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Asymptotic forms for jets from standing waves

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Standing gravity waves forced beyond the maximum height for perfect periodicity can produce vertical jets with sharp-pointed tips. In this paper, canonical forms for the wave crests are derived which display sharp cusps in the limit as the time t tends to infinity. The theoretical profile is in general quartic in the space coordinates, and can describe the smooth transition of a fairly low wave crest to a cusped form. There is no singularity as the surface slope passes through 45°.

1. Introduction

The forming of vertical jets in standing waves and in steep progressive waves meeting a vertical wall has been studied in several recent laboratory experiments (Chan & Melville 1988; Jiang, Perlin & Schultz 1998; Bredmose *et al.* 2000) and numerical computations (Cooker & Peregrine 1991; Longuet-Higgins 2001*a*,*b*; Longuet-Higgins & Dommermuth 2001). While the early, highly accelerated phase of the motion has received some attention and a certain degree of explanation (Longuet-Higgins & Oguz 1997; Cooker 2000; Longuet-Higgins 2001*a*,*b*), later stages of the flow, when the vertical acceleration at the crest has fallen almost to $-g$ and the tip of the jet is in free-fall, have yet to be represented analytically, even when the influences of surface tension and viscosity are theoretically absent.

Both the experiments and the more recent numerical calculations, especially those by Longuet-Higgins & Dommermuth (2001), strongly suggest that the tip of the jet can tend towards a sharp cusp with increasing time t. This implies the presence of cubic or higher-order terms in the expression for the velocity potential. Such an analytic form has been suggested previously for a plunging breaker in a progressive gravity wave (Longuet-Higgins 1980, § 10). Here we shall apply the same type of analysis to the simpler case of a symmetrical jet produced by a standing wave.

In all of the present calculation it is assumed that the flow is inviscid and irrotational and that surface tension can be ignored. Moreover the motion will be viewed in a free-fall frame of reference, so that there is no local effect of gravity. The solutions are to be valid asymptotically near the tip of the jet and as the time t tends to infinity.

As far as fourth order in the radius r, the procedure yields a family of canonical solutions with one free parameter; see § 4. These are illustrated by numerical computations in § 5, and a discussion follows in § 6.

2. General solution

Take rectangular coordinates as in figure 1, with the x-axis horizontal and the y-axis vertically upwards; r and θ are polar coordinates such that $(x, y) = r(\sin\theta, -\cos\theta)$.

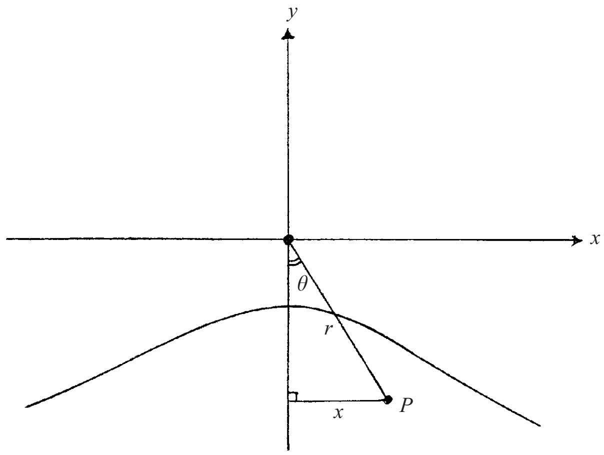

FIGURE 1. Rectangular and polar coordinates in a free-fall reference frame.

Consider a velocity potential

$$\phi = \tfrac{1}{2}Ar^2 \cos 2\theta + \tfrac{1}{3}Br^3 \cos 3\theta + \tfrac{1}{4}Cr^4 \cos 4\theta, \tag{2.1}$$

where A, B, C are functions of the time t. This expression is to represent a free-surface flow valid in the limit as $r/t \to 0$. In fact we shall take $A = 1/t$, with B and C proportional to higher inverse powers of the time t. The reference frame being inertial, the pressure p is given by

$$-p = \phi_t + \tfrac{1}{2}q^2 - F(t), \tag{2.2}$$

where q is the particle velocity and F an arbitrary function of t. On substitution from (2.1)

