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RESEARCH ARTICLE

Local delta invariants of weak del Pezzo surfaces with the anti-canonical degree ≥ 5

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Abstract

The delta invariant interprets the criterion for the K-(poly)stability of log terminal Fano varieties. In this paper, we determine local delta invariants for all weak del Pezzo surfaces with the anti-canonical degree ≥ 5.

1. Introduction

Throughout the paper, we work out over the complex number field $\mathbb{C}$. It is an important problem whether a log terminal Fano variety X is K-polystable. In order to interpret the criterion for the K-(poly)stability of X, the delta invariant $\delta(X)$ is introduced in [3, 10]. In fact, it is known by [3, 4, 10, 11, 12, 13] that

$$\delta(X) > 1 \iff X \text{ is K-stable } \iff X \text{ is K-polystable and } \sharp\mathrm{Aut}(X) < \infty.$$

The delta invariants of smooth del Pezzo surfaces are known in [2, §2]. On the other hand, it is hard to estimate the delta invariant for higher dimensional Fano varieties. In order to estimate the delta invariant, Abban and Zhuang introduced the local delta invariant instead of the delta invariant in [1]. They gave an important idea for reducing the estimation of local delta invariants to that of lower dimensional cases.

We recall the definition of the local delta invariant. Let X be a n-dimensional weak Fano variety with at worst log terminal singularities and let $p \in X$ be a closed point. Take a projective birational morphism $\sigma : Y \to X$ with smooth variety Y and a prime divisor E on Y. We call E a divisor over X. Let

$$A_X(E) := 1 + \mathrm{ord}_E(K_Y - \sigma^* K_X),$$

and we let

$$S(E) := \frac{1}{(-K_X)^n} \int_0^\tau \mathrm{vol}(\sigma^*(-K_X) - uE) du,$$

where τ is the pseudo effective threshold of E with respect to $-K_X$, that is,

$$\tau := \sup\{u \in \mathbb{Q}_{\geq 0} \mid \sigma^*(-K_X) - uE \text{ is big }\}.$$

The local delta invariant $\delta_p(X)$ of X at $p \in X$ is defined as follows:

$$\delta_p(X) := \inf \left\{ \frac{A_X(E)}{S(E)} \mid E \text{ is a prime divisor over } X \text{ with } p \in C_X(E) \right\}.$$

Moreover, the delta invariant $\delta(X)$ of X is given by

$$\delta(X) := \inf_{p \in X} \delta_p(X).$$

In this paper, for every weak del Pezzo surface S with the anti-canonical degree ≥ 5, we compute the local delta invariant at each $p \in S$. These results are important in the following aspects:

(1) As a directed corollary, we compute the delta invariant for a du Val del Pezzo surface $\overline{S}$ with the anti-canonical degree ≥ 5. Indeed, let $\sigma : S \to \overline{S}$ be its minimal resolution, then for each $\overline{p} \in \overline{S}$, we can immediately give

$$\delta_{\overline{p}}(\overline{S}) = \inf_{\substack{p \in S \\ \sigma(p)=\overline{p}}} \delta_p(S).$$

Therefore, we get $\delta(\overline{S}) = \delta(S)$ for any du Val del Pezzo surface $\overline{S}$ with the anti-canonical degree ≥ 5.

(2) The estimation of the local delta invariant of weak del Pezzo surfaces is useful for the K-stability of higher dimensional Fano varieties. In fact, the estimation of the local delta invariant of the quintic del Pezzo surfaces plays a crucial role in determining the K-stability of certain Fano 3-folds in [5, Lemma 24, 25]. Our results are useful for determining the K-stability of other higher dimensional Fano varieties.

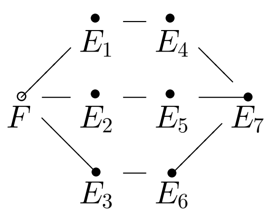
The (global) delta invariants of del Pezzo surfaces (the case $-K_S$ is ample) are exhibited in the book [2, Table 2.1]. Denisova [8] has also independently computed (global) delta invariants of all Du Val del Pezzo surfaces of anti-canonical degree ≥ 4.

Now, the main results can be stated as follows. Let S be a weak del Pezzo surface with the anti-canonical degree ≥ 5, let $E_1, \ldots, E_k$ be (-1)-curves in S, let $F_1, \ldots, F_r$ be (-2)-curves in S. We present the local delta invariants of weak del Pezzo surfaces with the anti-canonical degree ≥ 5. We note that each surface is uniquely determined by the configuration of (-1) and (-2) curves (see [9, §8]). We refer to some papers [6, 7, 9] for the basic properties of these surfaces.

At first, we present the local delta invariants of weak del Pezzo surfaces of degree 5. It is known that there exist 7 types of the configuration of negative curves.

Theorem 1.1. *The local delta invariant $\delta_p(S)$ of S at $p \in S$ is as follows.*

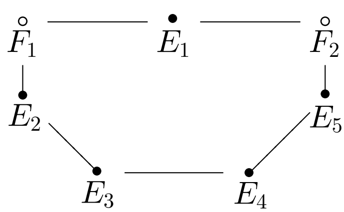
(1) If the configuration of negative curves on S is



then the local delta invariant $\delta_p(S)$ of S at $p \in S$ is as follows.

$p \in S$	F	$E_i \setminus F \ (i=1,2,3)$	$E_{i+3} \setminus E_i \ (i=1,2,3)$	E_7	$S \setminus (F \cup \bigcup_{i=1}^{7} E_i)$
$\delta_p(S)$	$\frac{15}{17}$	1	$\frac{15}{13}$	$\frac{15}{13}$	$\frac{4}{3}$

(2) If the configuration of negative curves on S is



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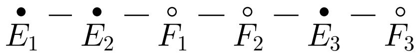
then the local delta invariant $\delta_p(S)$ of S at $p \in S$ is as follows.

$p \in S$	E_1	$F_1 \setminus E_1$, $F_2 \setminus E_1$	$E_2 \setminus F_1$, $E_5 \setminus F_2$	$E_3 \setminus E_2$, $E_4 \setminus E_5$	$S \setminus \bigcup_{i,j} (E_i \cup F_j)$
$\delta_p(S)$	$\frac{15}{19}$	$\frac{15}{17}$	1	$\frac{15}{13}$	$\frac{4}{3}$

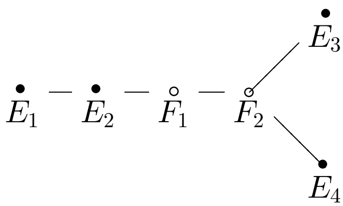
(3) If the configuration of negative curves on S is



then the local delta invariant $\delta_p(S)$ of S at $p \in S$ is as follows.

$p \in S$	$E_1 \setminus E_2$	$E_2 \setminus F_1$, $F_3 \setminus E_3$	$F_1 \setminus F_2$	$F_2 \setminus E_3$	E_3	$S \setminus \bigcup_{i,j} (E_i \cup F_j)$
$\delta_p(S)$	$\frac{15}{13}$	$\frac{15}{17}$	$\frac{15}{19}$	$\frac{5}{7}$	$\frac{15}{23}$	$\frac{30}{23}$

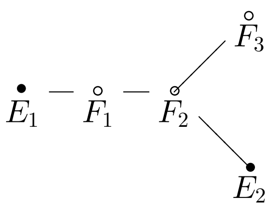
(4) If the configuration of negative curves on S is



then the local delta invariant $\delta_p(S)$ of S at $p \in S$ is as follows.

$p \in S$	$E_1 \setminus E_2$	$E_2 \setminus F_1$	$F_1 \setminus F_2$	F_2	$E_i \setminus F_2$ $(i = 3, 4)$	$S \setminus \bigcup_{i,j} (E_i \cup F_j)$
$\delta_p(S)$	$\frac{15}{13}$	$\frac{15}{17}$	$\frac{15}{19}$	$\frac{5}{7}$	$\frac{30}{31}$	$\frac{30}{23}$

(5) If the configuration of negative curves on S is



then the local delta invariant $\delta_p(S)$ of S at $p \in S$ is as follows.

$p \in S$	$E_1 \setminus F_1$	$F_1 \setminus F_2$	F_2	$F_3 \setminus F_2$	$E_2 \setminus F_2$	$S \setminus \bigcup_{i,j} (E_i \cup F_j)$
$\delta_p(S)$	$\frac{15}{16}$	$\frac{30}{43}$	$\frac{5}{9}$	$\frac{15}{19}$	$\frac{10}{13}$	$\frac{5}{4}$

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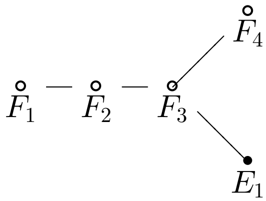
(6) *If the configuration of negative curves on S is*



then the local delta invariant $\delta_p(S)$ of S at $p \in S$ is as follows.

$p \in S$	$F_1 \setminus F_2$	$F_2 \setminus F_3$	F_3	$F_4 \setminus F_3$	$E_1 \setminus F_3$	$S \setminus (E_1 \cup \bigcup_j F_j)$
$\delta_p(S)$	$\frac{3}{4}$	$\frac{6}{11}$	$\frac{3}{7}$	$\frac{9}{13}$	$\frac{3}{5}$	$\frac{6}{5}$

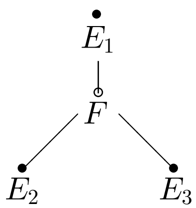
(7) *If S is a del Pezzo surface with the anti-canonical degree 5, then the local delta invariants $\delta_p(S)$ of S at $p \in S$ is as follows.*

$p \in S$	*p lies on a (-1) curve*	*p does NOT lies on any (-1) curve*
$\delta_p(S)$	$\frac{15}{13}$	$\frac{4}{3}$

We present the local delta invariants of weak del Pezzo surfaces with the anti-canonical degree 6. It is known that there exist 6 types of the configuration of negative curves ([7]).

Theorem 1.2. *The local delta invariant at $p \in S$ is as follows.*

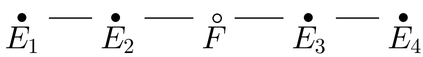
(1) *If the configuration of negative curves on S is*



then the local delta invariant $\delta_p(S)$ of S at $p \in S$ is as follows.

$p \in S$	$E_i \setminus F \ (i = 1, 2, 3)$	F	$S \setminus (\bigcup_i E_i \cup F)$
$\delta_p(S)$	$\frac{9}{10}$	$\frac{3}{4}$	$\frac{6}{5}$

(2) *If the configuration of negative curves on S is*



then the local delta invariant $\delta_p(S)$ of S at $p \in S$ is as follows.

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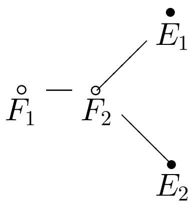
$p \in S$	$E_1 \setminus E_2$, $E_4 \setminus E_3$	E_2, E_3	$F \setminus (E_2 \cup E_3)$	$S \setminus (\bigcup_i E_i \cup F)$
$\delta_p(S)$	$\frac{9}{10}$	$\frac{9}{11}$	$\frac{9}{11}$	$\frac{9}{8}$

(3) If the configuration of negative curves on S is

then the local delta invariant $\delta_p(S)$ of S at $p \in S$ is as follows.

$p \in S$	$F_1 \setminus E_1$	E_1	$F_2 \setminus E_1$	$E_2 \setminus F_2$	$S \setminus \bigcup_{i,j} (E_i \cup F_j)$
$\delta_p(S)$	$\frac{9}{11}$	$\frac{9}{14}$	$\frac{3}{4}$	$\frac{9}{10}$	$\frac{9}{8}$

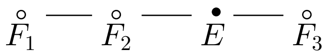
(4) If the configuration of negative curves on S is



then the local delta invariant $\delta_p(S)$ of S at $p \in S$ is as follows.

$p \in S$	$F_1 \setminus F_2$	F_2	$E_1 \setminus F_2$, $E_2 \setminus F_2$	$S \setminus \bigcup_{i,j} (E_i \cup F_j)$
$\delta_p(S)$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{4}{5}$	1

(5) If the configuration of negative curves on S is



then the local delta invariant $\delta_p(S)$ of S at $p \in S$ is as follows.

$p \in S$	$F_1 \setminus F_2$	$F_2 \setminus E$	E	$F_3 \setminus E$	$S \setminus (E \cup \bigcup_j F_j)$
$\delta_p(S)$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{1}{2}$	$\frac{3}{4}$	1

(6) If S is a del Pezzo surface with the anti-canonical degree 6*, then the local delta invariant $\delta_p(S)$ of S at $p \in S$ is as follows.*

$p \in S$	*p lies on a* (-1) *curve*	*p does NOT lies on any* (-1) *curve*
$\delta_p(S)$	1	$\frac{6}{5}$

We present the local delta invariants of weak del Pezzo surfaces with the anti-canonical degree 7. It is known that there exist 2 types of the configuration of negative curves ([7], [9, §8.4]).

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Theorem 1.3. *The local delta invariant at $p \in S$ is as follows.*

(1) *If the configuration of negative curves on S is*

$$\overset{\bullet}{E_1} \text{ --- } \overset{\bullet}{E_2} \text{ --- } \overset{\circ}{F}$$

then the local delta invariant $\delta_p(S)$ of S at $p \in S$ is as follows.

$p \in S$	$E_1 \setminus E_2$	E_2	$F \setminus E_2$	$S \setminus (E_1 \cup E_2 \cup F)$
$\delta_p(S)$	$\frac{21}{25}$	$\frac{21}{31}$	$\frac{7}{9}$	$\frac{21}{23}$

(2) *Let S be a del Pezzo surface with the anti-canonical degree 7. If the configuration of negative curves on S is*

$$\overset{\bullet}{E_1} \text{ --- } \overset{\bullet}{E_2} \text{ --- } \overset{\bullet}{E_3}$$

then the local delta invariant $\delta_p(S)$ of S at $p \in S$ is as follows.

$p \in S$	$(E_1 \cup E_3) \setminus E_2$	E_2	$S \setminus (E_1 \cup E_2 \cup E_3)$
$\delta_p(S)$	$\frac{21}{23}$	$\frac{21}{25}$	$\frac{21}{22}$

We present the local delta invariants for weak del Pezzo surfaces of the anti-canonical degree 8. Denote by $\pi : \Sigma_n \to \mathbb{P}^1$ the n-th Hirzebruch surface. Let C_0 be the section of π with $C_0^2 = -n$ and Γ the fiber of π. It is known that a weak del Pezzo surface of the anti-canonical degree 8 is either Σ_0, Σ_1 or Σ_2 ([7], [9, §8.4]).

Theorem 1.4. *Let S be a weak del Pezzo surface of the anti-canonical degree* 8.

(1) *If $S = \Sigma_2$, then for any point $p \in S$, it holds that*

$$\delta_p(S) = \frac{3}{4}.$$

(2) *If $S = \Sigma_1$, then for any point $p \in S$, it holds that*

$$\delta_p(S) = \begin{cases} \frac{6}{7} & \text{if } p \in C_0, \\ \frac{12}{13} & \text{if } p \in S \setminus C_0. \end{cases}$$

(3) *If $S = \Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$, then for any point $p \in S$, it holds that*

$$\delta_p(S) = 1.$$

Notation

In this paper, we tacitly use the following notations.

