J. Aust. Math. Soc. **110** (2021), 220–242
doi:10.1017/S1446788719000041

MONOTONE AND PSEUDO-MONOTONE EQUILIBRIUM PROBLEMS IN HADAMARD SPACES

HADI KHATIBZADEH✉ and VAHID MOHEBBI

(Received 28 September 2016; accepted 30 January 2019; first published online 11 March 2019)

Communicated by G. Willis

Abstract

As a continuation of previous work of the first author with Ranjbar [‘A variational inequality in complete CAT(0) spaces’, *J. Fixed Point Theory Appl.* **17** (2015), 557–574] on a special form of variational inequalities in Hadamard spaces, in this paper we study equilibrium problems in Hadamard spaces, which extend variational inequalities and many other problems in nonlinear analysis. In this paper, first we study the existence of solutions of equilibrium problems associated with pseudo-monotone bifunctions with suitable conditions on the bifunctions in Hadamard spaces. Then, to approximate an equilibrium point, we consider the proximal point algorithm for pseudo-monotone bifunctions. We prove existence of the sequence generated by the algorithm in several cases in Hadamard spaces. Next, we introduce the resolvent of a bifunction in Hadamard spaces. We prove convergence of the resolvent to an equilibrium point. We also prove $\triangle$-convergence of the sequence generated by the proximal point algorithm to an equilibrium point of the pseudo-monotone bifunction and also the strong convergence under additional assumptions on the bifunction. Finally, we study a regularization of Halpern type and prove the strong convergence of the generated sequence to an equilibrium point without any additional assumption on the pseudo-monotone bifunction. Some examples in fixed point theory and convex minimization are also presented.

2010 *Mathematics subject classification*: primary 47H05, 47J20, 47J25, 49J40; secondary 90C25, 65K10.

Keywords and phrases: Hadamard space, equilibrium problem, Halpern regularization, proximal point algorithm, pseudo-monotone bifunction, strong convergence, $\triangle$-convergence.

1. Introduction

Let (X, d) be a metric space. A geodesic from x to y is a map γ from the closed interval $[0, d(x, y)] \subset \mathbb{R}$ to X such that $\gamma(0) = x$, $\gamma(d(x, y)) = y$ and $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in [0, d(x, y)]$. The image of γ is called a geodesic (or metric) segment joining x to y. When it is unique, this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. Let X be a uniquely geodesic space and $Y \subset X$. The space Y is said to be convex if for any two points $x, y \in Y$ the geodesic joining x and y is contained in Y, that is, if $\gamma : [0, d(x, y)] \longrightarrow X$ is the geodesic such that $x = \gamma(0)$ and $y = \gamma(d(x, y))$, then $\gamma(t) \in Y$ for all $t \in [0, d(x, y)]$. The convex hull of Y (denoted by $\operatorname{conv}(Y)$) is the intersection of all convex subsets of X that contain Y. We recall the following lemma from [32], which will be used in the next section.

Lemma 1.1. *Let X be a unique geodesic metric space and let A be a subset of X. We set $C_0(A) = A$ and, for every integer $n \geq 0$, we let $C_{n+1}(A)$ be the union of all the geodesic segments in X that join pairs of points in $C_n(A)$. Then the geodesic convex hull* $\operatorname{conv}(A)$ *of A is given by*

$$\operatorname{conv}(A) = \bigcup_{n \geq 0} C_n(A).$$

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2 and x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$, where $d_{\mathbb{E}^2}$ is the usual metric in $\mathbb{R}^2$. A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom. Let Δ be a geodesic triangle in X and let $\bar{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y}).$$

A complete CAT(0) space is called a Hadamard space. The reader can consult [10, 11] to learn more about Hadamard spaces. We now collect some elementary facts about CAT(0) spaces, which will be used in the proofs of our main results.

Lemma 1.2. *Let X be a* CAT(0) *space and $x, y \in X$. Then, for each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that $d(x, z) = td(x, y)$ and $d(y, z) = (1 - t)\, d(x, y)$.*

Proof. See [18, Lemma 2.1(iv)]. □

We will use the notation $(1 - t)x \oplus ty$ for the unique point z satisfying the above condition.

Lemma 1.3. *Let X be a* CAT(0) *space. Then, for all $x, y, z \in X$ and $t, s \in [0, 1]$, we have:*

(i) $d((1 - t)x \oplus ty, z) \leq (1 - t)\, d(x, z) + t\, d(y, z)$;
(ii) $d((1 - t)x \oplus ty, (1 - s)x \oplus sy) = |t - s|\, d(x, y)$;
(iii) $d((1 - t)z \oplus tx, (1 - t)z \oplus ty) \leq t\, d(x, y)$;
(iv) $d^2((1 - t)x \oplus ty, z) \leq (1 - t)\, d^2(x, z) + t\, d^2(y, z) - t(1 - t)\, d^2(x, y)$.

Proof. For (i) see [18, Lemma 2.4], (ii) see [13], (iii) see [26, Lemma 3] and (iv) see [18, Lemma 2.5]. □

https://doi.org/10.1017/S1446788719000041 Published online by Cambridge University Press

Berg and Nikolaev in [7, 8] introduced the concept of quasi-linearization along the following lines. Let us formally denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then quasi-linearization is defined as a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \tfrac{1}{2}\{d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\}, \quad (a, b, c, d \in X).$$

It is easily seen that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a, b, c, d, x \in X$. We say that X satisfies the Cauchy–Schwarz inequality if $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \le d(a, b)d(c, d)$ for all $a, b, c, d \in X$. It is known [8, Corollary 3] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy–Schwarz inequality.

A Hadamard space X is called a flat Hadamard space if and only if the inequality in Part (iv) of Lemma 1.3 is an equality. A well-known result asserts that a flat Hadamard space is isometric to a closed convex subset of a Hilbert space. It is easy to check that in a flat Hadamard space X, for each $x, y, z, u \in X$ and $0 \le \lambda \le 1$,

$$\langle \overrightarrow{xy}, \overrightarrow{x(\lambda z \oplus (1 - \lambda)u)} \rangle = \lambda \langle \overrightarrow{xy}, \overrightarrow{xz} \rangle + (1 - \lambda)\langle \overrightarrow{xy}, \overrightarrow{xu} \rangle.$$

Let (X, d) be a Hadamard space, $\{x_n\}$ be a bounded sequence in X and $x \in X$. Let $r(x, \{x_n\}) = \limsup d(x, x_n)$. The asymptotic radius of $\{x_n\}$ is given by $r(\{x_n\}) = \inf\{r(x, \{x_n\}) \mid x \in X\}$ and the asymptotic center of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X \mid r(x, \{x_n\}) = r(\{x_n\})\}$. It is known that in a Hadamard space, $A(\{x_n\})$ is a singleton.

Definition 1.4. A sequence $\{x_n\}$ in a Hadamard space (X, d) $\triangle$-converges to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for each subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

We denote $\triangle$-convergence in X by $\xrightarrow{\triangle}$ and the (strong) metric convergence by $\to$.

Lemma 1.5. *Let X be a complete* CAT(0) *space. Then every bounded closed convex subset of X is $\triangle$-compact; that is, every bounded sequence in it has a $\triangle$-convergent subsequence.*

Proof. See [27, Proposition 3.6]. □

Let X be a complete CAT(0) space. Then every closed convex subset K of X is $\triangle$-closed in the sense that it contains all limit points of every $\triangle$-convergent sequence.

A function $f : X \to]-\infty, +\infty]$ is called:

(1) convex if and only if

$$f(\lambda x \oplus (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y), \quad \forall x, y \in X \text{ and } \lambda \in (0, 1);$$

(2) quasi-convex if and only if

$$f(\lambda x \oplus (1 - \lambda)y) \le \max\{f(x), f(y)\}, \quad \forall x, y \in X \text{ and } \lambda \in (0, 1);$$

(3) quasi-concave if and only if $-f$ is quasi-convex.

https://doi.org/10.1017/S1446788719000041 Published online by Cambridge University Press

A function $f : X \to]-\infty, +\infty]$ is called proper if and only if $D(f) := \{x \in X : f(x) < +\infty\} \neq \emptyset$. The function f is called lower semicontinuous (lsc for short) at $x \in D(f)$ if and only if

$$\liminf_{y \to x} f(y) \geq f(x)$$

and is called $\triangle$-lower semicontinuous ($\triangle$-lsc for short) at $x \in D(f)$ if and only if

$$\liminf_{n \to +\infty} f(x_n) \geq f(x)$$

for each sequence $x_n \xrightarrow{\triangle} x$. It is a well-known result that each convex and lsc function is $\triangle$-lsc (see [4, Lemma 3.2.3]).

Let $K \subset X$ be nonempty. A function $f : K \times K \to \mathbb{R}$ is called a bifunction. An equilibrium problem for f and K, briefly an EP(f; K), consists of finding $x^* \in K$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in K. \quad \text{(EP)}$$

The point x^* is called an equilibrium point. We denote the set of all equilibrium points for (EP) by $S(f; K)$. Each equilibrium problem EP(f, K) has a dual, a 'convex feasibility problem', CFP(f, K). It consists of finding $x^* \in K$ such that $f(x, x^*) \leq 0$ for all $x \in K$. Equilibrium problems extend and unify several problems in optimization, variational inequalities, fixed point theory and many other problems in nonlinear analysis. Henceforth, $K \subset X$ denotes a nonempty, closed and convex set unless explicitly stated otherwise. Take $o \in X$, where o is an arbitrary but fixed point (o is called a base point). The following conditions may be used throughout the paper; therefore, we denominate them as:

*P*1: $f(x, x) = 0$ for all $x \in K$;
*P*2: $f(\cdot, y) : K \to \mathbb{R}$ is upper semicontinuous for all $y \in K$;
*P*3: $f(x, \cdot) : K \to \mathbb{R}$ is convex and lower semicontinuous for all $x \in K$.

The function f is called monotone if and only if:

*P*4: $f(x, y) + f(y, x) \leq 0$ for all $x, y \in K$.

The function f is called pseudo-monotone if and only if:

*P*4*: whenever $f(x, y) \geq 0$ with $x, y \in K$, we have $f(y, x) \leq 0$.

The function f is called θ-undermonotone if and only if:

*P*4•: there exists $\theta \geq 0$ such that $f(x, y) + f(y, x) \leq \theta d^2(x, y)$ for all $x, y \in K$.