$$\begin{aligned} -p = r^2(\tfrac{1}{2}A^2 + \tfrac{1}{2}\dot{A}\cos 2\theta) + r^3(AB\cos\theta + \tfrac{1}{3}\dot{B}\cos 3\theta) \\ +r^4(\tfrac{1}{2}B^2 + AC\cos 2\theta + \tfrac{1}{4}\dot{C}\cos 4\theta) - F, \end{aligned} \tag{2.3}$$

terms of higher degree in r being neglected. Similarly for the rate of change of p following a particle we find

$$\begin{aligned} -\frac{\mathrm{D}p}{\mathrm{D}t} &= r^2[2A\dot{A} + (\tfrac{1}{2}\ddot{A} + A^3)\cos 2\theta] \\ &\quad + r^3[2(A^2B + A\dot{B} + \dot{A}B)\cos\theta + (2A^2B + \tfrac{1}{3}\ddot{B})\cos 3\theta] \\ &\quad + r^4[2B\dot{B} + (4AB^2 + 2\overline{A\dot{C} + \dot{A}C})\cos 2\theta \\ &\qquad + (5A^2C + AB^2 + \tfrac{1}{4}\ddot{C})\cos 4\theta] - \dot{F}. \end{aligned} \tag{2.4}$$

On the free surface, both p and $\mathrm{D}p/\mathrm{D}t$ are to vanish simultaneously. This can be ensured by specifying that

$$\frac{\mathrm{D}p}{\mathrm{D}t} = p\frac{\dot{F}}{F}(1 + r^3R + r^4S + O(r^5)), \tag{2.5}$$

where R and S are functions of θ and t only. By symmetry we may assume that

$$\left.\begin{aligned} R &= R_1\cos\theta + R_3\cos 3\theta, \\ S &= S_0 + S_2\cos 2\theta + S_4\cos 4\theta, \end{aligned}\right\} \tag{2.6}$$

where the R_n and S_n are functions of t only.

On equating coefficients of r^2 and $r^2\cos 2\theta$ in (2.5) we obtain

$$\frac{2A\dot{A}}{\frac{1}{2}A^2} = \frac{\frac{1}{2}\ddot{A} + A^3}{\frac{1}{2}\dot{A}} = \frac{\dot{F}}{F}, \tag{2.7}$$

of which the general solution was derived by Longuet-Higgins (1972). The particular solution in which we are interested here is

$$A = \frac{1}{t}, \quad F = \frac{F_0}{t^4}, \tag{2.8}$$

where F_0 is an arbitrary constant. Note that the first term in equation (2.1) represents a stagnation point flow:

$$\phi = \frac{1}{t}(y^2 - x^2), \tag{2.9}$$

which from (2.3) has a real free surface

$$x^2 = tF = F_0/t^3 \tag{2.10}$$

consisting of two parallel vertical planes, but only if $F_0/t^3 > 0$.

Next, on equating coefficients of $r^3\cos\theta$ and $r^3\cos 3\theta$ in equation (2.5) we obtain

$$\frac{2(A^2B + A\dot{B} + \dot{A}B) + \dot{F}R_1}{AB} = \frac{2A^2B + \frac{1}{3}\ddot{B} + \dot{F}R_3}{\frac{1}{3}\dot{B}} = \frac{\dot{F}}{F}. \tag{2.11}$$

These constitute two equations for R_1 and R_3 in terms of A, B and F, namely

$$\left.\begin{aligned} \dot{F}R_1 &= AB(\dot{F}/F) - 2(A^2B + A\dot{B} + \dot{A}B), \\ \dot{F}R_3 &= \tfrac{1}{3}\dot{B}(\dot{F}/F) - (2A^2B + \tfrac{1}{3}\ddot{B}). \end{aligned}\right\} \tag{2.12}$$

For example, if we choose $B = B_0/t^2$ where B_0 is a constant we find

$$\left.\begin{aligned} \dot{F}R_1 &= \frac{-4B_0}{t^4} - 2B_0\left(\frac{1}{t^4} - \frac{3}{t^4}\right) = 0, \\ \dot{F}R_3 &= \frac{8}{3}\frac{B_0}{t^4} - B_0\left(\frac{2}{t^4} + \frac{2}{t^4}\right) = -\frac{4}{3}\frac{B_0}{t^4}, \end{aligned}\right\} \tag{2.13}$$

and hence

$$R_1 = 0, \quad R_3 = \frac{B_0 t}{3F_0}. \tag{2.14}$$

Similarly by equating coefficients of $r^4, r^4\cos 2\theta$ and $r^4\cos 4\theta$ in equation (2.5) we can obtain three equations for S_0, S_2 and S_4 which, if $C = C_0/t^3$, yield

$$S_0 = -\frac{B_0^2}{2F_0}, \quad S_2 = \frac{B_0^2 - C_0^2}{F_0}, \quad S_4 = \frac{B_0^2 + 5C_0}{4F_0}. \tag{2.15}$$

Note that as t increases the term in r^3 in equation (2.5) will come to dominate the series. The term in r^4 is relatively small, since by hypothesis $r/t \ll 1$, and similarly for terms of higher order in r.