- The symbol $\sim$ means the linearly equivalence between Cartier divisors.
- We denote by H a general hyperplane of $\mathbb{P}^2$.
- We denote by $\overline{pq}$ the line on $\mathbb{P}^2$ passing through two distinct points $p, q \in \mathbb{P}^2$.
- We denote by $\mathrm{Bl}_{\{q_1,\cdots,q_k\}}\mathbb{P}^2$ the surface obtained by the composition of the blowing-ups at k distinct points $q_1, \cdots, q_k \in \mathbb{P}^2$.

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- Let $\sigma : Y \to X$ be a birational morphism between projective varieties. For a Cartier divisor D on X, we denote by $\sigma_*^{-1}D$ the proper transform of it.

2. Scheme of the proof

We fix the notations for this section. Let S be a weak del Pezzo surface with the anti-canonical degree ≥ 5, let $E_1, \ldots, E_k$ be (-1)-curves in S, let $F_1, \ldots, F_r$ be (-2)-curves in S and let p be a point in S.

We explain how to estimate $\delta_p(S)$. Fix a smooth curve C on S that passes through p. Let $\tau := \sup\{u \in \mathbb{Q}_{\geq 0} \mid -K_S - uC \text{ is big }\}$, let $P(u) + N(u)$ be the Zariski decomposition of $-K_S - uC$. By the definition of $\delta_p(S)$, we note that $\delta_p(S) \leq 1/S(C)$. So we explain how to estimate $\delta_p(S)$ from the below using these data and Abban-Zhuang Theory. Set

$$S(W^C_{\bullet,\bullet}, p) := \frac{2}{(-K_S)^2} \int_0^\tau (P(u) \cdot C) \cdot \mathrm{ord}_p N(u)|_C \, du + \frac{1}{(-K_S)^2} \int_0^\tau (P(u) \cdot C)^2 du.$$

Then it follows from [2, Theorem 1.106],

$$\delta_p(S) \geq \min\left\{\frac{1}{S(C)}, \frac{1}{S(W^C_{\bullet,\bullet}, p)}\right\}. \tag{1}$$

If p is contained in a (-1)-curve or (-2)-curve, we always choose C to be one of these curves. In many cases, these estimates actually compute $\delta_p(S)$. If p is not contained in any (-1)-curve and any (-2)-curve, then we have to consider the (ordinary) blowing up $\sigma : \widetilde{S} \to S$ at the point $p \in S$. Let Z be a exceptional curve over p, let $\widetilde{\tau} := \sup\{u \in \mathbb{Q}_{\geq 0} \mid \sigma^*(-K_S) - uZ \text{ is big }\}$ and let $\widetilde{P}(u) + \widetilde{N}(u)$ be the Zariski decomposition of $\sigma^*(-K_S) - uZ$. By the definition of $\delta_p(S)$, we note that $\delta_p(S) \leq 2/S(Z)$. For $q \in Z$, set

$$S(W^Z_{\bullet,\bullet}, q) := \frac{2}{(-K_S)^2} \int_0^{\widetilde{\tau}} (\widetilde{P}(u) \cdot Z) \cdot \mathrm{ord}_q \widetilde{N}(u)|_Z \, du + \frac{1}{(-K_S)^2} \int_0^{\widetilde{\tau}} (\widetilde{P}(u) \cdot Z)^2 du.$$

Then it follows from [2, Theorem 1.106],

$$\delta_p(S) \geq \min\left\{\frac{2}{S(E)}, \inf_{q \in Z} \frac{1}{S(W^Z_{\bullet,\bullet}, q)}\right\}. \tag{2}$$

These estimates compute $\delta_p(S)$ in every case except for one special case. In this special case, we use a (0)-curve C in S that passes through p to compute $\delta_p(S)$. Combining with what we already have, we get equality for $\delta_p(S)$.

So from now on, we just need to compute $\tau, P(u), N(u)$ and $S(W^C_{\bullet,\bullet}, p)$ for C being (-1)-curves or (-2)-curves passing through p. If p is not contained in any of these curves, we have to compute either $\widetilde{\tau}, \widetilde{P}(u), \widetilde{N}(u)$ and $S(W^Z_{\bullet,\bullet}, q)$ for exceptional curve Z or $\tau, P(u), N(u)$ and $S(W^C_{\bullet,\bullet}, p)$ for a (0)-curve C that passes through p. This will be done in the next sections.

In what follows, we will write every $\mathbb{R}$-divisor

$$D \sim_{\mathbb{R}} \sum_{i=1}^k a_i E_i + \sum_{j=1}^r b_j F_j$$

as $(a_1, \ldots, a_k, b_1, \ldots, b_r)$. To ease notation, we rewrite $(\overbrace{a, \cdots, a}^{l \text{ times}}, \overbrace{b, \cdots, b}^{m \text{ times}}, \overbrace{c, \cdots, c}^{n \text{ times}})$ as $(\overset{l}{a}, \overset{m}{b}, \overset{n}{c})$. Denote by A the intersection matrix of $E_1, \cdots, E_k, F_1, \cdots, F_r$:

$$A := \left(\begin{array}{ccc|ccc} E_1^2 & & E_kE_1 & F_1E_1 & & F_rE_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ E_1E_k & & E_k^2 & F_1E_k & & F_rE_k \\ \hline E_1F_1 & & E_kF_1 & F_1^2 & & F_rF_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ E_1F_r & & E_kF_r & F_1F_r & & F_rF_r \end{array}\right).$$

We mention that for a curve C being one of the curves $E_1, \cdots E_k, F_1, \cdots F_r$, we can immediately compute τ, $P(u)$, and $N(u)$ using $(a_1, \ldots, a_k, b_1, \ldots, b_r)$ and the matrix A, since (-1)-curves and (-2)curves generate the Kleiman–Mori cone if $K_S^2 \neq 8$.

3. The case of the anti-canonical degree 5

Let us use the assumptions and notations of Section 2. Suppose $K^2 = 5$

Proposition 3.1. *Suppose that the dual graph of the (-1)-curves and (-2)-curves on S is same as in Theorem 1.1 (1). Then the local delta invariant $\delta_p(S)$ is as follows.*

$p \in S$	F	$E_i \setminus F\ (i = 1, 2, 3)$	$E_{i+3} \setminus E_i\ (i = 1, 2, 3)$	E_7	$S \setminus (F \cup \bigcup_{i=1}^{7} E_i)$
$\delta_p(S)$	$\frac{15}{17}$	1	$\frac{15}{13}$	$\frac{15}{13}$	$\frac{4}{3}$

Proof. We recall the construction of S. Take non-colinear three points $q_0, q_1, q_3 \in \mathbb{P}^2$ and $q_2 \in \overline{q_1q_3} \setminus \{q_1, q_3\}$. Then S is obtained by $\rho : S = \mathrm{Bl}_{\{q_1,q_2,q_3,q_4\}}\mathbb{P}^2 \to \mathbb{P}^2$. Moreover, we have $F = \rho_*^{-1}\overline{q_1q_3}$, $E_1 = \rho^{-1}(q_1)$, $E_2 = \rho^{-1}(q_2)$, $E_3 = \rho^{-1}(q_3)$, $E_4 = \rho_*^{-1}(\overline{q_0q_1})$, $E_5 = \rho_*^{-1}(\overline{q_0q_2})$, $E_6 = \rho_*^{-1}(\overline{q_0q_3})$ and $E_7 = \rho^{-1}(q_0)$. We denote a divisor by $D = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, b)$. The intersection matrix of $\{E_1, E_2, E_3, E_4, E_5, E_6, E_7, F\}$ is

$$A := \left(\begin{array}{ccccccc|c} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \end{array}\right).$$

We note that $-K_S \sim (0, 0, 0, 1, 1, 1, 2, 0) = (\overset{3}{0}, \overset{3}{1}, 2, 0)$.
(1) The case $p \in F$. Set $C = F$, then we get $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$ $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$
$[0, 1]$	$(\overset{3}{0}, \overset{3}{1}, 2, -u)$	0	$5 - 2u^2$	$2u$
$[1, 2]$	$(\overset{3}{(1-u)}, \overset{3}{1}, 2, -u)$	$(u-1)(\overset{3}{1}, \overset{5}{0})$	$(4-u)(2-u)$	$3 - u$

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u	p	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 1]$	F	0
$[1, 2]$	$E_i \cap F\ (i = 1, 2, 3)$	$u - 1$
	$F \setminus \bigcup_{i=1}^{3} E_i$	0

Hence we get

$$S(F) = \frac{17}{15}, \quad S(W^F_{\bullet,\bullet}, p) = \begin{cases} \frac{11}{15} & \text{if } p \in E_i \cap F \text{ for } i = 1, 2, 3, \\ \frac{7}{15} & \text{if } p \in F \setminus \bigcup_{i=1}^{3} E_i. \end{cases}$$

Therefore, we have

$$\frac{A_S(F)}{S(F)} = \frac{15}{17} \geq \delta_p(S) \geq \min\left\{\frac{1}{S(F)}, \frac{1}{S(W^F_{\bullet,\bullet}, p)}\right\} = \frac{15}{17}$$

from (1). Thus, we have $\delta_p(S) = 15/17$ in this case.

(2) The case $p \in E_i \setminus (F \cup E_{i+3})$ for $i = 1, 2, 3$. Set $C = E_1$, then we get $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 1]$	$(-u, \overset{2}{0}, \overset{3}{1}, 2, -\frac{u}{2})$	$(\overset{7}{0}, \frac{u}{2})$	$5 - 2u - \frac{u^2}{2}$	$\frac{u+2}{2}$	0
$[1, 2]$	$(-u, \overset{2}{0}, 2 - u, \overset{2}{1}, 2, -\frac{u}{2})$	$(\overset{3}{0}, u - 1, \overset{3}{0}, \frac{u}{2})$	$\frac{1}{2}(6 - u)(2 - u)$	$\frac{4-u}{2}$	0

Hence we get $S(E_1) = 1$ and $S(W^{E_1}_{\bullet,\bullet}, p) = 19/30$. Therefore, we have

$$1 \geq \delta_p(S) \geq \min\left\{\frac{1}{S(E_1)}, \frac{1}{S(W^{E_1}_{\bullet,\bullet}, p)}\right\} = 1.$$

We can show $\delta_p(S) = 1$ for $p \in E_i \setminus (F \cup E_{i+3})(i = 2, 3)$ by the same calculation.

(3) The case $p \in E_i \setminus E_7$ $(i = 4, 5, 6)$. Set $C = E_4$, then we get $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$
$[0, 1]$	$(\overset{3}{0}, 1 - u, \overset{2}{1}, 2, 0)$	0	$5 - 2u - u^2$	$1 + u$
$[1, 2]$	$(2(1 - u), \overset{2}{0}, 1 - u, \overset{2}{1}, 3 - u, 1 - u)$	$(u - 1)(2, \overset{5}{0}, \overset{2}{1})$	$2(2 - u)^2$	$4 - 2u$

u	p	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 1]$	$E_4 \setminus E_7$	0
$[1, 2]$	$E_1 \cap E_4$	$2(u - 1)$
	$E_4 \setminus (E_1 \cup E_7)$	0

Hence we get

$$S(E_4) = \frac{13}{15}, \quad S(W^{E_4}_{\bullet,\bullet}, p) = \begin{cases} 1 & \text{if } p \in (E_1 \cap E_4), \\ \frac{11}{15} & \text{if } p \in E_4 \setminus (E_1 \cup E_7). \end{cases}$$

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Therefore, we have $\delta_p(S) = 15/13$ for $p \in E_4 \setminus (E_1 \cup E_7)$. If $\{p\} = E_1 \cap E_4$, we have $1 = S(E_1) \geq \delta_p(S)$ by the calculation in (2). Thus, we have

$$\delta_p(S) = \begin{cases} 1 & \text{if } p \in (E_1 \cap E_4), \\ \\ \frac{15}{13} & \text{if } p \in E_4 \setminus (E_1 \cup E_7). \end{cases}$$

We can show

$$\delta_p(S) = \begin{cases} 1 & \text{if } p \in (E_{i-3} \cap E_i), \\ \\ \frac{15}{13} & \text{if } p \in E_i \setminus (E_{i-3} \cup E_7), \end{cases}$$

for $i = 5, 6$ by the same calculation.

(4) The case $p \in E_7$. Set $C = E_7$, then we get $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$
$[0, 1]$	$(\overset{3}{0}, \overset{3}{1}, 2-u, 0)$	0	$5 - 2u - u^2$	$1 + u$
$[1, 2]$	$(2-u)(\overset{3}{0}, \overset{4}{1}, 0)$	$(u-1)(\overset{3}{0}, \overset{3}{1}, \overset{2}{0})$	$2(2-u)^2$	$4 - 2u$

u	p	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 1]$	E_7	0
$[1, 2]$	$E_i \cap E_7$ $(i = 4, 5, 6)$	$u - 1$
	$E_7 \setminus \bigcup_{i=4}^{6} E_i$	0

Hence we get

$$S(E_7) = \frac{13}{15}, \quad S(W^{E_7}_{\bullet,\bullet}, p) = \begin{cases} \frac{13}{15} & \text{if } p \in E_i \cap E_7 \text{ for } i = 4, 5, 6, \\ \\ \frac{11}{15} & \text{if } p \in E_7 \setminus \bigcup_{i=4}^{6} E_i. \end{cases}$$

Therefore, we have

$$\frac{15}{13} \geq \delta_p(S) \geq \min\left\{\frac{1}{S(E_7)}, \frac{1}{S(W^{E_7}_{\bullet,\bullet}, p)}\right\} = \frac{15}{13}.$$

from (1). Thus, we have $\delta_p(S) = 15/13$ in this case.

(5) The case $p \in S \setminus \left(F \cup \bigcup_{i=1}^{7} E_i\right)$. Consider a blowing up $\sigma\colon \widetilde{S} \to S$ at p. Let $\widetilde{F}$ and $\widetilde{E}_i$ be the proper transform of F and E_i, respectively. Put $G_i := (\rho\sigma)_*^{-1}\overline{\rho(p)q_i}$ for $i = 0, 1, 2, 3$. Then we have $\sigma^*(-K_S) - uZ \sim G_0 + G_2 + \widetilde{F} + \widetilde{E}_2 + (2-u)Z$ and $\widetilde{\tau} = 5/2$. The values $\widetilde{P}(u)$, $\widetilde{N}(u)$, $\widetilde{P}(u)^2$, $\widetilde{P}(u) \cdot Z$ and $\mathrm{ord}_q(\widetilde{N}(u)|_Z)$ are given by the following tables:

u	$\widetilde{P}(u)\&\widetilde{N}(u)$	$\widetilde{E}_2$	$\widetilde{F}$	G_0	G_1	G_2	G_3	Z
$[0, 2]$	$\widetilde{P}(u)$	1	1	1	0	1	0	$2-u$
	$\widetilde{N}(u)$	0	0	0	0	0	0	0
$[2, \frac{5}{2}]$	$\widetilde{P}(u)$	1	$3-u$	$5-2u$	$2-u$	$3-u$	$2-u$	$2-u$
	$\widetilde{N}(u)$	0	$u-2$	$2u-4$	$u-2$	$u-2$	$u-2$	0

u	$\widetilde{P}(u)^2$	$\widetilde{P}(u)\cdot Z$
$[0, 2]$	$5-u^2$	u
$[2, \frac{5}{2}]$	$(5-2u)^2$	$2(5-2u)$

u	p	$\mathrm{ord}_p(N(u)\vert_C)$
$[0, 2]$	Z	0
$[2, \frac{5}{2}]$	$Z\cap G_0$	$2(u-2)$
	$Z\cap G_i (i=1, 2, 3)$	$u-2$
	$Z\setminus\bigcup_{i=0}^3 G_i$	0

Therefore, we get

$$S(Z)=\frac{3}{2},\quad S(W^Z_{\bullet,\bullet}, q)=\begin{cases}\frac{11}{15} & \text{if } q\in Z\cap G_0,\\ \frac{7}{10} & \text{if } q\in Z\cap G_i \text{ for } i=1,2,3,\\ \frac{2}{3} & \text{if } q\in Z\setminus\bigcup_{i=0}^3 G_i.\end{cases}$$

Hence, we have

$$\frac{4}{3}\geq\delta_p(S)\geq\min\left\{\frac{2}{S(Z)}, \inf_{q\in Z}\frac{1}{S(W^Z_{\bullet,\bullet}, q)}\right\}=\frac{4}{3}$$

from (2). Thus, we have $\delta_p(S)=4/3$ in this case. □

Proposition 3.2. *Suppose that the dual graph of the* (-1)*-curves and* (-2)*-curves on* S *is same as in Theorem 1.1 (2). Then the local delta invariant* $\delta_p(S)$ *is as follows.*

$p\in S$	E_1	$F_1\setminus E_1$, $F_2\setminus E_1$	$E_2\setminus F_1$, $E_5\setminus F_2$	$E_3\setminus E_2$, $E_4\setminus E_5$	$S\setminus\bigcup_{i,j}(E_i\cup F_j)$
$\delta_p(S)$	$\frac{15}{19}$	$\frac{15}{17}$	1	$\frac{15}{13}$	$\frac{4}{3}$

Proof. We can assume that we get S from $\mathbb{P}^2$ as follows.

