VARIETIES WHOSE TOLERANCES ARE HOMOMORPHIC IMAGES OF THEIR CONGRUENCES

GÁBOR CZÉDLI✉ and EMIL W. KISS

Dedicated to Béla Csákány on his eightieth birthday

Abstract

The homomorphic image of a congruence is always a tolerance (relation) but, within a given variety, a tolerance is not necessarily obtained this way. By a Maltsev-like condition, we characterise varieties whose tolerances are homomorphic images of their congruences (TImC). As corollaries, we prove that the variety of semilattices, all varieties of lattices, and all varieties of unary algebras have TImC. We show that a congruence *n*-permutable variety has TImC if and only if it is congruence permutable, and construct an idempotent variety with a majority term that fails TImC.

1. Introduction

Let $\mathbf{A} = (A; F)$ be a (general) algebra. By a *tolerance* (*relation*) of $\mathbf{A}$ we mean a reflexive, symmetric, and compatible relation $\boldsymbol{\tau} \subseteq A^2$. Transitive tolerances are *congruences*. Tolerances, implicitly or explicitly, often played an important role in the theory of Maltsev (also written 'Mal'cev') conditions, for example in Czédli *et al.* [9] and Jónsson [15]. Tolerances are particularly useful in lattice theory; partly because the algebraic functions on a finite lattice are just the monotone functions preserving tolerances (see Kindermann [16]), and also because tolerances play a crucial role in decompositions of modular lattices into maximal complemented intervals (see Day and Herrmann [11] and Herrmann [14]).

There are two important ways to deal with tolerances. Following Chajda [1], Chajda *et al.* [6], and Czédli and Klukovits [10], one can describe them by their blocks. However, the present paper is devoted to a promising recent approach to tolerances: they can often be characterised as homomorphic images of congruences.

Clearly (see also Fried and Grätzer [12]), if $\varphi\colon \mathbf{B} \to \mathbf{A}$ is a surjective homomorphism and $\boldsymbol{\tau}$ is a tolerance of $\mathbf{B}$, then $\varphi(\boldsymbol{\tau}) = \{(\varphi(a), \varphi(b)) : (a, b) \in \boldsymbol{\tau}\}$ is a tolerance of $\mathbf{A}$. In particular, if $\boldsymbol{\vartheta}$ is a congruence of $\mathbf{B}$, then $\varphi(\boldsymbol{\vartheta})$ is a tolerance (but not necessarily a congruence) of $\mathbf{A}$. We are interested in varieties of algebras whose tolerances are homomorphic images of their congruences, (TImC). The TImC property holds in a variety $\mathcal{V}$ if for every $\mathbf{A} \in \mathcal{V}$ and each tolerance $\boldsymbol{\tau}$ of $\mathbf{A}$, there exist an algebra $\mathbf{B} \in \mathcal{V}$, a congruence $\boldsymbol{\vartheta}$ of $\mathbf{B}$, and a homomorphism $\varphi\colon \mathbf{B} \to \mathbf{A}$ such that $\boldsymbol{\tau} = \varphi(\boldsymbol{\vartheta})$. Notice that φ is necessarily surjective since $\boldsymbol{\tau}$ is reflexive.

Using an old construction discovered by the first author in [7], Czédli and Grätzer [8] proved that the variety of all lattices satisfies TImC. Some other varieties satisfying TImC have recently been found by Chajda *et al.* [4, 5]. In particular, we know from [5] that all varieties defined by so-called balanced identities satisfy TImC, and each algebra belongs to some variety satisfying TImC.

Our goal is to give a Maltsev-like characterisation of the TImC property. This characterisation enables us to find several new results stating that certain varieties, including all lattice varieties, all unary varieties, and the variety of semilattices, satisfy TImC. In the last section of the paper we initiate the investigation of the relationship between known Maltsev conditions and TImC.