3. The free surface

The free surface $p=0$ can be found by equating to zero the right-hand side of (2.3). If we now substitute

$$A=\frac{1}{t},\quad B=\frac{3\beta}{t^2},\quad C=\frac{4\gamma}{t^3} \tag{3.1}$$

in this expression we obtain

$$\begin{aligned}&\frac{r^2}{t^2}\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 2\theta\right)\\ &+\frac{r^3}{t^3}(3\beta\cos\theta-2\beta\cos 3\theta)\\ &+\frac{r^4}{t^4}\left(\tfrac{9}{2}\beta^2+4\gamma\cos 2\theta-3\gamma\cos 4\theta\right)=\frac{F_0}{t^4}.\end{aligned} \tag{3.2}$$

On replacing 1 by $(\cos^2\theta+\sin^2\theta)^n$ where appropriate we get

$$\begin{aligned}&\frac{r^2}{t^2}\sin^2\theta\\ &+\frac{r^3}{t^3}(\beta\cos^3\theta+9\beta\cos\theta\sin^2\theta)\\ &+\frac{r^4}{t^4}\left[\left(\tfrac{9}{2}\beta^2+\gamma\right)\cos^4\theta+9(\beta^2+2\gamma)\cos^2\theta\sin^2\theta+\left(\tfrac{9}{2}\beta^2-7\gamma\right)\sin^4\theta\right]=\frac{F_0}{t^4}.\end{aligned} \tag{3.3}$$

Now writing

$$(\xi,\eta)=(x/t,y/t)=(r/t)(\sin\theta,-\cos\theta) \tag{3.4}$$

we have

$$\xi^2-\beta(\eta^3+9\xi^2\eta)+\left[\left(\tfrac{9}{2}\beta^2+\gamma\right)\eta^4+9(\beta^2+2\gamma)\xi^2\eta^2+\left(\tfrac{9}{2}\beta^2-7\gamma\right)\xi^4\right]=\epsilon, \tag{3.5}$$

where

$$\epsilon=F_0/t^4. \tag{3.6}$$

The left-hand side of equation (3.5) being quadratic in ξ^2, with coefficients which are polynomials in η, for any given η we may solve for ξ^2. If either of the roots is non-negative we can then plot ξ versus η.

Equation (3.5) is asymptotically valid for sufficiently small values of ξ and η, that is to say for sufficiently large values of t when x and y are bounded, and for extremely small values of ϵ. Near the origin of (ξ,η), where the quartic terms are relatively small, we have approximately

$$\xi^2=\epsilon+\beta(\eta^3+9\xi^2\eta). \tag{3.7}$$

In the limit as $\epsilon\to 0$ this curve (or surface) has an upward- or downward-pointing cusp according as β is negative or positive 0. If $\epsilon<0$ the curves $p=0$ lie inside the cusped curve and if $\epsilon>0$ they lie outside. Since $\epsilon=F_0/t^4\to 0$ as $t\to\infty$ we see that the limiting curves are in general cusped.

Note that if all the fourth-order terms in equation (3.5) are neglected we obtain the lower-order approximation

