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Pull-back attractors for three-dimensional Navier–Stokes–Voigt equations in some unbounded domains

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We study the first initial–boundary-value problem for the three-dimensional non-autonomous Navier–Stokes–Voigt equations in an arbitrary (bounded or unbounded) domain satisfying the Poincaré inequality. The existence of a weak solution to the problem is proved by using the Faedo–Galerkin method. We then show the existence of a unique minimal finite-dimensional pull-back $\mathcal{D}_\sigma$-attractor for the process associated with the problem, with respect to a large class of non-autonomous forcing terms. We also discuss relationships between the pull-back attractor, the uniform attractor and the global attractor.

1. Introduction

Let Ω be a (bounded or unbounded) domain in $\mathbb{R}^3$ with boundary $\partial\Omega$. In this paper we study the long-time behaviour of solutions to the following three-dimensional (3D) non-autonomous Navier–Stokes–Voigt (sometimes written Voight) equations

$$\left.\begin{aligned} u_t - \nu\Delta u - \alpha^2\Delta u_t + (u\cdot\nabla)u + \nabla p = f, &\qquad x\in\Omega,\ t>\tau,\\ \nabla\cdot u = 0, &\qquad x\in\Omega,\ t>\tau,\\ u(x,t) = 0, &\qquad x\in\partial\Omega,\ t>\tau,\\ u(x,\tau) = u_0(x), &\qquad x\in\Omega, \end{aligned}\right\} \tag{1.1}$$

where $u = u(x,t) = (u_1,u_2,u_3)$ is the unknown velocity vector, $p = p(x,t)$ is the unknown pressure, $\nu > 0$ is the kinematic viscosity coefficient, α is a length-scale parameter characterizing the elasticity of the fluid, $f = f(x,t)$ is a given force field and u_0 is the initial velocity.

The system (1.1) models the dynamics of a Kelvin–Voigt viscoelastic incompressible fluid and was introduced by Oskolkov in [16] as a model of motion of linear, viscoelastic fluids. The system (1.1) was also proposed in [3] as a regularization, for small values of α, of the 3D Navier–Stokes equations for the sake of direct numerical simulations for both the period and the no-slip Dirichlet boundary conditions. In fact, if $\alpha = 0$, (1.1) becomes the classical 3D Navier–Stokes equations, and if $\nu = 0$, we get the inviscid simplified Bardina model [3].

In the past few years, the existence and long-time behaviour of solutions to the 3D Navier–Stokes–Voigt equations have attracted the attention of many mathe-

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maticians. For a given force field f that is time independent, the existence and uniqueness of solutions was investigated by Oskolkov in [16]. In [9,10], it was shown that the semigroup generated by (1.1) has a finite-dimensional global attractor. Recently, in [11, 12], Kalantarov and Titi improved this result, and proved the determining modes property and the Gevrey regularity of the global attractor. For an external force that is a translation bounded time-dependent function, the existence of a uniform attractor for (1.1) was proved very recently in [18]. However, to the best of our knowledge, all existing results for 3D Navier–Stokes–Voigt equations are devoted to the problem in bounded domains; only the work in [5] studied the problem in an unbounded two-dimensional domain.

The aim of this paper is to continue studying the long-time behaviour of weak solutions to (1.1) under a more general class of time-dependent external forces and in domains that are not necessarily bounded. We first prove the existence and uniqueness of weak solutions. To study the long-time behaviour of the solutions, we will use the theory of pull-back attractors that has been developed recently and has been shown to be very useful in understanding the dynamics of non-autonomous dynamical systems because it allows us to consider a larger class of non-autonomous forces than the theory of uniform attractors does. We will show the existence and estimates of the fractal dimension of a pull-back attractor for the process generated by the problem.

In order to study (1.1), we assume the following.

(H1) The domain Ω can be an arbitrary (bounded or unbounded) domain in $\mathbb{R}^3$, without any regularity assumption on its boundary $\partial\Omega$, provided that the Poincaré inequality holds on Ω: there exists $\lambda_1 > 0$ such that

$$\int_\Omega \phi^2 \,\mathrm{d}x \leqslant \frac{1}{\lambda_1}\int_\Omega |\nabla\phi|^2 \,\mathrm{d}x \quad \text{for all } \phi \in H_0^1(\Omega).$$

(H2) $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$ such that

$$\int_{-\infty}^t \mathrm{e}^{\sigma s}\|f(s)\|^2_{V'}\,\mathrm{d}s < +\infty \quad \text{for all } t \in \mathbb{R},$$

where $\sigma = \lambda_1\nu/(1+\alpha^2\lambda_1)$, λ_1 is the constant in the Poincaré inequality.

It is worth stressing that, by adding the regularizing $-\alpha^2\Delta u_t$ to the Navier–Stokes equations, (1.1) changes its parabolic character. In particular, (1.1) is globally well posed forwards and backwards in time and the associated process is only pull-back asymptotically compact, similarly to damped hyperbolic equations. This introduces some essential difficulty when proving the existence of a pull-back attractor. More difficulty arises due to the lack of compactness of the Sobolev embeddings, since the considered domain is unbounded. This leads to the fact that the decomposition method of the solutions used for Navier–Stokes–Voigt equations in bounded domains [11, 18] does not work here. To overcome these difficulties, we exploit the energy equations method introduced by Ball [2] to prove the pull-back asymptotic compactness of the process, and, as a consequence, the existence of a pull-back attractor. Such an approach has been used to prove the existence of pull-back attractors for 2D Navier–Stokes equations [4] and recently for the generalized

Korteweg–de Vries–Burgers equations [1]. Finally, we develop the method introduced by Ladyzhenskaya [13] to show that the pull-back attractor has a finite fractal dimension. In particular, the obtained results improve and extend some known results on the Navier–Stokes–Voigt equations in both bounded and unbounded domains.

The paper has the following structure. In § 2, we recall some auxiliary results on function spaces and inequalities for the nonlinear terms related to the Navier–Stokes–Voigt equations, and some abstract results on the existence and the fractal dimension of pull-back attractors. In § 3, we prove the existence and uniqueness of a weak solution to (1.1) by using the Faedo–Galerkin method. In §§ 4 and 5, following the general lines of the approach in [4, 14] for 2D Navier–Stokes equations, we prove the existence and fractal dimension estimates of a minimal unique pull-back attractor for the process associated with the problem. In the last section, we give relationships between the pull-back attractor obtained in § 4, the uniform attractor obtained recently in [18] and the global attractor obtained when the external force f is independent of the time variable t.

2. Preliminaries

2.1. Function spaces and inequalities for the nonlinear terms

Define

$$(u,v) := \int_\Omega \sum_{j=1}^{3} u_j v_j \, \mathrm{d}x, \qquad u=(u_1,u_2,u_3), v=(v_1,v_2,v_3) \in L^2(\Omega)^3,$$

and

$$((u,v)) := \int_\Omega \sum_{j=1}^{3} \nabla u_j \cdot \nabla v_j \, \mathrm{d}x, \quad u=(u_1,u_2,u_3), v=(v_1,v_2,v_3) \in H_0^1(\Omega)^3,$$

and the associated norms $|u|^2 := (u,u)$, $\|u\|^2 := ((u,u))$.

Let

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^3 \colon \nabla \cdot u = 0\}.$$

Denote by H the closure of $\mathcal{V}$ in $L^2(\Omega)^3$, and by V the closure of $\mathcal{V}$ in $H_0^1(\Omega)^3$. It follows that $V \subset H \equiv H' \subset V'$, where the injections are dense and continuous. We will use $\|\cdot\|_*$ for the norm in V', and $\langle\cdot,\cdot\rangle$ for the duality pairing between V and V'. Denote by P the Helmholtz–Leray orthogonal projection in $(H_0^1(\Omega))^3$ onto the space V.

In what follows, we will frequently use the inequalities listed below.

- *The Young inequality*

$$ab \leqslant \frac{\varepsilon}{p} a^p + \frac{1}{q\varepsilon^{1/(p-1)}} b^q \quad \text{for all } a,b,\varepsilon > 0, \text{ with } q = \frac{p}{p-1},\ 1 < p < +\infty.$$

- *The Ladyzhenskaya inequality* (*when* $n = 3$) (see, for example, [7])

$$\|u\|_{L^3} \leqslant c|u|^{1/2}\|u\|^{1/2} \quad \forall u \in V,$$
$$\|u\|_{L^4} \leqslant c|u|^{1/4}\|u\|^{3/4} \quad \forall u \in V.$$

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- *The Sobolev inequality* (see, for example, [17])

$$\|u\|_{L^6} \leqslant C\|u\| \quad \forall u \in V.$$

- *The Gagliardo–Nirenberg inequality* (see, for example, [17])

$$\|u\|_{L^{6/(3-2\varepsilon)}} \leqslant C|u|^{1-\varepsilon}\|u\|^{\varepsilon} \quad \forall 0 \leqslant \varepsilon \leqslant 1,\ u \in V.$$

We now define the trilinear form b by

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j \, \mathrm{d}x,$$

whenever the integrals make sense. It is easy to check that if $u, v, w \in V$, then

$$b(u,v,w) = -b(u,w,v). \tag{2.1}$$

Hence,

$$b(u,v,v) = 0 \quad \forall u, v \in V. \tag{2.2}$$

LEMMA 2.1 (Constantin and Foias [7] and Temam [17]). *If $n = 3$, then*

$$|b(u,v,w)| \leqslant \begin{cases} C|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|, \\ C\|u\|\|v\||w|^{1/2}\|w\|^{1/2}, \\ C\lambda_1^{1/4}\|u\|\|v\|\|w\|^{1/2} \end{cases} \quad \forall u, v, w \in V \tag{2.3}$$

and

$$|b(u,v,u)| \leqslant \sqrt{2}|u|\|u\|\|v\| \quad \forall u, v \in V. \tag{2.4}$$

Set $A\colon V \to V'$ by $\langle Au, v\rangle = ((u,v))$, $B\colon V \times V \to V'$ by $\langle B(u,v), w\rangle = b(u,v,w)$, $Bu = B(u,u)$. Then, $D(A) = H^2(\Omega) \cap V$ and $Au = -P\Delta u$ for all $u \in D(A)$.

2.2. Pull-back attractors

Let (X,d) be a metric space. For $A, B \subset X$, we define the Hausdorff semi-distance between A and B by

$$\operatorname{dist}(A,B) = \sup_{x\in A} \inf_{y \in B} d(x,y).$$

A process on X is a family of two-parameter mappings $\{U(t,\tau)\}$ in X having the properties

$$\begin{aligned} U(t,r)U(r,\tau) &= U(t,\tau) \quad \text{for all } t \geqslant r \geqslant \tau, \\ U(\tau,\tau) &= \mathrm{Id} \qquad\quad \text{for all } \tau \in \mathbb{R}. \end{aligned}$$

The process $\{U(t,\tau)\}$ is said to be continuous if $U(t,\tau)x_n \to U(t,\tau)x$, as $x_n \to x$ in X, for all $t \geqslant \tau$, $\tau \in \mathbb{R}$.

Suppose that $\mathcal{P}(X)$ is the family of all non-empty bounded subsets of X, and $\mathcal{D}$ is a non-empty class of parameterized sets $\hat{\mathcal{D}} = \{D(t)\colon t \in \mathbb{R}\} \subset \mathcal{P}(X)$.

Definition 2.2. The process $\{U(t,\tau)\}$ is said to be pull-back $\mathcal{D}$-asymptotically compact if for any $t \in \mathbb{R}$, any $\hat{\mathcal{D}} \in \mathcal{D}$, any sequence $\tau_n \to -\infty$ and any sequence $x_n \in D(\tau_n)$, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.

Definition 2.3. The family of bounded sets $\hat{\mathcal{B}} \in \mathcal{D}$ is called pull-back $\mathcal{D}$-absorbing for the process $U(t,\tau)$ if for any $t \in \mathbb{R}$ and any $\hat{\mathcal{D}} \in \mathcal{D}$, there exists $\tau_0 = \tau_0(\hat{\mathcal{D}}, t) \leqslant t$ such that

$$\bigcup_{\tau \leqslant \tau_0} U(t,\tau)D(\tau) \subset B(t).$$

Definition 2.4. A family $\hat{\mathcal{A}} = \{A(t) \colon t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is said to be a pull-back $\mathcal{D}$-attractor for $\{U(t,\tau)\}$ if the following hold.

(1) $A(t)$ is compact for all $t \in \mathbb{R}$.