(1) Let $\rho_1: S_1=\mathrm{Bl}_{\{q_1,q_2,q_3\}}\mathbb{P}^2\to\mathbb{P}^2$ be a blowing-up at non-colinear points q_1, q_2, q_3.
(2) Let q_4 be a point at which $\rho_1^{-1}(q_4)$ and $(\rho_1)_*^{-1}\overline{q_1q_2}$ meet. Take a blowing-up $\rho_2: S_2\to S_1$ at q_4. Then $S=S_2$. Put $\rho=\rho_1\rho_2: S\to\mathbb{P}^2$.

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Moreover, we have $E_1 = \rho_2^{-1}(q_4)$, $E_2 = \rho^{-1}(q_2)$, $E_3 = \rho_*^{-1}(\overline{q_2q_3})$, $E_4 = \rho^{-1}(q_3)$, $E_5 = \rho_*^{-1}(\overline{q_3q_1})$, $F_1 = \rho_*^{-1}(\overline{q_1q_2})$ and $F_2 = (\rho_2)_*^{-1}(\rho_1^{-1}(q_1))$. We denote a divisor $D = \sum_{i=1}^{5} a_iE_i + \sum_{j=1}^{2} b_jF_j \in \mathrm{Div}(S)$ $(a_i, b \in \mathbb{Z})$ by $D = (a_1, a_2, a_3, a_4, a_5, b_1, b_2)$. The intersection matrix of $\{E_1, E_2, E_3, E_4, E_5, F_1, F_2\}$ is

$$A := \left(\begin{array}{ccccc|cc} -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & -2 \end{array}\right).$$

We note that $-K_S \sim \sum_{i=1}^{5} E_i + \sum_{i=j}^{2} F_j = (1, 1, 1, 1, 1, 1, 1) = (\overset{7}{1})$.

(1) The case $p \in E_1$. Set $C = E_1$, then we get $\tau = 3$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$
$[0, 2]$	$(1-u, \overset{4}{1}, \overset{2}{(1-\frac{u}{2})})$	$(\overset{5}{0}, \overset{2}{\frac{u}{2}})$	$5-2u$	1
$[2, 3]$	$(1-u, 3-u, \overset{2}{1}, 3-u, \overset{2}{(2-u)})$	$(0, u-2, \overset{2}{0}, u-2, \overset{2}{(u-1)})$	$(3-u)^2$	$3-u$

u	p	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 2]$	$E_1 \cap F_j (j = 1, 2)$	$\frac{u}{2}$
	$E_1 \setminus \bigcup_{j=1}^{2} F_j$	0
$[2, 3]$	$E_1 \cap F_j (j = 1, 2)$	$u-1$
	$E_1 \setminus \bigcup_{j=1}^{2} F_j$	0

Therefore, we get

$$S(E_1) = \frac{19}{15}, \quad S(W^{E_1}_{\bullet,\bullet}, p) = \begin{cases} \frac{17}{15} & \text{if } p \in E_1 \cap F_j \text{ for } j = 1, 2, \\ \frac{7}{15} & \text{if } p \in E_1 \setminus \bigcup_{j=1}^{2} F_j. \end{cases}$$

Hence we have

$$\frac{15}{19} \geq \delta_p(S) \geq \min\left\{\frac{1}{S(E_1)}, \frac{1}{S(W^{E_1}_{\bullet,\bullet}, p)}\right\} = \frac{15}{19}$$

from (1). Thus, we have $\delta_p(S) = 15/19$ in this case.

(2) The case $p \in F_1 \setminus E_1$. Set $C = F_1$, then we get $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$
$[0, 1]$	$(\overset{5}{1}, 1-u, 1)$	0	$5-2u^2$	$2u$
$[1, 2]$	$(3-2u, 2-u, \overset{3}{1}, 1-u, 2-u)$	$(u-1)(2, 1, \overset{4}{0}, 1)$	$(2-u)(4-u)$	$3-u$

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u	p	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 1]$	$F_1 \setminus E_1$	0
$[1, 2]$	$F_1 \cap E_2$	$u-1$
	$F_1 \setminus (E_1 \cup E_2)$	0

Therefore, we get

$$S(F_1) = \frac{17}{15}, \quad S(W^{F_1}_{\bullet,\bullet}, p) = \begin{cases} 1 & \text{if } p \in F_1 \cap E_2, \\ \frac{11}{15} & \text{if } p \in F_1 \setminus (E_1 \cup E_2). \end{cases}$$

Hence we have

$$\frac{15}{17} \geq \delta_p(S) \geq \min\left\{\frac{1}{S(F_1)}, \frac{1}{S(W^{F_1}_{\bullet,\bullet}, p)}\right\} = \frac{15}{17}$$

from (1). Thus, we have $\delta_p(S) = 15/17$ in this case.
(3) The case $p \in E_2 \setminus F_1$. Set $C = E_2$, then we get $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$
$[0, 1]$	$(1, 1-u, \overset{3}{1}, 1-\frac{u}{2}, 1)$	$(\overset{5}{0}, \frac{u}{2}, 0)$	$5-2u-\frac{u^2}{2}$	$\frac{u+2}{2}$
$[1, 2]$	$(1, 1-u, 2-u, \overset{2}{1}, 1-\frac{u}{2}, 1)$	$(0, 0, u-1, \overset{2}{0}, \frac{u}{2}, 0)$	$\frac{1}{2}(6-u)(2-u)$	$\frac{4-u}{2}$

u	p	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 1]$	$E_2 \setminus F_1$	0
$[1, 2]$	$E_2 \cap E_3$	$u-1$
	$E_2 \setminus (F_1 \cup E_3)$	0

Therefore, we get

$$S(E_2) = 1, \quad S(W^{E_2}_{\bullet,\bullet}, p) = \begin{cases} \frac{13}{15} & \text{if } p \in E_2 \cap E_3, \\ \frac{19}{30} & \text{if } p \in E_2 \setminus (F_1 \cup E_3). \end{cases}$$

Hence we have

$$1 \geq \delta_p(S) \geq \min\left\{\frac{1}{S(E_2)}, \frac{1}{S(W^{E_2}_{\bullet,\bullet}, p)}\right\} = 1$$

from (1). Thus, we have $\delta_p(S) = 1$ in this case.
(4) The case $p \in E_3 \setminus E_2$. Set $C = E_3$, then we get $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$
$[0, 1]$	$(\overset{2}{1}, 1-u, \overset{4}{1})$	0	$5-2u-u^2$	$1+u$
$[1, 2]$	$(1, 3-2u, 1-u, 2-u, 1, 2-u, 1)$	$(u-1)(0, 2, 0, 1, 0, 1, 0)$	$2(2-u)^2$	$4-2u$

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u	p	$\mathrm{ord}_p(N(u)\|_C)$
$[0,1]$	$E_3 \setminus E_2$	0
$[1,2]$	$E_3 \cap E_4$	$2(u-1)$
	$E_3 \setminus (E_2 \cup E_4)$	0

Therefore, we get

$$S(E_3) = \frac{13}{15}, \quad S(W^{E_3}_{\bullet,\bullet}, p) = \begin{cases} \frac{13}{15} & \text{if } p \in E_3 \cap E_4, \\ \frac{11}{15} & \text{if } p \in E_3 \setminus (E_2 \cup E_4). \end{cases}$$

Hence we have

$$\frac{15}{13} \geq \delta_p(S) \geq \min\left\{\frac{1}{S(E_3)}, \frac{1}{S(W^{E_3}_{\bullet,\bullet}, p)}\right\} = \frac{15}{13}$$

from (1). Thus, we have $\delta_p(S) = 15/13$ in this case.

(5) The case $p \in S \setminus \left(\bigcup_{i,j}(E_i \cup F_j)\right)$. Consider a blowing up $\sigma : \widetilde{S} \to S$ at p. Let $\widetilde{E}_i$ and $\widetilde{F}_j$ be the proper transform of E_i and F_j, respectively. Put $G_i := (\rho\sigma)^{-1}_* \overline{\rho(p)q_i}$ for $i = 1, 2, 3$. Then we have $\sigma^*(-K_S) - uZ \sim \widetilde{F}_1 + \widetilde{E}_2 + G_2 + G_3 + (2-u)Z$ and $\widetilde{\tau} = 5/2$. The values $\widetilde{P}(u), \widetilde{N}(u), \widetilde{P}(u)^2, \widetilde{P}(u) \cdot Z$ and $\mathrm{ord}_q(\widetilde{N}(u)|_Z)$ are given by the following tables:

u	$\widetilde{P}(u)$&$\widetilde{N}(u)$	$\widetilde{E}_2$	$\widetilde{F}_1$	$\widetilde{F}_2$	G_1	G_2	G_3	Z
$[0,2]$	$\widetilde{P}(u)$	1	1	0	0	1	1	$2-u$
	$\widetilde{N}(u)$	0	0	0	0	0	0	0
$[2,\frac{5}{2}]$	$\widetilde{P}(u)$	1	$3-u$	$2-u$	$2(2-u)$	$3-u$	$5-2u$	$2-u$
	$\widetilde{N}(u)$	0	$u-2$	$u-2$	$u-2$	$2(u-2)$	$2(u-2)$	0

u	$\widetilde{P}(u)^2$	$\widetilde{P}(u) \cdot Z$
$[0,2]$	$5-u^2$	u
$[2,\frac{5}{2}]$	$(5-2u)^2$	$2(5-2u)$

u	q	$\mathrm{ord}_q(\widetilde{N}(u)\|_Z)$
$[0,2]$	Z	0
$[2,\frac{5}{2}]$	$Z \cap G_1$	$u-2$
	$Z \cap (G_2 \cup G_3)$	$2(u-2)$
	$Z \setminus (G_1 \cup G_2 \cup G_3)$	0

Therefore, we get

$$S(Z) = \frac{3}{2}, \quad S(W^{Z}_{\bullet,\bullet}, p) = \begin{cases} \frac{11}{15} & \text{if } q \in Z \cap G_1, \\ \frac{7}{10} & \text{if } q \in Z \cap (G_2 \cup G_3), \\ \frac{2}{3} & \text{if } q \in Z \setminus (G_1 \cup G_2 \cup G_3). \end{cases}$$

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Hence we have

$$\frac{4}{3} \geq \delta_p(S) \geq \min\left\{\frac{2}{S(Z)}, \inf_{q\in Z}\frac{1}{S(W^{Z}_{\bullet,\bullet}, q)}\right\} = \frac{4}{3}$$

from (2). Thus, we have $\delta_p(S) = 4/3$ in this case. □

Proposition 3.3. *Suppose that the dual graph of the* (-1)*-curves and* (-2)*-curves on* S *is same as in Theorem 1.1 (3). Then the local delta invariant* $\delta_p(S)$ *is as follows.*

$p \in S$	$E_1 \setminus E_2$	$E_2 \setminus F_1$, $F_3 \setminus E_3$	$F_1 \setminus F_2$	$F_2 \setminus E_3$	E_3	$S \setminus \bigcup_{i,j}(E_i \cup F_j)$
$\delta_p(S)$	$\frac{15}{13}$	$\frac{15}{17}$	$\frac{15}{19}$	$\frac{5}{7}$	$\frac{15}{23}$	$\frac{30}{23}$

Proof. We can assume that we get S from $\mathbb{P}^2$ as follows.

(1) Take two distinct points $q_1, q_4 \in \mathbb{P}^2$ and a line $l(\neq \overline{q_1q_4})$ passing through q_1. Let $\rho_1: S_1 = \mathrm{Bl}_{\{q_1,q_4\}}\mathbb{P}^2 \to \mathbb{P}^2$ be a blowing-up at points q_1, q_4, let $l_1 = (\rho_1)^{-1}_* l$ and let q_2 be a point at which l_1 and $\rho_1^{-1}(q_1)$ meet.

(2) Let $\rho_2: S_2 \to S_1$ be a blowing-up at q_2, let $l_2 = (\rho_2)^{-1}_* l_1$ and let q_3 be a point at which l_2 and $\rho_2^{-1}(q_2)$ meet.

(3) Let $\rho_3: S_3 \to S_2$ be a blowing-up at q_3. Then $S = S_3$. Put $\rho = \rho_1\rho_2\rho_3$.

Moreover, we have $E_1 = \rho^{-1}(q_4)$, $E_2 = \rho_*^{-1}(\overline{q_1q_4})$, $F_1 = (\rho_2\rho_3)^{-1}_*(\rho_1^{-1}(q_1))$, $F_2 = (\rho_3)^{-1}_*(\rho^{-1}(q_2))$, $E_3 = \rho_3^{-1}(q_3)$, $F_3 = \rho_*^{-1} l$. We denote $D = \sum_{i=1}^{3} a_iE_i + \sum_{j=1}^{3} b_jF_j \in \mathrm{Div}(S)$ $(a_i, b \in \mathbb{Z})$ by $D = (a_1, a_2, a_3, b_1, b_2, b_3)$. The intersection matrix of $\{E_1, E_2, E_3, F_1, F_2, F_3\}$ is

$$A := \left(\begin{array}{ccc|ccc} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 \\ \hline 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 \end{array}\right).$$

We note that $-K_S \sim 2E_1 + 3E_2 + 2F_1 + F_2 = (2, 3, 0, 2, 1, 0)$.

(1) The case $p \in E_1 \setminus E_2$. Set $C = E_1$, then we get $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u)\cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 1]$	$(2-u, 3, 0, 2, 1, 0)$	0	$5-2u-u^2$	$1+u$	0
$[1, 2]$	$(2-u)(1, 3, 0, 2, 1, 0)$	$(u-1)(0, 3, 0, 2, 1, 0)$	$2(2-u)^2$	$4-2u$	0

So we get $S(E_1) = 13/15$ and $S(W^{E_1}_{\bullet,\bullet}, p) = 11/15$. Thus, $\delta_p(S) = 15/13$ from (1).