$$\xi^2-\beta(\eta^3+9\xi^2\eta)=\epsilon \tag{3.8}$$

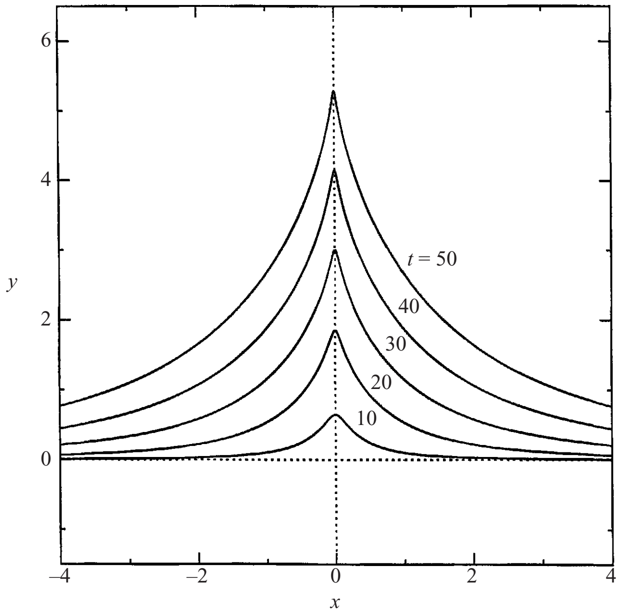

FIGURE 2. Surface profiles corresponding to equation (4.1) when $\delta = -1$ and $V = 1/9$.

with solution

$$\xi^2 = \frac{\epsilon + \beta\eta^3}{1 - 9\beta\eta}. \tag{3.9}$$

The corresponding profile will have a horizontal asymptote given by $\eta = (9\beta)^{-1}$.

4. Canonical forms

When $\beta < 0$, giving an upward-pointing cusp when $\epsilon \to 0$, we may by a suitable choice of velocity scale choose $\beta = -1$. (Setting $\beta = +1$ would simply reflect the profile in the horizontal axis, i.e. turn it upside down.) Then by a further choice of time scale we may set $\epsilon = \pm t^{-4}$. Thus equation (3.8) is reduced to the canonical form

$$x^2 = \delta t^{-2} - (y^3 + 9x^2 y)/t, \tag{4.1}$$

where $\delta = \pm 1$. Similarly equation (3.5) is reduced to

$$x^2 = \delta t^{-2} - (y^3 + 9x^2 y)/t + [(7\gamma - \tfrac{9}{2})x^4 - 9(2\gamma + 1)x^2 y^2 - (\gamma + \tfrac{9}{2})y^4]/t^2. \tag{4.2}$$

Exceptionally we may have $\beta = 0$, and then by choice of scales

$$x^2 = \delta t^{-2} + \delta'(7x^4 - 18x^2 y^2 - y^4)/t^2. \tag{4.3}$$

As always, these expressions are valid only when x/t and y/t are sufficiently small.

5. Numerical calculations

Consider first the approximation (4.1). The case $\delta = -1$ is shown in figure 2. For clarity, the profiles have been separated by adding successive vertical displacements $\Delta t/9$, where Δt is the time interval between profiles. Physically this is equivalent to

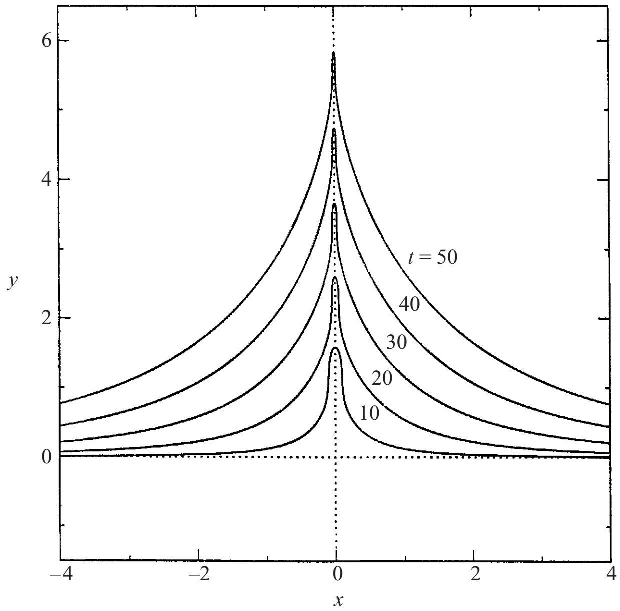

FIGURE 3. Surface profiles corresponding to equation (4.1) when $\delta = +1$ and $V = 1/9$.