(2) $\hat{\mathcal{A}}$ is invariant, i.e.

$$U(t,\tau)A(\tau) = A(t) \quad \text{for all } t \geqslant \tau.$$

(3) $\hat{\mathcal{A}}$ is pull-back $\mathcal{D}$-attracting, i.e.

$$\lim_{\tau \to -\infty} \operatorname{dist}(U(t,\tau)D(\tau), A(t)) = 0 \quad \text{for all } \hat{\mathcal{D}} \in \mathcal{D} \text{ and all } t \in \mathbb{R}.$$

(4) If $\{C(t) \colon t \in \mathbb{R}\}$ is another family of closed attracting sets, then $A(t) \subset C(t)$ for all $t \in \mathbb{R}$.

Theorem 2.5 (Li and Zhong [15]). *Let $\{U(t,\tau)\}$ be a continuous process such that $\{U(t,\tau)\}$ is pull-back $\mathcal{D}$-asymptotically compact. If there exists a family of pull-back $\mathcal{D}$-absorbing sets $\hat{\mathcal{B}} = \{B(t) \colon t \in \mathbb{R}\} \in \mathcal{D}$, then $\{U(t,\tau)\}$ has a unique pull-back $\mathcal{D}$-attractor $\hat{\mathcal{A}} = \{A(t) \colon t \in \mathbb{R}\}$ and*

$$A(t) = \bigcap_{s \leqslant t} \overline{\bigcup_{\tau \leqslant s} U(t,\tau)B(\tau)}.$$

We now recall some results on the estimates of the fractal dimension of pull-back $\mathcal{D}$-attractors.

Let V be a separable real Hilbert space. Given a compact set $K \subset V$, and $\varepsilon > 0$, we denote by $N_\varepsilon(K)$ the minimum number of open balls in V with radii ε that are necessary to cover K.

Definition 2.6. For any non-empty compact set $K \subset V$, the fractal dimension of K is the number

$$\dim_F(K) = \limsup_{\varepsilon \downarrow 0} \frac{\log(N_\varepsilon(K))}{\log(1/\varepsilon)}.$$

Let $V \subset H$ be a separable real Hilbert space such that the injection of V in H is continuous, and V is dense in H. We identify H with its topological dual H', and we consider V as a subspace of H', identifying $v \in V$ with the element $f_v \in H'$, defined by

$$f_v(h) = (v,h), \quad h \in H.$$

Let $F\colon V\times\mathbb{R}\to V'$ be a given family of nonlinear operators such that, for all $\tau\in\mathbb{R}$ and any $u_0\in V$, there exists a unique function $u(t)=u(t;\tau,u_0)$ satisfying

$$\left.\begin{aligned} u\in C([\tau,T];V),\quad F(u(t),t)\in L^1(\tau,T;V')\quad \text{for all } T>\tau,\\ \frac{\mathrm{d}u}{\mathrm{d}t}=F(u(t),t),\quad t>\tau,\\ u(\tau)=u_0.\end{aligned}\right\}\tag{2.5}$$

We define

$$U(t,\tau)u_0=u(t;\tau,u_0),\quad \tau\leqslant t,\ u_0\in V.$$

Let $T^*\in\mathbb{R}$ be fixed. We assume that there exists a family $\{A(t)\colon t\leqslant T^*\}$ of non-empty compact subsets of V satisfying the invariance property

$$U(t,\tau)A(\tau)=A(t)\quad \text{for all } \tau\leqslant t\leqslant T^*$$

and such that, for all $\tau\leqslant t\leqslant T^*$ and any $u_0\in A(\tau)$, there exists a continuous linear operator $L(t;\tau,u_0)\in\mathcal{L}(V)$ such that

$$\|U(t,\tau)\bar{u}_0-U(t,\tau)u_0-L(t;\tau,u_0)(\bar{u}_0-u_0)\|\leqslant\gamma(t-\tau,\|\bar{u}_0-u_0\|)\|\bar{u}_0-u_0\|\tag{2.6}$$

for all $\bar{u}_0\in A(\tau)$, where $\gamma\colon\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is a function such that $\gamma(s,\cdot)$ is non-decreasing for all $s\geqslant 0$, and

$$\lim_{r\to 0}\gamma(s,r)=0\quad \text{for any } s\geqslant 0.\tag{2.7}$$

We assume that, for all $t\leqslant T^*$, the mapping $F(\cdot,t)$ is Gâteaux differentiable in V, i.e. for any $u\in V$ there exists a continuous linear operator $F'(u,t)\in\mathcal{L}(V;V')$ such that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[F(u+\varepsilon v,t)-F(u,t)-\varepsilon F'(u,t)v]=0\in V'.$$

Moreover, we suppose that the mapping

$$F'\colon(u,t)\in V\times(-\infty,T^*]\mapsto F'(u,t)\in\mathcal{L}(V;V')$$

is continuous (thus, in particular, for each $t\leqslant T^*$, the mapping $F(\cdot,t)$ is continuously Fréchet differentiable in V).

Then, for all $\tau\leqslant T^*$ and $u_0,v_0\in V$, there exists a unique $v(t)=v(t;\tau,u_0,v_0)$, which is a solution of

$$\begin{aligned} v\in C([\tau,T];V)\quad \text{for all } \tau<T\leqslant T^*,\\ \frac{\mathrm{d}v}{\mathrm{d}t}=F'(U(t,\tau)u_0,t)v,\quad \tau<t<T^*,\\ v(\tau)=v_0.\end{aligned}$$

We make the assumption that

$$v(t;\tau,u_0,v_0)=L(t;\tau,u_0)v_0\quad \text{for all } \tau\leqslant t\leqslant T^*,\ u_0,v_0\in A(\tau).\tag{2.8}$$

The proof of the following result is a slight modification of the proof of [13, theorem 4.9], so we omit it.

Theorem 2.7. *Under the above assumptions, we suppose that*

$$\bigcup_{t\leqslant T^*} A(t) \text{ is relatively compact in } V.$$

Suppose that $F'^c(U(t,\tau)u_0,t) := F'(U(t,\tau)u_0,t) + F'^*(U(t,\tau)u_0,t)$ *satisfies the inequality*

$$(F'^c(t,\tau)v,v) \leqslant -h_0(t,\tau)|v|^2 + \sum_{k=1}^{m} h_{s_k}(t,\tau)\|v\|_{s_k}^2$$

for some numbers $s_k < 0$ $(k = 1,\ldots,m)$ *and some functions* $h_0, h_{s_k} \in L_{1,\mathrm{loc}}(\mathbb{R})$, $h_{s_k}(t,\tau) \geqslant 0$, $h_0(t,\tau) \geqslant 0$ *for all* $\tau \leqslant t \leqslant T^*$.

Then,

$$\dim_{\mathcal{H}}(A(t)) \leqslant \dim_F(A(t)) \leqslant N \quad \text{for all } t \in \mathbb{R},$$

where N *is such that*

$$-\bar{h}_0(t,\tau) + \sum_{k=1}^{m} \bar{h}_{s_k}(t,\tau) N^{s_k} < 0$$

for some $\tau < t < T^*$; *here*

$$\bar{h}_i(t) := \frac{1}{t-\tau}\int_\tau^t h_i(s)\,\mathrm{d}s.$$

3. Existence and uniqueness of weak solutions

We first give the definition of weak solutions to problem (1.1).

Definition 3.1. A function u is called a weak solution to problem (1.1) on the interval (τ, T) if

$$u \in C([\tau,T];V), \qquad \frac{\mathrm{d}u}{\mathrm{d}t} \in L^2(\tau,T;V),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) + \nu Au(t) + \alpha^2 Au'(t) + B(u(t),u(t)) = f(t) \text{ in } V'$$

$$\text{for almost everywhere (a.e.) } t \in (\tau,T),$$

$$u(\tau) = u_0.$$

Theorem 3.2. *For any* $u_0 \in V$ *and* $T > \tau$ *given, (1.1) has a unique weak solution* u *on* (τ,T).

Proof. We will prove the existence by using the Faedo–Galerkin method. Although this seems to be quite standard, we are not able to find the complete proof in the literature. Thus, we provide a full proof here for completeness.

Step 1 (constructing the Faedo–Galerkin approximations). Since V is separable and $\mathcal{V}$ is dense in V, there exists a sequence of linearly independent elements $\{w^1, w^2, \ldots\} \subset \mathcal{V}$ that is total in V. Define

$$V_m = \mathrm{span}\{w^1,\ldots,w^m\}$$

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and consider the projector

$$P^m u = \sum_{j=1}^{m} (u, w^j) w^j.$$

A function

$$u^m(t) = \sum_{j=1}^{m} c_j^m(t) w^j(x)$$

is an m-approximate solution of (1.1) if

$$\left\langle \frac{\partial u^m}{\partial t}, w^j \right\rangle + \nu \int_\Omega \nabla u^m \nabla w^j \,\mathrm{d}x + \alpha^2 \int_\Omega \nabla u_t^m \cdot \nabla w^j \,\mathrm{d}x + \int_\Omega (u^m \nabla) u^m w^j \,\mathrm{d}x = \langle f, w^j \rangle \tag{3.1}$$

for all $j = 1, \dots, m$, and

$$u^m(x, \tau) = \sum_{j=1}^{m} a^j w^j(x),$$

with $a^j = (u_0, w^j)$. By the Peano theorem, we get the existence of approximate solutions u^m on (τ, T).

Step 2 (establishing *a priori* estimates of u^m). Multiplying (3.1) by c_j^m, adding the resulting equations for j from 1 to m, and integrating from τ to t, we get that

$$\frac{1}{2} \int_\tau^t \frac{\partial}{\partial s} |u^m(s)|^2 \,\mathrm{d}s + \nu \int_\tau^t |\nabla u^m(s)|^2 \,\mathrm{d}s + \frac{\alpha^2}{2} \int_\tau^t \frac{\partial}{\partial s} |\nabla u^m(s)|^2 \,\mathrm{d}s = \int_\tau^t \langle f, u^m \rangle \,\mathrm{d}s.$$

Hence, this implies that

$$|u^m(t)|^2 + 2\nu \int_\tau^t |\nabla u^m(s)|^2 \,\mathrm{d}s + \alpha^2 |\nabla u^m(t)|^2 \leqslant \frac{1}{\nu} \|f\|^2_{L^2(\tau,t;V')} + \nu \|u^m\|^2_{L^2(\tau,t;V)} + |u_0|^2 + \alpha^2 |\nabla u_0|^2$$

or

$$|u^m(t)|^2 + \nu \int_\tau^t |\nabla u^m(s)|^2 \,\mathrm{d}s + \alpha^2 |\nabla u^m(t)|^2 \leqslant \frac{1}{\nu} \|f\|^2_{L^2(\tau,T;V')} + |u_0|^2 + \alpha^2 |\nabla u_0|^2.$$

This implies that $\{u^m\}$ is bounded in $L^\infty(\tau, T; V)$. Hence, it is easy to check that $\{Au^m\}$ and $\{Bu^m\}$ are bounded in $L^2(\tau, T; V')$.