2. Characterising TImC

Let $n \in \mathbb{N} = \{1, 2, \ldots\}$. We say that a variety $\mathcal{V}$ satisfies the *Maltsev-like condition* $M(n)$ if for any pair (f, g) of $2n$-ary terms such that the identity

$$f(x_0, x_0, x_1, x_1, \ldots, x_{n-1}, x_{n-1}) \approx g(x_0, x_0, x_1, x_1, \ldots, x_{n-1}, x_{n-1})$$

holds in $\mathcal{V}$, there exists a $4n$-ary term h such that the identities

$$\begin{aligned}
&f(x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}) \\
&\quad \approx h(x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}), \\
&g(x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}) \\
&\quad \approx h(y_0, x_0, y_1, x_1, \ldots, y_{n-1}, x_{n-1}, x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1})
\end{aligned}$$

also hold in $\mathcal{V}$. Our main goal is to prove the following two theorems.

THEOREM 2.1. *For an arbitrary variety $\mathcal{V}$ of algebras, the following two conditions are equivalent.*

(i)   *$\mathcal{V}$ satisfies TImC, that is, the tolerances of $\mathcal{V}$ are homomorphic images of its congruences.*

(ii)   *For all $n \in \mathbb{N}$, condition $M(n)$ holds in $\mathcal{V}$.*

For a variety $\mathcal{V}$, let $\mathfrak{A}(\mathcal{V})$ be the *clone algebra* of $\mathcal{V}$ introduced in Taylor [21, Definition 2.9]. It is a heterogeneous algebra consisting of the equivalence classes of all finitary terms of $\mathcal{V}$, where two terms, t_1 and t_2, are equivalent if and

only if the identity $t_1 \approx t_2$ holds in $\mathcal{V}$. This heterogeneous algebra is equipped with (heterogeneous) substitution operations and with constant operations assigning projections. In this terminology, $M(n)$ becomes a first-order formula in the language of $\mathfrak{A}(\mathcal{V})$, and $M(n)$ holds in $\mathcal{V}$ if and only if this first-order formula, also denoted by $M(n)$, holds in $\mathfrak{A}(\mathcal{V})$. Therefore, Theorem 2.1 characterises TImC by the countably infinite set $\{M(n) : n \in \mathbb{N}\}$ of first-order formulas in the language of clone algebras of varieties. This raises the question whether we really need infinitely many formulas. The answer and some additional information are given in the following statement.

THEOREM 2.2.

(i)   *For $n \in \mathbb{N}$, $M(n + 1)$ implies $M(n)$.*
(ii)   *For $n \in \mathbb{N}$, $M(n)$ does not imply $M(n + 1)$.*
(iii) *Assume that Σ is a finite set of first-order formulas in the language of clone algebras of varieties. Then Σ, that is, the conjunction of all members of Σ, is not equivalent to TImC.*

3. Proving our theorems

For $n \in \mathbb{N}$, the list $z_0, \ldots, z_{n-1}$ of the elements (or lists) z_i will often be denoted by $z_i\colon i < n$, and similar notation applies when n is an ordinal. Lists are concatenated by semicolons. This convention allows us to write terms in a concise form. For example, $M(n)$ in this notation is the following condition: if

$$f(x_i, x_i\colon i < n) \approx g(x_i, x_i\colon i < n), \tag{3.1}$$

then there exists a $4n$-ary term h such that

$$f(x_i, y_i\colon i < n) \approx h(x_i, y_i\colon i < n \,;\, x_i, y_i\colon i < n), \tag{3.2}$$
$$g(x_i, y_i\colon i < n) \approx h(y_i, x_i\colon i < n \,;\, x_i, y_i\colon i < n). \tag{3.3}$$

The algebra freely generated by X in a variety $\mathcal{V}$ will be denoted by $\mathbf{F}_{\mathcal{V}}(X)$. We consider only well-ordered free generating sets. Therefore, if κ and μ denote cardinal numbers and we write, say, $X = \{x_i, y_i\colon i < \kappa\,;\, z_j\colon j < \mu\}$, which is the same as $X = \{x_i : i < \kappa\} \cup \{y_i : i < \kappa\} \cup \{z_j : j < \mu\}$, then we always assume that $\{x_i : i < \kappa\}$, $\{y_i : i < \kappa\}$, and $\{z_j : j < \mu\}$ are pairwise disjoint and each of the equations $x_i = x_j$, $y_i = y_j$, and $z_i = z_j$ implies that $i = j$. However, if a_i and a_j are not necessarily free generators of a free algebra, then $a_i = a_j$ does not imply that $i = j$. Sometimes we allow 'formally infinitary' terms like $f(x_i, y_i\colon i < \kappa\,;\, z_j\colon j < \mu)$; they, of course, depend only on finitely many of their variables. The smallest congruence collapsing a and b is denoted by $\mathrm{con}(a, b)$. The idea of the following statement goes back to Maltsev [17] and Jónsson [15]; for the reader's convenience and also to demonstrate how our notation works, we give a short proof.