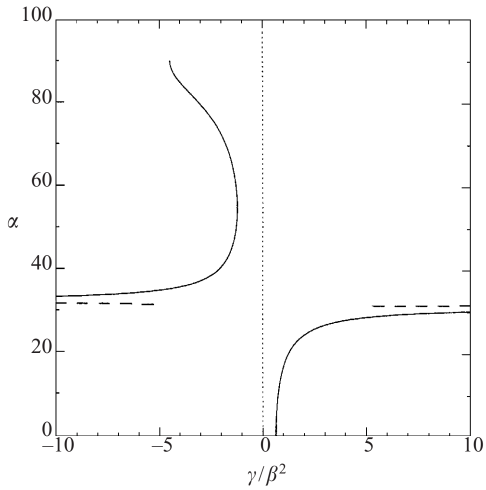

FIGURE 4. The inclination α to the horizontal of the asymptotes in the profile (4.2), as a function of γ (or γ/β^2).

imposing a uniform vertical velocity $V = 1/9$. The effect is to maintain the horizontal asymptote at the same level $y = 0$.

As t increases, the central part of the profile approaches the form of a semi-cubical parabola, with increasing accuracy.

The case $\delta = +1$ is shown in figure 3. The most noticeable difference between these profiles and those of figure 2 is that they can become very steep, indeed almost vertical, at points other than the central cusp. If a vertical velocity were not imposed, these curves would all be nested inside each other.

In order to classify the profiles corresponding to equation (4.2) it is convenient to

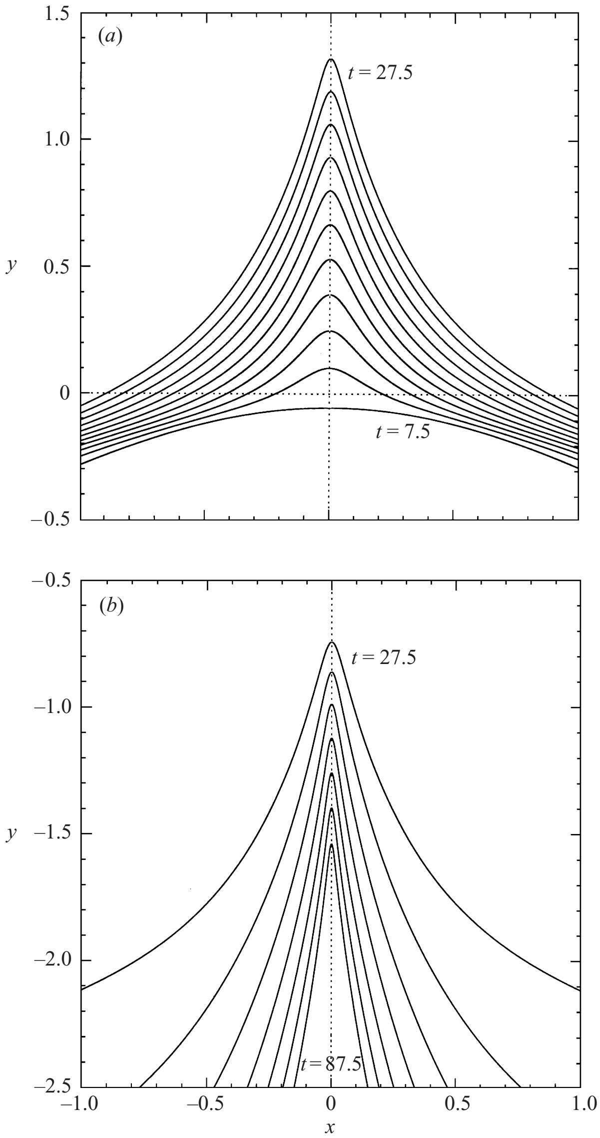

FIGURE 5. (*a*) Surface profiles corresponding to the canonical form (4.2) when $\delta = -1$ and $\gamma = -5$. The time interval Δt between successive profiles is 2. To separate the curves a vertical velocity $V = 0.06$ has been imposed. (*b*) Extension of (*a*) but with $\Delta t = 10$ and $V = -0.015$.

consider the asymptotes for large values of x and y (but small values of x/t and y/t). The directions of the asymptotes are found by equating to zero the fourth-order terms, hence

$$x^2/y^2 = [9(2\gamma+1) \pm \sqrt{\{81(2\gamma+1)^2 + (14\gamma-9)(2\gamma+9)\}}]/(14\gamma-9). \qquad (5.1)$$

The angle of slope

$$\alpha = |\arctan(y/x)| \qquad (5.2)$$

is plotted against γ (or more generally γ/β^2) in figure 4. In a central range, given by