Now, we prove the boundedness of $\{\mathrm{d}u^m/\mathrm{d}t\}$. Multiplying (3.1) by $\dot{c}_j^m(s)$, then adding the resulting equations for j and integrating from τ to t, we obtain

$$\int_\tau^t \left| \frac{\partial u^m}{\partial s} \right|^2 \mathrm{d}s + \nu \int_\tau^t \int_\Omega \nabla u^m \frac{\partial \nabla u^m}{\partial s} \,\mathrm{d}x \,\mathrm{d}s + \alpha^2 \int_\tau^t |\nabla u_s^m|^2 \,\mathrm{d}s + \int_\tau^t b(u^m, u^m, u_s^m) \,\mathrm{d}s = \int_\tau^t \langle f, u_s^m \rangle \,\mathrm{d}s.$$

Using Cauchy's and Ladyzhenskaya's inequalities, we have that

$$\begin{aligned}
&\int_\tau^t \left|\frac{\partial u^m}{\partial s}\right|^2 \mathrm{d}s + \frac{\nu}{2}\int_\tau^t \frac{\partial}{\partial s}|\nabla u^m|^2\,\mathrm{d}s + \alpha^2\int_\tau^t |\nabla u_s^m|^2\,\mathrm{d}s\\
&\quad = \int_\tau^t \langle f, u_s^m\rangle\,\mathrm{d}s + \int_\tau^t b(u^m, u_s^m, u^m)\,\mathrm{d}s\\
&\quad \leqslant \|f\|_{L^2(\tau,t;V')}\cdot\|u_s^m\|_{L^2(\tau,t;V)} + \int_\tau^t |u^m|_{L^4}^2|\nabla u_s^m|\,\mathrm{d}s\\
&\quad \leqslant \frac{1}{\alpha^2}\|f\|^2_{L^2(\tau,T;V')} + \frac{\alpha^2}{4}\|u_s^m\|^2_{L^2(\tau,t;V)} + c\int_\tau^t |u^m|^{1/2}|\nabla u^m|^{3/2}|\nabla u_s^m|\,\mathrm{d}s\\
&\quad \leqslant \frac{1}{\alpha^2}\|f\|^2_{L^2(\tau,T;V')} + \frac{\alpha^2}{4}\|u_s^m\|^2_{L^2(\tau,t;V)} + c\int_\tau^t |\nabla u^m||\nabla u_s^m|\,\mathrm{d}s\\
&\quad \leqslant \frac{1}{\alpha^2}\|f\|^2_{L^2(\tau,T;V')} + \frac{\alpha^2}{4}\|u_s^m\|^2_{L^2(\tau,t;V)}\\
&\qquad\quad + c\left(\int_\tau^t |\nabla u^m|^2\,\mathrm{d}s\right)^{1/2}\cdot\left(\int_\tau^t |\nabla u_s^m|^2\,\mathrm{d}s\right)^{1/2}\\
&\quad \leqslant \frac{1}{\alpha^2}\|f\|^2_{L^2(\tau,T;V')} + \frac{\alpha^2}{4}\|u_s^m\|^2_{L^2(\tau,T;V)}\\
&\qquad\quad + c\|u^m\|^2_{L^2(\tau,T;V)} + \frac{\alpha^2}{4}\|u_s^m\|^2_{L^2(\tau,T;V)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&2\int_\tau^t \left|\frac{\partial u^m}{\partial s}\right|^2 \mathrm{d}s + \alpha^2\int_\tau^t |\nabla u_s^m|^2\,\mathrm{d}s + \nu|\nabla u^m(t)|^2\\
&\qquad \leqslant \frac{2}{\alpha^2}\|f\|^2_{L^2(\tau,T;V')} + c\|u^m\|^2_{L^2(\tau,T;V)} + \nu|\nabla u_0|^2 \quad \text{for all } \tau\leqslant t\leqslant T.
\end{aligned}$$

Since $\{u^m\}$ is bounded in $L^\infty(\tau,T;V)$, $\{\mathrm{d}u^m/\mathrm{d}t\}$ is bounded in $L^2(\tau,T;V)$.

Step 3 (passing to the limit). Let ψ be a continuously differentiable function on $[\tau,T]$, with $\psi(T)=0$. Then, we have

$$\begin{aligned}
&\int_\tau^T \left\langle\frac{\partial u^m}{\partial t}, w^j\right\rangle\psi\,\mathrm{d}t + \nu\int_\tau^T\int_\Omega \nabla u^m\nabla w^j\psi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \alpha^2\int_\tau^T\int_\Omega \nabla u_t^m\nabla w^j\psi\,\mathrm{d}x\,\mathrm{d}t + \int_\tau^T\int_\Omega (u^m\nabla)u^m w^j\psi\,\mathrm{d}x\,\mathrm{d}t = \int_\tau^T\langle f, w^j\rangle\psi\,\mathrm{d}t
\end{aligned} \tag{3.2}$$

for all $j=1,\dots,\infty$. From the above estimates, we can extract a subsequence of $\{u^m\}$, also denoted by $\{u^m\}$, such that

$$\begin{aligned}
u^m &\rightharpoonup u \quad \text{in } L^\infty(\tau,T;V),\\
\frac{\mathrm{d}u^m}{\mathrm{d}t} &\rightharpoonup \frac{\mathrm{d}u}{\mathrm{d}t} \quad \text{in } L^2(\tau,T;V).
\end{aligned}$$

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Now, we need to prove that

$$\int_\tau^T \int_\Omega (u^m\nabla)u^m w^j \psi \,\mathrm{d}x\,\mathrm{d}t \to \int_\tau^T \int_\Omega (u\nabla)uw^j \psi \,\mathrm{d}x\,\mathrm{d}t.$$

For any ball $\mathcal{O}$ included in Ω we have that

$$H^1(\mathcal{O}) \subset\subset L^p(\mathcal{O}) \quad \forall p < 6,$$

$$u^m|_{\mathcal{O}} \text{ belongs to a bounded set of } L^\infty(\tau, T; H^1(\mathcal{O})),$$

$$\left.\frac{\mathrm{d}u^m}{\mathrm{d}t}\right|_{\mathcal{O}} \text{ belongs to a bounded set of } L^2(\tau, T; L^p(\mathcal{O})).$$

This shows that $\{u^m|_{\mathcal{O}}\}$ is relatively compact in $L^2(\tau, T; L^p(\mathcal{O}))$, or we can extract a subsequence of $\{u^m|_{\mathcal{O}}\}$, also denoted by $\{u^m|_{\mathcal{O}}\}$, that converges in $L^2(\tau, T; L^p(\mathcal{O}))$ for all balls $\mathcal{O} \subset \Omega$. So, $u^m \to u$ in $L^2(\tau, T; L^p_{\mathrm{loc}}(\Omega))$.

Hence, for any j, $\Omega' = \operatorname{supp} w_j \cap \Omega$, $u^m \to u$ in $L^2(\tau, T; L^p(\Omega'))$. We have that

$$\begin{aligned}
&\left|\int_\tau^T \int_\Omega (u^m\nabla)u^m w^j \psi \,\mathrm{d}x\,\mathrm{d}t - \int_\tau^T \int_\Omega (u\nabla)uw^j \psi \,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad = \left|\int_\tau^T \int_{\Omega'} (u^m\nabla)u^m w^j \psi \,\mathrm{d}x\,\mathrm{d}t - \int_\tau^T \int_{\Omega'} (u\nabla)uw^j \psi \,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad = \left|\int_\tau^T (b(u^m, u^m, w^j) - b(u, u, w^j))\psi \,\mathrm{d}t\right| \\
&\quad = \left|\int_\tau^T (b(u^m, u^m - u, w^j) + b(u^m - u, u, w^j))\psi \,\mathrm{d}t\right| \\
&\quad = \left|\int_\tau^T (b(u^m, w^j, u^m - u) + b(u^m - u, w^j, u))\psi \,\mathrm{d}t\right| \\
&\quad \leqslant \left|\int_\tau^T (b(u^m - u, w^j, u))\psi \,\mathrm{d}t\right| + \left|\int_\tau^T (b(u^m, w^j, u^m - u))\psi \,\mathrm{d}t\right| \\
&\quad \leqslant c\int_\tau^T \|u^m - u\|_{L^3(\Omega')}(\|u^m\|_{L^6(\Omega')} + \|u\|_{L^6(\Omega')})\psi \,\mathrm{d}t \\
&\quad \leqslant c\left(\int_\tau^T \|u^m - u\|^2_{L^3(\Omega')} \,\mathrm{d}t\right)^{1/2} \\
&\qquad\quad \times \left[\left(\int_\tau^T \|u^m\|^2_{L^6(\Omega')} \,\mathrm{d}t\right)^{1/2} + \left(\int_\tau^T \|u^m\|^2_{L^6(\Omega')} \,\mathrm{d}t\right)^{1/2}\right] \\
&\quad \leqslant c\|u^m - u\|_{L^2(\tau,T;L^3(\Omega'))} \to 0 \quad (\text{as } m \to \infty).
\end{aligned}$$

From (3.2), passing to the limit when $m \to \infty$, we obtain that for all

$$\psi \in C_0^\infty(\tau, T), \qquad \varphi = \sum_{k=1}^\infty c^k w^k \in V,$$

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we have

$$\int_\tau^T \left\langle \frac{\partial u}{\partial t}, \varphi \right\rangle \psi \,\mathrm{d}t + \nu \int_\tau^T \int_\Omega \nabla u \nabla \varphi \psi \,\mathrm{d}x \,\mathrm{d}t + \alpha^2 \int_\tau^T \int_\Omega \nabla u_t \nabla \varphi \psi \,\mathrm{d}x \,\mathrm{d}t + \int_\tau^T \int_\Omega (u\nabla) u \varphi \psi \,\mathrm{d}x \,\mathrm{d}t = \int_\tau^T \langle f, \varphi \rangle \psi \,\mathrm{d}t.$$

This implies that u satisfies the first equation in (1.1) in the weak sense.

STEP 4 (proving that $u \in C([\tau, T]; V)$ and $u(\tau) = u_0$). Let

$$\{u_n\} \in C^1([\tau, T]; V)$$

be a sequence such that

$$u_n \to u \quad \text{in } L^\infty(\tau, T; V),$$
$$\frac{\mathrm{d}u_n}{\mathrm{d}t} \to \frac{\mathrm{d}u}{\mathrm{d}t} \quad \text{in } L^2(\tau, T; V).$$

We have that

$$\|u_n(t) - u_m(t)\|^2 = \|u_n(t_0) - u_m(t_0)\|^2 + 2\int_{t_0}^t \langle \nabla u_n(s) - \nabla u_m(s), \nabla u_n'(s) - \nabla u_m'(s) \rangle \,\mathrm{d}s.$$

Choosing t_0 such that

$$\|u_n(t_0) - u_m(t_0)\|^2 = \frac{1}{T-\tau} \int_\tau^T \|u_n(t) - u_m(t)\|^2 \,\mathrm{d}t,$$

we have that

$$\begin{aligned}
\int_\Omega |\nabla u_n(t) - \nabla u_m(t)|^2 \,\mathrm{d}x
&= \frac{1}{T-\tau} \int_\tau^T \int_\Omega |\nabla u_n(t) - \nabla u_m(t)|^2 \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + 2\int_{t_0}^t \int_\Omega (\nabla u_n(s) - \nabla u_n(s)) \cdot (\nabla u_n'(s) - \nabla u_n'(s)) \,\mathrm{d}x \,\mathrm{d}s \\
&\leqslant \frac{1}{T-\tau} \int_\tau^T \int_\Omega |\nabla u_n(t) - \nabla u_m(t)|^2 \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + 2\biggl(\int_{t_0}^t \int_\Omega |\nabla u_n(s) - \nabla u_m(s)|^2 \,\mathrm{d}x \,\mathrm{d}s \biggr)^{1/2} \\
&\qquad \times \biggl(\int_{t_0}^t \int_\Omega |\nabla u_n'(s) - \nabla u_m'(s)|^2 \,\mathrm{d}x \,\mathrm{d}s \biggr)^{1/2} \\
&\leqslant \frac{1}{T-\tau} \|\nabla u_n - \nabla u_m\|^2_{L^2(\tau,T;H)} \\
&\quad + 2\|\nabla u_n - \nabla u_m\|_{L^2(\tau,T;H)} \cdot \|\nabla u_n' - \nabla u_m'\|_{L^2(\tau,T;H)}.
\end{aligned}$$

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Hence, $\{u_n\}$ is a Cauchy sequence in $C([\tau,T];V)$. Thus, $u_n \to v$ in $C([\tau,T];V)$. On the other hand, $u_n \to u$ in V for a.e. $t \in [\tau,T]$. Therefore, $u = v$ (except on a set of zero measure), and this leads to $u \in C([\tau,T];V)$.