(2) The case $p \in E_2 \setminus F_1$. Set $C = E_2$, then we get $\tau = 3$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u)\cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$
$[0, 1]$	$(2, 3-u, 0, 2-\frac{2}{3}u, 1-\frac{u}{3}, 0)$	$(0, 0, 0, \frac{2}{3}u, \frac{u}{3}, 0)$	$5-2u-\frac{u^2}{3}$	$\frac{3+u}{3}$
$[1, 3]$	$(3-u)(1, 1, 0, \frac{2}{3}, \frac{1}{3}, 0)$	$(u-1, 0, 0, \frac{2}{3}u, \frac{u}{3}, 0)$	$\frac{2}{3}(3-u)^2$	$2-\frac{2}{3}u$

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u	p	$\mathrm{ord}_p(N(u)\|_C)$
$[0,1]$	$E_2 \setminus F_1$	0
$[1,3]$	$E_2 \cap E_1$	$u-1$
	$E_2 \setminus (E_1 \cup F_1)$	0

Therefore, we get

$$S(E_2) = \frac{17}{15}, \quad S(W^{E_2}_{\bullet,\bullet}, p) = \begin{cases} \frac{13}{15} & \text{if } p \in E_2 \cap E_1, \\ \frac{23}{45} & \text{if } p \in E_2 \setminus (E_1 \cup F_1). \end{cases}$$

Thus, we have $\delta_p(S) = 15/17$ from (1).

(3) The case $p \in F_1 \setminus F_2$. Set $C = F_1$, then we get $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$
$[0,1]$	$(2,3,0,2-u,1-\frac{u}{2},0)$	$(\overset{4}{0},\frac{u}{2},0)$	$5-\frac{3}{2}u^2$	$\frac{3u}{2}$
$[1,2]$	$(2,4-u,0,2-u,1-\frac{u}{2},0)$	$(0,u-1,0,0,\frac{u}{2},0)$	$\frac{1}{2}(2-u)(6+u)$	$\frac{2+u}{2}$

u	p	$\mathrm{ord}_p(N(u)\|_C)$
$[0,1]$	$F_1 \setminus F_2$	0
$[1,2]$	$F_1 \cap E_2$	$u-1$
	$F_1 \setminus (E_2 \cup F_2)$	0

Therefore, we get

$$S(F_1) = \frac{19}{15}, \quad S(W^{F_1}_{\bullet,\bullet}, p) = \begin{cases} \frac{17}{15} & \text{if } p \in F_1 \cap E_2, \\ \frac{23}{30} & \text{if } p \in F_1 \setminus (E_2 \cup F_2). \end{cases}$$

Thus, we have $\delta_p(S) = 15/19$ from (1).

(4) The case $p \in F_2 \setminus E_3$. Set $C = F_2$, then we get $\tau = 3$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$
$[0,1]$	$(2,3,0,2-\frac{u}{2},1-u,0)$	$(\overset{3}{0},\frac{u}{2},\overset{2}{0})$
$[1,2]$	$(2,3,2(1-u),2-\frac{u}{2},(1\overset{2}{-}u))$	$(\overset{2}{0},2(u-1),\frac{u}{2},0,u-1)$
$[2,3]$	$(2,5-u,2(1-u),3-u,(1\overset{2}{-}u))$	$(0,u-2,2(u-1),u-1,0,u-1)$

u	$P(u)^2$	$P(u) \cdot C$
$[0,1]$	$5-\frac{3}{2}u^2$	$\frac{3u}{2}$
$[1,2]$	$\frac{1}{2}(u^2-8u+14)$	$2-\frac{u}{2}$
$[2,3]$	$(3-u)^2$	$3-u$

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u	p	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 1]$	$F_2 \cap F_1$	$\frac{u}{2}$
	$F_2 \setminus (E_3 \cup F_1)$	0
$[1, 2]$	$F_2 \cap F_1$	$\frac{u}{2}$
	$F_2 \setminus (E_3 \cup F_1)$	0
$[2, 3]$	$F_2 \cap F_1$	$u-1$
	$F_2 \setminus (E_3 \cup F_1)$	0

Therefore, we get

$$S(F_2) = \frac{7}{5}, \quad S(W^{F_2}_{\bullet,\bullet}, p) = \begin{cases} \frac{23}{30} & \text{if } p \in F_2 \cap F_1, \\ \frac{8}{15} & \text{if } p \in F_2 \setminus (E_3 \cup F_1). \end{cases}$$

Thus, we have $\delta_p(S) = 5/7$ from (1).

(5) The case $p \in E_3$. Set $C = E_3$, then we get $\tau = 4$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$
$[0, 3]$	$(2, 3, -u, 2-\frac{u}{3}, 1-\frac{2u}{3}, -\frac{u}{2})$	$(\overset{3}{0}, \frac{u}{3}, \frac{2u}{3}, \frac{u}{2})$	$5-2u+\frac{u^2}{6}$	$\frac{6-u}{6}$
$[3, 4]$	$(2, 6-u, -u, 4-u, 2-u, -\frac{u}{2})$	$(0, u-3, 0, u-2, u-1, \frac{u}{2})$	$\frac{1}{2}(4-u)^2$	$\frac{4-u}{2}$

u	p	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 3]$	$E_3 \cap F_2$	$\frac{2u}{3}$
	$E_3 \cap F_3$	$\frac{u}{2}$
	$E_3 \setminus (F_2 \cup F_3)$	0
$[3, 4]$	$E_3 \cap F_2$	$u-1$
	$E_3 \cap F_3$	$\frac{u}{2}$
	$E_3 \setminus (F_2 \cup F_3)$	0

Therefore, we get

$$S(E_3) = \frac{23}{15}, \quad S(W^{E_3}_{\bullet,\bullet}, p) = \begin{cases} \frac{7}{5} & \text{if } p \in E_3 \cap F_2, \\ \frac{17}{15} & \text{if } p \in E_3 \cap F_3, \\ \frac{11}{30} & \text{if } p \in E_3 \setminus (F_2 \cup F_3). \end{cases}$$

Thus, we have $\delta_p(S) = 15/23$ from (1).

(6) The case $p \in F_3 \setminus E_3$. Set $C = F_3$, then we get $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$ and $P(u) \cdot C$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$
$[0, 1]$	$(2, 3, 0, 2, 1, -u)$	0	$5-2u^2$	$2u$
$[1, 2]$	$(2, 3, 3-3u, 3-u, 3-2u, -u)$	$(u-1)(\overset{2}{0}, 3, 1, 2, 0)$	$(4-u)(2-u)$	$3-u$

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Moreover, $\mathrm{ord}_p(N(u)|_C)=0$. Therefore, we get $S(F_3)=17/15$ and $S(W^{F_3}_{\bullet,\bullet},p)=17/15$. Thus, we have $\delta_p(S)=15/17$ from (1).

(7) The case $p\in S\setminus\left(\bigcup_{i,j}(E_i\cup F_j)\right)$. Consider a blowing up $\sigma:\widetilde{S}\to S$ at p. Let $\widetilde{E}_i$ and $\widetilde{F}_j$ be the proper transform of E_i and F_j, respectively. Take two (-1)-curves $G_1:=(\rho\sigma)^{-1}_*(\overline{\rho\sigma(p)q_4})$ and $G_2:=(\rho\sigma)^{-1}_*(\overline{\rho\sigma(p)q_1})$ on $\widetilde{S}$. Since $\overline{\rho\sigma(p)q_4}+\overline{\rho\sigma(p)q_1}+l\in|-K_{\mathbb{P}^2}|$, we have $\sigma^*(-K_S)-uE\sim\widetilde{E}_3+\widetilde{F}_1+\widetilde{F}_2+\widetilde{F}_3+G_1+G_2+(2-u)Z$ and $\widetilde{\tau}=5/2$. The values $\widetilde{P}(u)$, $\widetilde{N}(u)$, $\widetilde{P}(u)^2$, $\widetilde{P}(u)\cdot Z$ and $\mathrm{ord}_q(\widetilde{N}(u)|_Z)$ are given by the following tables:

u	$\widetilde{P}(u)\&\widetilde{N}(u)$	$\widetilde{E}_3$	$\widetilde{F}_1$	$\widetilde{F}_2$	$\widetilde{F}_3$	G_1	G_2	Z
$[0,2]$	$\widetilde{P}(u)$	1	1	1	1	1	1	$2-u$
	$\widetilde{N}(u)$	0	0	0	0	0	0	0
$[2,\frac{5}{2}]$	$\widetilde{P}(u)$	1	$5-2u$	$3-u$	$3-u$	$5-2u$	$7-3u$	$2-u$
	$\widetilde{N}(u)$	0	$2(u-2)$	$u-2$	$u-2$	$2(u-2)$	$3(u-2)$	0

u	$\widetilde{P}(u)^2$	$\widetilde{P}(u)\cdot Z$
$[0,2]$	$5-u^2$	u
$[2,\frac{5}{2}]$	$(5-2u)^2$	$2(5-2u)$

u	q	$\mathrm{ord}_q(\widetilde{N}(u)|_Z)$
$[0,2]$	Z	0
$[2,\frac{5}{2}]$	$Z\cap G_1$	$2(u-2)$
	$Z\cap G_2$	$3(u-2)$
	$Z\setminus(G_1\cup G_2)$	0

Therefore, we get

$$S(Z)=\frac{3}{2},\quad S(W^Z_{\bullet,\bullet},p)=\begin{cases}\frac{11}{15} & \text{if } q\in Z\cap G_1,\\ \frac{23}{30} & \text{if } q\in Z\cap G_2,\\ \frac{2}{3} & \text{if } q\in Z\setminus(G_1\cup G_2).\end{cases}$$

Hence we have

$$\frac{4}{3}\geq\delta_p(S)\geq\min\left\{\frac{2}{S(E)},\frac{1}{S(W^E_{\bullet,\bullet},p)}\right\}=\frac{30}{23}.$$

We also calculate $S(G_2)$. Take $u\in\mathbb{R}_{\geq 0}$. Let $\widetilde{P}(u)+\widetilde{N}(u)$ be the Zariski decomposition of $\sigma^*(-K_S)-uG_2$. The values $\widetilde{P}(u)$, $\widetilde{N}(u)$ and $\widetilde{P}(u)^2$ are given by the following tables:

u	$\widetilde{P}(u)\&\widetilde{N}(u)$	$\widetilde{E}_2$	$\widetilde{E}_3$	$\widetilde{F}_1$	$\widetilde{F}_2$	$\widetilde{F}_3$	G_1	G_2	Z
$[0,\frac{3}{2}]$	$\widetilde{P}(u)$	0	1	$1-\frac{2}{3}u$	$1-\frac{1}{3}u$	1	1	$1-u$	$2-u$
	$\widetilde{N}(u)$	0	0	$\frac{2}{3}u$	$\frac{1}{3}u$	0	0	0	u
$[\frac{3}{2},2]$	$\widetilde{P}(u)$	$3-2u$	1	$3-2u$	$2-u$	1	1	$1-u$	$2-u$
	$\widetilde{N}(u)$	$2u-3$	0	$2(u-1)$	$u-1$	0	0	0	u

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u	$\widetilde{P}(u)^2$
$[0, \frac{3}{2}]$	$5-4u+\frac{2u^2}{3}$
$[\frac{3}{2}, 2]$	$2(2-u)^2$

Therefore, we get $S(G_2)=23/30$ by the definition of $S(G_2)$. Hence we have $30/23 \geq \delta_p(S)$. Therefore, we get $\delta_p(S)=30/23$. □

Proposition 3.4. *Suppose that the dual graph of the* (-1)*-curves and* (-2)*-curves on* S *is same as in Theorem 1.1 (4). Then the local delta invariant* $\delta_p(S)$ *is as follows.*

$p\in S$	$E_1\setminus E_2$	$E_2\setminus F_1$	$F_1\setminus F_2$	F_2	$E_i\setminus F_2$ $(i=3,4)$	$S\setminus\bigcup_{i,j}(E_i\cup F_j)$
$\delta_p(S)$	$\frac{15}{13}$	$\frac{15}{17}$	$\frac{15}{19}$	$\frac{5}{7}$	$\frac{30}{31}$	$\frac{30}{23}$

Proof. We can assume that we get S from $\mathbb{P}^2$ as follows.

(1) Take three distinct co-linear points $q_1, q_3, q_4 \in \mathbb{P}^2$ and a line $l(\neq \overline{q_1q_3})$ passing through q_1. Let $\rho_1{:}S_1 = \mathrm{Bl}_{\{q_1,q_3,q_4\}}\mathbb{P}^2 \to \mathbb{P}^2$ be a blowing-up at points q_1, q_3, q_4, and let $q_2 \in S_1$ be a point at which of $(\rho_1)_*^{-1}l$ and $\rho_1^{-1}(q_1)$ meet.
(2) Let $\rho_2{:}S_2 \to S_1$ be a blowing-up at q_2. Then $S=S_2$. Put $\rho=\rho_1\rho_2$.

Moreover, we have $E_1=\rho_*^{-1}l$, $E_2=\rho_2^{-1}(q_2)$, $F_1=(\rho_2)_*^{-1}(\rho_1^{-1}(q_1))$, $F_2=(\rho)_*^{-1}(\overline{q_1q_3})$, $E_3=\rho^{-1}(q_3)$, $E_4=\rho^{-1}(q_4)$. We denote $D=\sum_{i=1}^4 a_iE_i+\sum_{j=1}^2 b_jF_j \in \mathrm{Div}(S)$ $(a_i, b\in\mathbb{Z})$ by $D=(a_1,a_2,a_3,a_4,b_1,b_2)$. The intersection matrix of $\{E_1,E_2,E_3,E_4,F_1,F_2\}$ is

$$A := \left(\begin{array}{cccc|cc} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{array}\right).$$

We note that

$$-K_S \sim 2E_1+3E_2+2F_1+F_2=\left(2,3,\overset{2}{0},2,1\right).$$

(1) The case $p\in E_1\setminus E_2$. Set $C=E_1$, then we get $\tau=2$. The values $P(u)$, $N(u)$, $P(u)^2$ and $P(u)\cdot C$ is given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$	$\mathrm{ord}_p(N(u)\|_C)$
$[0,1]$	$(2-u,3,\overset{2}{0},2,1)$	0	$5-2u-u^2$	$1+u$	0
$[1,2]$	$(2-u)(1,3,\overset{2}{0},2,1)$	$(u-1)(0,3,\overset{2}{0},2,1)$	$2(2-u)^2$	$4-2u$	0

So we get $S(E_1)=13/15$ and $S(W_{\bullet,\bullet}^{E_1},p)=11/15$. Thus, $\delta_p(S)=15/13$ from (1).
(2) The case $p\in E_2\setminus F_1$. Set $C=E_2$, then we get $\tau=3$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u)\cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$
$[0,1]$	$(2, 3-u, 0, 0, 2-\frac{2}{3}u, 1-\frac{u}{3})$	$(\overset{4}{0}, \frac{2}{3}u, \frac{1}{3}u)$	$5-2u-\frac{u^2}{3}$	$\frac{3+u}{3}$
$[1,3]$	$(3-u)(1, 1, 0, 0, \frac{2}{3}, \frac{1}{3})$	$(u-1, 3, 0, 0, \frac{2}{3}u, \frac{u}{3})$	$\frac{2}{3}(3-u)^2$	$2-\frac{2}{3}u$

u	p	$\mathrm{ord}_p(N(u)\vert_C)$
$[0,1]$	$E_2 \setminus F_1$	0
$[1,3]$	$E_2 \cap E_1$	$u-1$
	$E_2 \setminus (E_1 \cup F_1)$	0