The function f is called coercive if and only if:

*P*5: Let $o \in X$ be the base point. Then, for any sequence $\{x_k\} \subset K$ satisfying $\lim d(x_k, o) = +\infty$, there exist $u \in K$ and $n_0 \in \mathbb{N}$ such that $f(x_n, u) \leq 0$ for all $n \geq n_0$.

https://doi.org/10.1017/S1446788719000041 Published online by Cambridge University Press

Equilibrium problems for monotone and pseudo-monotone bifunctions have been extensively studied in Hilbert, Banach as well as in topological vector spaces by many authors (see [9, 12, 16, 21, 23] and many other references). Recently some authors have studied variational inequalities, monotone inclusions and equilibrium problems in Hadamard manifolds (see [6, 15, 28, 29, 31, 35, 37]). In order to extend and unify related results from Hilbert spaces and Hadamard manifolds as well as extend some recent results on variational inequalities and minimization problems in Hadamard spaces (see [25, 33, 36]), we study monotone and pseudo-monotone equilibrium problems in the Hadamard space setting.

The paper has been organized as follows. Following the introduction, we present some well-known lemmas in the Hadamard space framework. In Section 2, we study the existence of solutions of equilibrium problems. In Section 3, in order to approximate an equilibrium point, we use an auxiliary problem. Existence of solutions of the auxiliary problem is not guaranteed for bifunctions under the usual assumptions $P1, P2, P3, P4, P4^{*}, P4^{\bullet}$ in general Hadamard spaces (see the related explanations in the first part of Section 3). So, in this section, we study the existence of solutions of the auxiliary problem in several special cases. Section 4 is devoted to introducing the resolvent operator for pseudo-monotone bifunctions and its strong convergence to an equilibrium point. In Section 5, we prove Δ-convergence of the proximal point algorithm for pseudo-monotone bifunctions in Hadamard spaces. Since the strong convergence (convergence in metric) does not occur even in Hilbert space, in Section 6, we prove strong convergence of a regularized version of the sequence of Halpern type in Hadamard spaces. Finally, in Section 7, some examples and applications will be presented. Now we present some lemmas that we need in the next section.

Lemma 1.6. *With conditions $P1$, $P2$ and $P3$, every solution of* $\mathrm{CFP}(f, K)$ *solves* $\mathrm{EP}(f, K)$.

Proof. See [22, Lemma 2.4]. □

Corollary 1.7. *If f satisfies $P1$, $P2$, $P3$ and $P4^{*}$, then* $\mathrm{EP}(f, K)$ *and* $\mathrm{CFP}(f, K)$ *have the same solution set.*

The following lemma is the KKM lemma in complete CAT(0) spaces. It has been proved on finite-dimensional Hadamard manifolds in [15]. The proof is similar for complete CAT(0) spaces, but, for completeness of the paper, we rewrite the proof in complete CAT(0) spaces.

Lemma 1.8. *Suppose that X is a complete* CAT(0) *space and $K \subset X$. Let $G : K \to 2^{K}$ be a mapping such that for each $x \in K$, $G(x)$ is $\triangle$-closed. Suppose that:*

(i) *for all $x_1, \ldots, x_m \in K$, $\mathrm{conv}(\{x_1, \ldots, x_m\}) \subset \bigcup_{i=1}^{m} G(x_i)$;*

(ii) *there exists $x_0 \in K$ such that $G(x_0)$ is $\triangle$-compact,*

then $\bigcap_{x \in K} G(x) \neq \varnothing$.

https://doi.org/10.1017/S1446788719000041 Published online by Cambridge University Press

Proof. Take $x_1, \ldots, x_m \in K$ and define $D(\{x_1, \ldots, x_m\}) := \bigcup_{i=1}^m D_i$, where $D_1 = \{x_1\}$ and, for any $2 \le j \le n$, $D_j = \{z \in \gamma_{x_j,y} \,|\, y \in D_{j-1}\}$, where $\gamma_{x_j,y}$ is the geodesic joining x_j to some $y \in D_{j-1}$. Therefore, $D(\{x_1, \ldots, x_m\})$ is a closed subset of $\mathrm{conv}(\{x_1, \ldots, x_m\})$. Let $y_1 = x_1$ and, for $k = 2, \ldots, m$, choose $y_k \in D_k \subseteq D(\{x_1, \ldots, x_m\})$. Then y_k can be written as

$$y_k = \gamma(t_k), \tag{1.1}$$

where $t_k \in [0, 1]$ and γ is the geodesic joining x_k to some $y_{k-1} \in D_{k-1}$. To each x_i, we associate a corresponding vertex e_i of the simplex $\sigma = \langle e_1, \ldots, e_m\rangle \subset \mathbb{R}^{m+1}$. Let $T : \sigma \to D(\{x_1, \ldots, x_m\})$ be the mapping defined by induction as follows: for $\lambda_1 = e_1$, define $T(\lambda_1) = x_1$ and, for $1 < k \le m$, if $\lambda_k \in \langle e_1, \ldots, e_k\rangle \backslash \langle e_1, \ldots, e_{k-1}\rangle$, then $\lambda_k = t_k e_k + (1 - t_k)\lambda_{k-1}$ for some $t_k \in (0, 1]$ and $\lambda_{k-1} \in \langle e_1, \ldots, e_{k-1}\rangle$. Also, we define $T(\lambda_k) = \gamma_k(t_k)$, where γ_k is the geodesic joining x_k to $T(\lambda_{k-1})$ and t_k is the unique element in $[0, 1]$ such that $T(\lambda_k) = \gamma_k(t_k)$.

The equality (1.1) shows that $T(\sigma)$ coincides with $D(\{x_1, \ldots, x_m\})$. Now we show that T is continuous. For any $j = 1, 2$, let $\lambda^j = \sum_{i=1}^m t_i^j e_i \in \sigma$ for some sequences $\{t_i^j\}_{i=1}^m \subset [0, 1]$ satisfying $\sum_{i=1}^m t_i^j = 1$. By definition, we have that $T(\lambda^j) = \gamma_m^j(t_m^j)$, where γ_m^j joins x_m to $T(\sum_{i=1}^{m-1} t_i^j e_i)$. Now let $L := \mathrm{diam}(D(\langle x_1, \ldots, x_m\rangle))$; applying in turn parts (ii) and (iii) of Lemma 1.3,

$$\begin{aligned} d(T(\lambda^1), T(\lambda^2)) &\le d(\gamma_m^1(t_m^1), \gamma_m^1(t_m^2)) + d(\gamma_m^1(t_m^2), \gamma_m^2(t_m^2)) \\ &\le |t_m^1 - t_m^2|\, d\Big(x_m, T\Big(\sum_{i=1}^{m-1} t_i^1 e_i\Big)\Big) + d\Big(T\Big(\sum_{i=1}^{m-1} t_i^1 e_i\Big), T\Big(\sum_{i=1}^{m-1} t_i^2 e_i\Big)\Big) \\ &\le L|t_m^1 - t_m^2| + d\Big(T\Big(\sum_{i=1}^{m-1} t_i^1 e_i\Big), T\Big(\sum_{i=1}^{m-1} t_i^2 e_i\Big)\Big). \end{aligned}$$

By recursion, we obtain that $d(T(\lambda^1), T(\lambda^2)) \le L \sum_{i=1}^m |t_i^1 - t_i^2|$.

This shows the continuity of T. Consider the closed sets $\{E_i\}_{i=1}^m$ defined by $E_i := T^{-1}(D(\{x_1, \ldots, x_m\}) \cap G(x_i))$. Let us prove that for every $I \subset \{1, \ldots, m\}$,

$$\mathrm{conv}(\{e_i \,|\, i \in I\}) \subset \bigcup_{i \in I} E_i.$$

Indeed, let $\lambda = \sum_{j=1}^k t_{i_j} e_{i_j} \in \mathrm{conv}(\{e_{i_1}, \ldots, e_{i_k}\})$ with $\{t_{i_j}\} \subset [0, 1]$ such that $\sum_{j=1}^k t_{i_j} = 1$. Since, by the hypothesis,

$$T(\lambda) \in D(\{x_{i_1}, \ldots, x_{i_k}\}) \subseteq \mathrm{conv}(\{x_{i_1}, \ldots, x_{i_k}\}) \subseteq \bigcup_{n=1}^k G(x_{i_n}),$$

there exists $j \in \{1, \ldots, k\}$ for which $T(\lambda) \in G(x_{i_j}) \bigcap D(\{x_{i_1}, \ldots, x_{i_k}\})$ and, consequently, $\lambda \in E_{i_j}$. By applying the KKM lemma to the family $\{E_i\}_{i=1}^m$, we get existence of a point $\hat{\lambda} \in \mathrm{conv}(\{e_1, \ldots, e_m\})$ such that $\hat{\lambda} \in \bigcap_{i=1}^m E_i$, so $T(\hat{\lambda}) \in \bigcap_{i=1}^m G(x_i)$. We have already proved that the family of Δ-closed sets $\{G(x) \cap G(x_0)\}_{x \in K}$ has the finite intersection property. Since $G(x_0)$ is Δ-compact, it follows that $\bigcap_{x \in K} G(x) = \bigcap_{x \in K}(G(x_0) \cap G(x)) \neq \emptyset$. □

https://doi.org/10.1017/S1446788719000041 Published online by Cambridge University Press

2. Existence of solutions

In this section, we are going to study existence of the solutions to equilibrium problems in complete CAT(0) spaces. In [21], Iusem *et al.* proved the existence of solutions to pseudo-monotone equilibrium problems in Hilbert spaces. Now we want to extend their results to Hadamard spaces. We assume that X is a Hadamard space and $K \subset X$ is nonempty, closed and convex. Let $o \in K$ be the base point. For each $n \in \mathbb{N}$, set $K_n = \{x \in K \mid d(o, x) \leq n\}$. Then $K_n \neq \emptyset$ for all $n \in \mathbb{N}$. Suppose that f satisfies $P1$, $P2$ and $P3$. We define, for each $y \in K$,

$$L_f(n, y) := \{x \in K_n \mid f(y, x) \leq 0\}.$$

By applying Lemma 1.6 with K_n instead of K, we conclude that $\bigcap_{y \in K_n} L_f(n, y) \subseteq \{x \in K_n \mid f(x, y) \geq 0, \forall y \in K_n\}$, that is, each solution of the convex feasibility problem restricted to K_n is a solution of the equilibrium problem restricted to K_n. Let $K_n^\circ \subset K$ be the intersection of K with the open ball of radius n around o, that is, $K_n^\circ = \{x \in K \mid d(o, x) < n\}$. We need the following technical lemmas for the existence result.

Lemma 2.1. *Let f satisfy $P1$, $P2$ and $P3$. If for some $n \in \mathbb{N}$ and some $\bar{x} \in \bigcap_{y \in K_n} L_f(n, y)$ there exists $\bar{y} \in K_n^\circ$ such that $f(\bar{x}, \bar{y}) \leq 0$, then $f(\bar{x}, y) \geq 0$ for all $y \in K$.*

Proof. This is a trivial extension of [22, Lemma 3.7] to geodesic spaces. □

Definition 2.2. A function $f : K \times K \to \mathbb{R}$ is called properly quasi-monotone if for every finite set A of K and every $y \in \mathrm{conv}(A)$, $\min_{x \in A} f(x, y) \leq 0$.