LEMMA 3.1. *Let $X = \{x_i, y_i\colon i < \kappa\,;\, z_j\colon j < \mu\}$ be a nonempty set, let $\mathcal{V}$ be a variety, and denote by ϑ the congruence $\bigvee\{\mathrm{con}(x_i, y_i) : i < \kappa\}$ of $\mathbf{F}_{\mathcal{V}}(X)$. Let f and g be*

terms over X. Then ϑ collapses the elements $f(x_i, y_i \colon i < \kappa \,; z_j \colon j < \mu)$ and $g(x_i, y_i \colon i < \kappa \,; z_j \colon j < \mu)$ of $\mathbf{F}_{\mathcal{V}}(X)$ if and only if $\mathcal{V}$ satisfies the identity $f(x_i, x_i \colon i < \kappa \,; z_j \colon j < \mu) \approx g(x_i, x_i \colon i < \kappa \,; z_j \colon j < \mu)$.

PROOF. If the identity holds, then

$$f(x_i, y_i \colon i < \kappa \,; z_j \colon j < \mu) \,\vartheta\, f(x_i, x_i \colon i < \kappa \,; z_j \colon j < \mu)$$
$$= g(x_i, x_i \colon i < \kappa \,; z_j \colon j < \mu) \,\vartheta\, g(x_i, y_i \colon i < \kappa \,; z_j \colon j < \mu),$$

and the transitivity of ϑ applies. Conversely, if ϑ collapses the two elements of $\mathbf{F}_{\mathcal{V}}(X)$ in question, then let $Y = \{x_i \colon i < \kappa \,; z_j \colon j < \mu\}$, and consider the unique homomorphism $\varphi \colon \mathbf{F}_{\mathcal{V}}(X) \to \mathbf{F}_{\mathcal{V}}(Y)$ such that $\varphi(x_i) = \varphi(y_i) = x_i$ for $i < \kappa$ and $\varphi(z_j) = z_j$ for $j < \mu$. Then $\vartheta \subseteq \mathrm{Ker}(\varphi)$ since φ collapses the pairs that generate ϑ. Hence

$$f(x_i, x_i \colon i < \kappa \,; z_j \colon j < \mu) = \varphi(f(x_i, y_i \colon i < \kappa \,; z_j \colon j < \mu))$$
$$= \varphi(g(x_i, y_i \colon i < \kappa \,; z_j \colon j < \mu))$$
$$= g(x_i, x_i \colon i < \kappa \,; z_j \colon j < \mu),$$

which implies that the required identity holds in $\mathcal{V}$. $\square$

The following auxiliary statement follows from Chajda [2, Lemma 1.7]; it also follows easily from the observation that the tolerances of $\mathbf{A}$ are just the symmetric subalgebras of $\mathbf{A}^2$ containing the diagonal $\{(a, a) : a \in A\}$.

LEMMA 3.2. *Assume that τ is the smallest tolerance of $\mathbf{A} = (A; F)$ that contains the pairs (a_i, b_i) for $i < \kappa$. Let $(d, e) \in A^2$, and assume that $C = \{c_j : j < \mu\} \subseteq A$ generates $\mathbf{A}$. Then $(d, e) \in \tau$ if and only if there is a term h such that*

$$d = h(a_i, b_i \colon i < \kappa \,; c_j \colon j < \mu) \quad and \quad e = h(b_i, a_i \colon i < \kappa \,; c_j \colon j < \mu).$$