Therefore, we get

$$S(E_2) = \frac{17}{15}, \quad S(W^{E_2}_{\bullet,\bullet}, p) = \begin{cases} \frac{13}{15} & \text{if } p \in E_2 \cap E_1, \\ \frac{23}{45} & \text{if } p \in E_2 \setminus (E_1 \cup F_1). \end{cases}$$

Thus, we have $\delta_p(S) = 15/17$ from (1).
(3) The case $p \in F_1 \setminus F_2$. Set $C = F_1$, then we get $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u)\cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$
$[0,1]$	$(2, 3, \overset{2}{0}, 2-u, 1-\frac{u}{2})$	$(\overset{5}{0}, \frac{u}{2})$	$5-\frac{3}{2}u^2$	$\frac{3u}{2}$
$[1,2]$	$(2, 4-u, \overset{2}{0}, 2-u, 1-\frac{u}{2})$	$(0, u-1, \overset{3}{0}, \frac{u}{2})$	$\frac{1}{2}(2-u)(6+u)$	$\frac{2+u}{2}$

u	p	$\mathrm{ord}_p(N(u)\vert_C)$
$[0,1]$	$F_1 \setminus F_2$	0
$[1,2]$	$F_1 \cap E_2$	$u-1$
	$F_1 \setminus (E_2 \cup F_2)$	0

Therefore, we get

$$S(F_1) = \frac{19}{15}, \quad S(W^{F_1}_{\bullet,\bullet}, p) = \begin{cases} \frac{17}{15} & \text{if } p \in F_1 \cap E_2, \\ \frac{23}{30} & \text{if } p \in F_1 \setminus (E_2 \cup F_2). \end{cases}$$

Thus, we have $\delta_p(S) = 15/19$ from (1).
(4) The case $p \in F_2$. Set $C = F_2$, then we get $\tau = 3$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u)\cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$
$[0,1]$	$(2, 3, \overset{2}{0}, 2-\frac{u}{2}, 1-u)$	$(\overset{4}{0}, \frac{u}{2}, 0)$	$5-\frac{3}{2}u^2$	$\frac{3u}{2}$
$[1,2]$	$(2, 3, \overset{2}{(1-u)}, 2-\frac{u}{2}, 1-u)$	$(\overset{2}{0}, \overset{2}{(u-1)}, \frac{u}{2}, 0)$	$\frac{1}{2}(u^2-8u+14)$	$2-\frac{u}{2}$
$[2,3]$	$(2, 5-u, \overset{2}{(1-u)}, 3-u, 1-u)$	$(0, u-2, \overset{3}{(u-1)}, 0)$	$(3-u)^2$	$3-u$

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u	p	$\mathrm{ord}_p(N(u)\|_C)$
$[0,1]$	$F_2 \cap F_1$	$\frac{u}{2}$
	$F_2 \cap E_i(i=3,4)$	0
	$F_2 \setminus (F_1 \cup E_3 \cup E_4)$	0
$[1,2]$	$F_2 \cap F_1$	$\frac{u}{2}$
	$F_2 \cap E_i(i=3,4)$	$u-1$
	$F_2 \setminus (F_1 \cup E_3 \cup E_4)$	0
$[2,3]$	$F_2 \cap F_1$	$u-1$
	$F_2 \cap E_i(i=3,4)$	$u-1$
	$F_2 \setminus (F_1 \cup E_3 \cup E_4)$	0

Therefore, we get

$$S(F_2)=\frac{7}{5},\quad S(W^{F_2}_{\bullet,\bullet},p)=\begin{cases}\frac{19}{15} & \text{if } p\in(F_1\cap F_2)\,,\\ \frac{31}{30} & \text{if } p\in F_2\cap E_i(i=3,4),\\ \frac{8}{15} & \text{if } p\in F_2\setminus(F_1\cup E_3\cup E_4).\end{cases}$$

Thus, we have $\delta_p(S)=5/7$ from (1).
(5) The case $p\in E_3\setminus F_2$. Set $C=E_3$, then we get $\tau=2$. The values $P(u)$, $N(u)$, $P(u)^2$ and $P(u)\cdot C$ is given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$
$[0,\frac{3}{2}]$	$(2,3,-u,0,2-\frac{u}{3},1-\frac{2}{3}u)$	$(\overset{4}{0},\frac{u}{3},\frac{2}{3}u)$	$5-2u-\frac{u^2}{3}$	$1+\frac{u}{3}$
$[\frac{3}{2},2]$	$(2,3,-u,3-2u,3-u,3-2u)$	$(\overset{3}{0},2u-3,u-1,2(u-1))$	$8-6u+u^2$	$3-u$

Moreover, $\mathrm{ord}_p(N(u)|_C)=0$. Therefore, we get $S(E_3)=31/30$ and $S(W^{E_3}_{\bullet,\bullet},p)=19/30$. Thus, we have $\delta_p(S)=30/31$ from (1).
(6) The case $p\in S\setminus\left(\bigcup_{i,j}(E_i\cup F_j)\right)$. Set $C=\rho_*^{-1}\overline{\rho(p)q_1}$. We note that $C\in|\rho^*H-E_2-F_1|$ and $C\sim E_1+E_2$. Hence we have $-K_S-uC\sim(2-u)E_1+(3-u)E_2+2F_1+F_2=\left(2-u,3-u,\overset{2}{0},2,1\right)$ and $\widetilde{\tau}=2$. The values $P(u)$, $N(u)$, $P(u)^2$ and $P(u)\cdot C$ is given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$
$[0,\frac{3}{2}]$	$(2-u,3-u,\overset{2}{0},2-\frac{2}{3}u,1-\frac{u}{3})$	$(\overset{4}{0},\frac{2}{3}u,\frac{u}{3})$	$5-4u+\frac{2u^2}{3}$	$2-\frac{2}{3}u$
$[\frac{3}{2},2]$	$(2-u)(1,3,\overset{2}{0},2,1)$	$(0,2u-3,\overset{2}{0},2(u-1),u-1)$	$2(2-u)^2$	$2(2-u)$

Moreover, $\mathrm{ord}_p(N(u)|_C)=0$. Therefore, we get $S(C)=23/30$ and $S(W^{C}_{\bullet,\bullet},p)=22/30$. Thus, we have $\delta_p(S)=30/23$ from (1). □

Proposition 3.5. *Suppose that the dual graph of the* (-1)*-curves and* (-2)*-curves on* S *is same as in Theorem 1.1 (5). Then the local delta invariant* $\delta_p(S)$ *is as follows.*

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$p \in S$	$E_1 \setminus F_1$	$F_1 \setminus F_2$	F_2	$F_3 \setminus F_2$	$E_2 \setminus F_2$	$S \setminus \bigcup_{i,j}(E_i \cup F_j)$
$\delta_p(S)$	$\frac{15}{16}$	$\frac{30}{43}$	$\frac{5}{9}$	$\frac{15}{19}$	$\frac{10}{13}$	$\frac{5}{4}$

Proof. We can assume that we get S from $\mathbb{P}^2$ as follows.

(1) Take two distinct points $q_1, q_4 \in \mathbb{P}^2$. Let $\rho_1 : S_1 = \mathrm{Bl}_{\{q_1,q_4\}}\mathbb{P}^2 \to \mathbb{P}^2$ be the composition of blowing-ups at points q_1, q_4 and let $q_2 \in S_1$ be the point at which $(\rho_1)_*^{-1}(\overline{q_1q_4})$ and $\rho_1^{-1}(q_1)$ meet.
(2) Let $\rho_2 : S_2 \to S_1$ be a blowing-up at q_2. Take a point

$$q_3 \in \rho_2^{-1}(q_2) \setminus \left((\rho_1\rho_2)_*^{-1}(\overline{q_1q_4}) \cup (\rho_2)_*^{-1}\left(\rho_1^{-1}(q_1)\right)\right).$$

(3) Let $\rho_3 : S_3 \to S_2$ be a blowing-up at q_3. Then $S = S_3$. Put $\rho = \rho_1\rho_2\rho_3$.

Moreover, we have $E_1 = (\rho_2\rho_3)_*^{-1}(\rho_1^{-1}(q_4))$, $F_1 = \rho_*^{-1}(\overline{q_1q_4})$, $F_2 = (\rho_3)_*^{-1}(\rho_2^{-1}(q_2))$, $F_3 = (\rho_2\rho_3)_*^{-1}(\rho_1^{-1}(q_1))$, $E_2 = \rho_3^{-1}(q_3)$. by $D = (a_1, a_2, b_1, b_2, b_3)$. The intersection matrix of $\{E_1, E_2, F_1, F_2, F_3\}$ is

$$A := \left(\begin{array}{cc|ccc} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ \hline 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right).$$

We note that $-K_S \sim 2E_1 + 3E_2 + 3F_1 + 4F_2 + 2F_3 = (2, 3, 3, 4, 2)$.
(1) The case $p \in E_1 \setminus F_1$. Set $C = E_1$, then we get $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$	$\mathrm{ord}_p(N(u)\vert_C)$
$[0, 2]$	$(2-u, 3, 3-\frac{3}{4}u, 4-\frac{u}{2}, 2-\frac{u}{4})$	$(0, 0, \frac{3}{4}u, \frac{u}{2}, \frac{u}{4})$	$5-2u-\frac{u^2}{4}$	$1+\frac{u}{4}$	0

Therefore, we get $S(E_1) = 16/15$ and $S(W^{E_1}_{\bullet,\bullet}, p) = 4/5$. Thus, we have $\delta_p(S) = 15/16$.
(2) The case $p \in F_1 \setminus F_2$. Set $C = F_1$, then we get $\tau = 3$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$
$[0, 1]$	$(2, 3, 3-u, 4-\frac{2u}{3}, 2-\frac{u}{3})$	$(0, 0, 0, \frac{2}{3}u, \frac{u}{3})$	$5-\frac{4}{3}u^2$	$\frac{4u}{3}$
$[1, \frac{3}{2}]$	$(3-u, 3, 3-u, 4-\frac{2u}{3}, 2-\frac{u}{3})$	$(u-1, 0, 0, \frac{2}{3}u, \frac{u}{3})$	$6-2u-\frac{u^2}{3}$	$1+\frac{u}{3}$
$[\frac{3}{2}, 3]$	$(3-u)(1, 2, 1, 2, 1)$	$(u-1, 2u-3, 0, 2(u-1), u-1)$	$(3-u)^2$	$3-u$

u	p	$\mathrm{ord}_p(N(u)\vert_C)$
$[0, 1]$	$F_1 \setminus F_2$	0
$[1, \frac{3}{2}]$	$F_1 \cap E_1$	$u-1$
	$F_1 \setminus (E_1 \cup F_2)$	0
$[\frac{3}{2}, 3]$	$F_1 \cap E_1$	$u-1$
	$F_1 \setminus (E_1 \cup F_2)$	0

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Therefore, we get

$$S(F_1)=\frac{43}{30},\quad S(W^{F_1}_{\bullet,\bullet},p)=\begin{cases}\frac{16}{15} & \text{if } p\in(F_1\cap E_1),\\ \frac{49}{90} & \text{if } p\in F_1\setminus(E_1\cup F_2).\end{cases}$$

Thus, we have $\delta_p(S)=30/43$.

(3) The case $p\in F_2$. Set $C=F_2$, then we get $\tau=3$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u)\cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$
$[0,1]$	$(2,3,3-\frac{u}{2},(4-u),2-\frac{u}{2})$	$(0,0,\frac{u}{2},0,\frac{u}{2})$	$5-u^2$	u
$[1,2]$	$(2,4-u,3-\frac{u}{2},(4-u),2-\frac{u}{2})$	$(0,u-1,\frac{u}{2},0,\frac{u}{2})$	$6-2u$	1
$[2,3]$	$(4-u)(1,1,1,1,\frac{1}{2})$	$(u-2,\frac{(u-1)}{2},0,\frac{u}{2})$	$8-4u+\frac{u^2}{2}$	$2-\frac{u}{2}$

u	p	$\mathrm{ord}_p(N(u)\vert_C)$
$[0,1]$	$F_2\cap E_2$	0
	$F_2\cap F_1$	$\frac{u}{2}$
	$F_2\cap F_3$	$\frac{u}{2}$
	$F_2\setminus(E_2\cup F_1\cup F_3)$	0
$[1,2]$	$F_2\cap E_2$	$u-1$
	$F_2\cap F_1$	$\frac{u}{2}$
	$F_2\cap F_3$	$\frac{u}{2}$
	$F_2\setminus(E_2\cup F_1\cup F_3)$	0
$[2,3]$	$F_2\cap E_2$	$u-1$
	$F_2\cap F_1$	$u-1$
	$F_2\cap F_3$	$\frac{u}{2}$
	$F_2\setminus(E_2\cup F_1\cup F_3)$	0

Therefore, we get

$$S(F_2)=\frac{9}{5},\quad S(W^{F_2}_{\bullet,\bullet},p)=\begin{cases}\frac{13}{10} & \text{if } p\in(F_2\cap F_3),\\ \frac{43}{30} & \text{if } p\in(F_2\cap F_1),\\ \frac{19}{15} & \text{if } p\in(F_2\cap E_2),\\ \frac{2}{5} & \text{if } p\in F_2\setminus(E_2\cup F_1\cup F_3).\end{cases}$$

Thus, we have $\delta_p(S)=5/9$.

(4) The case $p\in E_2\setminus F_2$. Set $C=E_2$, then we get $\tau=3$. The values $P(u)$, $N(u)$, $P(u)^2$ and $P(u)\cdot C$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$
$[0,2]$	$(2,3-u,3-\frac{u}{2},(4-u),2-\frac{u}{2})$	$(0,0,\frac{u}{2},u,\frac{u}{2})$	$5-2u$	1
$[2,3]$	$(3-u)(2,1,2,2,1)$	$(2u-4,0,2u-3,2(u-1),u-1)$	$(3-u)^2$	$3-u$

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Moreover, $\mathrm{ord}_p(N(u)|_C) = 0$. Therefore, we get $S(E_2) = 19/15$ and $S(W^{E_2}_{\bullet,\bullet}, p) = 7/15$. Thus, we have $\delta_p(S) = 15/19$ from (1).
(5) The case $p \in F_3 \setminus F_2$. Set $C = F_3$, then we get $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$ and $P(u) \cdot C$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$
$[0, \frac{3}{2}]$	$(2, 3, 3 - \frac{u}{3}, (4 - \frac{2}{3}u), 2 - u)$	$(0, 0, \frac{u}{3}, \frac{2}{3}u, 0)$	$5 - \frac{4}{3}u^2$	$\frac{4u}{3}$
$[\frac{3}{2}, 2]$	$(2, 2(3 - u), 4 - u, 2(3 - u), 2 - u)$	$(0, 2u - 3, u - 1, 2(u - 1), 0)$	$4\,(2 - u)$	2

Moreover, $\mathrm{ord}_p(N(u)|_C) = 0$. Therefore, we get $S(F_3) = 13/10$ and $S(W^{F_3}_{\bullet,\bullet}, p) = 4/5$. Thus, we have $\delta_p(S) = 10/13$ from (1).
(6) The case $p \in S \setminus \left(\bigcup_{i,j}(E_i \cup F_j)\right)$. Let $C := \rho_*^{-1}\overline{\rho(p)q_1}$. We note that $C \in |\rho^*H - E_2 - F_2 - F_3|$ and $C \sim E_1 + E_2 + F_1 + F_2$. Hence we have $-K_S - uC \sim (2-u)E_1 + (3-u)E_2 + (3-u)F_1 + (4-u)F_2 + 2F_3$ and $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 2]$	$(2 - u, 3 - u, 3 - \frac{5}{4}u, 4 - \frac{3}{2}u, 2 - \frac{3}{4}u)$	$\frac{u}{4}\overset{2}{(0, 1, 2, 3)}$	$5 - 4u + \frac{3u^2}{4}$	$2 - \frac{3}{4}u$	0

Therefore, we get $S(C) = 4/5$ and $S(W^C_{\bullet,\bullet}, p) = 7/10$. We have $\delta_p(S) = 5/4$. □

Proposition 3.6. *Suppose that the dual graph of the (-1)-curves and (-2)-curves on S is same as in Theorem 1.1 (6). Then the local delta invariant $\delta_p(S)$ is as follows.*

$p \in S$	$F_1 \setminus F_2$	$F_2 \setminus F_3$	F_3	$F_4 \setminus F_3$	$E_1 \setminus F_3$	$S \setminus (E_1 \cup \bigcup_j F_j)$
$\delta_p(S)$	$\frac{3}{4}$	$\frac{6}{11}$	$\frac{3}{7}$	$\frac{9}{13}$	$\frac{3}{5}$	$\frac{6}{5}$

Proof. We can assume that we get S from $\mathbb{P}^2$ as follows.