Lemma 2.3. *If f satisfies one of the following conditions:*

(i) *$f(\cdot, y)$ is quasi-concave for all $y \in K$ and $P1$ holds;*
(ii) *$f(x, \cdot)$ is quasi-convex for all $x \in K$ and $P1$ and $P4^*$ hold,*

then f is properly quasi-monotone.

Proof. Let $A = \{x_0, x_1, \dots, x_k\} \subset K$ and $y \in \mathrm{conv}(A)$ be arbitrary. By Lemma 1.1, there is an integer $n \geq 0$ such that $y \in C_n(A)$. We will show that $\min_{x \in A} f(x, y) \leq 0$. Suppose, to the contrary, $f(x, y) > \lambda > 0$ for all $x \in A$.

(i) By quasi-convexity of $-f(\cdot, y)$ and the definition of $C_1(A)$,

$$-f(u, y) \leq \max\{-f(x, y) \mid x \in A\} < -\lambda < 0, \quad \forall u \in C_1(A).$$

Again by the definition of $C_2(A)$,

$$-f(v, y) \leq \max\{-f(u, y) \mid u \in C_1(A)\} \leq -\lambda < 0, \quad \forall v \in C_2(A)$$

and finally by induction we get $0 = -f(y, y) \leq -\lambda < 0$, which is a contradiction.

(ii) By using $P4^*$, we have $f(y, x_i) \leq 0$ for all $i \in \{0, \dots, k\}$. Now if $f(y, x_i) = 0$ for some i, then again by using $P4^*$, we have $f(x_i, y) \leq 0$, which is a contradiction. Therefore, $f(y, x_i) < 0$ for all $0 \leq i \leq k$. Next, by the quasi-convexity of $f(y, \cdot)$ and Lemma 1.1 and similar reasoning to that in part (i), we get a contradiction, which proves the lemma. □

https://doi.org/10.1017/S1446788719000041 Published online by Cambridge University Press

Theorem 2.4. *Suppose that f is properly quasi-monotone and $P1$, $P2$, $P3$ and $P5$ hold; then* $\mathrm{EP}(f, K)$ *admits a solution.*

Proof. Let $n \in \mathbb{N}$ be arbitrary; we are going to use Lemma 1.8 with K_n instead of K and $G(y) := L_f(n, y)$. Therefore, we must check the validity of its hypotheses. First we verify condition (i) of Lemma 1.8.

Take $x_0, x_1, \ldots, x_k \in K_n$ and $\bar{x} \in \mathrm{conv}(\{x_0, x_1, \ldots, x_k\})$. We must verify that $\bar{x} \in \bigcup_{i=0}^{k} L_f(n, x_i)$, that is, $\bar{x} \in K_n$ and $f(x_i, \bar{x}) \le 0$ for some i. Since K_n is convex, $\bar{x} \in K_n$ and the rest of this fact follows from the properly quasi-monotonicity assumption, which guarantees that $\min_{0 \le i \le k} f(x_i, \bar{x}) \le 0$.

Now we verify condition (ii) of Lemma 1.8. Since $f(y, \cdot)$ is convex and lower semicontinuous, $G(y) = L_f(n, y) = \{x \in K_n \mid f(y, x) \le 0\}$ is closed and convex. Also, $G(y)$ is bounded, because it is contained in K_n. Hence, by Lemma 1.5, $G(y)$ is $\triangle$-compact for all $y \in K$. Therefore, we are within the hypotheses of Lemma 1.8 and we can conclude that $\bigcap_{y \in K_n} L_f(n, y) \neq \emptyset$ for each $n \in \mathbb{N}$, so that for each $n \in \mathbb{N}$ we may choose $x_n \in \bigcap_{y \in K_n} L_f(n, y)$. We distinguish two cases.

(i) There is $n \in \mathbb{N}$ such that $d(o, x_n) < n$. In this case $x_n \in K_n^\circ$ solves $\mathrm{EP}(f, K)$ by Lemma 2.1.

(ii) $d(o, x_n) = n$ for all $n \in \mathbb{N}$. In this case $P5$ ensures the existence of $u \in K$ and $n_0 > 0$ such that $f(x_n, u) \le 0$ for all $n \ge n_0$. Take $n' \ge n_0$ such that $d(o, u) < n'$; then $f(x_{n'}, u) \le 0$ and $u \in K_{n'}^\circ$. Again, $x_{n'}$ turns out be a solution of $\mathrm{EP}(f, K)$ by Lemma 2.1. □

Theorem 2.5. *Let f satisfy $P1$, $P2$, $P3$ and $P4^*$; then* $\mathrm{EP}(f, K)$ *has a solution if and only if $P5$ holds.*

Proof. $\Rightarrow$ Take $x^* \in S(f, K)$; then $f(x^*, y) \ge 0$ for all $y \in K$. By $P4^*$, we have $f(y, x^*) \le 0$ for all $y \in K$. Hence, $P5$ holds.

$\Leftarrow$ Now, by Lemma 2.3, $P1$ and $P4^*$ imply that f is properly quasi-monotone; then, by Theorem 2.4, $\mathrm{EP}(f, K)$ has a solution if $P5$ holds. □

The following theorem also shows the existence of solutions for some equilibrium problems. It has been essentially proved on finite-dimensional Hadamard manifolds in [15, Theorem 3.2] and we rewrite the proof in Hadamard spaces.

Theorem 2.6. *Let $f : K \times K \to \mathbb{R}$ be a bifunction such that:*

(i) *for any $x \in K$, $f(x, x) \ge 0$;*

(ii) *for every $x \in K$, the set $\{y \in K \mid f(x, y) < 0\}$ is convex;*

(iii) *for every $y \in K$, $x \mapsto f(x, y)$ is $\triangle$-upper semicontinuous;*

(iv) *there exist a $\triangle$-compact set $L \subseteq X$ and a point $y_0 \in L \cap K$ such that $f(x, y_0) < 0$ for all $x \in K \backslash L$;*

then there exists a point $x_0 \in L \cap K$ satisfying $f(x_0, y) \ge 0$ for all $y \in K$.

https://doi.org/10.1017/S1446788719000041 Published online by Cambridge University Press

Proof. The mapping $G : K \to 2^K$ is defined by $G(y) := \{x \in K \mid f(x, y) \geq 0\}$ for each $y \in K$. Since $f(\cdot, y)$ is $\triangle$-upper semicontinuous, $G(y)$ is $\triangle$-closed for all $y \in K$. In turn by condition (iv) there exists a point $y_0 \in K$ such that $G(y_0) \subseteq L$, so $G(y_0)$ is $\triangle$-compact. We are going to use Lemma 1.8; thus, we must prove that for every $y_1, \ldots, y_m \in K$, we have $\operatorname{conv}(\{y_1, \ldots, y_m\}) \subset \bigcup_{i=1}^m G(y_i)$.

To this end, suppose to the contrary that there exists a point x' such that $x' \in \operatorname{conv}(\{y_1, \ldots, y_m\})$ but $x' \notin \bigcup_{i=1}^m G(y_i)$, that is,

$$f(x', y_i) < 0, \quad 1 \leq i \leq m.$$

This implies that for all $1 \leq i \leq m$, we have $y_i \in \{y \in K \mid f(x', y) < 0\}$. Since $\{y \in K \mid f(x', y) < 0\}$ is convex, we have $x' \in \operatorname{conv}(\{y_1, \ldots, y_m\}) \subseteq \{y \in K \mid f(x', y) < 0\}$, but, by (i), we have $f(x', x') \geq 0$, which is a contradiction. So, Lemma 1.8 applies and there exists $x_0 \in K$ such that $x_0 \in \bigcap_{y \in K} G(y)$ with $x_0 \in G(y_0) \subseteq L \cap K$. In other words, there exists a point $x_0 \in L \cap K$ satisfying $f(x_0, y) \geq 0$ for all $y \in K$. □

3. An auxiliary problem

In this section, we consider the proximal point scheme for pseudo-monotone equilibrium problems in Hadamard spaces to approximate an equilibrium point. The proximal point algorithm for a pseudo-monotone bifunction $f : K \times K \to \mathbb{R}$ generates the sequence $\{x_k\}$ which is given by the following process. Given $x_0 \in X$ arbitrary, inductively for $x_{k-1} \in K$, select x_k so that it satisfies in the following inequality:

$$f(x_k, y) + \lambda_{k-1}\langle \overrightarrow{x_{k-1}x_k}, \overrightarrow{x_k y}\rangle \geq 0, \quad \forall y \in K, \tag{3.1}$$

where $\{\lambda_k\}$ is a positive sequence. When X is a Hilbert space, f is θ-undermonotone and $\lambda_k > \theta$ for all $k \in \mathbb{N}$, Iusem and Sosa in [23] proved existence and uniqueness of the sequence generated by (3.1). They also proved the weak convergence of the sequence to an equilibrium point of f when f is a pseudo-monotone bifunction. Unfortunately, we cannot obtain existence of the sequence $\{x_k\}$ defined by (3.1) in general Hadamard spaces for each bifunction f with the usual conditions $P1$, $P2$, $P3$, $P4$, $P4^*$ and $P4^\bullet$ discussed in Section 2. In [25], the first author and Ranjbar proved the existence of the sequence defined by (3.1) and its Δ-convergence for a bifunction $f(x, y) = \langle \overrightarrow{Txx}, \overrightarrow{xy}\rangle$, where $T : X \to X$ is a nonexpansive mapping. In this section, we study the existence of the sequence given by (3.1) in some other cases. In order to prove existence and uniqueness of the sequence $\{x_k\}$ satisfying (3.1), consider the bifunction $\tilde{f}$ which is defined by

$$\tilde{f}(x, y) = f(x, y) + \lambda\langle \overrightarrow{\bar{x}x}, \overrightarrow{xy}\rangle, \tag{3.2}$$

where $\bar{x} \in X$ and f is a bifunction that satisfies $P1$, $P2$, $P3$ and $P4^\bullet$ and $\lambda > \theta$. First we prove the uniqueness of the sequence $\{x_k\}$ satisfying (3.1). Assume that both x' and x'' solve $\mathrm{EP}(\tilde{f}, K)$. Note that

$$0 \leq \tilde{f}(x', x'') = f(x', x'') + \lambda\langle \overrightarrow{\bar{x}x'}, \overrightarrow{x'x''}\rangle,$$
$$0 \leq \tilde{f}(x'', x') = f(x'', x') + \lambda\langle \overrightarrow{\bar{x}x''}, \overrightarrow{x''x'}\rangle.$$

https://doi.org/10.1017/S1446788719000041 Published online by Cambridge University Press

By summing both sides of the above inequalities,

$$0 \le f(x', x'') + f(x'', x') - \lambda d^2(x', x'') \le (\theta - \lambda) d^2(x', x'').$$

Since $\lambda > \theta$, we deduce that $x' = x''$.