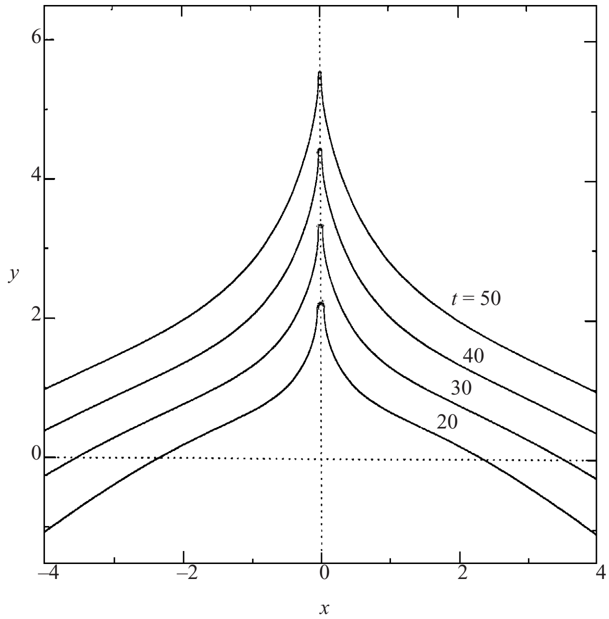

FIGURE 6. Surface profiles corresponding to the canonical form (4.2) when $\delta = +1$ and $\gamma = -5$. Time interval $\Delta t = 10$. Imposed velocity $V = 1/9$.

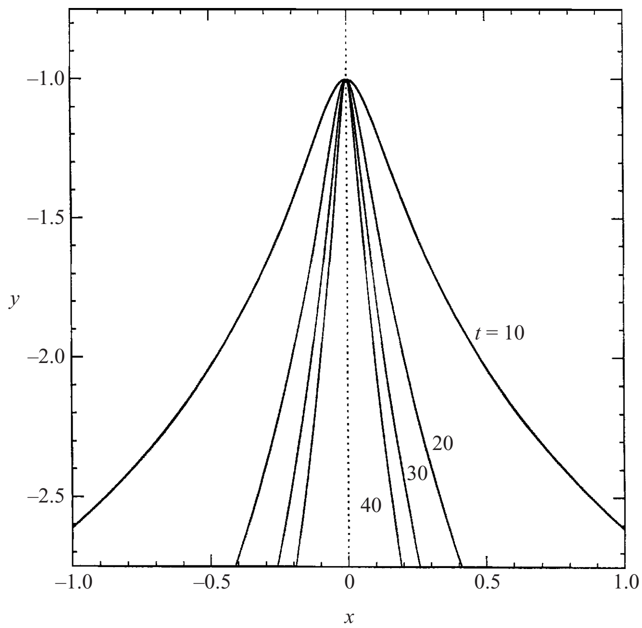

FIGURE 7. Surface profiles corresponding to the exceptional case $\beta = 0$, equation (4.3), when $\delta = -1, \delta' = -1$. Time interval between profiles: $\Delta t = 10$. Imposed velocity $V = 0$.

$-1.227 < \gamma < 0.643$, there are no real asymptotes. On the other hand as $\gamma \to \pm\infty$, the angle γ tends to a finite value: $\alpha = 31.68°$ (see below).

A typical example, when $\gamma = -5$, is shown in figures 5(*a*) and 5(*b*). In figure 5(*a*), in order to separate the curves, a vertical velocity $V = 0.05$ has been imposed. The