PROOF OF THEOREM 2.1. In order to prove that part (i) implies part (ii), assume that $n \in \mathbb{N}$, that $\mathcal{V}$ satisfies TImC, and that f and g are $2n$-ary terms such that the identity $f(x_i, x_i \colon i < n) \approx g(x_i, x_i \colon i < n)$ holds in $\mathcal{V}$. Let $X = \{x_i, y_i \colon i < n\}$ and $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(X)$. Define τ as the tolerance generated by $\{(x_i, y_i) : i < n\}$. By assumption, there exists a $\mathbf{B} \in \mathcal{V}$, a congruence ϑ of $\mathbf{B}$, and a surjective homomorphism $\mathbf{B} \to \mathbf{F}$ such that $\tau = \varphi(\vartheta)$. We can pick elements $a_i, b_i \in B$ such that, for $i < n$, $(a_i, b_i) \in \vartheta$, $\varphi(a_i) = x_i$, and $\varphi(b_i) = y_i$. Since

$$f(a_i, b_i \colon i < n) \,\vartheta\, f(a_i, a_i \colon i < n) = g(a_i, a_i \colon i < n) \,\vartheta\, g(a_i, b_i \colon i < n),$$

by applying φ we conclude that $(f(x_i, y_i \colon i < n), g(x_i, y_i \colon i < n)) \in \tau$. Therefore, applying Lemma 3.2 with $C = X$ and using the fact that an equation of two terms on the free generators is an identity that holds in $\mathcal{V}$, we obtain a $2n$-ary term h such that identities (3.2) and (3.3) hold in $\mathcal{V}$. Hence part (i) implies part (ii).

To prove the converse implication, assume that $M(n)$ holds in $\mathcal{V}$ for all $n \in \mathbb{N}$, $\mathbf{A} = (A; F) \in \mathcal{V}$, and $\tau = \{(a_i, b_i) : i < \kappa\}$ is a tolerance of $\mathbf{A}$. Here κ is an ordinal.

By reflexivity, $A = \{a_i : i < \kappa\}$, but usually this is a redundant enumeration of A. We will find a congruence preimage $\boldsymbol{\vartheta}$ of $\boldsymbol{\tau}$ in two steps: we construct first a 'free' tolerance preimage $\boldsymbol{\varrho}$ of $\boldsymbol{\tau}$, and then a congruence preimage $\boldsymbol{\vartheta}$ of $\boldsymbol{\varrho}$.

Let $Y = \{u_i, v_i : i < \kappa\}$, $\mathbf{G} = \mathbf{F}_{\mathcal{V}}(Y)$, and let $\boldsymbol{\varrho}$ be the tolerance generated by $\{(u_i, v_i) : i < \kappa\}$. Consider the surjective homomorphism $\varphi \colon \mathbf{G} \to \mathbf{A}$ such that $\varphi(u_i) = a_i$ and $\varphi(v_i) = b_i$ for $i < \kappa$. Since $\varphi(\boldsymbol{\varrho})$ is a tolerance of $\mathbf{A}$ and contains the pairs $(a_i, b_i) = (\varphi(u_i), \varphi(v_i))$, we have that $\boldsymbol{\tau} \subseteq \varphi(\boldsymbol{\varrho})$. To show the converse inclusion, take a pair in $\varphi(\boldsymbol{\varrho})$. It is of the form $(t_1(a_i, b_i : i < \kappa), t_2(a_i, b_i : i < \kappa))$ where t_1 and t_2 are terms and $(t_1(u_i, v_i : i < \kappa), t_2(u_i, v_i : i < \kappa)) \in \boldsymbol{\varrho}$. Applying Lemma 3.2 with $C = Y$, we obtain a term t_3 such that

$$t_1(u_i, v_i : i < \kappa) = t_3(u_i, v_i : i < \kappa ; u_i, v_i : i < \kappa),$$
$$t_2(u_i, v_i : i < \kappa) = t_3(v_i, u_i : i < \kappa ; u_i, v_i : i < \kappa).$$

Since φ transfers these two equations to

$$t_1(a_i, b_i : i < \kappa) = t_3(a_i, b_i : i < \kappa ; a_i, b_i : i < \kappa),$$
$$t_2(a_i, b_i : i < \kappa) = t_3(b_i, a_i : i < \kappa ; a_i, b_i : i < \kappa),$$

it follows (directly or from Lemma 3.2) that $(t_1(a_i, b_i : i < \kappa), t_2(a_i, b_i : i < \kappa)) \in \boldsymbol{\tau}$. Therefore, $\boldsymbol{\tau} = \varphi(\boldsymbol{\varrho})$. In this first step we did not use condition $M(n)$.