LEMMA 3.1. *Let f satisfy $P1$, $P2$, $P3$ and $P4^\bullet$ and $\lambda > \theta$; then $\tilde{f}$ satisfies $P4$ and $P5$.*

PROOF. First we prove that $\tilde{f}$ satisfies $P4$. Note that

$$\begin{aligned}\tilde{f}(x,y) + \tilde{f}(y,x) &= f(x,y) + f(y,x) + \lambda\langle\overrightarrow{\bar{x}x}, \overrightarrow{xy}\rangle + \lambda\langle\overrightarrow{\bar{x}y}, \overrightarrow{yx}\rangle \\ &= f(x,y) + f(y,x) - \lambda d^2(x,y) \le (\theta - \lambda) d^2(x,y) \le 0.\end{aligned}$$

Now we show that $\tilde{f}$ satisfies $P5$. Take and fix $o \in X$, then take a sequence $\{x_k\}$ such that $\lim d(o, x_k) = +\infty$ and let $u = P_K(\bar{x})$, where $P_K : X \longrightarrow K$ is the projection map onto K. Since, by [17, Theorem 2.2], we have $\langle\overrightarrow{\bar{x}u}, \overrightarrow{x_k u}\rangle \le 0$,

$$\begin{aligned}\tilde{f}(x_k, u) &= f(x_k, u) + \lambda\langle\overrightarrow{\bar{x}x_k}, \overrightarrow{x_k u}\rangle \\ &= f(x_k, u) + \lambda\langle\overrightarrow{\bar{x}u}, \overrightarrow{x_k u}\rangle + \lambda\langle\overrightarrow{u x_k}, \overrightarrow{x_k u}\rangle \\ &\le f(x_k, u) - \lambda d^2(u, x_k) \le -f(u, x_k) + \theta d^2(u, x_k) - \lambda d^2(u, x_k) \\ &= -f(u, x_k) - (\lambda - \theta) d^2(u, x_k),\end{aligned} \tag{3.3}$$

where in the second inequality we have used the θ-undermonotonicity of f. Now take z in the domain of $f(u, \cdot)$ and $t \in \mathbb{R}$ with $t < f(u, z)$; since $f(u, \cdot)$ is convex, proper and lower semicontinuous, by [2, Lemma 3.2], there are $v \in X$ and a real number $t < s \le f(u, z)$ such that

$$f(u, y) \ge \frac{1}{s - t}\langle\overrightarrow{vz}, \overrightarrow{vy}\rangle + s, \quad \forall y \in K;$$

therefore, by setting $y = x_k$, we have $-f(u, x_k) \le (-1/(s - t))\langle\overrightarrow{vz}, \overrightarrow{vx_k}\rangle - s$. Now, by the Cauchy–Schwarz inequality,

$$-f(u, x_k) \le \frac{1}{s - t} d(v, z)\, d(v, x_k) - s \le \frac{1}{s - t} d(v, z)\, d(u, x_k) + \frac{1}{s - t} d(v, z)\, d(v, u) - s. \tag{3.4}$$

By replacing (3.4) in (3.3),

$$\tilde{f}(x_k, u) \le d(x_k, u)\left[\frac{1}{s - t} d(v, z) - (\lambda - \theta)\, d(u, x_k)\right] + \frac{1}{s - t} d(v, z)\, d(v, u) - s. \tag{3.5}$$

Since $\lambda - \theta > 0$ and $\lim d(x_k, o) = +\infty$, so that $\lim d(x_k, u) = +\infty$, it follows easily from (3.5) that $\lim \tilde{f}(x_k, u) = -\infty$ as $k \to +\infty$, so that $\tilde{f}(x_k, u) \le 0$ for sufficiently large k. Therefore, $\tilde{f}$ satisfies $P5$. □

PROPOSITION 3.2. *Let f satisfy $P1$, $P2$, $P3$ and $P4^\bullet$ and $\lambda > \theta$. If $\tilde{f}(x, \cdot)$ is convex for all $x \in K$, then* $\mathrm{EP}(\tilde{f}, K)$ *has a unique solution.*

https://doi.org/10.1017/S1446788719000041 Published online by Cambridge University Press

Proof. It is clear that $\tilde{f}$ satisfies $P1$, $P2$ and $P3$. Also, Lemma 3.1 shows that $\tilde{f}$ satisfies $P4$ and $P5$. Hence, Theorem 2.5 implies that $\tilde{f}$ has a solution. Uniqueness of the solution has already been proved. □

By Proposition 3.2, if $\tilde{f}(x,\cdot)$ is convex for all $x \in K$, then $\mathrm{EP}(\tilde{f},K)$ has a unique solution, but, since $y \mapsto \langle \overrightarrow{\bar{x}x}, \overrightarrow{xy}\rangle$ is not convex in general unless in flat Hadamard spaces, the conditions of Proposition 3.2 are satisfied in these spaces and we have the following corollary.

Corollary 3.3. *Let f satisfy $P1$, $P2$, $P3$ and $P4^\bullet$ and $\lambda > \theta$. If X is a flat Hadamard space, then* $\mathrm{EP}(\tilde{f},K)$ *has a unique solution.*

If the function $y \mapsto \langle \overrightarrow{\bar{x}x}, \overrightarrow{xy}\rangle$ is convex, existence of a solution for $\tilde{f}$ is ensured by the usual conditions on the bifunction f. But in general $y \mapsto \langle \overrightarrow{\bar{x}x}, \overrightarrow{xy}\rangle$ is not convex in Hadamard spaces. In the following theorems we try to overcome this problem and prove the existence of solutions for $\tilde{f}$ in some special cases.

In order to prove existence of an equilibrium point for $\tilde{f}$ when f is cyclic monotone, we recall the definition of cyclic monotonicity of bifunctions from [19] and a lemma that we need to prove the main result.

Definition 3.4. The function $f : K \times K \to \mathbb{R}$ is said to be cyclic monotone if and only if for each $n \in \mathbb{N}$ and each $x_1, x_2, \ldots, x_n \in X$,

$$f(x_1,x_2) + f(x_2,x_3) + \cdots + f(x_n,x_1) \le 0.$$

Lemma 3.5. *Suppose that $f : K \times K \to \mathbb{R}$ is monotone and $P1$ is satisfied. Also, f is convex with respect to the second variable and upper hemi-continuous (upper semicontinuous along geodesics) with respect to the first variable. Let $\bar{x} \in K$; then the following are equivalent:*

(i) *there exists $x \in K$ such that $f(z,x) + \langle \overrightarrow{x\bar{x}}, \overrightarrow{xz}\rangle \le 0$ for all $z \in K$;*

(ii) *there exists $x \in K$ such that $f(x,z) \ge \langle \overrightarrow{x\bar{x}}, \overrightarrow{xz}\rangle$ for all $z \in K$.*

Proof. (ii) ⇒ (i) is trivial by the monotonicity of f. We prove (i) ⇒ (ii). For all $z \in K$ and $0 \le t \le 1$, take $z_t = tz \oplus (1-t)x$. By convexity of f with respect to the second argument,

$$\begin{aligned}
0 = f(z_t,z_t) &\le t f(z_t,z) + (1-t) f(z_t,x) \le t f(z_t,z) + (1-t)\langle \overrightarrow{\bar{x}x}, \overrightarrow{xz_t}\rangle \\
&= t f(z_t,z) + \frac{1-t}{2}\{d^2(\bar{x},z_t) - d^2(x,z_t) - d^2(\bar{x},x)\} \\
&= t f(z_t,z) + \frac{1-t}{2}\{t\, d^2(\bar{x},z) + (1-t)\, d^2(\bar{x},x) - t(1-t)\, d^2(z,x) - t^2\, d^2(z,x) - d^2(\bar{x},x)\} \\
&= t f(z_t,z) + \frac{t(1-t)}{2}\{d^2(\bar{x},z) - d^2(\bar{x},x) - d^2(z,x)\}.
\end{aligned}$$

Therefore,

$$f(z_t,z) \ge (1-t)\langle \overrightarrow{x\bar{x}}, \overrightarrow{xz}\rangle.$$

https://doi.org/10.1017/S1446788719000041 Published online by Cambridge University Press

Letting $t \to 0$, by upper hemi-continuity of f with respect to the first argument,

$$f(x, z) \geq \langle \overrightarrow{x\bar{x}}, \overrightarrow{xz} \rangle, \quad \forall z \in K,$$

as desired. □

THEOREM 3.6. *Let $f : K \times K \to \mathbb{R}$ be a cyclic monotone bifunction which satisfies $P1$, $P2$ and $P3$. Then $\tilde{f}$ has a solution.*

PROOF. Without loss of generality from now to the end of this section, we take $\lambda = 1$ in $\mathrm{EP}(\tilde{f}, K)$. By a similar argument to [19, Propositions 5.1 and 5.2], $f(z, x) \leq g(x) - g(z)$, where g is a convex and lower semicontinuous function on X. By [2, Theorem 4.2], for a given $\bar{x} \in K$ there exists exactly one $x \in K$ such that

$$g(x) - g(z) \leq \langle \overrightarrow{\bar{x}x}, \overrightarrow{xz} \rangle, \quad \forall z \in K;$$

then

$$f(z, x) + \langle \overrightarrow{x\bar{x}}, \overrightarrow{xz} \rangle \leq 0, \quad \forall z \in K.$$

Now Lemma 3.5 implies the required result. □

Now we want to prove existence of an equilibrium point for $\tilde{f}$ when f satisfies a cyclic pseudo-monotonicity condition. In [20], cyclic pseudo-monotonicity was defined for pseudo-monotone operators. In [24], we defined it for pseudo-monotone bifunctions as follows.

(i) f is called n-pseudo-monotone if the following implication holds:

$$f(x_1, x_2) \geq 0, \quad f(x_2, x_3) \geq 0, \ldots, f(x_{n-1}, x_n) \geq 0 \Longrightarrow f(x_n, x_1) \leq 0;$$

(ii) f is called cyclic pseudo-monotone if f is n-pseudo-monotone for all $n \in \mathbb{N}$.

In order to prove the existence of a solution for $\tilde{f}$, we define a stronger version of cyclic pseudo-monotonicity as follows.

DEFINITION 3.7. The function f is called n-pseudo-monotone of type (I) if the following implication holds:

$$f(x_1, x_2) + f(x_2, x_3) + \cdots + f(x_{n-2}, x_{n-1}) \leq f(x_1, x_n) \Longrightarrow f(x_n, x_{n-1}) \leq 0.$$

The function f is called cyclic pseudo-monotone of type (I) if it is n-pseudo-monotone of type (I) for each $n \geq 3$.