We now prove that $u(\tau) = u_0$. Since u is a weak solution of (1.1), choosing $\psi \in C^\infty[\tau,T]$, with $\psi(T) = 0$, for all $\phi \in V$ we have that

$$\int_\tau^T \int_\Omega \frac{\partial u}{\partial t}\phi\psi \,\mathrm{d}x\,\mathrm{d}t + \nu \int_\tau^T \int_\Omega \nabla u \nabla\phi\psi \,\mathrm{d}x\,\mathrm{d}t + \alpha^2 \int_\tau^T \int_\Omega \nabla u_t \nabla\phi\psi \,\mathrm{d}x\,\mathrm{d}t + \int_\tau^T \int_\Omega (u\nabla)u\phi\psi \,\mathrm{d}x\,\mathrm{d}t = \int_\tau^T \langle f,\phi\rangle\psi \,\mathrm{d}t.$$

Integrating by parts, we obtain

$$-\int_\Omega u(\tau)\phi\psi(\tau)\,\mathrm{d}x - \int_\tau^T \int_\Omega u\frac{\partial}{\partial t}(\phi\psi)\,\mathrm{d}x\,\mathrm{d}t + \nu \int_\tau^T \int_\Omega \nabla u\nabla\phi\psi\,\mathrm{d}x\,\mathrm{d}t + \alpha^2\biggl(-\int_\Omega \nabla u(\tau)\nabla\phi\psi(\tau)\,\mathrm{d}x - \int_\tau^T \int_\Omega \nabla u\frac{\partial}{\partial t}(\nabla\phi\psi)\,\mathrm{d}x\,\mathrm{d}t\biggr) + \int_\tau^T \int_\Omega (u\nabla)u\phi\psi\,\mathrm{d}x\,\mathrm{d}t = \int_\tau^T \langle f,\phi\rangle\psi\,\mathrm{d}t. \tag{3.3}$$

Treating the approximate solution u^m similarly, we have that

$$-\int_\Omega u^m(\tau)\phi\psi(\tau)\,\mathrm{d}x - \int_\tau^T \int_\Omega u^m\frac{\partial}{\partial t}(\phi\psi)\,\mathrm{d}x\,\mathrm{d}t + \nu \int_\tau^T \int_\Omega \nabla u^m\nabla\phi\psi\,\mathrm{d}x\,\mathrm{d}t + \alpha^2\biggl(-\int_\Omega \nabla u^m(\tau)\nabla\phi\psi(\tau)\,\mathrm{d}x - \int_\tau^T \int_\Omega \nabla u^m\frac{\partial}{\partial t}(\nabla\phi\psi)\,\mathrm{d}x\,\mathrm{d}t\biggr) + \int_\tau^T \int_\Omega (u^m\nabla)u^m\phi\psi\,\mathrm{d}x\,\mathrm{d}t = \int_\tau^T \langle f,\phi\rangle\psi\,\mathrm{d}t.$$

Passing to the limit when $m \to \infty$ we obtain

$$-\int_\Omega u_0\phi\psi(\tau)\,\mathrm{d}x - \int_\tau^T \int_\Omega u\frac{\partial}{\partial t}(\phi\psi)\,\mathrm{d}x\,\mathrm{d}t + \nu \int_\tau^T \int_\Omega \nabla u\nabla\phi\psi\,\mathrm{d}x\,\mathrm{d}t + \alpha^2\biggl(-\int_\Omega \nabla u_0\nabla\phi\psi(\tau)\,\mathrm{d}x - \int_\tau^T \int_\Omega \nabla u\frac{\partial}{\partial t}(\nabla\phi\psi)\,\mathrm{d}x\,\mathrm{d}t\biggr) + \int_\tau^T \int_\Omega (u\nabla)u\phi\psi\,\mathrm{d}x\,\mathrm{d}t = \int_\tau^T \langle f,\phi\rangle\psi\,\mathrm{d}t. \tag{3.4}$$

From (3.3) and (3.4), we have that

$$\int_\Omega u(\tau)\phi\psi(\tau)\,\mathrm{d}x + \alpha^2\int_\Omega \nabla u(\tau)\nabla\phi\psi(\tau)\,\mathrm{d}x = \int_\Omega u_0\phi\psi(\tau)\,\mathrm{d}x + \alpha^2\int_\Omega \nabla u_0\nabla\phi\psi(\tau)\,\mathrm{d}x. \tag{3.5}$$

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It is easy to see that

$$((u,v))_2 = (u,v) + \alpha^2((u,v))$$

is an inner product in V, which is equivalent to the usual inner product $((\cdot,\cdot))$. From (3.5), we have that

$$\psi(\tau)((u(\tau),\phi))_2 = \psi(\tau)((u_0,\phi))_2 \quad \text{for all } \psi \in C^\infty[\tau,T],\ \phi \in V.$$

This implies that $u(\tau) = u_0$ in V.

Step 5 (uniqueness and continuous dependence on initial data). Assume that u_1, u_2 are two weak solutions of (1.1), with initial data $u_{0_1}, u_{0_2} \in V$. Denote $u = u_1 - u_2$; then, $u \in C([\tau,T];V)$ and, for all test functions ϕ, we have that

$$\left\langle \frac{\mathrm{d}u}{\mathrm{d}t}, \phi \right\rangle + \nu(\nabla u, \nabla\phi) + \alpha^2 \left\langle \frac{\mathrm{d}\nabla u}{\mathrm{d}t}, \nabla\phi \right\rangle + b(u_1,u_1,\phi) - b(u_2,u_2,\phi) = 0.$$

Choosing $\phi = u$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \nu|\nabla u|^2 + \frac{\alpha^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla u|^2 + b(u_1,u_1,u) - b(u_2,u_2,u) = 0.$$

Hence,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}&(|u|^2 + \alpha^2|\nabla u|^2) + 2\nu|\nabla u|^2 \\
&= 2(-b(u_1,u_1,u) + b(u_2,u_2,u)) \\
&= -2b(u,u_2,u) \\
&= -2\sum_{i,j=1}^{3} \int_\Omega u^i \frac{\partial u_2^j}{\partial x^i} u^j \,\mathrm{d}x \\
&\leqslant 2\sum_{i,j=1}^{3} \left(\int_\Omega |u^i|^4 \,\mathrm{d}x\right)^{1/4} \left(\int_\Omega |\nabla u_2^j|^2 \,\mathrm{d}x\right)^{1/2} \left(\int_\Omega |u^j|^4 \,\mathrm{d}x\right)^{1/4} \\
&\leqslant 2|\nabla u_2| \cdot \|u\|_4^2 \\
&\leqslant 2c|\nabla u_2| \cdot |u|^{1/2} \cdot |\nabla u|^{3/2} \\
&\leqslant c|u|^{1/2} \cdot |\nabla u|^{3/2} \\
&\leqslant c(\nu)(|u|^{1/2})^4 + 2\nu(|\nabla u|^{3/2})^{4/3} \\
&\leqslant c|u|^2 + 2\nu|\nabla u|^2.
\end{aligned}$$

This implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}(|u|^2 + \alpha^2|\nabla u|^2) \leqslant c|u|^2 \leqslant c(|u|^2 + \alpha^2|\nabla u|^2).$$

By Gronwall's inequality, we get the desired result. □

4. Existence of pull-back $\mathcal{D}_\sigma$-attractors

Due to theorem 3.2, we can define a process $U(t,\tau)\colon V \to V$ by

$$U(t,\tau)u_0 = u(t;\tau,u_0), \quad \tau \leqslant t,\ u_0 \in V,$$

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where $u(t) = u(t; \tau, u_0)$ is the unique weak solution of (1.1) with the initial datum $u(\tau) = u_0$.

From now on, we denote $\sigma = \lambda_1 \nu/(1 + \alpha^2 \lambda_1)$.

Let $\mathcal{R}_\sigma$ be the set of all functions $r\colon \mathbb{R} \to (0, +\infty)$ such that

$$\lim_{t \to -\infty} \mathrm{e}^{\sigma t} r^2(t) = 0,$$

and denote by $\mathcal{D}_\sigma$ the class of all families $\hat{D} = \{D(t); t \in \mathbb{R}\} \subset \mathcal{P}(V)$ such that $D(t) \subset \bar{B}(0, r_{\hat{D}}(t))$, for some $r_{\hat{D}}(t) \in \mathcal{R}_\sigma$, where $\bar{B}(0, r_{\hat{D}}(t))$ denotes the closed ball in V centred at zero with radius $r_{\hat{D}}(t)$.

LEMMA 4.1. *Let $\{u_{0_n}\} \subset V$ be a sequence converging weakly in V to an element u_0 in V. Then,*

$$U(t, \tau, u_{0_n}) \rightharpoonup U(t, \tau, u_0) \quad \text{weakly in } V \text{ for all } \tau \geqslant t, \tag{4.1}$$

$$U(t, \tau, u_{0_n}) \rightharpoonup U(t, \tau, u_0) \quad \text{weakly in } L^2(\tau, T; V) \text{ for all } \tau \geqslant t. \tag{4.2}$$

Proof. Let $u_n(t) = U(t, \tau, u_{0_n})$, $u(t) = U(t, \tau, u_0)$. As in the proof of theorem 3.2, we have, for all $T \geqslant \tau$, that

$$\{u_n\} \text{ is bounded in } L^\infty(\tau, T; V) \tag{4.3}$$

and

$$\{u_n'\} \text{ is bounded in } L^2(\tau, T; V).$$

Then, for all $v \in V$,

$$((u_n(t + a) - u_n(t), v)) = \int_t^{t+a} \langle u_n'(s), v \rangle \, \mathrm{d}s \leqslant \|v\| a^{1/2} \|u_n'\|_{L^2(\tau, T, V)} \leqslant C_T \|v\| a^{1/2}, \tag{4.4}$$

where C_T is positive and independent of n. Then, for $v = u_n(t + a) - u_n(t)$, which belongs to V for almost every t, from (4.3) we have that

$$\|u_n(t + a) - u_n(t)\|^2 \leqslant C_T a^{1/2} \|u_n(t + a) - u_n(t)\|.$$

Hence,

$$\int_\tau^{T-a} \|u_n(t + a) - u_n(t)\|^2 \, \mathrm{d}t \leqslant C_T a^{1/2} \int_\tau^{T-a} \|u_n(t + a) - u_n(t)\| \, \mathrm{d}t. \tag{4.5}$$

Using Cauchy's inequality and (4.3), we deduce from (4.5) that

$$\int_\tau^{T-a} \|u_n(t + a) - u_n(t)\|^2 \, \mathrm{d}t \leqslant \tilde{C}_T a^{1/2}$$

for another positive constant $\tilde{C}_T$ independent of n. Therefore,

$$\lim_{a \to 0} \sup_n \int_\tau^{T-a} \|u_n(t + a) - u_n(t)\|^2_{H^1(\Omega_r)} \, \mathrm{d}t = 0 \tag{4.6}$$

for all $r > 0$, where $\Omega_r = \{x \in \Omega \colon |x| < r\}$. Moreover, from (4.3),

$$\{u_n|_{\Omega_r}\} \text{ is bounded in } L^\infty(\tau, T; H^1(\Omega_r))$$

for all $r > 0$. Now, consider a truncation function $\rho \in C^1(\mathbb{R}^+)$, with $\rho(s) = 1$ in $[0,1]$ and $\rho(s) = 0$ in $[2,+\infty)$. For each $r > 0$, define $v_{n,r}(x) = \rho(|x|^2/r^2)u_n(x)$ for $x \in \Omega_{2r}$. Then, from (4.6), we have that

$$\lim_{a\to 0}\sup_n \int_\tau^{T-a} \|v_{n,r}(t+a) - v_{n,r}(t)\|^2_{H^1(\Omega_{2r})}\,\mathrm{d}t = 0 \quad \text{for all } T > \tau,\ r > 0,$$

while from (4.3) we have that $\{v_{n,r}\}$ is bounded in $L^\infty(\tau,T;H_0^1(\Omega_{2r}))$ for all $T > \tau$, $r > 0$. Thus, by applying the Compactness Lemma, we obtain that

$$\{v_{n,r}\} \text{ is relatively compact in } L^2(\tau,T;H_0^1(\Omega_{2r})) \text{ for all } T > \tau,\ r > 0.$$

It follows that

$$\{u_n|_{\Omega_r}\} \text{ is relatively compact in } L^2(\tau,T;H_0^1(\Omega_{2r})) \text{ for all } T > \tau,\ r > 0.$$

Then, by a diagonal process, we can extract a subsequence $\{u_{n'}\}$ such that

$$\left.\begin{aligned} &u_{n'} \to \tilde{u} \quad \text{weakly-* in } L^\infty_{\text{loc}}(\mathbb{R};V),\\ &u_{n'} \to \tilde{u} \quad \text{strongly in } L^2_{\text{loc}}(\mathbb{R};H_0^1(\Omega_r)),\ r > 0,\end{aligned}\right\} \tag{4.7}$$

for some $\tilde{u} \in L^\infty_{\text{loc}}(\mathbb{R},\Omega)$. The convergences (4.7) allow us to pass to the limit in the equation for $u_{n'}$, to find that $\tilde{u}$ is a weak solution of (1.1) with $\tilde{u}(\tau) = u_0$. By the uniqueness of the solutions, we must have that $\tilde{u} = u$. Then, by a contradiction argument we deduce that the whole sequence $\{u_n\}$ converges to u in the sense of (4.7). This proves (4.2).