(1) Take a point $q_1 \in \mathbb{P}^2$ and a line l passing through q_1. Let $\rho_1 : S_1 = \mathrm{Bl}_{\{q_1\}}\mathbb{P}^2 \to \mathbb{P}^2$ be the blowing-up at point q_1, and let $q_2 \in S_1$ be the point at which $(\rho_1)_*^{-1}l$ and $\rho_1^{-1}(q_1)$ meet.
(2) Let $\rho_2 : S_2 \to S_1$ be a blowing-up at q_2 and let $q_3 \in S_2$ be the point at which $(\rho_1\rho_2)_*^{-1}l$ and $\rho_2^{-1}(q_2)$ meet.
(3) Let $\rho_3 : S_3 \to S_2$ be a blowing-up at q_3. Take a point
$$q_4 \in \rho_3^{-1}(q_3) \setminus \left((\rho_1\rho_2\rho_3)_*^{-1}l \cup (\rho_3)_*^{-1}\left(\rho_2^{-1}(q_2)\right)\right).$$
(4) Let $\rho_4 : S_4 \to S_3$ be the blowing-up at q_4. Then $S = S_4$. Put $\rho = \rho_1\rho_2\rho_3\rho_4$.

Moreover, we have $E_1 = \rho_4^{-1}(q_4)$, $F_1 = (\rho_2\rho_3\rho_4)_*^{-1}(\rho_1^{-1}(q_1))$, $F_2 = (\rho_3\rho_4)_*^{-1}(\rho_2^{-1}(q_2))$, $F_3 = (\rho_4)_*^{-1}(\rho_3^{-1}(q_3))$, $F_4 = \rho_*^{-1}l$. We denote $D = a_1E_1 + \sum_{j=1}^4 b_jF_j \in \mathrm{Div}(S)$ $(a_i, b \in \mathbb{Z})$ by $D = (a_1, b_1, b_2, b_3, b_4)$. The intersection matrix of $\{E_1, F_1, F_2, F_3, F_4\}$ is
$$A := \left(\begin{array}{c|cccc} -1 & 0 & 0 & 1 & 0 \\ \hline 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right).$$

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We note that $-K_S \sim 5E_1 + 2F_1 + 4F_2 + 6F_3 + 3F_4 = (5, 2, 4, 6, 3)$.
(1) The case $p \in F_1 \setminus F_2$. Set $C = F_1$, then we get $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$ and $P(u) \cdot C$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$	$\mathrm{ord}_p(N(u)\vert_C)$
$[0, 2]$	$(5, 2-u, 4-\frac{3}{4}u, 6-\frac{u}{2}, 3-\frac{1}{4}u)$	$(0, 0, \frac{3}{4}u, \frac{u}{2}, \frac{1}{4}u)$	$\frac{5}{4}(4-u^2)$	$\frac{5u}{4}$	0

Therefore, we get $S(F_1) = 4/3$ and $S(W^{F_1}_{\bullet,\bullet}, p) = 11/6$. Thus, we have $\delta_p(S) = 3/4$.
(2) The case $p \in F_2 \setminus F_3$. Set $C = F_2$, then we get $\tau = 4$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$
$[0, \frac{3}{2}]$	$(5, 2-\frac{u}{2}, 4-u, 6-\frac{2}{3}u, 3-\frac{u}{3})$	$(0, \frac{u}{2}, 0, \frac{2}{3}u, \frac{u}{3})$	$\frac{5}{6}(6-u^2)$	$\frac{5}{6}u$
$[\frac{3}{2}, 4]$	$(4-u)(2, \frac{1}{2}, 1, 2, 1)$	$(2u-3, \frac{u}{2}, 0, 2(u-1), u-1)$	$\frac{1}{2}(4-u)^2$	$\frac{4-u}{2}$

u	p	$\mathrm{ord}_p(N(u)\vert_C)$
$[0, \frac{3}{2}]$	$F_2 \cap F_1$	$\frac{u}{2}$
	$F_2 \setminus (F_1 \cup F_3)$	0
$[\frac{3}{2}, 4]$	$F_2 \cap F_1$	$\frac{u}{2}$
	$F_2 \setminus (F_1 \cup F_3)$	0

Therefore, we get

$$S(F_2) = \frac{11}{6}, \quad S(W^{F_2}_{\bullet,\bullet}, p) = \begin{cases} \frac{4}{3} & \text{if } p \in (F_2 \cap F_1), \\ \frac{5}{12} & \text{if } p \in F_2 \setminus (F_1 \cup F_3). \end{cases}$$

Thus, we have $\delta_p(S) = 6/11$.
(3) The case $p \in F_3$. Set $C = F_3$, then we get $\tau = 6$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$
$[0, 1]$	$(5, 2-\frac{1}{3}u, 4-\frac{2u}{3}, (6-u), 3-\frac{u}{2})$	$(0, \frac{u}{3}, \frac{2u}{3}, 0, \frac{u}{2})$	$5-\frac{5}{6}u^2$	$\frac{5}{6}u$
$[1, 6]$	$(6-u, 2-\frac{u}{3}, 4-\frac{2u}{3}, (6-u), 3-\frac{u}{2})$	$(u-1, \frac{u}{3}u, \frac{2u}{3}, 0, \frac{u}{2})$	$\frac{(6-u)^2}{6}$	$\frac{6-u}{6}$

u	p	$\mathrm{ord}_p(N(u)\vert_C)$
$[0, 1]$	$F_3 \cap F_2$	$\frac{2u}{3}$
	$F_3 \cap F_4$	$\frac{u}{2}$
	$F_3 \cap E_1$	0
	$F_3 \setminus (F_2 \cup F_4 \cup E_1)$	0
$[1, 6]$	$F_3 \cap F_2$	$\frac{2u}{3}$
	$F_3 \cap F_4$	$\frac{u}{2}$
	$F_3 \cap E_1$	$u-1$
	$F_3 \setminus (F_2 \cup F_4 \cup E_1)$	0

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Therefore, we get

$$S(F_3) = \frac{7}{3}, \quad S(W^{F_3}_{\bullet,\bullet}, p) = \begin{cases} \frac{11}{6} & \text{if } p \in (F_3 \cap F_2), \\ \frac{13}{9} & \text{if } p \in (F_3 \cap F_4), \\ \frac{5}{3} & \text{if } p \in (F_3 \cap E_1), \\ \frac{5}{18} & \text{if } p \in F_3 \setminus (F_2 \cup F_4 \cup E_1). \end{cases}$$

Thus, we have $\delta_p(S) = 3/7$.
(4) The case $p \in F_4 \setminus F_3$. Set $C = F_4$, then we get $\tau = 3$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$
$[0, \frac{4}{3}]$	$(5, 2 - \frac{u}{4}, 4 - \frac{u}{2}, 6 - \frac{3}{4}u, 3 - u)$	$(0, \frac{u}{4}, \frac{u}{2}, \frac{3}{4}u, 0)$	$\frac{5}{4}(4 - u^2)$	$\frac{5}{4}u$
$[\frac{4}{3}, 3]$	$(3 - u)(3, 1, 2, 3, 1)$	$(3u - 4, u - 1, 2(u - 1), 3(u - 1), 0)$	$(3 - u)^2$	$3 - u$

Moreover, we have $\mathrm{ord}_p(N(u)|_C) = 0$. Therefore, we get $S(F_4) = 13/9$ and $S(W^{F_4}_{\bullet,\bullet}, p) = 5/9$. Thus, we have $\delta_p(S) = 9/13$.
(5) The case $p \in E_1 \setminus F_3$. Set $C = E_1$, then we get $\tau = 5$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 5]$	$(5 - u)(1, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}, \frac{3}{5})$	$(0, \frac{2}{5}u, \frac{4}{5}u, \frac{6}{5}u, \frac{3}{5}u)$	$\frac{(5-u)^2}{5}$	$\frac{5-u}{5}$	0

Therefore, we get $S(E_1) = 5/3$ and $S(W^{E_1}_{\bullet,\bullet}, p) = 1/3$. Thus, we have $\delta_p(S) = 3/5$.
(6) The case $p \in S \setminus \left(E_1 \bigcup_j F_j\right)$.
Let $C := \rho_*^{-1}\overline{\rho(p)q_1}$. We note that $C \sim 2E_1 + F_2 + 2F_3 + F_4$. Hence we have $-K_S - uC \sim (5 - 2u)E_1 + 2F_1 + (4 - u)F_2 + (6 - 2u)F_3 + (3 - u)F_4 = (5 - 2u, 2, 4 - u, 6 - 2u, 3 - u)$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$	$\mathrm{ord}_p(N(u)\|_C)$
$[0, \frac{5}{2}]$	$(5 - 2u)(1, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}, \frac{3}{5})$	$(0, \frac{4}{5}u, \frac{3}{5}u, \frac{2}{5}u, \frac{1}{5}u)$	$\frac{1}{5}(5 - 2u)^2$	$\frac{5-2u}{5}$	0

Therefore, we get $S(C) = 5/6$ and $S(W^C_{\bullet,\bullet}, p) = 1/6$. Thus, we have $\delta_p(S) = 6/5$. □

At the end of this section, we introduce the local delta invariants of del Pezzo surface of degree 5. Since the computation of local delta invariants of the surface is essentially done in [2, Lemma 2.11], we omit the proof.

Proposition 3.7. *Let S be the del Pezzo surface with the anti-canonical degree* 5*. Then, for a point $p \in S$, it holds that*

$$\delta_p(S) = \begin{cases} \frac{15}{13} & \text{if } p \text{ lies on a } (-1)\text{-curve}, \\ \frac{4}{3} & \text{if } p \text{ does not lies on all } (-1)\text{-curves}. \end{cases}$$

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4. The case of the anti-canonical degree 6

Let us use the assumptions and notations of Section 2. Suppose $K^2 = 6$.

Proposition 4.1. *Suppose that the dual graph of the* (-1)*-curves and* (-2)*-curves on* S *is same as in Theorem 1.2 (1). Then the local delta invariant* $\delta_p(S)$ *is as follows.*

$p \in S$	$E_i \setminus F\ (i = 1, 2, 3)$	F	$S \setminus (\bigcup_i E_i \cup F)$
$\delta_p(S)$	$\frac{9}{10}$	$\frac{3}{4}$	$\frac{6}{5}$

Proof. We can assume that we get S from $\mathbb{P}^2$ as follows. Take three colinear points $q_1, q_2, q_3 \in \mathbb{P}^2$ and the line l passing through these points. Then we have $\rho: S = \mathrm{Bl}_{\{q_1,q_2,q_3\}}\mathbb{P}^2 \to \mathbb{P}^2$. Moreover, we have $E_i := \rho^{-1}(q_i)$ $(i = 1, 2, 3)$ and $F = \rho_*^{-1} l$. We denote $D = \sum_{i=1}^{3} a_i E_i + bF \in \mathrm{Div}(S)$ $(a_i, b \in \mathbb{Z})$ by $D = (a_1, a_2, a_3, b)$. The intersection matrix of $\{E_1, E_2, E_3, F\}$ is

$$A := \left(\begin{array}{ccc|c} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ \hline 1 & 1 & 1 & -2 \end{array}\right).$$

We note that $-K_S \sim 2E_1 + 2E_2 + 2E_3 + 3F = (2, 2, 2, 3)$.

(1) The case $p \in E_1$. Set $C = E_1$, then we get $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$
$[0, 2]$	$(2 - u, 2, 2, 3 - \frac{u}{2})$	$(0, 0, 0, \frac{u}{2})$	$6 - 2u - \frac{u^2}{2}$	$\frac{2+u}{2}$

u	p	$\mathrm{ord}_p(N(u)\vert_C)$
$[0, 2]$	$E_1 \cap F$	$\frac{u}{2}$
	$E_1 \setminus F$	0

Therefore, we get

$$S(E_1) = \frac{10}{9}, \quad S(W^{E_1}_{\bullet,\bullet}, p) = \begin{cases} \frac{7}{9} & \text{if } p \in E_1 \setminus F, \\ \frac{4}{3} & \text{if } p \in E_1 \cap F. \end{cases}$$

Thus, we have

$$\delta_p(S) \begin{cases} = \frac{9}{10} & \text{if } p \in E_1 \setminus F, \\ \geq \frac{3}{4} & \text{if } p \in E_1 \cap F. \end{cases}$$

For $i = 2, 3$, one can show

$$\delta_p(S) \begin{cases} = \frac{9}{10} & \text{if } p \in E_i \setminus F, \\ \geq \frac{3}{4} & \text{if } p \in E_i \cap F, \end{cases}$$

by the same calculation.