First we prove that the new condition is stronger than that of n-pseudo-monotonicity.

THEOREM 3.8. *If $f : K \times K \to \mathbb{R}$ is n-pseudo-monotone of type (I) and P_1 is satisfied, then f is n-pseudo-monotone.*

https://doi.org/10.1017/S1446788719000041 Published online by Cambridge University Press

Proof. Take $x_1, x_2, \dots, x_n \in K$ and let $f(x_1, x_2) \ge 0$, $f(x_2, x_3) \ge 0, \dots, f(x_{n-1}, x_n) \ge 0$. Condition $P1$ and n-pseudo-monotonicity of type (I) imply that P_4^* and hence

$$f(x_{n-1}, x_{n-2}) + f(x_{n-2}, x_{n-3}) + \cdots + f(x_2, x_1) \le f(x_{n-1}, x_n).$$

Now n-pseudo-monotonicity of type (I) implies that $f(x_n, x_1) \le 0$, as desired. □

Proposition 3.9. *Assume that* $f : K \times K \to \mathbb{R}$ *is cyclic pseudo-monotone of type (I) and there are* $u, v \in K$ *such that* $f(u, v) > 0$. *Then:*

(i) *there exists a function* $g : K \to \mathbb{R}$ *such that* $f(x, y) \ge g(y) - g(x)$;
(ii) *if* $f(\cdot, y)$ *is concave for each* $y \in K$, *then* g *is convex.*

Proof. (i) Let $f : K \times K \to \mathbb{R}$ be n-pseudo-monotone of type (I) and $f(u, v) > 0$. Note that the definition of n-pseudo-monotone of type (I) implies that $f(x_1, x_2) + f(x_2, x_3) + \cdots + f(x_{n-2}, v) > f(x_1, u)$ for each $x_1, x_2, \dots, x_{n-2} \in K$. Now we define $\varphi(x_1) = \inf_{x_2,\dots,x_{n-2},n\ge3}\{f(x_1, x_2) + f(x_2, x_3) + \cdots + f(x_{n-2}, v)\} \ge f(x_1, u)$. Hence, we have $f(x_1, x_2) + \inf_{x_3,\dots,x_{n-2},n\ge3}\{f(x_2, x_3) + \cdots + f(x_{n-2}, v)\} \ge \inf_{x_2,\dots,x_{n-2},n\ge3}\{f(x_1, x_2) + \cdots + f(x_{n-2}, v)\}$; therefore, $f(x_1, x_2) \ge \varphi(x_1) - \varphi(x_2)$. Taking $g = -\varphi$, we have $f(x, y) \ge g(y) - g(x)$.

(ii) If, for all $y \in K$, $f(\cdot, y)$ is concave, then for all $\lambda \in (0, 1)$ and $z_1, z_2 \in K$,

$$\begin{aligned}
\varphi(\lambda z_1 + (1-\lambda)z_2) &= \inf_{x_2,\dots,x_{n-2},n\ge3}\{f(\lambda z_1 + (1-\lambda)z_2, x_2) + \cdots + f(x_{n-2}, v)\}\\
&\ge \lambda \inf_{x_2,\dots,x_{n-2},n\ge3}\{f(z_1, x_2) + \cdots + f(x_{n-2}, v)\}\\
&\quad + (1-\lambda)\inf_{x_2,\dots,x_{n-2},n\ge3}\{f(z_2, x_2) + \cdots + f(x_{n-2}, v)\}\\
&= \lambda\varphi(z_1) + (1-\lambda)\varphi(z_2).
\end{aligned}$$

Therefore, $g = -\varphi$ is convex. □

Theorem 3.10. *Let* f *be a cyclic pseudo-monotone bifunction of type (I) which satisfies* $P2$. *If* f *is concave with respect to the first argument and there exist* $u, v \in K$ *such that* $f(u, v) > 0$, *then* $\mathrm{EP}(\tilde{f}, K)$ *has a solution.*

Proof. By Proposition 3.9, $f(x, y) \ge g(y) - g(x)$, where g is convex and lower semicontinuous. Therefore, by [2, Theorem 4.2], for each $\bar{x} \in K$ there exists $x \in K$ such that

$$g(y) - g(x) \ge \langle \overrightarrow{x\bar{x}}, \overrightarrow{xy} \rangle, \quad \forall y \in K.$$

This implies that $\mathrm{EP}(\tilde{f}, K)$ has a solution. □

Problem. We have already proved existence of a solution for $\mathrm{EP}(\tilde{f}, K)$ or equivalently existence of a sequence which satisfies (3.1) by imposing some conditions on the monotone or pseudo-monotone bifunction f and the Hadamard space X, but we do not know whether the problem (3.2) has a solution without these extra conditions.

https://doi.org/10.1017/S1446788719000041 Published online by Cambridge University Press

4. Convergence of resolvent

Now consider a monotone bifunction $f : K \times K \to \mathbb{R}$. Assume that for each $\lambda > 0$ and $\bar{x} \in K$, the equilibrium problem for $\tilde{f}$ (see (3.2)) has a solution that is unique. This unique solution is denoted by $J_\lambda^f \bar{x}$ and it is called the resolvent of f of order $\lambda > 0$ at $\bar{x}$. The resolvent J_λ^f or briefly J_λ for monotone bifunctions in Hilbert and Banach spaces has been introduced by Mansour *et al.* in [30] (see also [19]). In Hadamard spaces, we proved existence of the resolvent in some special cases in the previous section. In the following theorem we prove that J_λ is firmly nonexpansive and then prove that for each $x \in X$, $J_\lambda x$ converges strongly to an equilibrium point of f as $\lambda \to 0$ if $S(f, K) \neq \emptyset$. First we recall the definitions of firmly nonexpansive and quasi-firmly nonexpansive mappings.

Definition 4.1. A mapping $T : X \to X$ is called firmly nonexpansive if and only if

$$\langle \overrightarrow{xy}, \overrightarrow{TxTy} \rangle \geq d^2(Tx, Ty), \quad \forall x, y \in X.$$

The mapping T is called quasi-firmly nonexpansive if $\mathrm{Fix}(T) \neq \emptyset$, where $\mathrm{Fix}(T)$ is the set of all fixed points of T and

$$\langle \overrightarrow{xp}, \overrightarrow{Txp} \rangle \geq d^2(Tx, p), \quad \forall x \in X$$

for each $p \in \mathrm{Fix}(T)$.

Proposition 4.2. *Let $f : K \times K \to \mathbb{R}$ be a bifunction and $\lambda > 0$ be such that $J_\lambda x$ exists.*

(i) *If f is monotone, then the mapping $x \mapsto J_\lambda x$ is firmly nonexpansive.*
(ii) *If f is pseudo-monotone and $S(f, K) \neq \emptyset$, then $J_\lambda x$ is quasi-firmly nonexpansive.*

Proof. (i) First suppose that f is monotone. Take two points $x, z \in X$. We have

$$f(J_\lambda x, y) + \lambda \langle \overrightarrow{xJ_\lambda x}, \overrightarrow{J_\lambda xy} \rangle \geq 0, \quad \forall y \in K \tag{4.1}$$

and also

$$f(J_\lambda z, y) + \lambda \langle \overrightarrow{zJ_\lambda z}, \overrightarrow{J_\lambda zy} \rangle \geq 0, \quad \forall y \in K. \tag{4.2}$$

Now, letting $y = J_\lambda z$ in (4.1) and $y = J_\lambda x$ in (4.2) and then summing the latest inequalities, by the monotonicity of f,

$$\langle \overrightarrow{xJ_\lambda x}, \overrightarrow{J_\lambda xJ_\lambda z} \rangle + \langle \overrightarrow{zJ_\lambda z}, \overrightarrow{J_\lambda zJ_\lambda x} \rangle \geq 0.$$

By a straightforward computation and using quasi-inner product properties,

$$\langle \overrightarrow{xz}, \overrightarrow{J_\lambda xJ_\lambda z} \rangle \geq d^2(J_\lambda x, J_\lambda z),$$

from which follows the desired result. Also, the last inequality implies nonexpansiveness of J_λ by the Cauchy–Schwarz inequality.

(ii) If f is pseudo-monotone, then set $y = p \in S(f, K)$ in (4.1); since $f(J_\lambda x, p) \leq 0$,

$$\langle \overrightarrow{xJ_\lambda x}, \overrightarrow{J_\lambda xp} \rangle \geq 0,$$

which implies that

$$\langle \overrightarrow{xp}, \overrightarrow{J_\lambda xp} \rangle \geq d^2(J_\lambda x, p).$$

□

https://doi.org/10.1017/S1446788719000041 Published online by Cambridge University Press

It is easy to see that in the two cases of Proposition 4.2, $S(f,K) = \mathrm{Fix}(J_\lambda)$.

Before the main result of this section, we need to prove the Kadec–Klee property in Hadamard spaces.

PROPOSITION 4.3. *Suppose that x_n is Δ-convergent to x and there exists $y \in X$ such that* $\limsup d(x_n, y) \le d(x, y)$*; then x_n converges strongly to x.*

PROOF. By the definition and properties of quasi-linearization,

$$\begin{aligned} d^2(x_n, x) &= \langle \overrightarrow{x_n x}, \overrightarrow{x_n x} \rangle = \langle \overrightarrow{x_n x}, \overrightarrow{x_n y} \rangle + \langle \overrightarrow{x_n x}, \overrightarrow{y x} \rangle \\ &= \langle \overrightarrow{x_n y}, \overrightarrow{x_n y} \rangle + \langle \overrightarrow{y x}, \overrightarrow{x_n y} \rangle + \langle \overrightarrow{x_n x}, \overrightarrow{y x} \rangle \\ &= d^2(x_n, y) + 2\langle \overrightarrow{y x}, \overrightarrow{x_n x} \rangle - d^2(x, y). \end{aligned}$$

Taking limsup when $n \to +\infty$, by [1, Theorem 2.6] and the hypotheses,

$$\limsup d^2(x_n, x) \le \limsup d^2(x_n, y) - d^2(x, y) \le 0,$$

as desired. □

THEOREM 4.4. *Let $f : K \times K \to \mathbb{R}$ be a monotone bifunction that satisfies $P1, P3$, Δ-upper semicontinuity with respect to the first argument and $S(f,K) \neq \emptyset$. If for each $\lambda > 0$ and $x \in K$, $J_\lambda x$ exists, then, as $\lambda \to 0$, $J_\lambda x$ converges strongly to $p \in S(f,K)$, which is the nearest point of $S(f,K)$ to x.*