Now, from the strong convergence in (4.7) we also have that $u_n(t)$ converges strongly in $H_0^1(\Omega_r)$ to $u(t)$ for a.e. $t \geqslant \tau$ and all $r > 0$. Hence, for all $v \in \mathcal{V}$,

$$((u_n(t),v)) \to ((u(t),v)) \quad \text{for a.e. } t \in \mathbb{R}.$$

Moreover, from (4.3) and (4.4), we see that $\{(u_n(t),v)\}$ is equibounded and equicontinuous on $[\tau,T]$ for all $T > 0$. Therefore,

$$((u_n(t),v)) \to ((u(t),v)) \quad \forall t \in \mathbb{R},\ \forall v \in \mathcal{V}.$$

Finally, (4.1) follows from the fact that $\mathcal{V}$ is dense in V. □

THEOREM 4.2. *Suppose that $f \in L^2_{\text{loc}}(\mathbb{R};V')$ exists such that*

$$\int_{-\infty}^t \mathrm{e}^{\sigma\xi}\|f(\xi)\|_*^2\mathrm{d}\xi < +\infty \quad \textit{for all } t \in \mathbb{R}.$$

Then, there exists a unique pull-back $\mathcal{D}_\sigma$-attractor $\hat{\mathcal{A}} = \{A(t)\colon t \in \mathbb{R}\}$ for the process $U(t,\tau)$ associated with (1.1).

Proof. Let $\tau \in \mathbb{R}$, $u_0 \in V$ be fixed, and define

$$u(t) = u(t;\tau,u_0) = U(t,\tau)u_0 \quad \text{for all } t \geqslant \tau.$$

We introduce two new Hilbert norms in V as follows:

$$\begin{aligned} [u]_1^2 &:= \nu\|u\|^2 - \tfrac{1}{2}\sigma(|u|^2 + \alpha^2\|u\|^2),\\ [u]_2^2 &:= |u|^2 + \alpha^2\|u\|^2,\end{aligned}$$

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which are equivalent to the usual norm $\|\cdot\|$ in V. We will check the two conditions in theorem 2.5.

(i) *The process* $U(t,\tau)$ *has a family* $\hat{B}$ *of pull-back* $\mathcal{D}_\sigma$*-absorbing sets.*

From

$$\left\langle \frac{\mathrm{d}u}{\mathrm{d}t}, v \right\rangle + \nu((u,v)) + \alpha^2((u_t,v)) + b(u,u,v) = \langle f, v\rangle,$$

choosing $v(r) = \mathrm{e}^{\sigma r}u(r)$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}&(\mathrm{e}^{\sigma r}|u(r)|^2) + \alpha^2\frac{\mathrm{d}}{\mathrm{d}r}(\mathrm{e}^{\sigma r}\|u(r)\|^2) + 2\nu\mathrm{e}^{\sigma r}\|u(r)\|^2\\ &= \sigma\mathrm{e}^{\sigma r}|u(r)|^2 + \alpha^2\sigma\mathrm{e}^{\sigma r}\|u(r)\|^2 + 2\mathrm{e}^{\sigma r}\langle f(r), u(r)\rangle\\ &\leqslant \frac{\sigma}{\lambda_1}\mathrm{e}^{\sigma r}\|u(r)\|^2 + \alpha^2\sigma\mathrm{e}^{\sigma r}\|u(r)\|^2 + 2\mathrm{e}^{\sigma r}\|f(r)\|_*\|u(r)\|\\ &\leqslant \nu\mathrm{e}^{\sigma r}\|u(r)\|^2 + \frac{1}{\nu}\mathrm{e}^{\sigma r}\|f(r)\|_*^2 + \nu\mathrm{e}^{\sigma r}\|u(r)\|^2.\end{aligned}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}r}(\mathrm{e}^{\sigma r}|u(r)|^2 + \alpha^2\mathrm{e}^{\sigma r}\|u(r)\|^2) \leqslant \frac{1}{\nu}\mathrm{e}^{\sigma r}\|f(r)\|_*^2.$$

Integrating from τ to t we get that

$$\mathrm{e}^{\sigma t}(|u(t)|^2 + \alpha^2\|u(t)\|^2) \leqslant \mathrm{e}^{\sigma\tau}(|u(\tau)|^2 + \alpha^2\|u(\tau)\|^2) + \frac{1}{\nu}\int_\tau^t \mathrm{e}^{\sigma r}\|f(r)\|_*^2\,\mathrm{d}r$$

or

$$|U(t,\tau)u_0|^2 + \alpha^2\|U(t,\tau)u_0\|^2 \leqslant \mathrm{e}^{\sigma(\tau-t)}[u_0]_2^2 + \frac{\mathrm{e}^{-\sigma t}}{\nu}\int_\tau^t \mathrm{e}^{\sigma r}\|f(r)\|_*^2\,\mathrm{d}r.$$

This implies that

$$\|U(t,\tau)u_0\|^2 \leqslant \frac{1}{\alpha^2}\mathrm{e}^{\sigma(\tau-t)}[u_0]_2^2 + \frac{\mathrm{e}^{-\sigma t}}{\nu\alpha^2}\int_\tau^t \mathrm{e}^{\sigma r}\|f(r)\|_*^2\,\mathrm{d}r. \tag{4.8}$$

Denote by $R_\sigma^2(t)$ the non-negative number given for each $t\in\mathbb{R}$ by

$$R_\sigma^2(t) = \frac{2\mathrm{e}^{-\sigma t}}{\nu\alpha^2}\int_{-\infty}^t \mathrm{e}^{\sigma r}\|f(r)\|_*^2\,\mathrm{d}r, \tag{4.9}$$

and consider the family $\hat{B}_\sigma$ of the balls in V defined by

$$B_\sigma(t) = \{v\in V\colon \|v\| \leqslant R_\sigma(t)\}. \tag{4.10}$$

It is straightforward to check that $\hat{B_\sigma} = \{B_\sigma(t)\colon t\in\mathbb{R}\}\in\mathcal{D}_\sigma$ and, moreover, is a family of pull-back $\mathcal{D}_\sigma$-absorbing sets for the process $U(t,\tau)$.

(ii) $U(t,\tau)$ *is pull-back* $\mathcal{D}_\sigma$*-asymptotically compact.*

We fix $\hat{D}\in\mathcal{D}_\sigma$, $t\in\mathbb{R}$, a sequence $\tau_n\to-\infty$ and a sequence $u_{0_n}\in D(\tau_n)$. We have to prove that from the sequence $\{U(t,\tau_n)u_{0_n}\}$ we can extract a subsequence that converges in V.

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As the family $\hat{B}_\sigma$ is pull-back $\mathcal{D}_\sigma$-absorbing, for each integer $k \geqslant 0$, there exists a $\tau_{\hat{D}}(k)$ such that

$$U(t-k, \tau-k, D(\tau-k)) \subset B_\sigma(t-k) \quad \text{for all } \tau \leqslant \tau_{\hat{D}}(k).$$

Now, observe that, for $\tau \leqslant \tau_{\hat{D}}(k) + k$,

$$U(t-k, \tau, D(\tau)) \subset B_\sigma(t-k).$$

It is not difficult to conclude that there exist a subsequence

$$\{(\tau_{n'}, u_{0_{n'}})\} \subset \{(\tau_n, u_{0_n})\}$$

and a sequence $\{w_k; k \geqslant 0\} \subset V$ such that, for all $k \geqslant 0$ and $w_k \in B_\sigma(t-k)$,

$$U(t-k, \tau_{n'})u_{0_{n'}} \rightharpoonup w_k \quad \text{weakly in } V.$$

Since $V \subset H$ continuously,

$$U(t-k, \tau_{n'})u_{0_{n'}} \rightharpoonup w_k \quad \text{weakly in } H.$$

Observe that

$$\begin{aligned} w_0 &= \operatorname*{weak-lim}_{n' \to \infty} U(t, \tau_{n'})u_{0_{n'}} \\ &= \operatorname*{weak-lim}_{n' \to \infty} U(t, t-k)U(t-k, \tau_{n'})u_{0_{n'}} \\ &= U(t, t-k)\Big(\operatorname*{weak-lim}_{n' \to \infty} U(t-k, \tau_{n'})u_{0_{n'}}\Big), \end{aligned}$$

i.e.

$$U(t, t-k)w_k = w_0 \quad \text{for all } k \geqslant 0.$$

Then, by the lower semi-continuity of the norm,

$$[w_0]_2 \leqslant \liminf_{n' \to \infty} [U(t, \tau_{n'})u_{0_{n'}}]_2.$$

So, if we now also prove that

$$[w_0]_2 \geqslant \limsup_{n' \to \infty} [U(t, \tau_{n'})u_{0_{n'}}]_2,$$

then we have that

$$\lim_{n' \to \infty} [U(t, \tau_{n'})u_{0_{n'}}]_2 = [w_0]_2,$$

and this, together with the weak convergence, implies the strong convergence in V of $U(t, \tau_{n'})u_{0_{n'}}$ to w_0.

For all $\tau \in \mathbb{R}$, $t \geqslant \tau$, $u_0 \in V$, from

$$\Big\langle \frac{\mathrm{d}u}{\mathrm{d}t}, v \Big\rangle + \nu((u, v)) + \alpha^2((u_t, v)) + b(u, u, v) = \langle f, v \rangle,$$

choosing $v = \mathrm{e}^{\sigma(r-t)}u$, we get that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}(\mathrm{e}^{\sigma(r-t)}|u(r)|^2) &+ \alpha^2 \frac{\mathrm{d}}{\mathrm{d}r}(\mathrm{e}^{\sigma(r-t)}\|u(r)\|^2) + 2\nu\mathrm{e}^{\sigma(r-t)}\|u(r)\|^2 \\ &= \sigma\mathrm{e}^{\sigma(r-t)}|u(r)|^2 + \alpha^2\sigma\mathrm{e}^{\sigma(r-t)}\|u(r)\|^2 + 2\mathrm{e}^{\sigma(r-t)}\langle f(r), u(r) \rangle. \end{aligned}$$

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Integrating from τ to t, we obtain

$$\begin{aligned}
&|u(t)|^2 + \alpha^2\|u(t)\|^2 \\
&\quad = \mathrm{e}^{\sigma(\tau-t)}(|u(\tau)|^2 + \alpha^2\|u(\tau)\|^2) \\
&\qquad + 2\int_\tau^t \mathrm{e}^{\sigma(r-t)}(\langle f(r), u(r)\rangle - (\nu\|u(r)\|^2 - \tfrac{1}{2}\sigma|u(r)|^2 - \tfrac{1}{2}\alpha^2\sigma\|u(r)\|^2))\,\mathrm{d}r
\end{aligned}$$

or

$$[U(t,\tau)u_0]_2^2 = \mathrm{e}^{\sigma(\tau-t)}[u_0]_2^2 + 2\int_\tau^t \mathrm{e}^{\sigma(r-t)}(\langle f(r), U(r,\tau)u_0\rangle - [U(r,\tau)u_0]_1^2)\,\mathrm{d}r.$$

Thus, for all $k \geqslant 0$, $\tau_{n'} \leqslant k$, we have that

$$\begin{aligned}
[U(t,\tau_{n'})u_{0_{n'}}]_2^2 &= [U(t,t-k)U(t-k,\tau_{n'})u_{0_{n'}}]_2^2 \\
&= \mathrm{e}^{-\sigma k}[U(t-k,\tau_{n'})u_{0_{n'}}]_2^2 \\
&\quad + 2\int_{t-k}^t \mathrm{e}^{\sigma(r-t)}\langle f(r), U(r,t-k)U(t-k,\tau_{n'})u_{0_{n'}}\rangle\,\mathrm{d}r \\
&\quad - 2\int_{t-k}^t \mathrm{e}^{\sigma(r-t)}[U(r,t-k)U(t-k,\tau_{n'})u_{0_{n'}}]_1^2\,\mathrm{d}r.
\end{aligned}$$

By $\tau \leqslant \tau_{\hat{D}}(k) + k$, $k \geqslant 0$,

$$U(t-k,\tau,D(\tau)) \subset B_\sigma(t-k),$$

we have that

$$\limsup_{n'\to\infty}(\mathrm{e}^{-\sigma k}[U(t-k,\tau_{n'})u_{0_{n'}}]_2^2) \leqslant \mathrm{e}^{-\sigma k}\left(\frac{1}{\lambda_1} + \alpha^2\right)R_\sigma^2(t-k) \quad \forall k \geqslant 0.$$