Next, let $\{(c_j, d_j) : j < \mu\} = \boldsymbol{\varrho}$, $X = \{x_j, y_j : j < \mu\}$, and $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(X)$. Consider the congruence $\boldsymbol{\vartheta} = \bigvee \{\mathrm{con}(x_j, y_j) : j < \mu\}$ of $\mathbf{F}$, and let $\psi \colon \mathbf{F} \to \mathbf{G}$ be the surjective homomorphism defined by $\psi(x_j) = c_j$ and $\psi(y_j) = d_j$ for $j < \mu$. Clearly, $\boldsymbol{\varrho} \subseteq \psi(\boldsymbol{\vartheta})$. To prove the converse inclusion, take a pair in $\boldsymbol{\vartheta}$. It is of the form $(p(x_j, y_j : j < \mu), q(x_j, y_j : j < \mu)) \in \boldsymbol{\vartheta}$, where p and q are terms. We have to prove that the pair

$$(p(c_j, d_j : j < \mu), q(c_j, d_j : j < \mu)) = (\psi(p(x_j, y_j : j < \mu)), \psi(q(x_j, y_j : j < \mu)))$$

belongs to $\boldsymbol{\varrho}$. We obtain from Lemma 3.1 that

$$\text{the identity } p(x_j, x_j : j < \mu) \approx q(x_j, x_j : j < \mu) \text{ holds in } \mathcal{V}. \tag{3.4}$$

Since the terms p and q depend on finitely many variables and the original ordering of variables is irrelevant, we can assume that $p(x_j, y_j : j < \mu) \approx f(x_i, y_i : i < n)$ and $q(x_j, y_j : j < \mu) \approx g(x_i, y_i : i < n)$ hold in $\mathcal{V}$ for some $n \in \mathbb{N}$ and some $2n$-ary terms f and g. Hence (3.4) turns into

$$f(x_i, x_i : i < n) \approx g(x_i, x_i : i < n) \text{ holds in } \mathcal{V},$$

and all we have to prove is that

$$(f(c_i, d_i : i < n), g(c_i, d_i : i < n)) \in \boldsymbol{\varrho}. \tag{3.5}$$

Let h be a $4n$-ary term provided by $M(n)$. Since

$$f(c_i, d_i : i < n) = h(c_i, d_i : i < n ; c_i, d_i : i < n),$$
$$g(c_i, d_i : i < n) = h(d_i, c_i : i < n ; c_i, d_i : i < n),$$

and h preserves $\boldsymbol{\varrho}$, (3.5) follows. This shows that $\boldsymbol{\varrho} = \psi(\boldsymbol{\vartheta})$.

Finally, the composite map $\varphi \circ \psi \colon \mathbf{F} \to \mathbf{A}$, $z \mapsto \varphi(\psi(z))$ is a surjective homomorphism, and $(\varphi \circ \psi)(\boldsymbol{\vartheta}) = \varphi(\psi(\boldsymbol{\vartheta})) = \varphi(\boldsymbol{\varrho}) = \boldsymbol{\tau}$. That is, $\mathcal{V}$ satisfies TImC.   □

PROOF OF THEOREM 2.2. To prove part (i), assume that $M(n + 1)$ holds in a variety $\mathcal{V}$, and f and g are $2n$-ary terms such that the identity $f(x_i, x_i \colon i < n) \approx g(x_i, x_i \colon i < n)$ holds in $\mathcal{V}$. By adding two fictitious variables, we define two $2(n + 1)$-ary terms as follows: $f^*(x_i, y_i \colon i \le n) = f(x_i, y_i \colon i < n)$ and $g^*(x_i, y_i \colon i \le n) = g(x_i, y_i \colon i < n)$. Since $f^*(x_i, x_i \colon i \le n) \approx g^*(x_i, x_i \colon i \le n)$ clearly holds in $\mathcal{V}$, $M(n + 1)$ gives a $4(n + 1)$-ary term h^* such that the identities