PROOF. Take $p \in S(f,K)$. By Proposition 4.2, $d(J_\lambda x, p) \le d(x, p)$. Therefore, $\{J_\lambda x\}$ is bounded. Suppose that there is a sequence λ_n converging to 0 such that $J_{\lambda_n} x$ Δ-converges to q. By Δ-upper semicontinuity of f and (4.1), we get $f(q, y) \ge 0$ for each $y \in K$ and hence $q \in S(f,K)$. Note, by monotonicity of f, $f(J_\lambda x, p) \le 0$ for all $p \in S(f,K)$. Therefore, (4.1) implies that $\langle \overrightarrow{x J_\lambda x}, \overrightarrow{J_\lambda x p} \rangle \ge 0$. Hence,

$$d^2(J_\lambda x, x) \le \langle \overrightarrow{x J_\lambda x}, \overrightarrow{x p} \rangle, \quad \forall p \in S(f,K).$$

By the Cauchy–Schwarz inequality,

$$d(x, J_\lambda x) \le d(x, p), \quad \forall p \in S(f,K). \tag{4.3}$$

Now taking $\lambda = \lambda_n$ and taking liminf when $n \to +\infty$, since $d(x, \cdot)$ is convex and continuous, so Δ-lower semicontinuous,

$$d(x, q) \le d(x, p), \quad \forall p \in S(f,K).$$

Therefore, $q = P_{S(f,K)} x$. This proves the Δ-convergence of $J_\lambda x$ to $P_{S(f,K)} x$ as $\lambda \to 0$. By (4.3),

$$d(x, J_\lambda x) \le d(x, P_{S(f,K)} x).$$

Now, by Proposition 4.3, $J_\lambda x$ converges strongly to $P_{S(f,K)} x$ as $\lambda \to 0$. □

REMARK. Theorem 4.4 is true also for pseudo-monotone bifunctions if $J_\lambda x$ exists. But since by Proposition 3.2, the condition $\lambda > \theta \ge 0$ is essential for existence of a solution to $\tilde{f}$ and therefore existence of $J_\lambda x$, we explored Theorem 4.4 only for monotone bifunctions.

https://doi.org/10.1017/S1446788719000041 Published online by Cambridge University Press

5. Proximal point algorithm

In this section, we study the convergence of the proximal point method for equilibrium problems when the bifunction f satisfies $P1$, $P2$, $P3$, $P4^*$ and $P4^\bullet$ by assuming existence of a sequence that satisfies (3.1). For computational and numerical purposes and since the existence of the sequence satisfying (3.1) is not guaranteed in general, we consider an inexact version of (3.1). Let θ be the undermonotonicity constant of f. Take a sequence of regularization parameters $\{\lambda_k\} \subset (\theta, \bar{\lambda}]$ for some $\bar{\lambda} > \theta$. Take $x_0 \in X$ and construct the sequence $\{x_k\} \subset K$ as follows.

Given x_k, we take y_k such that $d(x_k, y_k) \leq e_k$ and in turn x_{k+1} as the unique solution of problem $\mathrm{EP}(f_k, K)$, where $f_k : K \times K \to \mathbb{R}$ is defined as

$$f_k(x, y) = f(x, y) + \lambda_k \langle \overrightarrow{y_k x}, \overrightarrow{xy} \rangle, \tag{5.1}$$

where $\sum_{k=1}^{\infty} e_k < +\infty$. Throughout this section, we assume that $\sum_{k=1}^{\infty} e_k < +\infty$.

Lemma 5.1. *Consider* $\mathrm{EP}(f, K)$*, where f satisfies $P1$, $P2$, $P3$, $P4^*$ and $P4^\bullet$. If* $\mathrm{EP}(f, K)$ *has a solution, then the sequence $\{x_k\}$, which is generated by (5.1), is bounded and* $\lim d(x_k, x_{k+1}) = 0$.

Proof. Take $x^* \in S(f, K)$. Note that $f(x_{k+1}, x^*) + \lambda_k \langle \overrightarrow{y_k x_{k+1}}, \overrightarrow{x_{k+1} x^*} \rangle \geq 0$. Since $f(x_{k+1}, x^*) \leq 0$, we have $\langle \overrightarrow{y_k x_{k+1}}, \overrightarrow{x_{k+1} x^*} \rangle \geq 0$, which implies that

$$d^2(x_{k+1}, x^*) + d^2(y_k, x_{k+1}) \leq d^2(y_k, x^*).$$

Therefore, we conclude that

$$\begin{aligned} d^2(x_{k+1}, x^*) + d^2(y_k, x^*) + d^2(x_{k+1}, x^*) - 2\langle \overrightarrow{y_k x^*}, \overrightarrow{x_{k+1} x^*} \rangle &\leq d^2(y_k, x^*) \\ \Longrightarrow 2d^2(x_{k+1}, x^*) \leq 2\langle \overrightarrow{y_k x^*}, \overrightarrow{x_{k+1} x^*} \rangle &\leq 2d(y_k, x^*) d(x_{k+1}, x^*) \\ \Longrightarrow d(x_{k+1}, x^*) \leq d(y_k, x^*) &\leq d(x_k, x^*) + d(x_k, y_k). \end{aligned}$$

Hence,

$$d(x_{k+1}, x^*) \leq d(x_k, x^*) + e_k. \tag{5.2}$$

Therefore, $\lim d(x_k, x^*)$ exists.

Also, from $d^2(x_{k+1}, x^*) + d^2(y_k, x_{k+1}) \leq d^2(y_k, x^*)$,

$$\begin{aligned} d^2(x_{k+1}, x^*) + d^2(x_k, y_k) + d^2(x_{k+1}, x_k) - 2\langle \overrightarrow{y_k x_k}, \overrightarrow{x_{k+1} x_k} \rangle \\ \leq d^2(x_k, y_k) + d^2(x_k, x^*) + 2\langle \overrightarrow{y_k x_k}, \overrightarrow{x_k x^*} \rangle. \end{aligned}$$

Thus,

$$d^2(x_{k+1}, x_k) \leq d^2(x_k, x^*) - d^2(x_{k+1}, x^*) + 2\langle \overrightarrow{y_k x_k}, \overrightarrow{x_{k+1} x^*} \rangle.$$

By the Cauchy–Schwarz inequality,

$$d^2(x_{k+1}, x_k) \leq d^2(x_k, x^*) - d^2(x_{k+1}, x^*) + 2e_k d(x_{k+1}, x^*).$$

Since $\lim d(x_k, x^*)$ exists and $\sum_{k=1}^{\infty} e_k < +\infty$, $\lim d(x_{k+1}, x_k) = 0$. □

https://doi.org/10.1017/S1446788719000041 Published online by Cambridge University Press

Theorem 5.2. *Consider* EP(f, K)*, where* f *satisfies* $P1$*,* $P3$*,* $P4^*$ *and* $P4^\bullet$*. If* $f(\cdot, y)$ *is* $\triangle$*-upper semicontinuous for all* $y \in K$ *and* EP(f, K) *has a solution, then the sequence* $\{x_k\}$ *generated by (5.1) is* $\triangle$*-convergent to some solution of* EP(f, K)*.*

Proof. Fix $y \in K$. Since x_{k+1} solves EP(f_k, K),

$$\begin{aligned}0 \le f_k(x_{k+1}, y) &= f(x_{k+1}, y) + \lambda_k \langle \overrightarrow{y_k x_{k+1}}, \overrightarrow{x_{k+1} y} \rangle \\ &\le f(x_{k+1}, y) + \lambda_k d(y_k, x_{k+1}) d(x_{k+1}, y) \\ &\le f(x_{k+1}, y) + \lambda_k [d(y_k, x_k) + d(x_k, x_{k+1})]\, d(x_{k+1}, y).\end{aligned}$$

Since $\{\lambda_k\}$ and $\{x_k\}$ are bounded and by Lemma 5.1, $\lim d(x_{k+1}, x_k) = 0$,

$$0 \le \liminf f(x_k, y), \qquad \forall y \in K. \tag{5.3}$$

On the other hand, since $\{x_k\}$ is bounded and K is closed and convex, there exist a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ and $x' \in K$ such that $x_{k_i} \xrightarrow{\triangle} x'$. Now, since $f(\cdot, y)$ is $\triangle$-upper semicontinuous for all $y \in K$,

$$0 \le \liminf f(x_k, y) \le \limsup f(x_{k_i}, y) \le f(x', y)$$

for all $y \in K$, so that $x' \in S(f, K)$.

It remains to prove that there exists only one $\triangle$-cluster point of $\{x_k\}$. Let x', x'' be two $\triangle$-cluster points of $\{x_k\}$, so that there exist two subsequences $\{x_{k_i}\}$ and $\{x_{k_j}\}$ of $\{x_k\}$ whose $\triangle - \lim$ points are x' and x'', respectively. We have already proved that x' and x'' are solutions of EP(f, K). In turn by (5.2), we can assume that $\lim d(x_k, x') = \delta_1$ and $\lim d(x_k, x'') = \delta_2$. On the other hand,

$$2\langle \overrightarrow{x_{k_i} x_{k_j}}, \overrightarrow{x'' x'} \rangle = d^2(x_{k_i}, x') + d^2(x_{k_j}, x'') - d^2(x_{k_i}, x'') - d^2(x_{k_j}, x').$$

Letting $i \to +\infty$ and then $j \to +\infty$, we get $\lim_{j \to +\infty} \lim_{i \to +\infty} \langle \overrightarrow{x_{k_i} x_{k_j}}, \overrightarrow{x'' x'} \rangle = 0$. Also, we can write the left-hand side of the above statement as

$$2\langle \overrightarrow{x_{k_i} x_{k_j}}, \overrightarrow{x'' x'} \rangle = 2\langle \overrightarrow{x_{k_i} x'}, \overrightarrow{x'' x'} \rangle + 2\langle \overrightarrow{x' x''}, \overrightarrow{x'' x'} \rangle + 2\langle \overrightarrow{x'' x_{k_j}}, \overrightarrow{x'' x'} \rangle.$$

By taking lim sup in the above and using [1, Theorem 2.6], we conclude that $d^2(x', x'') \le 0$ and hence $x' = x''$. This establishes that the set of $\triangle$-cluster points of $\{x_k\}$ is a singleton. □

Definition 5.3. A bifunction $f : K \times K \to \mathbb{R}$ is called strongly monotone if there exists $\alpha > 0$ such that $f(x, y) + f(y, x) \le -\alpha\, d^2(x, y)$ for all $x, y \in K$.

Also, a bifunction $f : K \times K \to \mathbb{R}$ is called strongly pseudo-monotone if there exists $\beta > 0$ such that if $f(x, y) \ge 0$, then $f(y, x) \le -\beta\, d^2(x, y)$ for all $x, y \in K$.

It is obvious that if f is strongly monotone, then f is strongly pseudo-monotone.

