcal{Q}$ such that $S\phi X = \mu\phi X$, (6.7) gives

$$\lambda\phi X + \alpha\lambda\mu\phi X = \nabla_\xi X - g(A\nabla_\xi X, \xi)A\xi - g(A\nabla_\xi X, N)AN + \alpha S\nabla_\xi X. \tag{6.8}$$

On the other hand, if we change X to ϕX in (6.7), then it follows that

$$\phi\nabla_\xi X - g(A\phi\nabla_\xi X, \xi)A\xi - g(A\phi\nabla_\xi X, N) + \alpha S\phi\nabla_\xi X = \phi S\phi X + \alpha S\phi S\phi X.$$

This can be arranged as

$$\phi\nabla_\xi X - g(\nabla_\xi X, AN)A\xi + g(\nabla_\xi X, A\xi)AN - \alpha S\phi\nabla_\xi X = -\mu X - \alpha\lambda\mu X.$$

By applying the structure tensor ϕ to both sides, we have

$$\nabla_\xi X - g(\nabla_\xi X, AN)AN - g(\nabla_\xi X, A\xi)A\xi + \alpha\phi S\phi\nabla_\xi X = \mu\phi X + \alpha\lambda\mu\phi X. \tag{6.9}$$

From(6.8) and (6.9) it follows that

$$(\lambda - \mu)\phi X = \alpha S\nabla_\xi X - \alpha\phi S\phi\nabla_\xi X.$$

Taking the inner product with ϕX, we have

$$\lambda - \mu = (\lambda - \mu)g(\phi X, \phi X) = \alpha(\mu + \lambda)g(\nabla_\xi X, \phi X). \tag{6.10}$$

Here we want to get the information about the formula $g(\nabla_\xi X, \phi X)$. In order to do this, we consider $SX = \lambda X$, $X \in \mathcal{Q}$ in (6.3) such that $S\phi X = \mu\phi X$, $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$. Then it follows that

$$\lambda\phi X + \alpha\lambda\mu\phi X = \nabla_\xi X - g(A\nabla_\xi X, \xi)A\xi - g(A\nabla_\xi X, N)AN + \alpha S\nabla_\xi X.$$

Applying ϕX to both sides of (6.4), $X \in \mathcal{Q}$, and using Lemma 6.1, it follows that

$$\lambda(1 + \alpha\mu) = g(\nabla_\xi X, \phi X) + \alpha\mu g(\nabla_\xi X, \phi X) = (1 + \alpha\mu)g(\nabla_\xi X, \phi X).$$

So we consider two subcases.

Subcase 2.1. $1 + \alpha\mu \neq 0$. In this subcase, $g(\nabla_\xi X, \phi X) = \lambda$. So from (6.10) it follows that

$$\lambda - \mu = \alpha(\lambda + \mu)g(\nabla_\xi X, \phi X) = \lambda(\lambda + \mu).$$

Then

$$\alpha\lambda^3 - \lambda^2 + 2\alpha\lambda + 1 = 0.$$

Accordingly, in such a subcase, M has at most five distinct constant principal curvatures α, 0, and the solution of the cubic equation mentioned above.

Subcase 2.2. $1 + \alpha\mu = 0$. In this subcase, $\mu = -\frac{1}{\alpha} = \frac{\alpha\lambda+2}{2\lambda-\alpha}$. Then $(\alpha^2 + 2)\lambda = -\alpha$. This means that $\lambda = -\frac{\alpha}{\alpha^2+2}$ and $\mu = -\frac{1}{\alpha}$. So M has four distinct constant principal curvatures given by α, 0, $-\frac{\alpha}{\alpha^2+2}$, and $-\frac{1}{\alpha}$ with multiplicities 1, 2, $m-2$, and $m-2$, respectively. The expression of the shape operator becomes

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -\frac{\alpha}{\alpha^2+2} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\frac{\alpha}{\alpha^2+2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{\alpha} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & -\frac{1}{\alpha} \end{bmatrix}.$$

Accordingly, in such a subcase, by Lemma 3.3, M has four distinct constant principal curvatures given by α, 0, $-\frac{\alpha}{\alpha^2+2}$, and $-\frac{1}{\alpha}$ with multiplicities 1, 2, $m-2$, and $m-2$, respectively.

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