$$
\begin{aligned}
f(x_i, y_i \colon i < n) &\approx f^*(x_i, y_i \colon i \le n) \approx h^*(x_i, y_i \colon i \le n \,;\, x_i, y_i \colon i \le n), \\
g(x_i, y_i \colon i < n) &\approx g^*(x_i, y_i \colon i \le n) \approx h^*(y_i, x_i \colon i \le n \,;\, x_i, y_i \colon i \le n)
\end{aligned}
\tag{3.6}
$$

hold in $\mathcal{V}$. Define a $4n$-ary term h by letting $h(u_i, v_i \colon i < n \,;\, x_i, y_i \colon i < n)$ be $h^*(u_i, v_i \colon i < n \,;\, u_{n-1}, v_{n-1} \,;\, x_i, y_i \colon i < n \,;\, x_{n-1}, y_{n-1})$. It follows from (3.6) that, with this h, (3.2) and (3.3) hold in $\mathcal{V}$. Hence $M(n)$ holds in $\mathcal{V}$, proving part (i) of Theorem 2.2.

To prove part (ii), we construct a variety $\mathcal{V}$ generated by an algebra $\mathbf{A} = (A; f, g)$ such that $M(n)$ holds but $M(n + 1)$ fails in $\mathcal{V}$. Let $A = \{0, 1, \ldots, 2n + 2\}$, and denote $A \setminus \{0\}$ by A^+. We define a $2(n + 1)$-ary operation f on A by the following rule:

$$
f(a_i, b_i \colon i \le n) = \begin{cases} 1 & \text{if } \{a_i, b_i \colon i \le n\} = A^+, \\ 0 & \text{otherwise.} \end{cases}
$$

Similarly, we also define a $2(n + 1)$-ary operation g on A as follows:

$$
g(a_i, b_i \colon i \le n) = \begin{cases} 2 & \text{if } \{a_i, b_i \colon i \le n\} = A^+, \\ 0 & \text{otherwise.} \end{cases}
$$

This way we have defined $\mathbf{A} = (A; f, g)$ and $\mathcal{V}$.

The identity $f(x_i, x_i \colon i \le n) \approx g(x_i, x_i \colon i \le n)$ clearly holds in $\mathcal{V}$ since both sides induce the constant $A^{n+1} \to \{0\}$ map in $\mathbf{A}$. Suppose for a contradiction that $M(n + 1)$ holds in $\mathcal{V}$. Then there exists a $4(n + 1)$-ary term h such that (3.2) and (3.3) hold in $\mathcal{V}$ with the above-defined f and g. Then h is not a projection since neither f nor g is projection. Therefore the term h has an outermost operation, which is either f or g. If the outermost operation is f, then the term function $h_{\mathbf{A}}$, induced by h on $\mathbf{A}$, cannot take the value 2, whence (3.3) fails in $\mathbf{A}$. Similarly, if the outermost operation is g, then (3.2) fails by the 1-2 symmetry. Therefore, $M(n + 1)$ fails in $\mathcal{V}$.

To show that $M(n)$ holds in $\mathcal{V}$, observe that any two $2n$-ary terms that are not projections are equivalent in $\mathcal{V}$ since they induce the same constant map $A^{2n} \to \{0\}$ in $\mathbf{A}$. Therefore, for any two $2n$-ary terms $f^\diamond$ and $g^\diamond$, either none of them is a projection and we can let $h^\diamond(u_i, v_i \colon i < n \,;\, x_i, y_i \colon i < n) = f^\diamond(x_i, y_i \colon i < n)$, or both are projections and we can trivially find an appropriate $h^\diamond$. This proves that $M(n)$ holds in $\mathcal{V}$.

Next, to prove part (iii), suppose for a contradiction that TImC is equivalent to a finite Σ. We can assume that $\Sigma = \{\sigma\}$ is a singleton since otherwise we can form the

conjunction of all members of Σ. Taking Theorem 2.1 into account, we obtain that σ is equivalent to $\{M(k) : k \in \mathbb{N}\}$. Notice that, by introducing unary relations instead of components and replacing heterogeneous operations by usual relations, heterogeneous algebras can easily be described by usual relational systems. Thus the compactness theorem is valid for heterogeneous algebras, and we conclude that there is a finite set $S \subseteq \mathbb{N}$ such that $\{M(k) : k \in S\}$ implies σ. Let n be the largest element of S. By part (i), $M(n)$ in itself implies σ and, therefore, TImC. Hence, again by Theorem 2.1, $M(n)$ implies $M(n + 1)$, which contradicts part (ii). $\qquad\square$

4. Applications

Next, we give some consequences of Theorem 2.1. In Czédli and Grätzer [8] it is proved that the variety of all lattices satisfies TImC. Theorem 2.1 yields the much stronger statement that every variety of lattices satisfies this property.