On the other hand, as $U(t-k,\tau_{n'})u_{0_{n'}} \rightharpoonup w_k$ weakly in V, we have that

$$U(r,t-k)U(t-k,\tau_{n'})u_{0_{n'}} \rightharpoonup U(r,t-k)w_k \quad \text{weakly in } L^2(t-k,t;V).$$

Taking into account that, in particular, $\mathrm{e}^{\sigma(r-t)}f(r) \in L^2(t-k,t;V')$, we obtain

$$\begin{aligned}
&\limsup_{n'\to\infty}\int_{t-k}^t \mathrm{e}^{\sigma(r-t)}\langle f(r), U(r,t-k)U(t-k,\tau_{n'})u_{0_{n'}}\rangle\,\mathrm{d}r \\
&\qquad\qquad = \int_{t-k}^t \mathrm{e}^{\sigma(r-t)}\langle f(r), U(r,t-k)w_k\rangle\,\mathrm{d}r.
\end{aligned}$$

Moreover, as

$$\left(\int_{t-k}^t \mathrm{e}^{\sigma(r-t)}[v(r)]_1^2\,\mathrm{d}r\right)^{1/2}$$

defines a norm in $L^2(t-k,t;V)$, which is equivalent to the usual one, we also obtain

$$\begin{aligned}
&\int_{t-k}^t \mathrm{e}^{\sigma(r-t)}[U(r,t-k)w_k]_1^2\,\mathrm{d}r \\
&\qquad\qquad \leqslant \liminf_{n'\to\infty}\int_{t-k}^t \mathrm{e}^{\sigma(r-t)}[U(r,t-k)U(t-k,\tau_{n'})u_{0_{n'}}]_1^2\,\mathrm{d}r.
\end{aligned}$$

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Then, we easily obtain

$$\begin{aligned}\limsup_{n'\to\infty}&[U(t,\tau_{n'})u_{0_{n'}}]_2^2\\ &\leqslant \mathrm{e}^{-\sigma k}\left(\frac{1}{\lambda_1}+\alpha^2\right)R_\sigma^2(t-k)\\ &\quad + 2\int_{t-k}^{t}\mathrm{e}^{\sigma(r-t)}(\langle f(r),U(r,t-k)w_k\rangle - [U(r,t-k)w_k]_1^2)\,\mathrm{d}r.\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}[w_0]_2^2 &= [U(t,t-k)w_k]_2^2\\ &= \mathrm{e}^{-\sigma k}[w_k]_2^2 + 2\int_{t-k}^{t}\mathrm{e}^{\sigma(r-t)}\langle f(r),U(r,t-k)w_k\rangle\,\mathrm{d}r\\ &\quad - 2\int_{t-k}^{t}\mathrm{e}^{\sigma(r-t)}[U(r,t-k)w_k]_1^2\,\mathrm{d}r.\end{aligned}$$

Then,

$$\begin{aligned}\limsup_{n'\to\infty}[U(t,t-\tau_{n'})u_{0_{n'}}]_2^2 &\leqslant \mathrm{e}^{-\sigma k}\left(\frac{1}{\lambda_1}+\alpha^2\right)R_\sigma^2(t-k) + [w_0]_2^2 - \mathrm{e}^{-\sigma k}[w_k]_2^2\\ &\leqslant \mathrm{e}^{-\sigma k}\left(\frac{1}{\lambda_1}+\alpha^2\right)R_\sigma^2(t-k) + [w_0]_2^2,\end{aligned}$$

and thus, taking into account that

$$\mathrm{e}^{-\sigma k}R_\sigma^2(t-k) = \frac{2\mathrm{e}^{-\sigma t}}{\nu}\int_{-\infty}^{t-k}\mathrm{e}^{\sigma r}\|f(r)\|_*^2\,\mathrm{d}r \to 0$$

when $k\to+\infty$, we easily obtain

$$\limsup_{n'\to\infty}[U(t,\tau_{n'})u_{0_{n'}}]_2^2 \leqslant [w_0]_2^2.$$

This completes the proof. □

5. Fractal dimension estimates of the pull-back $\mathcal{D}_\sigma$-attractor

Observe that (1.1) can be written in the form (2.5) by taking

$$F(u,t) = -\nu Au(t) - \alpha^2 Au_t(t) - Bu(t) + f(t). \tag{5.1}$$

Then, it follows immediately that for all $t\in\mathbb{R}$ the mapping $F(\cdot,t)$ is Gâteaux differentiable in V, with

$$F'(u,t)v = -\nu Av - \alpha^2 Av_t - B(u,v) - B(v,u),\quad u,v\in V,$$

and that the mapping $F'\colon (u,t)\in V\times\mathbb{R}\mapsto F'(u,t)\in\mathcal{L}(V;V')$ is continuous.

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Evidently, for any $\tau \in \mathbb{R}$, $u_0, v_0 \in V$, there exists a unique solution $v(t) = v(t; \tau, u_0, v_0)$ of the problem

$$\left.\begin{aligned} &v \in C([\tau, T]; V),\\ \frac{\mathrm{d}v}{\mathrm{d}t} = -\nu Av(t) - \alpha^2 Av_t(t) - B(U(t,\tau)u_0, v(t)) &- B(v(t), U(t,\tau)u_0), \quad \tau < t,\\ &v(\tau) = v_0. \end{aligned}\right\} \tag{5.2}$$

From now on, we suppose that

$$f \in L^\infty(-\infty, T^*; V') \quad \text{for some } T^* \in \mathbb{R}. \tag{5.3}$$

Definition 5.1. We say that a process $U(t,\tau)$ is (uniform in the past) pull-back asymptotically compact if there exists T^* such that, for any sequence

$$\{(t_n, \tau_n)\}_{n \geqslant 1} \subset \mathbb{R}^2$$

satisfying

$$\tau_n \leqslant t_n \leqslant T^*, \quad n \geqslant 1, \quad \lim_{n\to\infty}(t_n - \tau_n) = +\infty,$$

and for any bounded sequence $\{u_{0_n}\}_{n\geqslant 1} \subset V$ such that $\{U(t_n, \tau_n)u_{0_n}\}_{n\geqslant 1}$ has a convergent subsequence in V.

Lemma 5.2. *Suppose that $f \in L^2_{\text{loc}}(\mathbb{R}; V')$ and satisfies (5.3). Then. $U(t,\tau)$ is (uniform in the past) pull-back asymptotically compact.*

Proof. We define, for each $n \geqslant 1$,

$$f_n(t) = \begin{cases} f(t+\tau_n) & \text{if } t < T^* - \tau_n,\\ 0 & \text{if } t > T^* - \tau_n. \end{cases}$$

Evidently, $f_n \in L^\infty(\mathbb{R}; V')$, with

$$\|f_n\|_{L^\infty(\mathbb{R},V')} \leqslant \|f\|_{L^\infty(-\infty,T^*;V')} \quad \text{for any } n \geqslant 1. \tag{5.4}$$

We define $U_{f_n}(s,0)u_{0_n} = v_n(s)$, $s \geqslant 0$, with v_n the unique solution of

$$\begin{gathered} v_n \in L^\infty(0,T;V) \quad \text{for all } T > 0,\\ \frac{\mathrm{d}v_n(s)}{\mathrm{d}s} = -\nu Av_n(s) - \alpha^2 Av_{ns}(s) - B(v_n(s)) + f_n(s), \quad s > 0,\\ v_n(0) = u_{0_n}. \end{gathered}$$

Then, it is not difficult to see that

$$U(s+\tau_n, \tau_n)u_{0_n} = U_{f_n}(s,0)u_{0_n} \quad \text{for any } s \in [0, T^* - \tau_n]. \tag{5.5}$$

In fact, if we define $w_n(s) = U(s+\tau_n, \tau_n)u_{0_n}$, $s \geqslant 0$, we have that w_n is the solution of

$$\begin{gathered} w_n \in L^\infty(0,T;V) \quad \text{for all } T > 0,\\ \frac{\mathrm{d}w_n(s)}{\mathrm{d}s} = -\nu Aw_n(s) - \alpha^2 Aw_{ns}(s) - B(w_n(s)) + f(s+\tau_n), \quad s > 0,\\ w_n(0) = u_{0_n}, \end{gathered}$$

and then, taking into account that $f_n(s) = f(s + \tau_n)$ for $s \in (0, T^* - \tau_n)$, we immediately obtain $v_n(s) = w_n(s)$ for all $s \in [0, T^* - \tau_n]$.

From (5.5), taking $s = t_n - \tau_n$, we obtain

$$U(t_n, \tau_n)u_{0_n} = U_{f_n}(t_n - \tau_n, 0)u_{0_n} \quad \text{for any } n \geqslant 1. \tag{5.6}$$

Now, taking into account (5.4) and (5.6), the assertion in the lemma is deduced by arguments similar to the ones in the proof of [8, lemma 3.3]. □

LEMMA 5.3. *Suppose that conditions (H1), (H2) and (5.3) hold. Then, the pull-back $\mathcal{D}_\sigma$-attractor $\hat{\mathcal{A}}$ obtained in theorem 4.2 satisfies*

$$\bigcup_{\tau \leqslant T^*} \mathcal{A}(\tau) \quad \textit{is relatively compact in } V.$$

Proof. We denote $M = \|f\|_{L^\infty(-\infty, T^*; V')}$. Using the notation introduced in (4.9), for any $t \leqslant T^*$, we have that

$$(R_\sigma(t))^2 \leqslant \frac{2M^2 \mathrm{e}^{-\sigma t}}{\nu\alpha^2} \int_{-\infty}^{t} \mathrm{e}^{\sigma\xi}\,\mathrm{d}\xi = \frac{2M^2}{\nu\sigma\alpha^2} = R^2,$$

and, consequently, for the ball $B_\sigma(t)$ defined by (4.10) we obtain that

$$B^* := \bigcup_{\tau \leqslant T^*} B_\sigma(\tau) \quad \text{is bounded in } V. \tag{5.7}$$

We denote by $\mathcal{M}$ the set of all $y \in V$ for which there exist the sequence

$$\{(t_n, \tau_n)\}_{n \geqslant 1} \subset \mathbb{R}^2$$

satisfying

$$\tau_n \leqslant t_n \leqslant T^*, \quad n \geqslant 1, \quad \lim_{n \to \infty} (t_n - \tau_n) = +\infty,$$

and the sequence $\{u_{0_n}\}_{n \geqslant 1} \subset B^*$, such that

$$\lim_{n \to \infty} \|U(t_n, \tau_n)u_{0_n} - y\| = 0.$$

First, observe that

$$\mathcal{A}(t) \subset \mathcal{M} \quad \text{for all } t \leqslant T^*. \tag{5.8}$$

In fact, by the definition of $\hat{\mathcal{A}}$, if $t \leqslant T^*$ and $y \in \mathcal{A}(t)$, there exist a sequence $\tau_n \leqslant t$ and a sequence $u_{0_n} \in B_\sigma(\tau_n) \subset B^*$ such that $\lim_{n \to \infty} \|U(t, \tau_n)u_{0_n} - y\| = 0$. Consequently, taking $t_n = t$ for all $n \geqslant 1$, we obtain that $y \in \mathcal{M}$.

On the other hand, $\mathcal{M}$ is the relatively compact subset of V. In fact, if $\{y_k\}_{k \geqslant 1} \subset \mathcal{M}$ is a given sequence, for each $k \geqslant 1$ we can take a pair $(t_k, \tau_k) \in \mathbb{R}^2$ and an element $u_{0_k} \in B^*$ such that $t_k \leqslant T^*$, $t_k - \tau_k \geqslant k$ and $\|U(t_k, \tau_k)u_{0_k} - y_k\| \leqslant (1/k)$. Then, by lemma 5.2, $\{U(t_k, \tau_k)u_{0_k}\}_{k \geqslant 1}$ has a subsequence, denoted by $\{U(t_k, \tau_k)u_{0_k}\}_{k \geqslant 1}$, convergent to $y \in V$. We have that

$$\lim_{k \to +\infty} \|y_k - y\| \leqslant \lim_{k \to +\infty} \|U(t_k, \tau_k)u_{0_k} - y_k\| + \lim_{k \to +\infty} \|U(t_k, \tau_k)u_{0_k} - y\| = 0.$$

It is immediate to show that we can extract a subsequence from $\{y_k\}_{k \geqslant 1}$ that converges in V.