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(2) The case $p \in F$. Set $C = F$, then we get $\tau = 3$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$
$[0, 1]$	$(2, 2, 2, 3-u)$	0	$6-2u^2$	$2u$
$[1, 3]$	$(3-u)(\overset{4}{1})$	$(u-1)(\overset{3}{1}, 0)$	$(3-u)^2$	$3-u$

u	p	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 1]$	F	0
$[1, 3]$	$F \cap E_i \ (i = 1, 2, 3)$	$u-1$
	$F \setminus \bigcup_{i=1}^{3} E_i$	0

Therefore, we get $S(F) = 4/3$ by the definition of $S(F)$. Hence we get $3/4 \geq \delta_p(S)$ for any $p \in F$. If $p \in F \cap \bigcup_{i=1,2,3} E_i$, then we have $\delta_p(S) \geq 3/4$ by (1). Hence we get $\delta_p(S) = 3/4$ at $p \in F \cap \bigcup_{i=1,2,3} E_i$. If $p \in F \setminus \bigcup_{i=1,2,3} E_i$, then we have $S(W^F_{\bullet,\bullet}, p) = 10/9$. Hence, we have

$$\frac{3}{4} \geq \delta_p(S) \geq \min\left\{\frac{1}{S(F)}, \frac{1}{S(W^F_{\bullet,\bullet}, p)}\right\} = \frac{3}{4}$$

at a point $p \in F \setminus \bigcup_{i=1,2,3} E_i$. Thus, we have $\delta_p(S) = 3/4$ for any $p \in F$.
(3) The case $p \in S \setminus \left(\bigcup_i E_i \cup F\right)$. Consider a blowing up $\sigma : \widetilde{S} \to S$ at p. Let $\widetilde{E}_i$ and $\widetilde{F}$ be the proper transform of E_i and F, respectively. Take three (-1)-curves $G_i := (\rho\sigma)^{-1}_{*}(\overline{\rho\sigma(p)q_i})$ for $i = 1, 2, 3$. We note that $\sigma^*(-K_S) \sim G_1 + G_2 + G_3 + 3Z$. Hence, we have $\sigma^*(-K_S) - uZ \sim G_1 + G_2 + G_3 + (3-u)Z$ and $\widetilde{\tau} = 3$. The values $\widetilde{P}(u)$, $\widetilde{N}(u)$, $\widetilde{P}(u)^2$, $\widetilde{P}(u) \cdot Z$ and $\mathrm{ord}_q(\widetilde{N}(u)|_Z)$ are given by the following tables:

u	$\widetilde{P}(u)$&$\widetilde{N}(u)$	G_1	G_2	G_3	Z
$[0, 2]$	$\widetilde{P}(u)$	1	1	1	$3-u$
	$\widetilde{N}(u)$	0	0	0	0
$[2, 3]$	$\widetilde{P}(u)$	$3-u$	$3-u$	$3-u$	$3-u$
	$\widetilde{N}(u)$	$u-2$	$u-2$	$u-2$	0

u	$\widetilde{P}(u)^2$	$\widetilde{P}(u) \cdot Z$
$[0, 2]$	$6-u^2$	u
$[2, 3]$	$2(3-u)^2$	$2(3-u)$

u	q	$\mathrm{ord}_q(\widetilde{N}(u)\|_Z)$
$[0, 2]$	Z	0
$[2, 3]$	$Z \cap G_i \ (i = 1, 2, 3)$	$u-2$
	$Z \setminus (G_1 \cup G_2 \cup G_3)$	0

Therefore, we get

$$S(Z) = \frac{5}{3}, \quad S(W^Z_{\bullet,\bullet}, q) = \begin{cases} \frac{7}{9} & \text{if } q \in Z \cap G_i, \\ \frac{2}{3} & \text{if } q \in Z \setminus (G_1 \cup G_2 \cup G_3). \end{cases}$$

Thus, we have $\delta_p(S) = 6/5$ from Corollary 1. □

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Proposition 4.2. *Suppose that the dual graph of the* (-1)*-curves and* (-2)*-curves on* S *is same as in Theorem 1.2 (2). Then the local delta invariant* $\delta_p(S)$ *is as follows.*

$p \in S$	$E_1 \setminus E_2$, $E_4 \setminus E_3$	E_2, E_3	$F \setminus (E_2 \cup E_3)$	$S \setminus (\bigcup_i E_i \cup F)$
$\delta_p(S)$	$\frac{9}{10}$	$\frac{9}{11}$	$\frac{9}{11}$	$\frac{9}{8}$

Proof. We denote $D = \sum_{i=1}^{4} a_iE_i + bF \in \mathrm{Div}(S)$ $(a_i, b \in \mathbb{Z})$ by $D = (a_1, a_2, a_3, a_4, b)$. The intersection matrix of $\{E_1, E_2, E_3, E_4, F\}$ is

$$A := \left(\begin{array}{cccc|c} -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ \hline 0 & 1 & 1 & 0 & -2 \end{array}\right).$$

We note that $-K_S \sim 2E_1 + 3E_2 + E_3 + 2F = (2, 3, 1, 0, 2)$.

(1) The case $p \in E_1 \setminus E_2$. Set $C = E_1$, then we get $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 1]$	$(2-u, 3, 1, 0, 2)$	0	$(6-2u-u^2)$	$1+u$	0
$[1, 2]$	$(2-u, 5-2u, 1, 0, 3-u)$	$(u-1)(0, 2, \overset{2}{0}, 1)$	$5-2u$	$3-u$	0

Therefore, we get $S(E_1) = 10/9$ and $S(W^{E_1}_{\bullet,\bullet}, p) = 7/9$. Hence we have $\delta_p(S) = 9/10$. We can check $\delta_p(S) = 9/10$ for $p \in E_4 \setminus E_3$ by the same calculation.

(2) The case $p \in E_2$. Set $C = E_2$, then we get $\tau = 3$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$
$[0, 1]$	$(2, 3-u, 1, 0, 2-\frac{u}{2})$	$(0, 0, 0, 0, \frac{u}{2})$	$6-2u-\frac{u^2}{2}$	$1+\frac{u}{2}$
$[1, 2]$	$(3-u, 3-u, 1, 0, 2-\frac{u}{2})$	$(u-1, 0, 0, 0, \frac{u}{2})$	$7-4u+\frac{u^2}{2}$	$2-\frac{u}{2}$
$[2, 3]$	$(3-u)(1, 1, 1, 0, 1)$	$(u-1, 0, u-2, 0, u-1)$	$(3-u)^2$	$3-u$

u	p	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 1]$	$E_2 \cap E_1$	0
	$E_2 \cap F$	$\frac{u}{2}$
	$E_2 \setminus (E_1 \cup F)$	0
$[1, 2]$	$E_2 \cap E_1$	$u-1$
	$E_2 \cap F$	$\frac{u}{2}$
	$E_2 \setminus (E_1 \cup F)$	0
$[2, 3]$	$E_2 \cap E_1$	$u-1$
	$E_2 \cap F$	$u-1$
	$E_2 \setminus (E_1 \cup F)$	0

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Therefore, we get

$$S(E_2)=\frac{11}{9},\quad S(W^{E_2}_{\bullet,\bullet},p)=\begin{cases}1 & \text{if } p\in E_2\cap E_1,\\ \frac{11}{9} & \text{if } p\in E_2\cap F,\\ \frac{7}{12} & \text{if } p\in E_2\setminus(E_1\cup F).\end{cases}$$

Thus, we have $\delta_p(S)=9/11$. We can check $\delta_p(S)=9/11$ for $p\in E_3$ by the same calculation.
(3) The case $p\in F\setminus(E_2\cup E_3)$. Set $C=F$, then we get $\tau=2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u)\cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$	$\mathrm{ord}_p(N(u)\vert_C)$
$[0,1]$	$(2,3,1,0,2-u)$	0	$6-2u^2$	$2u$	0
$[1,2]$	$(2,4-u,2-u,0,2-u)$	$(u-1)(0,1,1,0,0)$	$4(2-u)$	2	0

Therefore, we get $S(F)=11/9$ and $S(W^F_{\bullet,\bullet},p)=8/9$. Thus, we have $\delta_p(S)=9/11$.
(4) The case $p\in S\setminus\left(\bigcup_i E_i\cup F\right)$. Let $L\in|E_1+E_2|$ be a smooth irreducible curve. Set $C=L$, then we get $\tau=2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u)\cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$	$\mathrm{ord}_p(N(u)\vert_C)$
$[0,2]$	$(2-u,3-u,1,0,2-\frac{u}{2})$	$(0,0,0,0,\frac{u}{2})$	$\frac{(u-2)(u-6)}{2}$	$\frac{4-u}{2}$	0

Therefore, we get $S(L)=8/9$ and $S(W^L_{\bullet,\bullet},p)=7/9$. Thus, we have $\delta_p(S)=9/8$. □

Proposition 4.3. *Suppose that the dual graph of the (-1)-curves and (-2)-curves on S is same as in Theorem 1.2 (3). Then the local delta invariant $\delta_p(S)$ is as follows.*

$p\in S$	$F_1\setminus E_1$	E_1	$F_2\setminus E_1$	$E_2\setminus F_2$	$S\setminus\bigcup_{i,j}(E_i\cup F_j)$
$\delta_p(S)$	$\frac{9}{11}$	$\frac{9}{14}$	$\frac{3}{4}$	$\frac{9}{10}$	$\frac{9}{8}$

Proof. We denote $D=\sum_{i=1,2}a_iE_i+\sum_{j=1,2}b_jF_j\in\mathrm{Div}(S)$ $(a_i,b_j\in\mathbb{Z})$ by $D=(a_1,a_2,b_1,b_2)$. The intersection matrix of $\{E_1,E_2,F_1,F_2\}$ is

$$A:=\left(\begin{array}{cc|cc}-1&0&1&1\\0&-1&0&1\\\hline 1&0&-2&0\\1&1&0&-2\end{array}\right).$$

We note that $-K_S\sim 4E_1+2E_2+2F_1+3F_2=(4,2,2,3)$.
(1) The case $p\in F_1\setminus E_1$. Set $C=F_1$, then we get $\tau=2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u)\cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$	$\mathrm{ord}_p(N(u)\vert_C)$
$[0,1]$	$(4,2,2-u,3)$	0	$6-2u^2$	$2u$	0
$[1,2]$	$(2(3-u),2,2-u,4-u)$	$(u-1)(2,0,0,1)$	$4(2-u)$	2	0

Therefore, we get $S(F_1)=11/9$ and $S(W^{F_1}_{\bullet,\bullet},p)=8/9$. Thus, we have $\delta_p(S)=9/11$.

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(2) The case $p \in E_1$. Set $C = E_1$, then we get $\tau = 4$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$
$[0, 2]$	$(4 - u, 2, 2 - \frac{u}{2}, 3 - \frac{u}{2})$	$(0, 0, \frac{u}{2}, \frac{u}{2})$	$6 - 2u$	1
$[2, 4]$	$(4 - u, 4 - u, 2 - \frac{u}{2}, 4 - u)$	$(0, u - 2, \frac{u}{2}, u - 1)$	$\frac{(4-u)^2}{2}$	$\frac{4-u}{2}$

u	p	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 2]$	$E_1 \cap F_1$	$\frac{u}{2}$
	$E_1 \cap F_2$	$\frac{u}{2}$
	$E_1 \setminus (F_1 \cup F_2)$	0
$[2, 4]$	$E_1 \cap F_1$	$\frac{u}{2}$
	$E_1 \cap F_2$	$u - 1$
	$E_1 \setminus (F_1 \cup F_2)$	0

Therefore, we get

$$S(E_1) = \frac{14}{9}, \quad S(W^{E_1}_{\bullet,\bullet}, p) = \begin{cases} 1 & \text{if } p \in E_1 \cap F_1, \\ \frac{10}{9} & \text{if } p \in E_1 \cap F_2, \\ \frac{4}{9} & \text{if } p \in E_1 \setminus \bigcup_{j=1}^{2} F_j. \end{cases}$$

Thus, we have $\delta_p(S) = 9/14$.

(3) The case $p \in F_2 \setminus E_1$. Set $C = F_2$, then we get $\tau = 3$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$
$[0, 1]$	$(4, 2, 2, 3 - u)$	0	$6 - 2u^2$	$2u$
$[1, 3]$	$(3 - u)(2, 1, 1, 1)$	$(u - 1)(2, 1, 1, 0)$	$(3 - u)^2$	$3 - u$

u	p	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 1]$	$F_2 \setminus E_1$	0
$[1, 3]$	$F_2 \cap E_2$	$u - 1$
	$F_2 \setminus (E_1 \cup E_2)$	0

Therefore, we get

$$S(F_2) = \frac{4}{3}, \quad S(W^{F_2}_{\bullet,\bullet}, p) = \begin{cases} \frac{16}{9} & \text{if } p \in F_2 \cap E_2, \\ \frac{4}{3} & \text{if } p \in F_2 \setminus (E_1 \cup E_2). \end{cases}$$

Thus, we have

$$\delta_p(S) \begin{cases} = \frac{3}{4} & \text{if } p \in F_2 \setminus (E_1 \cup E_2), \\ \leq \frac{3}{4} & \text{if } \{p\} = F_2 \cap E_2. \end{cases}$$

(4) The case $p \in E_2$. Set $C = E_2$, then we get $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$
$[0, 2]$	$(4, 2-u, 2, 3-\frac{u}{2})$	$\overset{3}{(0, \frac{u}{2})}$	$6 - 2u - \frac{u^2}{2}$	$1 + \frac{u}{2}$

u	p	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 2]$	$E_2 \cap F_2$	$\frac{u}{2}$
	$E_2 \setminus F_2$	0

Therefore, we get

$$S(E_2) = \frac{10}{9}, \quad S(W^{E_2}_{\bullet,\bullet}, p) = \begin{cases} \frac{4}{3} & \text{if } p \in E_2 \cap F_2, \\ \frac{7}{9} & \text{if } p \in E_2 \setminus F_2. \end{cases}$$

Thus, we have

$$\delta_p(S) \begin{cases} \geq \frac{3}{4} & \text{if } \{p\} = F_2 \cap E_2, \\ = \frac{9}{10} & \text{if } p \in E_2 \setminus F_2. \end{cases}$$

By (3), we have $3/4 \geq \delta_p(S)$ for $\{p\} = F_2 \cap E_2$. Therefore, we get $\delta_p(S) = 3/4$ for $\{p\} = F_2 \cap E_2$.
(5) The case $p \in S \setminus (E_1 \cup E_2 \cup F_1 \cup F_2)$. Let $L \in |E_1 + E_2 + F_2|$ be a smooth irreducible curve. Set $C = L$, then we get $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 2]$	$(4-u, 2-u, 2-\frac{u}{2}, 3-u)$	$(0, 0, \frac{u}{2}, 0)$	$\frac{(u-2)(u-6)}{2}$	$\frac{4-u}{2}$	0

Therefore, we get $S(L) = 8/9$ and $S(W^{L}_{\bullet,\bullet}, p) = 7/9$. Thus, we have $\delta_p(S) = 9/8$. □

Proposition 4.4. *Suppose that the dual graph of the (-1)-curves and (-2)-curves on S is same as in Theorem 1.2 (4). Then the local delta invariant $\delta_p(S)$ is as follows.*

$p \in S$	$F_1 \setminus F_2$	F_2	$E_1 \setminus F_2, E_2 \setminus F_2$	$S \setminus \bigcup_{i,j} (E_i \cup F_j)$
$\delta_p(S)$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{4}{5}$	1

Proof. We denote $D = \sum_{i=1,2} a_i E_i + \sum_{j=1,2} b_j F_j \in \mathrm{Div}(S)$ $(a_i, b_j \in \mathbb{Z})$ by $D = (a_1, a_2, b_1, b_2)$. The intersection matrix of $\{E_1, E_2, F_1, F_2\}$ is

$$A := \left(\begin{array}{cc|cc} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ \hline 0 & 0 & -2 & 1 \\ 1 & 1 & 1 & -2 \end{array}\right).$$

We note that $-K_S \sim 3E_1 + 3E_2 + 2F_1 + 4F_2 = (3, 3, 2, 4)$.