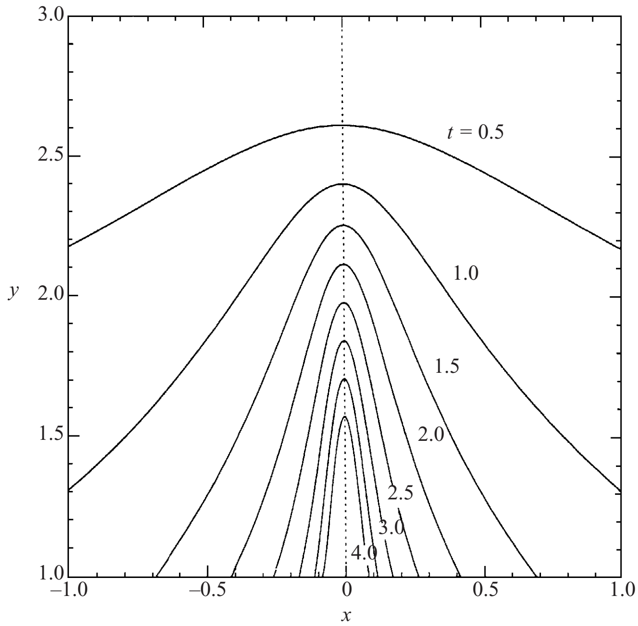

FIGURE 8. Numerically calculated profiles of wave crests generated at time $t = 0$ by imposing the vertical velocity (6.1) on a flat water surface. Here $C = 1$. Enlargement of figure 9 of Longuet-Higgins & Dommermuth (2001), seen in a free-fall reference frame.

successive profiles seem to represent the transition of a rounded wave crest into an upwards vertical jet. It is noticeable that the maximum slope of each curve passes through 45° without any singular behaviour with respect to the time t.

The sequence is extended to larger times t in figure 5(*b*), and here, for the sake of clarity we have imposed a uniform downwards velocity $V = -0.01$. As $t \to \infty$ we see the development of a sharp cusp.

Similarly when $\delta = 1$ and $\gamma = -5$ we obtain the sequence of surface profiles shown in figure 6. The curves are steeper, as in figure 4. The profile corresponding to $t = 10$ is not shown, since that joins with another branch coming from above the crest to yield an unphysical solution.

When $\delta = 1$ and $\gamma = 5$ the curves are again uninteresting.

Lastly we consider the exceptional case $\beta = 0$, given by (4.3). The only physically interesting case is when $\delta = -1$. The profiles are then as in figure 7. No vertical velocity V has been imposed. For large values of x and y the curves approach the asymptotes

$$7x^4 - 18x^2y^2 - y^4 = 0, \tag{5.3}$$

that is to say

$$x^2/y^2 = (9 + \sqrt{88})/7 = 2.6258 \tag{5.4}$$

or

$$y/x = \pm 0.6171 = \pm \arctan 31.68°. \tag{5.5}$$

In each of figures 2, 3, 5, 6 and 7 the necessary condition that x/t and y/t shall be small can be seen to be satisfied.

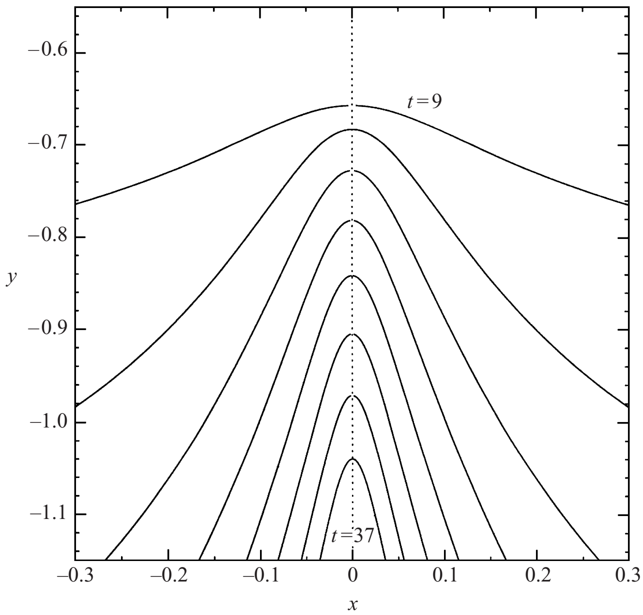

FIGURE 9. Enlargement of the crests in figures 5(a) and 5(b), but with superposed velocity $V = -0.020$. These correspond to equation (4.2) with $\delta = -1, \gamma = -5$. Time interval between profiles, $\Delta t = 4$.

6. Discussion

Like the limiting Stokes corner flow in progressive surface waves, and its extension to progressive waves of near-limiting height (Longuet-Higgins & Fox 1977, 1978) the expressions given above are valid only asymptotically, in this case in the limit as $t \to \infty$ and $r/t \to 0$. The functions $B(t)$ and $C(t)$ in §§2 and 3 are to some extent arbitrary, and must be determined, in any particular case, by fitting the inner flow which they represent to the outer flow in the rest of the fluid. The fitting must of course be carried out in both the space and time domains.