As $\mathcal{M}$ is the relatively compact subset of V, taking into account that $\mathcal{A}(t) \subset \mathcal{M}$ for all $t \leqslant T^*$, we obtain that

$$\bigcup_{\tau \leqslant T^*} \mathcal{A}(\tau) \quad \text{is relatively compact in } V.$$

□

LEMMA 5.4. *Suppose that conditions (H1), (H2) and (5.3) hold. Then, the process* $U(t,\tau)$ *associated with (1.1) satisfies the quasi-differentiability properties (2.6), (2.7) and (2.8), with* $v(t) = v(t; \tau, u_0, v_0)$ *defined by (5.2).*

Proof. By (5.3) and lemma 5.3, there exists a constant $C > 1$ such that

$$\|f\|^2_{L^\infty(-\infty,T^*;V')} \leqslant C\nu^3 \quad \text{and} \quad \|u_0\|^2 \leqslant C\nu^2 \quad \text{for all } u_0 \in \bigcup_{\tau \leqslant T^*} \mathcal{A}(\tau). \tag{5.9}$$

We fix $\tau \leqslant T^*$, $u_0, \bar{u}_0 \in K(\tau)$, and we define $u(t) = U(t,\tau)u_0$, $\bar{u}(t) = U(t,\tau)\bar{u}_0$ and that $v(t)$ is the solution of (5.2), with $v_0 = \bar{u}_0 - u_0$. Let $z(t)$ be defined by

$$z(t) = \bar{u}(t) - u(t) - v(t), \quad t \geqslant \tau. \tag{5.10}$$

Evidently, $z(t)$ satisfies

$$\left.\begin{aligned} z \in L^\infty(\tau, T; V) \cap C([\tau, T]; V) \quad \text{for all } \tau < T, \\ \frac{\mathrm{d}}{\mathrm{d}t} z = -\nu A z - \alpha^2 A z_t - B(\bar{u}, \bar{u}) + B(u,u) + B(u,v) + B(v,u), \quad \tau < t, \\ z(\tau) = 0. \end{aligned}\right\} \tag{5.11}$$

We define

$$w(t) = \bar{u}(t) - u(t), \qquad \tau \leqslant t. \tag{5.12}$$

By using (2.2), we have that

$$B(\bar{u}, \bar{u}) - B(u,u) - B(u,v) - B(v,u) = B(w,w) + B(u,z) + B(z,u),$$

and, consequently,

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|z(t)|^2 + \alpha^2\|z(t)\|^2) + \nu\|z(t)\|^2 \\ = -b(z(t), u(t), z(t)) - b(w(t), w(t), z(t)), \quad \tau < t. \end{aligned} \tag{5.13}$$

But, taking into account (2.4), we have that

$$|b(z(t), u(t), z(t))| \leqslant \sqrt{2}|z(t)|\|z(t)\|\|u(t)\| \leqslant \frac{\nu}{2}\|z(t)\|^2 + \frac{1}{\nu}|z(t)|^2\|u(t)\|^2, \tag{5.14}$$

$$|b(w(t), w(t), z(t))| = |b(w(t), z(t), w(t))| \leqslant \frac{\nu}{2}\|z(t)\|^2 + \frac{1}{\nu}|w(t)|^2\|w(t)\|^2. \tag{5.15}$$

Thus, we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(|z(t)|^2+\alpha^2\|z(t)\|^2)&\\ &\leqslant \frac{2}{\nu}|z(t)|^2\|u(t)\|^2+\frac{2}{\nu}|w(t)|^2\|w(t)\|^2\\ &\leqslant \frac{2}{\nu(1+\lambda_1\alpha^2)}(|z(t)|^2+\alpha^2\|z(t)\|^2)\|u(t)\|^2+\frac{2}{\nu}|w(t)|^2\|w(t)\|^2.\end{aligned}$$

We integrate from τ to t, and apply the fact that $z(\tau)=0$, to obtain

$$\begin{aligned}&|z(t)|^2+\alpha^2\|z(t)\|^2\\ &\quad\leqslant \frac{2}{\nu(1+\alpha^2\lambda_1)}\int_\tau^t(|z(s)|^2+\alpha^2\|z(s)\|^2)\|u(s)\|^2\,\mathrm{d}s+\frac{2}{\nu}\int_\tau^t|w(s)|^2\|w(s)\|^2\,\mathrm{d}s,\end{aligned}$$

and consequently, by Gronwall's inequality,

$$|z(t)|^2+\alpha^2\|z(t)\|^2\leqslant\frac{2}{\nu}\int_\tau^t|w(s)|^2\|w(s)\|^2\,\mathrm{d}s\exp\left(\frac{2}{\nu(1+\alpha^2\lambda_1)}\int_\tau^t\|u(s)\|^2\,\mathrm{d}s\right). \tag{5.16}$$

On the other hand, integrating from τ to t the equality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|u(t)|^2+\alpha^2\|u(t)\|^2)+\nu\|u(t)\|^2=\langle f(t),u(t)\rangle,$$

we have that

$$\begin{aligned}|u(t)|^2+\alpha^2\|u(t)\|^2+2\nu\int_\tau^t\|u(s)\|^2\,\mathrm{d}s&\\ &\leqslant |u_0|^2+\alpha^2\|u_0\|^2+2\int_\tau^t\|f(s)\|_*\|u(s)\|\,\mathrm{d}s\\ &\leqslant \frac{1+\alpha^2\lambda_1}{\lambda_1}\|u_0\|^2+\frac{1}{\nu}\int_\tau^t\|f(s)\|_*^2\,\mathrm{d}s+\nu\int_\tau^t\|u(s)\|^2\,\mathrm{d}s.\end{aligned}$$

This implies that

$$|u(t)|^2+\alpha^2\|u(t)\|^2+\nu\int_\tau^t\|u(s)\|^2\,\mathrm{d}s\leqslant\frac{1+\alpha^2\lambda_1}{\lambda_1}\|u_0\|^2+\frac{1}{\nu}\int_\tau^t\|f(s)\|_*^2\,\mathrm{d}s \tag{5.17}$$

for all $\tau\leqslant t\leqslant T^*$. Taking into account (5.9), we easily deduce that, in particular,

$$\int_\tau^t\|u(s)\|^2\,\mathrm{d}s\leqslant C\nu\left(\frac{1+\alpha^2\lambda_1}{\lambda_1}+t-\tau\right)\quad\text{for all }\tau\leqslant t\leqslant T^*. \tag{5.18}$$

Also, from $w(t)=\bar{u}(t)-u(t)$, $w(t)$ satisfies

$$\begin{gathered}w\in L^\infty(\tau,T;V),\\ \frac{\mathrm{d}}{\mathrm{d}t}(w+\alpha^2Aw)=-\nu Aw-B(\bar{u})+B(u),\\ w(\tau)=\bar{u}_0-u_0.\end{gathered}$$

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This implies that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|w(t)|^2+\alpha^2\|w(t)\|^2)+\nu\|w(t)\|^2 &= b(u(t),u(t),w(t))-b(\bar{u}(t),\bar{u}(t),w(t))\\ &= -b(w(t),u(t),w(t))\\ &\leqslant \sqrt{2}|w(t)|\|w(t)\|\|u(t)\|\\ &\leqslant \frac{\nu}{2}\|w(t)\|^2+\frac{1}{\nu}|w(t)|^2\|u(t)\|^2\end{aligned}$$

for all $\tau \leqslant t$. So,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(|w(t)|^2+\alpha^2\|w(t)\|^2)+\nu\|w(t)\|^2 &\leqslant \frac{2}{\nu}|w(t)|^2\|u(t)\|^2\\ &\leqslant \frac{2}{\nu(1+\alpha^2\lambda_1)}(|w(t)|^2+\alpha^2\|w(t)\|^2)\|u(t)\|^2.\end{aligned}$$

In particular,

$$\begin{aligned}&|w(t)|^2+\alpha^2\|w(t)\|^2\\ &\quad\leqslant |w(\tau)|^2+\alpha^2\|w(\tau)\|^2+\frac{2}{\nu(1+\alpha^2\lambda_1)}\int_\tau^t(|w(s)|^2+\alpha^2\|w(s)\|^2)\|u(s)\|^2\,\mathrm{d}s.\end{aligned}$$

Then, by Gronwall's inequality,

$$|w(t)|^2+\alpha^2\|w(t)\|^2 \leqslant (|w(\tau)|^2+\alpha^2\|w(\tau)\|^2)\exp\left(\frac{2}{\nu(1+\alpha^2\lambda_1)}\int_\tau^t\|u(s)\|^2\,\mathrm{d}s\right)$$

for all $\tau \leqslant t$, and thus

$$|w(t)|^2+\alpha^2\|w(t)\|^2 \leqslant \frac{1+\alpha^2\lambda_1}{\lambda_1}\|w(\tau)\|^2\exp\left(2C\left(\frac{1}{\lambda_1}+\frac{1}{1+\alpha^2\lambda_1}(t-\tau)\right)\right) \tag{5.19}$$

for all $\tau \leqslant t \leqslant T^*$.

We also have, for all $\tau \leqslant t \leqslant T^*$, that

$$\begin{aligned}&\nu\int_\tau^t\|w(s)\|\,\mathrm{d}s\\ &\quad\leqslant |w(\tau)|^2+\alpha^2\|w(\tau)\|^2+\frac{2}{\nu}\int_\tau^t|w(s)|^2\|u(s)\|^2\,\mathrm{d}s\\ &\quad\leqslant \frac{1+\alpha^2\lambda_1}{\lambda_1}\|w(\tau)\|^2\\ &\qquad+\frac{2}{\nu}\int_\tau^t\frac{1+\alpha^2\lambda_1}{\lambda_1}\|w(\tau)\|^2\exp\left(2C\left(\frac{1}{\lambda_1}+\frac{1}{1+\alpha^2\lambda_1}(s-\tau)\right)\right)\|u(s)\|^2\,\mathrm{d}s\\ &\quad\leqslant \frac{1+\alpha^2\lambda_1}{\lambda_1}\|w(\tau)\|^2\\ &\qquad\times\left(1+\frac{2}{\nu}\exp\left(2C\left(\frac{1}{\lambda_1}+\frac{1}{1+\alpha^2\lambda_1}(t-\tau)v\right)\right)\int_\tau^t\|u(s)\|^2\,\mathrm{d}s\right)\end{aligned}$$

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$$\leqslant \frac{1+\alpha^2\lambda_1}{\lambda_1}\|w(\tau)\|^2 \times \left(1+2C\left(\frac{1+\alpha^2\lambda_1}{\lambda_1}+t-\tau\right)\exp\left(2C\left(\frac{1}{\lambda_1}+\frac{1}{1+\alpha^2\lambda_1}(t-\tau)\right)\right)\right). \tag{5.20}$$

From (5.16), (5.18), (5.19), (5.20) and the fact that $C>1$, we easily obtain

$$|z(t)|^2+\alpha^2\|z(t)\|^2 \leqslant \frac{2C(\lambda_1+2)}{\nu^2}\left(\frac{1+\alpha^2\lambda_1}{\lambda_1}\right)^2\|w(\tau)\|^4\left(\frac{1+\alpha^2\lambda_1}{\lambda_1}+t-\tau\right) \times \exp\left(6C\left(\frac{1}{\lambda_1}+\frac{1}{1+\alpha^2\lambda_1}(t-\tau)\right)\right)$$

for all $\tau \leqslant t \leqslant T^*$, i.e. (2.6)–(2.8) hold, with

$$\gamma(s,r)=\frac{r\sqrt{2C(\lambda_1+2)}}{\alpha\nu}\left(\frac{1+\alpha^2\lambda_1}{\lambda_1}\right)\sqrt{\frac{1+\alpha^2\lambda_1}{\lambda_1}+s}\exp\left(3C\left(\frac{1}{\lambda_1}+\frac{1}{1+\alpha^2\lambda_1}s\right)\right).$$

□

THEOREM 5.5. *Suppose that conditions (H1), (H2) and (5.3) hold. Then, the pull-back $\mathcal{D}_\sigma$-attractor $\hat{\mathcal{A}}=\{A(t)\colon t\in\mathbb{R}\}$ of the process $U(t,\tau)$ associated with (1.1) satisfies*

$$d_F(A(t)) \leqslant 2+\frac{C(\lambda_1+\alpha^2)^2\|f\|^4_{L^\infty(-\infty,T^*;V')}}{\nu^6\alpha^6\sigma^2} \quad \textit{for all } t\in\mathbb{R}. \tag{5.21}$$