(1) The case $p \in F_1 \setminus F_2$. Set $C = F_1$, then we get $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 2]$	$(3, 3, 2-u, 4-\frac{u}{2})$	$(0, 0, 0, \frac{u}{2})$	$\frac{3(2-u)(2+u)}{2}$	$\frac{3u}{2}$	0

Therefore, we get $S(F_1) = 4/3$ and $S(W^{F_1}_{\bullet,\bullet}, p) = 1$. Thus, we have $\delta_p(S) = 3/4$.
(2) The case $p \in F_2$. Set $C = F_2$, then we get $\tau = 4$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$
$[0, 1]$	$(3, 3, 2-\frac{u}{2}, 4-u)$	$(0, 0, \frac{u}{2}, 0)$	$6-\frac{3u^2}{2}$	$\frac{3u}{2}$
$[1, 4]$	$(4-u)(1, 1, \frac{1}{2}, 1)$	$(u-1, u-1, \frac{u}{2}, 0)$	$\frac{(4-u)^2}{2}$	$2-\frac{u}{2}$

u	p	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 1]$	$F_2 \cap F_1$	$\frac{u}{2}$
	$F_2 \cap (E_1 \cup E_2)$	0
	$F_2 \setminus (F_1 \cup E_1 \cup E_2)$	0
$[1, 4]$	$F_2 \cap F_1$	$\frac{u}{2}$
	$F_2 \cap (E_1 \cup E_2)$	$u-1$
	$F_2 \setminus (F_1 \cup E_1 \cup E_2)$	0

Therefore, we get

$$S(F_2) = \frac{5}{3}, \quad S(W^{F_2}_{\bullet,\bullet}, p) = \begin{cases} \frac{4}{3} & \text{if } p \in F_2 \cap F_1, \\ \frac{5}{4} & \text{if } p \in F_2 \cap (E_1 \cup E_2), \\ \frac{1}{2} & \text{if } p \in F_2 \setminus (F_1 \cup E_1 \cup E_2). \end{cases}$$

Thus, we have $\delta_p(S) = 3/5$ in this case.
(3) The case $p \in E_1 \setminus F_2$. Set $C = E_1$, then we get $\tau = 3$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$	$\mathrm{ord}_p(N(u)\|_C)$
$[0, \frac{3}{2}]$	$(3-u, 3, 2-\frac{u}{3}, 4-\frac{2}{3}u)$	$(0, 0, \frac{u}{3}, \frac{2}{3}u)$	$6-2u-\frac{u^2}{3}$	$1+\frac{u}{3}$	0
$[\frac{3}{2}, 3]$	$(3-u)(1, 2, 1, 2)$	$(0, 2u-3, u-1, 2(u-1))$	$(3-u)^2$	$3-u$	0

Therefore, we get $S(E_1) = 5/4$ and $S(W^{E_1}_{\bullet,\bullet}, p) = 7/12$. Thus, we have $\delta_p(S) = 4/5$.
(4) The case $p \in S \setminus (E_1 \cup E_2 \cup F_1 \cup F_2)$. Let $L \in |E_1 + E_2 + F_2|$ be a smooth irreducible curve. Set $C = L$, then we get $\tau = 3$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

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u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$	$\mathrm{ord}_p(N(u)\|_C)$
$[0,3]$	$(3-u)(1,1,\frac{2}{3},\frac{4}{3})$	$(0,0,\frac{2u}{3},\frac{u}{3})$	$\frac{2(3-u)^2}{3}$	$\frac{2(3-u)}{3}$	0

Therefore, we get $S(L)=1$ and $S(W^{L}_{\bullet,\bullet},p)=2/3$. Thus, we have $\delta_p(S)=1$. □

Proposition 4.5. *Suppose that the dual graph of the (-1)-curves and (-2)-curves on S is same as in Theorem 1.2 (5). Then the local delta invariant $\delta_p(S)$ is as follows.*

$p\in S$	$F_1\setminus F_2$	$F_2\setminus E$	E	$F_3\setminus E$	$S\setminus(E\cup\bigcup_j F_j)$
$\delta_p(S)$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{1}{2}$	$\frac{3}{4}$	1

Proof. We denote $D=aE+\sum_{j=1,2,3}b_jF_j\in\mathrm{Div}(S)$ $(a,b_j\in\mathbb{Z})$ by $D=(a,b_1,b_2,b_3)$. The intersection matrix of $\{E,F_1,F_2,F_3\}$ is

$$A:=\left(\begin{array}{c|ccc}-1&0&1&1\\\hline 0&-2&1&0\\1&1&-2&0\\1&0&0&-2\end{array}\right).$$

We note that $-K_S\sim 6E+2F_1+4F_2+3F_3=(6,2,4,3)$.
(1) The case $p\in F_1\setminus F_2$. Set $C=F_1$, then we get $\tau=2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u)\cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$	$\mathrm{ord}_p(N(u)\|_C)$
$[0,2]$	$(6,2-u,4-\frac{u}{2},3)$	$(0,0,\frac{u}{2},0)$	$\frac{3(4-u^2)}{2}$	$\frac{3u}{2}$	0

Therefore, we get $S(F_1)=4/3$ and $S(W^{F_1}_{\bullet,\bullet},p)=1$. Thus, we have $\delta_p(S)=3/4$.
(2) The case $p\in F_2\setminus E$. Set $C=F_2$, then we get $\tau=4$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u)\cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$
$[0,1]$	$(6,2-\frac{u}{2},4-u,3)$	$(0,\frac{u}{2},0,0)$	$6-\frac{3u^2}{2}$	$\frac{3u}{2}$
$[1,4]$	$(4-u)(2,\frac{1}{2},1,1)$	$(2(u-1),\frac{u}{2},0,u-1)$	$\frac{(4-u)^2}{2}$	$2-\frac{u}{2}$

u	p	$\mathrm{ord}_p(N(u)\|_C)$
$[0,1]$	$F_2\cap F_1$	$\frac{u}{2}$
	$F_2\setminus(F_1\cup E)$	0
$[1,4]$	$F_2\cap F_1$	$\frac{u}{2}$
	$F_2\setminus(F_1\cup E)$	0

Therefore, we get

$$S(F_2)=\frac{5}{3},\quad S(W^{F_2}_{\bullet,\bullet},p)=\begin{cases}\frac{4}{3} & \text{if } p\in F_2\cap F_1,\\ \frac{1}{2} & \text{if } p\in F_2\setminus(F_1\cup E).\end{cases}$$

Thus, we have $\delta_p(S)=3/5$.

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(3) The case $p \in E$. Set $C = E$, then we get $\tau = 6$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u)\cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$
$[0, 6]$	$(6-u)(1, \frac{1}{3}, \frac{2}{3}, \frac{1}{2})$	$(0, \frac{u}{3}, \frac{2u}{3}, \frac{u}{2})$	$\frac{1}{6}(6-u^2)$	$\frac{1}{6}(6-u)$

u	p	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 6]$	$E \cap F_2$	$\frac{2u}{3}$
	$E \cap F_3$	$\frac{u}{2}$
	$E \setminus (F_2 \cup F_3)$	0

Therefore, we get

$$S(E) = 2, \quad S(W^E_{\bullet,\bullet}, p) = \begin{cases} \frac{5}{3} & \text{if } p \in E \cap F_2, \\ \frac{4}{3} & \text{if } p \in E \cap F_3, \\ \frac{1}{3} & \text{if } p \in E \setminus (F_2 \cup F_3). \end{cases}$$

Thus, we have $\delta_p(S) = 1/2$ in this case.

(4) The case $p \in F_3 \setminus E$. Set $C = F_3$, then we get $\tau = 3$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u)\cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 1]$	$(6, 2, 4, 3-u)$	0	$6-2u^2$	$2u$	0
$[1, 3]$	$(3-u)(3, 1, 2, 1)$	$(u-1)(3, 1, 2, 0)$	$(3-u)^2$	$3-u$	0

Therefore, we get $S(F_3) = 4/3$ and $S(W^{F_3}_{\bullet,\bullet}, p) = 2/3$. Thus, we have $\delta_p(S) = 3/4$.

(5) The case $p \in S \setminus (E \cup F_1 \cup F_2 \cup F_3)$. Let $L \in |2E + F_2 + F_3|$ be a smooth irreducible curve. Set $C = L$, then we get $\tau = 3$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u)\cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u)\cdot C$	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 3]$	$(3-u)(2, \frac{2}{3}, \frac{4}{3}, 1)$	$(0, \frac{2u}{3}, \frac{u}{3}, 0)$	$\frac{2(3-u)^2}{3}$	$\frac{2(3-u)}{3}$	0

Therefore, we get $S(L) = 1$ and $S(W^L_{\bullet,\bullet}, p) = 2/3$. Thus, we have $\delta_p(S) = 1$. □

At the end of this section, we introduce the local delta invariants of del Pezzo surface of degree 6. We omit the proof.

Proposition 4.6. *Let S be the del Pezzo surface with the anti-canonical degree 6. The local delta invariant $\delta_p(S)$ of S at $p \in S$ is as follows.*

$p \in S$	*p lies on a (-1) curve*	*p does NOT lies on any (-1) curve*
$\delta_p(S)$	1	$\frac{6}{5}$

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5. The case of the anti-canonical degree 7

Let us use the assumptions and notations of Section 2. Suppose $K^2 = 7$.

Proposition 5.1. *Suppose that the dual graph of the* (-1)*-curves and* (-2)*-curves on* S *is same as in Theorem 1.3 (1). Then the local delta invariant* $\delta_p(S)$ *is as follows.*

$p \in S$	$E_1 \setminus E_2$	E_2	$F \setminus E_2$	$S \setminus (E_1 \cup E_2 \cup F)$
$\delta_p(S)$	$\frac{21}{25}$	$\frac{21}{31}$	$\frac{7}{9}$	$\frac{21}{23}$

Proof. We denote $D = \sum_{i=1,2} a_i E_i + F \in \mathrm{Div}(S)$ $(a_i, b \in \mathbb{Z})$ by $D = (a_1, a_2, b)$. The intersection matrix of $\{E_1, E_2, F\}$ is

$$A := \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -2 & 1 \\ \hline 0 & 1 & -2 \end{array}\right).$$

We note that $-K_S \sim 3E_1 + 4E_2 + 2F = (3, 4, 2)$.
(1) The case $p \in E_1 \setminus E_2$. Set $C = E_1$, then we get $\tau = 3$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 1]$	$(3-u, 4, 2)$	0	$7 - 2u - u^2$	$1 + u$	0
$[1, 3]$	$(3-u)(1, 2, 1)$	$(u-1)(0, 2, 1)$	$(3-u)^2$	$3 - u$	0

Therefore, $S(E_1) = 25/21$ and $S(W^{E_1}_{\bullet,\bullet}, p) = 15/21$. Thus, we have $\delta_p(S) = 21/25$.
(2) The case $p \in E_2$. Set $C = E_2$, then we get $\tau = 4$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following tables:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$
$[0, 1]$	$(3, 4-u, 2-\frac{u}{2})$	$(0, 0, \frac{u}{2})$	$7 - 2u - \frac{u^2}{2}$	$1 + \frac{u}{2}$
$[1, 4]$	$(4-u)(1, 1, \frac{1}{2})$	$(u-1, 0, \frac{u}{2})$	$2\left(2 - \frac{u}{2}\right)^2$	$2 - \frac{u}{2}$

u	p	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 1]$	$E_2 \cap E_1$	0
	$E_2 \cap F$	$\frac{u}{2}$
	$E_2 \setminus (F \cup E_1)$	0
$[1, 4]$	$E_2 \cap E_1$	$u - 1$
	$E_2 \cap F$	$\frac{u}{2}$
	$E_2 \setminus (F \cup E_1)$	0

Therefore, we get

$$S(E_2) = \frac{31}{21}, \quad S(W^{E_2}_{\bullet,\bullet}, p) = \begin{cases} \frac{25}{21} & \text{if } p \in E_2 \cap E_1, \\ \frac{9}{7} & \text{if } p \in E_2 \cap F, \\ \frac{23}{42} & \text{if } p \in E_2 \setminus (F \cup E_1). \end{cases}$$

Thus, we have $\delta_p(S) = 21/31$ for $p \in E_2$.
(3) The case $p \in F \setminus E_2$. Set $C = F$, then we get $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 1]$	$(3, 4, 2-u)$	0	$7-2u^2$	$2u$	0
$[1, 2]$	$(3, 5-u, 2-u)$	$(u-1)(0, 1, 0)$	$8-2u-u^2$	$1+u$	0

Therefore, we get $S(F) = 9/7$ and $S(W^F_{\bullet,\bullet}, p) = 23/21$. Thus, we have $\delta_p(S) = 7/9$.
(4) The case $p \in S \setminus (E_1 \cup E_2 \cup F)$. Let $L \in |E_1 + E_2|$ be a smooth irreducible curve.
Set $C = L$, then we get $\tau = 3$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 2]$	$(3-u, 4-u, 2-\frac{u}{2})$	$(0, 0, \frac{u}{2})$	$7-4u+\frac{u^2}{2}$	$2-\frac{u}{2}$	0
$[2, 3]$	$(3-u)(1, 2, 1)$	$(0, u-2, u-1)$	$(3-u)^2$	$3-u$	0

Since we get $S(L) = 23/21$ and $S(W^L_{\bullet,\bullet}, p) = 15/21$, we have $\delta_p(S) = 21/23$. □

At the end of this section, we introduce the local delta invariants of del Pezzo surface of degree 7. We omit the proof.

Proposition 5.2. *Suppose that the dual graph of the* (-1)*-curves on* S *is same as in Theorem 1.3 (2). Then the local delta invariant* $\delta_p(S)$ *is as follows.*

$p \in S$	$(E_1 \cup E_3) \setminus E_2$	E_2	$S \setminus (E_1 \cup E_2 \cup E_3)$
$\delta_p(S)$	$\frac{21}{23}$	$\frac{21}{25}$	$\frac{21}{22}$

6. The case of the anti-canonical degree 8

Let us use assumptions and notations of Section 2. Suppose $K^2 = 8$.

Proposition 6.1. *If* $S = \Sigma_2$*, then for any point* $p \in S$*, it holds that*

$$\delta_p(S) = \frac{3}{4}.$$

Proof. We denote $D = aC_0 + b\Gamma \in \mathrm{Div}(S)$ $(a, b \in \mathbb{Z})$ by $D = (a, b)$. The intersection matrix of $\{C_0, \Gamma\}$ is

$$A := \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}.$$

We note that $-K_S \sim 2C_0 + 4\Gamma = (2, 4)$.

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(1) The case $p \in C_0$. Set $C = C_0$, then we get $\tau = 2$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 2]$	$(2-u, 4)$	0	$2(4-u^2)$	$2u$	0

Therefore, we get $S(C_0) = 4/3$ and $S(W^{C_0}_{\bullet,\bullet}, p) = 4/3$. Thus, we have $\delta_p(S) = 3/4$.

(2) The case $p \in S \setminus C_0$. Let Γ be the fiber of π passing through p. Set $C = \Gamma$, then we get $\tau = 4$. The values $P(u)$, $N(u)$, $P(u)^2$, $P(u) \cdot C$ and $\mathrm{ord}_p(N(u)|_C)$ are given by the following table:

u	$P(u)$	$N(u)$	$P(u)^2$	$P(u) \cdot C$	$\mathrm{ord}_p(N(u)\|_C)$
$[0, 4]$	$(2-\frac{u}{2}, 4-u)$	$(\frac{u}{2}, 0)$	$2(2-\frac{u}{2})^2$	$2-\frac{u}{2}$	0

Therefore, we get $S(\Gamma) = 4/3$ and $S(W^{\Gamma}_{\bullet,\bullet}, p) = 2/3$. Thus, we have $\delta_p(S) = 3/4$. □

At the end of this section, we introduce the local delta invariants of del Pezzo surface of degree 8. We omit the proof.

Proposition 6.2. *If $S = \Sigma_1$, then for any point $p \in S$, it holds that*

$$\delta_p(S) = \begin{cases} \frac{6}{7} & \text{if } p \in C_0, \\ \frac{12}{13} & \text{if } p \in S \setminus C_0. \end{cases}$$

If $S = \Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$, then for any point $p \in S$, it holds that $\delta_p(S) = 1$.

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