COROLLARY 4.1. *Assume that* $\mathcal{V}$ *is a variety with the following properties.*

(i) $\mathcal{V}$ *has two binary terms,* $\vee$ *and* $\wedge$*, that satisfy the lattice axioms.*

(ii)   *For each operation symbol* f*, say n-ary,* $\mathcal{V}$ *satisfies the identity*

$$f(x_i \wedge y_i \colon i < n) \wedge f(x_i \colon i < n) \approx f(x_i \wedge y_i \colon i < n).$$

(*In other words, all operations are monotone with respect to the lattice reduct.*)

Then $\mathcal{V}$ *satisfies TImC.*

In virtue of this corollary, every variety of lattices satisfies TImC. So does every variety of lattices with involution; see, for example, Chajda and Czédli [3] for the definition.

PROOF OF COROLLARY 4.1. Assume that f and g are $2n$-ary terms such that identity (3.1) holds in $\mathcal{V}$. Define a $4n$-ary term h as follows:

$$h(u_i, v_i \colon i < n \, ; \, x_i, y_i \colon i < n) = f(x_i \wedge u_i, y_i \wedge v_i \colon i < n) \vee g(x_i \wedge v_i, y_i \wedge u_i \colon i < n).$$

Using (3.1) and the assumption that the terms of $\mathcal{V}$ are monotone with respect to the lattice order $\leq$ induced by $\wedge$ and $\vee$, we conclude that

$$\begin{aligned}
h(x_i, y_i \colon i < n \, ; \, x_i, y_i \colon i < n) &\approx f(x_i \wedge x_i, y_i \wedge y_i \colon i < n) \vee g(x_i \wedge y_i, y_i \wedge x_i \colon i < n) \\
&\approx f(x_i, y_i \colon i < n) \vee f(x_i \wedge y_i, x_i \wedge y_i \colon i < n) \\
&\approx f(x_i, y_i \colon i < n)
\end{aligned}$$

holds in $\mathcal{V}$. Similarly,

$$\begin{aligned}
h(y_i, x_i \colon i < n \, ; \, x_i, y_i \colon i < n) &\approx f(x_i \wedge y_i, y_i \wedge x_i \colon i < n) \vee g(x_i \wedge x_i, y_i \wedge y_i \colon i < n) \\
&\approx g(x_i \wedge y_i, x_i \wedge y_i \colon i < n) \vee g(x_i, y_i \colon i < n) \\
&\approx g(x_i, y_i \colon i < n).
\end{aligned}$$

Therefore, $M(n)$ holds in $\mathcal{V}$, and Theorem 2.1 applies. $\qquad\square$

TABLE 1. Defining h in the proof of Corollary 4.2.

Occurs in f		Occurs in g		Occurs in h			
x_i	y_i	x_i	y_i	u_i	v_i	x_i	y_i
−	−	−	−	−	−	−	−
+	−	+	−	−	−	+	−
−	+	−	+	−	−	−	+
+	−	−	+	+	−	−	−
−	+	+	−	−	+	−	−
+	+	+	+	−	−	+	+
+	+	+	−	−	+	+	−
+	+	−	+	+	−	−	+
+	−	+	+	+	−	+	−
−	+	+	+	−	+	−	+

The next three corollaries exemplify how to apply Theorem 2.1 for varieties in which the terms and identities are easy to handle. While the proof above allowed us to enrich the lattice structure with further monotone operations, the next proof seems not to allow a similar enrichment.