Proof. We rewrite (1.1) in the form

$$\hat{u}_t=-\frac{\nu}{\alpha^2}\hat{u}+\frac{\nu}{\alpha^2}G^{-2}\hat{u}-G^{-1}B(G^{-1}\hat{u},G^{-1}\hat{u})+G^{-1}f, \tag{5.22}$$

where $G^2=I+\alpha^2A$ and $\hat{u}=Gu$. The equation of linear variations corresponding to (5.22) has the form

$$w_t=C(t)w, \tag{5.23}$$

where

$$C(t)w=-\frac{\nu}{\alpha^2}w+\frac{\nu}{\alpha^2}G^{-2}w-G^{-1}B(G^{-1}w,G^{-1}\hat{u})-G^{-1}B(G^{-1}\hat{u},G^{-1}w).$$

Now, we consider the quadratic form

$$(C(t)w,w)=-\frac{\nu}{\alpha^2}|w|^2+\frac{\nu}{\alpha^2}|G^{-1}w|^2-b(G^{-1}w,G^{-1}\hat{u},G^{-1}w).$$

By using inequality (2.3) and the inequality $\|G^{-1}u\| \leqslant (1/\alpha)|u|$, we get that

$$|b(G^{-1}w,G^{-1}\hat{u},G^{-1}w)| \leqslant C|G^{-1}w|^{1/2}\|G^{-1}w\|^{3/2}\|G^{-1}\hat{u}\| \leqslant \frac{C}{\alpha^{5/2}}|G^{-1}w|^{1/2}|w|^{3/2}|\hat{u}|.$$

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Employing Young's inequality, with $p = 4/3$, $e = 2\nu/3\alpha^2$, and the fact that on the pull-back attractor $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$ the estimate $|\hat{u}(t)| \leqslant (\lambda_1 + \alpha^2)^{1/2} R$ for all $t \leqslant T^*$, where

$$R^2 = \frac{2\|f\|^2_{L^\infty(-\infty,T^*;V')}}{\nu\sigma\alpha^2}$$

holds as in the proof of lemma 5.3, we obtain

$$|b(G^{-1}w, G^{-1}\hat{u}, G^{-1}w)| \leqslant \frac{\nu}{2\alpha^2}|w|^2 + \frac{C(\lambda_1+\alpha^2)^2R^4}{\alpha^4\nu^3}|G^{-1}w|^2.$$

Due to the last inequality, the quadratic form $(C(t)w, w)$ gives the estimate

$$(C(t)w, w) \leqslant -\frac{\nu}{2\alpha^2}|w|^2 + \left(\frac{\nu}{\alpha^2} + \frac{C(\lambda_1+\alpha^2)^2R^4}{\alpha^4\nu^3}\right)|G^{-1}w|^2.$$

Thus, we can use theorem 2.7, with

$$h_0(t,\tau) = \frac{\nu}{2\alpha^2}, \quad s_1 = -1, \quad h_{s_1}(t,\tau) = \frac{\nu}{\alpha^2} + \frac{C(\lambda_1+\alpha^2)^2R^4}{\alpha^4\nu^3}, \quad h_{s_k} = 0, \quad k \geqslant 2.$$

So

$$\bar{h}_0(t,\tau) = \frac{\nu}{2\alpha^2}, \quad s_1 = -1, \quad \bar{h}_{s_1}(t,\tau) = \frac{\nu}{\alpha^2} + \frac{C(\lambda_1+\alpha^2)^2R^4}{\alpha^4\nu^3} \quad \bar{h}_{s_k} = 0, \quad k \geqslant 2.$$

We choose N such that

$$-\bar{h}_0(t) + \sum_{k=1}^{m} \bar{h}_{s_k}(t)N^{s_k} < 0,$$

i.e.

$$-\frac{\nu}{2\alpha^2} + \left(\frac{\nu}{\alpha^2} + \frac{C(\lambda_1+\alpha^2)^2R^4}{\alpha^4\nu^3}\right)N^{-1} < 0$$

or

$$N^{-1} < \frac{\nu}{2\alpha^2} : \left(\frac{\nu}{\alpha^2} + \frac{C(\lambda_1+\alpha^2)^2R^4}{\alpha^4\nu^3}\right) = \frac{\nu^4\alpha^2}{2(\nu^4\alpha^2 + C(\lambda_1+\alpha^2)^2R^4)},$$

so

$$N > 2 + \frac{C(\lambda_1+\alpha^2)^2R^4}{\nu^4\alpha^2}.$$

Therefore, we have that

$$d_F(A(t)) \leqslant 2 + \frac{C(\lambda_1+\alpha^2)^2R^4}{\nu^4\alpha^2} = 2 + \frac{C(\lambda_1+\alpha^2)^2\|f\|^4_{L^\infty(-\infty,T^*;V')}}{\nu^6\alpha^6\sigma^2} \quad \text{for all } t \leqslant T^*.$$

Finally, since $U(t,\tau)$ is Lipschitz in $A(\tau)$, it follows that $d_F A(t)$ is bounded for every $t \geqslant \tau$ by the same bound. □

6. Relationships between pull-back attractors, uniform attractors and global attractors

First, we note that all results in the paper are still valid for the two-dimensional non-autonomous Navier–Stokes–Voigt equations in unbounded domains, with obvious changes in the proofs (for example, replacing Ladyzhenskaya's inequality in the case $N = 3$ by the one in the case $N = 2$, using the estimates of $b(u, u, v)$ when $N = 2$, etc). We now discuss the relationships between the pull-back attractor obtained in theorem 4.2, the uniform attractor obtained in [18] and the global attractor obtained when the external force f does not depend on the time variable t.

6.1. A relationship between pull-back attractors and global attractors

We now briefly consider the matter of the existence of a global attractor when the function f does not depend on the time variable t, i.e. in the autonomous case.

In this case, we define a continuous semigroup $S(t)\colon V \to V$ by

$$S(t)u_0 = u(t),$$

where $u(t)$ is the unique weak solution to (1.1) corresponding to the initial datum u_0. It is easy to see that

$$S(t)u_0 = U(t, 0)u_0 = U(t + \tau, \tau)u_0 \quad \text{for any } \tau \in \mathbb{R}.$$

We recall that the compact set $\mathcal{A}$ is said to be a global attractor for $S(t)$ if it is invariant (i.e. $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geqslant 0$) and attracts every bounded subset B of V, i.e.

$$\operatorname{dist}(S(t)B, \mathcal{A}) \to 0 \quad \text{as } t \to +\infty.$$

Now, assume that $f \in V'$. From (4.8), we obtain that the ball

$$B_0 = \left\{ u \in V \colon \|u\| \leqslant \frac{2\|f\|_*^2}{\nu\sigma\alpha^2} \right\}$$

is a bounded absorbing set for $S(t)$, i.e. for any bounded subset B there exists $T(B)$ such that $S(t)B \subset B_0$ when $t \geqslant T(B)$.

On the other hand, for any $t_n \to +\infty$ and $u_n \in B$, the sequence

$$S(t_n)u_n = U(t_n, 0)u_n = U(0, -t_n)u_n$$

is relatively compact in V by (ii) in the proof of theorem 4.2. Hence, $S(t)$ is asymptotically compact.

Then, it follows from standard theorems (see, for example, [6]) that the semigroup $S(t)$ possesses a connected global attractor $\mathcal{A}$ in V. Moreover, we formally deduce from theorem 5.5 that

$$\dim_F \mathcal{A} \leqslant 2 + \frac{C(\lambda_1 + \alpha^2)^2\|f\|_*^4}{\nu^6\alpha^6\sigma^2}.$$

Thus, even in the autonomous case, our results extend the recent results for Navier–Stokes–Voigt equations in bounded domains [11] to the case of unbounded domains.

6.2. A relationship between pull-back attractors and uniform attractors

First, we recall the concept of kernel sections. The kernel $\mathcal{K}$ of the process $U(t,\tau)$ consists of all bounded complete trajectories of the process $U(t,\tau)$ and can be written as

$$\mathcal{K} = \{u(\cdot) \mid U(t,\tau)u(\tau) = u(t),\ \operatorname{dist}(u(t), u(0)) \leqslant C_u\ \forall t \geqslant \tau,\ \tau \in \mathbb{R}\}.$$

The set $\mathcal{K}(s) = \{u(s) \colon u(\cdot) \in \mathcal{K}\}$ is said to be the kernel section at time $t = s$, $s \in \mathbb{R}$.

We now assume that the domain Ω is bounded and the external force f satisfies the following condition.

(H2′) $f \in L^2_b(\mathbb{R}; H)$, the set of translation bounded functions (see [6, 18]).

Denote by $\mathcal{H}_w(f)$ the closure of $\{f(\cdot + h) \mid h \in \mathbb{R}\}$ in $L^2_{\mathrm{loc}}(\mathbb{R}; H)$ with the weak topology. It is known that $\mathcal{H}_w(f)$ is weakly compact in $L^2_{\mathrm{loc}}(\mathbb{R}; H)$. By theorem 3.2, for each external force $\sigma \in \mathcal{H}_w(f)$ given, (1.1) has a unique weak solution $U_\sigma(t,\tau)u_\tau$ subject to the initial datum u_τ. Thus, we get a family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(f)}$ associated with (1.1). The following result was proved in [18].

Theorem 6.1. *Assume conditions (H1), (H2′) hold. Then, the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(f)}$ has a uniform attractor $\mathcal{A}_{\mathcal{H}_w(f)}$ in V. Moreover,*

$$\mathcal{A}_{\mathcal{H}_w(f)} = \bigcup_{\sigma\in\mathcal{H}_w(f)} \mathcal{K}_\sigma(s) \quad \forall s \in \mathbb{R},$$

where $\mathcal{K}_\sigma(s)$ is the kernel section at $t = s$ of the kernel $\mathcal{K}_\sigma$ of the process $\{U_\sigma(t,\tau)\}$, with symbol $\sigma \in \mathcal{H}_w(f)$.

Obviously, (H2′) implies (H2). For (1.1), it is proved in theorem 4.2 that for any $\sigma \in \mathcal{H}_w(f)$ the process $\{U_\sigma(t,\tau)\}$ has a pull-back attractor $\hat{\mathcal{A}}_\sigma = \{A_\sigma(t) \colon t \in \mathbb{R}\}$ in V. Moreover, we have the following.

Theorem 6.2. *Assume conditions (H1), (H2′) hold. Then, for any $\sigma \in \mathcal{H}_w(f)$, the process $U_\sigma(t,\tau)$ has a pull-back attractor $\hat{\mathcal{A}}_\sigma = \{A_\sigma(t) \colon t \in \mathbb{R}\}$ in V, and*

$$A_\sigma(s) = \mathcal{K}_\sigma(s), \qquad \bigcup_{\sigma\in\mathcal{H}_w(f)} A_\sigma(s) = \mathcal{A}_{\mathcal{H}_w(f)} \quad \forall s \in \mathbb{R},$$

where $\mathcal{A}_{\mathcal{H}_w(f)}$ is the uniform attractor of (1.1), $\mathcal{K}_\sigma$ is the kernel of the process $U_\sigma(t,\tau)$.

Proof. Since $\hat{\mathcal{A}}_\sigma$ is pull-back attracting and since $A_\sigma(s)$ is compact, we have that

$$\mathcal{K}_\sigma(s) \subset A_\sigma(s) \quad \text{for any } s \in \mathbb{R}.$$

On the other hand, by the definition of $\mathcal{K}_\sigma(s)$ and the invariance of $\hat{\mathcal{A}}_\sigma$, we have that

$$A_\sigma(s) \subset \mathcal{K}_\sigma(s) \quad \text{for any } s \in \mathbb{R}.$$

So, we have that

$$A_\sigma(s) = \mathcal{K}_\sigma(s) \quad \text{for any } s \in \mathbb{R}. \tag{6.1}$$

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Then, by (6.1) and theorem 6.1,

$$\mathcal{A}_{\mathcal{H}_w(f)} = \bigcup_{\sigma \in \mathcal{H}_w(f)} \mathcal{K}_\sigma(s) = \bigcup_{\sigma \in \mathcal{H}_w(f)} A_\sigma(s) \quad \forall s \in \mathbb{R}.$$

The proof is complete. □

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