Theorem 5.4. *Consider* EP(f, K)*, where* f *satisfies* $P1$*,* $P2$*,* $P3$*,* $P4^*$*,* $P4^\bullet$ *and* $S(f, K) \neq \emptyset$*. If any one of the following conditions is satisfied:*

https://doi.org/10.1017/S1446788719000041 Published online by Cambridge University Press

(i) *f is strongly pseudo-monotone;*
(ii) *$f(x, \cdot)$ is strongly convex for all $x \in K$;*
(iii) *$f(\cdot, y)$ is strongly concave for all $y \in K$,*

then the sequence $\{x_k\}$ generated by (5.1) is strongly convergent to a point of $S(f, K)$.

PROOF. Take $x^* \in S(f, K)$. In each part, we show that x_k converges strongly to $x^* \in S(f, K)$.

(i) Since $f(x^*, x_k) \geq 0$, by assumption there is $\beta > 0$ such that $f(x_k, x^*) \leq -\beta\, d^2(x_k, x^*)$ for all $k \in \mathbb{N}$. Next, by (5.3) in the proof of Theorem 5.2, we have $\liminf f(x_k, x^*) \geq 0$. Therefore, by taking liminf,

$$0 \leq \liminf f(x_k, x^*) \leq \liminf(-\beta\, d^2(x_k, x^*)) = -\beta \limsup d^2(x_k, x^*)$$

and hence we deduce that $x_k \longrightarrow x^*$.

(ii) Let $\lambda \in (0, 1)$ and set $w_k = \lambda x_k \oplus (1 - \lambda)x^*$ for all $k \in \mathbb{N}$. Since $f(x_k, \cdot)$ is strongly convex,

$$\begin{aligned} 0 &\leq f(x_k, w_k) + \lambda_{k-1}\langle \overrightarrow{y_{k-1}x_k}, \overrightarrow{x_k w_k}\rangle \\ &\leq \lambda f(x_k, x_k) + (1-\lambda) f(x_k, x^*) - \lambda(1-\lambda)\, d^2(x_k, x^*) \\ &\quad + (1-\lambda)\lambda_{k-1}\langle \overrightarrow{y_{k-1}x_k}, \overrightarrow{x_k x^*}\rangle. \end{aligned}$$

Hence, we have $\lambda\, d^2(x_k, x^*) \leq \lambda_{k-1}\langle \overrightarrow{y_{k-1}x_k}, \overrightarrow{x_k x^*}\rangle$. By using the Cauchy–Schwarz inequality, we get $\lambda\, d(x_k, x^*) \leq \lambda_{k-1}\, d(y_{k-1}, x_k) \leq \lambda_{k-1}(d(y_{k-1}, x_{k-1}) + d(x_{k-1}, x_k)) \leq \lambda_{k-1}(e_{k-1} + d(x_{k-1}, x_k))$. Now, from Lemma 5.1, we conclude that $x_k \longrightarrow x^*$.

(iii) Let $\lambda \in (0, 1)$ and set $w_k = \lambda x_k \oplus (1 - \lambda)x^*$ for all $k \in \mathbb{N}$. Since $f(\cdot, x^*)$ is strongly concave,

$$\lambda f(x_k, x^*) + (1-\lambda) f(x^*, x^*) + \lambda(1-\lambda)\, d^2(x_k, x^*) \leq f(w_k, x^*) \leq 0.$$

Now we get $f(x_k, x^*) \leq -(1 - \lambda)\, d^2(x_k, x^*)$. Next, by (5.3) in the proof of Theorem 5.2 and taking liminf,

$$0 \leq \liminf f(x_k, x^*) \leq -(1-\lambda) \limsup d^2(x_k, x^*)$$

and hence we deduce that the sequence $\{x_k\}$ is strongly convergent to $x^* \in S(f, K)$. □

6. Halpern regularization

Let $K \subseteq X$ be closed and convex, $f : K \times K \to \mathbb{R}$ be a bifunction and suppose that θ is the undermonotonicity constant of f. Take a sequence of regularization parameters $\{\lambda_k\} \subset (\theta, \bar{\lambda}]$ for some $\bar{\lambda} > \theta$ and $x_0 \in X$. Consider the following Halpern regularization of the proximal point algorithm for the equilibrium problem:

$$\begin{cases} f(y_k, y) + \lambda_{k-1}\langle \overrightarrow{x_{k-1}y_k}, \overrightarrow{y_k y}\rangle \geq 0, \quad \forall y \in K, \\ x_k = \alpha_k u \oplus (1 - \alpha_k) y_k, \end{cases} \tag{6.1}$$

https://doi.org/10.1017/S1446788719000041 Published online by Cambridge University Press

where $u \in X$ and the sequence $\{\alpha_k\} \subset (0,1)$ satisfies $\lim \alpha_k = 0$ and $\sum_{k=1}^{+\infty} \alpha_k = +\infty$. We will prove the strong convergence of the sequence generated by (6.1) to a solution of $\mathrm{EP}(f, K)$ when the bifunction f satisfies $P1$, $P2$, $P3$, $P4^*$ and $P4^\bullet$. In fact, we prove that $x_k \to x^* = \mathrm{Proj}_{S(f,K)} u$. First we give an essential lemma that we need in the sequel. For a proof, the reader can see [34, Lemma 2.6].

LEMMA 6.1. *Let $\{s_k\}$ be a sequence of nonnegative real numbers, $\{a_k\}$ be a sequence of real numbers in $(0,1)$ with $\sum_{k=1}^{\infty} a_k = +\infty$ and $\{t_k\}$ be a sequence of real numbers. Suppose that*

$$s_{k+1} \leq (1 - a_k)s_k + a_k t_k, \quad \forall k \in \mathbb{N}.$$

If $\limsup t_{k_n} \leq 0$ for every subsequence $\{s_{k_n}\}$ of $\{s_k\}$ satisfying $\liminf(s_{k_n+1} - s_{k_n}) \geq 0$, then $\lim s_k = 0$.

THEOREM 6.2. *Suppose that f satisfies $P1, P3$, $P4^*$, $P4^\bullet$ and $S(f, K) \neq \emptyset$. If $f(\cdot, y)$ is $\triangle$-upper semicontinuous for all $y \in K$, then $\{x_k\}$ converges strongly to $\mathrm{Proj}_{S(f,K)} u$, where $\{x_k\}$ is the sequence generated by (6.1).*

PROOF. Since $S(f, K)$ is closed and convex, we may assume that $x^* = \mathrm{Proj}_{S(f,K)} u$ exists. Note that $f(y_k, x^*) + \lambda_{k-1}\langle \overrightarrow{x_{k-1}y_k}, \overrightarrow{y_k x^*}\rangle \geq 0$. In turn, since by $P4^*$ every element of $S(f, K)$ solves $\mathrm{CFP}(f, K)$, we have $f(y_k, x^*) \leq 0$ and thus $\langle \overrightarrow{x_{k-1}y_k}, \overrightarrow{y_k x^*}\rangle \geq 0$, that is,

$$d^2(x^*, x_{k-1}) - d^2(x^*, y_k) - d^2(x_{k-1}, y_k) \geq 0. \tag{6.2}$$

Hence, $d(x^*, y_k) \leq d(x^*, x_{k-1})$. On the other hand, by (6.1),

$$\begin{aligned} d(x^*, x_k) &\leq \alpha_k\, d(x^*, u) + (1 - \alpha_k) d(x^*, y_k) \\ &\leq \alpha_k\, d(x^*, u) + (1 - \alpha_k) d(x^*, x_{k-1}) \\ &\leq \max\{d(x^*, u), d(x^*, x_{k-1})\} \leq \cdots \leq \max\{d(x^*, u), d(x^*, x_0)\}. \end{aligned}$$

Therefore, $\{x_k\}$ is bounded. Since $d(x^*, y_k) \leq d(x^*, x_{k-1})$ for all $k \in \mathbb{N}$, $\{y_k\}$ is bounded. Now, by (6.1),

$$d^2(x_{k+1}, x^*) \leq (1 - \alpha_{k+1})\, d^2(y_{k+1}, x^*) + \alpha_{k+1}\, d^2(u, x^*) - \alpha_{k+1}(1 - \alpha_{k+1})\, d^2(u, y_{k+1}).$$

Since, by (6.2), $d^2(x^*, y_{k+1}) \leq d^2(x^*, x_k)$,

$$d^2(x_{k+1}, x^*) \leq (1 - \alpha_{k+1})\, d^2(x_k, x^*) + \alpha_{k+1}\, d^2(u, x^*) - \alpha_{k+1}(1 - \alpha_{k+1})\, d^2(u, y_{k+1}).$$

In the sequel, we show that $d(x_{k+1}, x^*) \to 0$. By Lemma 6.1, it suffices to show that $\limsup(d^2(u, x^*) - (1 - \alpha_{k_n+1})\, d^2(u, y_{k_n+1})) \leq 0$ for every subsequence $\{d^2(x_{k_n}, x^*)\}$ of $\{d^2(x_k, x^*)\}$ satisfying $\liminf(d^2(x_{k_n+1}, x^*) - d^2(x_{k_n}, x^*)) \geq 0$.

https://doi.org/10.1017/S1446788719000041 Published online by Cambridge University Press

For this, suppose that $\{d^2(x_{k_n}, x^*)\}$ is a subsequence of $\{d^2(x_k, x^*)\}$ such that $\liminf(d^2(x_{k_n+1}, x^*) - d^2(x_{k_n}, x^*)) \geq 0$. Then

$$\begin{aligned}
0 &\leq \liminf(d^2(x^*, x_{k_n+1}) - d^2(x^*, x_{k_n})) \\
&\leq \liminf(\alpha_{k_n+1}\, d^2(x^*, u) + (1-\alpha_{k_n+1})\, d^2(x^*, y_{k_n+1}) - d^2(x^*, x_{k_n})) \\
&= \liminf(\alpha_{k_n+1}(d^2(x^*, u) - d^2(x^*, y_{k_n+1})) + d^2(x^*, y_{k_n+1}) - d^2(x^*, x_{k_n})) \\
&\leq \limsup \alpha_{k_n+1}(d^2(x^*, u) - d^2(x^*, y_{k_n+1})) + \liminf(d^2(x^*, y_{k_n+1}) - d^2(x^*, x_{k_n})) \\
&= \liminf(d^2(x^*, y_{k_n+1}) - d^2(x^*, x_{k_n})) \\
&\leq \limsup(d^2(x^*, y_{k_n+1}) - d^2(x^*, x_{k_n})) \leq 0.
\end{aligned}$$

Therefore, we conclude that $\lim(d^2(x^*, y_{k_n+1}) - d^2(x^*, x_{k_n})) = 0$ and hence, by (6.2), we get $\lim d^2(x_{k_n}, y_{k_n+1}) = 0$.