At first sight it appears that by adding suitable terms of order r^5 or higher powers on the right of equation (2.5) one could proceed indefinitely to higher approximations. However, there is no guarantee that the process will converge. It may well be more convenient to adopt a low-order approximation which is certainly valid asymptotically in the prescribed limits.

There is strong evidence that the asymptotic expressions for the surface profile given in §§3 and 4 above do indeed describe the development of jets in some recent numerical calculations. For example Longuet-Higgins & Dommermuth (2001) have shown by precise calculation that if an upwards vertical velocity v of the form

$$v = C(gk)^{-1/2} \cos kx \tag{6.1}$$

is imparted at time $t = 0$ to an infinitely deep body of fluid, where k is a horizontal wavenumber, g denotes gravity and C is a constant of order 1, then the crest of the wave so generated tends to develop a sharp upwards jet. Figure 8 is an enlargement of the wave crest, seen in a free-fall frame of reference; a displacement $\frac{1}{2}g(t-t_0)^2$ has been added to the computed wave profile, where t_0 is taken equal to 2.5. This may be compared with figure 9, which is an enlargement of the crests in figures 5(a) and 5(b), but with a superposed uniform velocity $V = -0.015$. The profiles in figure 9 represent equation (4.2) when $\delta = -1$ and $\gamma = -5$. The similarity of figures 8 and 9 is striking. No special attempt has been made to select the optimum value of γ for figure 8.

A further interesting feature of the canonical form (4.2) is that it provides a smooth transition for a wave crest with maximum slope less than 45° to pass to a profile having maximum slope much greater than 45° (as is shown in figures 5*a* and 6, for example) without having to pass through a time-singularity such as occurs in the Dirichlet hyperbola (Longuet-Higgins 1972) at time $t = 0$. For further discussion and references see Longuet-Higgins & Dommermuth (2001).

The limiting forms described above may apply not only to steep, progressive waves meeting a vertical wall but also to the bow waves of certain ships.

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REFERENCES

Bredmose, H., Brocchini, M., Peregrine, D. H. & Thais, L. Experimental investigation and numerical modelling of steep forced water waves. *Euromech Colloquium 416, Genova, Italy, 17–20 Sept. 2000.*

Chan, E. S. & Melville, W. K. 1988 Deep-water plunging wave pressures on a vertical plane wall. *Proc. R. Soc. Lond.* A **417**, 95–131.

Cooker, M. J. 2001 Violently erupting free-surface jets. *Proc. IUTAM Symp. on Free Surface Flows, Birmingham, UK, 10–14 July 2000.* Kluwer.

Cooker, M. J. & Peregrine, D. H. 1991 Violent surface motion as near-breaking waves meet a wall. In *Breaking Waves* (ed. M. L. Banner & R. H. J. Grimshaw), pp. 291–297. Springer.

Jiang, L., Perlin, M. & Schultz, W. W. 1998 Period tripling and energy dissipation of breaking standing waves. *J. Fluid Mech.* **369**, 273–299.

Longuet-Higgins, M. S. 1972 A class of exact, time-dependent, free-surface flows. *J. Fluid Mech.* **55**, 529–543.

Longuet-Higgins, M. S. 1980 On the forming of sharp corners at a free surface. *Proc. R. Soc. Lond.* A **371**, 453–478.

Longuet-Higgins, M. S. 2001*a* Vertical jets from standing waves: the bazooka effect. In *IUTAM Symp. on Free Surface Flows, Birmingham, UK, 10–14 July 2000*, pp. 195–204. Kluwer.

Longuet-Higgins, M. S. 2001*b* Vertical jets from standing waves in deep water. *Proc. R. Soc.* A **457**, 495–510.

Longuet-Higgins, M. S. & Dommermuth, D. G. 2001 On the breaking of standing waves by falling jets. *Phys. Fluids* **13**, 1652–1659.

Longuet-Higgins, M. S. & Fox, M. J. H. 1977 Theory of the almost-highest wave: The inner solution. *J. Fluid Mech.* **80**, 721–741.

Longuet-Higgins, M. S. & Fox, M. J. H. 1978 Theory of the almost-highest wave. II. Matching and analytic extension. *J. Fluid Mech.* **85**, 769–786.

Longuet-Higgins, M. S. & Oguz, H. N. 1997 Critical jets in surface waves and collapsing cavities. *Phil. Trans. R. Soc. Lond.* A **355**, 625–639.