On the other hand, there are a subsequence $\{y_{k_{n_i}+1}\}$ of $\{y_{k_n+1}\}$ and $p \in K$ such that $y_{k_{n_i}+1} \xrightarrow{\triangle} p$ and

$$\limsup(d^2(u, x^*) - (1-\alpha_{k_n+1})\, d^2(u, y_{k_n+1})) = \lim(d^2(u, x^*) - (1-\alpha_{k_{n_i}+1})\, d^2(u, y_{k_{n_i}+1})).$$

Since $y_{k_{n_i}+1} \xrightarrow{\triangle} p$,

$$\begin{aligned}
0 &\leq \limsup(f(y_{k_{n_i}+1}, y) + \lambda_{k_{n_i}} \langle \overrightarrow{x_{k_{n_i}} y_{k_{n_i}+1}}, \overrightarrow{y_{k_{n_i}+1} y} \rangle) \\
&\leq \limsup(f(y_{k_{n_i}+1}, y) + \lambda_{k_{n_i}} d(x_{k_{n_i}}, y_{k_{n_i}+1}) d(y_{k_{n_i}+1}, y)) \leq f(p, y)
\end{aligned}$$

for all $y \in K$. Therefore, $p \in S(f, K)$. Now, since $x^* = \mathrm{Proj}_{S(f,K)} u$,

$$\limsup(d^2(u, x^*) - (1-\alpha_{k_n+1})\, d^2(u, y_{k_n+1})) \leq d^2(u, x^*) - d^2(u, p) \leq 0.$$

Therefore, Lemma 6.1 shows that

$$d(x_{n+1}, x^*) \to 0.$$

Also, (6.2) implies that $d(y_{n+1}, x^*) \to 0$, which is the desired result. □

7. Applications to fixed point theory and convex minimization

In this short section, we present two examples of equilibrium problems in Hadamard spaces.

(1) Let X be a Hadamard space. The mapping $T : X \to X$ is called a pseudo-contraction if and only if

$$\langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle \leq d^2(x, y), \quad \forall x, y \in X.$$

It is easy to check that if T is a pseudo-contraction, then $f(x, y) = \langle \overrightarrow{Txx}, \overrightarrow{xy} \rangle$ is a monotone bifunction. If T is nonexpansive, which is a stronger condition, then $J_\lambda^f = J_\lambda^T$, where J_λ^T is the resolvent of T (see [4, 5, 25]). Now the results of this paper are applicable to find and approximate an equilibrium point of f, which is a solution of the variational inequality for T.

https://doi.org/10.1017/S1446788719000041 Published online by Cambridge University Press

(2) To solve the constraint minimization problem

$$\min_{x\in K} \varphi(x),$$

where the constraint set K is a convex and closed subset of a Hadamard space X and $\varphi : X \to]-\infty, +\infty]$ is a convex, proper and lower semicontinuous function, we can consider the monotone bifunction $f(x, y) = \varphi(y) - \varphi(x)$ on $K \times K$. It is easy to see that $J^f_\lambda = J^\varphi_\lambda$, where J^φ_λ is the resolvent of φ (see [3, 4]). Then the methods discussed in Sections 4–6 are applicable to approximate an equilibrium point of f, which is a minimum point of φ on K. In fact, Theorem 6.2 in the case $f(x, y) = \varphi(y) - \varphi(x)$ extends [14, Theorem 3.2], where the sequence λ_n is constant.

There are several examples of convex functions and minimization in Hadamard spaces. Some of those are the energy functional on a Hadamard space and computation of the median and mean for a finite family of points in a Hadamard space. For more examples and explanations, the interested reader can consult [3, 4].

Acknowledgement

The authors are grateful to the referee for careful reading and valuable comments and corrections.

References

[1] B. Ahmadi Kakavandi, 'Weak topologies in complete CAT(0) spaces', *Proc. Amer. Math. Soc.* **141** (2013), 1029–1039.

[2] B. Ahmadi Kakavandi and M. Amini, 'Duality and subdifferential for convex function on CAT(0) metric spaces', *Nonlinear Anal.* **73** (2010), 3450–3455.

[3] M. Bacak, 'The proximal point algorithm in metric spaces', *Israel J. Math.* **194** (2013), 689–701.

[4] M. Bacak, *Convex Analysis Optimization in Hadamard Spaces*, De Gruyter Series in Nonlinear Analysis and Applications, 22 (De Gruyter, Berlin, 2014).

[5] M. Bacak and S. Reich, 'The asymptotic behavior of a class of nonlinear semigroups in Hadamard spaces', *J. Fixed Point Theory Appl.* **16** (2014), 189–202.

[6] G. C. Bento, O. P. Ferreira and P. R. Oliveira, 'Local convergence of the proximal point method for a special class of nonconvex functions on Hadamard manifolds', *Nonlinear Anal.* **73** (2010), 564–572.

[7] I. D. Berg and I. G. Nikolaev, 'On a distance between directions in an Alexandrov space of curvature ≤K', *Michigan Math. J.* **45** (1998), 275–289.

[8] I. D. Berg and I. G. Nikolaev, 'Quasilinearization and curvature of Alexandrov spaces', *Geom. Dedicata* **133** (2008), 195–218.

[9] M. Bianchi and S. Schaible, 'Generalized monotone bifunctions and equilibrium problems', *J. Optim. Theory Appl.* **90** (1996), 31–43.

[10] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-positive Curvature* (Springer, Berlin, 1999).

[11] D. Burago, Y. Burago and S. Ivanov, *A Course in Metric Geometry*, Graduate Studies in Mathematics, 33 (American Mathematical Society, Providence, RI, 2001).

[12] O. Chadli, Z. Chbani and H. Riahi, 'Equilibrium problems with generalized monotone bifunctions and applications to variational inequalities', *J. Optim. Theory Appl.* **105** (2000), 299–323.

[13] P. Chaoha and A. Phon-on, 'A note on fixed point sets in CAT(0) spaces', *J. Math. Anal. Appl.* **320** (2006), 983–987.

[14] P. Cholamjiak, ‘The modified proximal point algorithm in CAT(0) spaces’, *Optim. Lett.* **9** (2015), 1401–1410.
[15] V. Colao, G. López, G. Marino and V. Martín-Márquez, ‘Equilibrium problems in Hadamard manifolds’, *J. Math. Anal. Appl.* **388** (2012), 61–77.
[16] P. L. Combetes and S. A. Hirstoaga, ‘Equilibrium programming in Hilbert spaces’, *J. Nonlinear Convex Anal.* **6** (2005), 117–136.
[17] H. Dehghan and J. Rooin, ‘A characterization of metric projection in Hadamard spaces with applications’. arXiv:1311.4174.
[18] S. Dhompongsa and B. Panyanak, ‘On △-convergence theorems in CAT(0) spaces’, *Comput. Math. Appl.* **56** (2008), 2572–2579.
[19] N. Hadjisavvas and H. Khatibzadeh, ‘Maximal monotonicity of bifunctions’, *Optimization* **59** (2010), 147–160.
[20] N. Hadjisavvas, S. Schaible and N. C. Wong, ‘Pseudomonotone operators: a survey of the theory and its applications’, *J. Optim. Theory Appl.* **152** (2012), 1–20.
[21] A. N. Iusem, G. Kassay and W. Sosa, ‘On certain conditions for the existence of solutions of equilibrium problems’, *Math. Program. Ser. B* **116** (2009), 259–273.
[22] A. N. Iusem and W. Sosa, ‘New existence results for equilibrium problems’, *Nonlinear Anal.* **52** (2003), 621–635.
[23] A. N. Iusem and W. Sosa, ‘On the proximal point method for equilibrium problems in Hilbert spaces’, *Optimization* **59** (2010), 1259–1274.
[24] H. Khatibzadeh, V. Mohebbi and M. H. Alizadeh, ‘On the cyclic pseudomonotonicity and the proximal point algorithm’, *Numer. Algebra Control Optim.* **8** (2018), 441–449.
[25] H. Khatibzadeh and S. Ranjbar, ‘A variational inequality in complete CAT(0) spaces’, *J. Fixed Point Theory Appl.* **17** (2015), 557–574.
[26] W. A. Kirk, ‘Geodesic geometry and fixed point theory. II’, in: *Int. Conf. Fixed Point Theory and Applications* (Yokohama Publishers, Yokohama, Japan, 2004), 113–142.
[27] W. A. Kirk and B. Panyanak, ‘A concept of convergence in geodesic spaces’, *Nonlinear Anal.* **68** (2008), 3689–3696.
[28] C. Li, G. Lopez and V. Martin-Marquez, ‘Monotone vector fields and the proximal point algorithm on Hadamard manifolds’, *J. Lond. Math. Soc.* **79** (2009), 663–683.
[29] C. Li, G. Lopez, V. Martin-Marquez and J. H. Wang, ‘Resolvent of set valued monotone vector fields in Hadamard manifolds’, *Set-Valued Anal.* **19** (2011), 361–383.
[30] M. A. Mansour, Z. Chbani and H. Riahi, ‘Recession bifunction and solvability of noncoercive equilibrium problems’, *Commun. Appl. Anal.* **7** (2003), 369–377.
[31] M. A. Noor and K. I. Noor, ‘Some algorithms for equilibrium problems on Hadamard manifolds’, *J. Inequal. Appl.* (2012), 2012:230, 8 pp.
[32] A. Papadopoulos, *Metric Spaces, Convexity and Nonpositive Curvature*, IRMA Lectures in Mathematics and Theoretical Physics, 6 (European Mathematical Society (EMS), Zürich, 2005).
[33] E. A. Papa Quiroz and P. R. Oliveira, ‘Proximal point method for minimizing quasiconvex locally Lipschitz functions on Hadamard manifolds’, *Nonlinear Anal.* **75** (2012), 5924–5932.
[34] S. Saejung and P. Yotkaew, ‘Approximation of zeros of inverse strongly monotone operators in Banach spaces’, *Nonlinear Anal.* **75** (2012), 742–750.
[35] G. Tang and Y. Xiao, ‘A note on the proximal point algorithm for pseudomonotone variational inequalities on Hadamard manifolds’, *Adv. Nonlinear Var. Inequal.* **18** (2015), 58–69.
[36] G. Tang, L. Zhou and N. Huang, ‘The proximal point algorithm for pseudomonotone variational inequalities on Hadamard manifolds’, *Optim. Lett.* **7**(4) (2013), 779–790.
[37] J. H. Wang, G. Lopez, V. Martin-Marquez and C. Li, ‘Monotone and accretive vector fields on Riemannian manifolds’, *J. Optim. Theory Appl.* **146** (2010), 691–708.

https://doi.org/10.1017/S1446788719000041 Published online by Cambridge University Press

HADI KHATIBZADEH, Department of Mathematics, University of Zanjan,
PO Box 45195-313, Zanjan, Iran
e-mail: hkhatibzadeh@znu.ac.ir

VAHID MOHEBBI, Department of Mathematics, University of Zanjan,
PO Box 45195-313, Zanjan, Iran
e-mail: mohebbi@znu.ac.ir