COROLLARY 4.2. *The variety of semilattices satisfies TImC.*

PROOF. Up to equivalence, each semilattice term is characterised by the variables occurring in it. Assume that f and g are $2n$-ary terms such that (3.1) holds in the variety $\mathcal{V}$ of semilattices. We define an appropriate $4n$-ary semilattice term $h(u_i, v_i: i < n; x_i, y_i: i < n)$ by specifying which variables occur in it. This is done for each i separately by Table 1; notice that, by (3.1), f contains at least one of x_i and y_i if and only if so does g. Thus we obtain an h witnessing that $M(n)$ holds in $\mathcal{V}$, and Theorem 2.1 applies. □

The next statement is a particular case of the result in [5] on balanced varieties. The proof we give here is entirely different from that in [5].

COROLLARY 4.3. *For each tolerance $\boldsymbol{\tau}$ of an algebra* **A***, there exist an algebra* **B***, a congruence $\boldsymbol{\vartheta}$ of* **B***, and a homomorphism φ:* **B** $\to$ **A** *such that $\varphi(\boldsymbol{\vartheta}) = \boldsymbol{\tau}$.*

PROOF. Let $\mathcal{V}$ be the class of all algebras similar to (of the same type as) **A**. The corollary asserts that $\mathcal{V}$ satisfies TImC. Let f and g be $2n$-ary terms such that (3.1) holds in $\mathcal{V}$. Let $f^* = f^*(z_j: j < s)$ be the term we obtain from f by distinguishing its variables. For example, if $f(x_0, y_0) = ((y_0 x_0)(x_0 y_0))x_0$ (in the language of one binary operation), then $f^*(z_i: i < 5) = ((z_0 z_1)(z_2 z_3))z_4$. Define g^* analogously. Since only trivial identities hold in $\mathcal{V}$, the terms $f(x_i, x_i: i < n)$ and $g(x_i, x_i: i < n)$ are the same (equal sequences of symbols) and, moreover, $f^* = g^*$. Let $h^* = f^*$.

TABLE 2. Defining h in the proof of Corollary 4.3.

z_j in f	z_j in g	z_j in h
x_i	x_i	x_i
y_i	y_i	y_i
x_i	y_i	u_i
y_i	x_i	v_i

TABLE 3. Defining h in the proof of Corollary 4.4.

f depends on	g depends on	$h(u_i, v_i: i < n\,; x_i, y_i: i < n)$
x_j	x_j	$f^\bullet(x_j)$
x_j	y_j	$f^\bullet(u_j)$
y_j	x_j	$f^\bullet(v_j)$
y_j	y_j	$f^\bullet(y_j)$

By substituting one of the elements of $\{u_i, v_i, x_i, y_i: i < n\}$ for z_j in h^* according to Table 2, we clearly obtain a term $h(u_i, v_i: i < n\,; x_i, y_i: i < n)$ witnessing that $M(n)$ holds in $\mathcal{V}$. □

A variety is *unary* if all of its basic operations are at most unary.

COROLLARY 4.4. *Every unary variety satisfies TImC.*

PROOF. We modify the proof of Corollary 4.3 as follows. Assume that (3.1) holds in $\mathcal{V}$. Since every term of $\mathcal{V}$ depends on at most one variable, there exist a j and a unary term $f^\bullet$ such that $f(x_i, y_i: i < n) \approx f^\bullet(x_j)$ or $f(x_i, y_i: i < n) \approx f^\bullet(y_j)$ holds in $\mathcal{V}$. Similarly, there exist a k and a unary term $g^\bullet$ such that $g(x_i, y_i: i < n) \approx g^\bullet(x_k)$ or $g(x_i, y_i: i < n) \approx g^\bullet(y_k)$ holds in $\mathcal{V}$.

Assume first that $j \neq k$. Then (3.1) yields that $f^\bullet(x_j) \approx g^\bullet(x_k)$ holds in $\mathcal{V}$, and so $f(x_i, y_i: i < n)$ and $g(x_i, y_i: i < n)$ induce the same constant function on each $\mathbf{A} \in \mathcal{V}$. Hence we can define h by $h(u_i, v_i: i < n\,; x_i, y_i: i < n) = f(x_i, y_i: i < n)$.

Secondly, assume that $j = k$. Then (3.1) yields that $f^\bullet(x_j) \approx g^\bullet(x_j)$ holds in $\mathcal{V}$. Clearly, we can define h according to Table 3. □

A systematic survey of known varieties with TImC is not pursued in this paper. We note that, as opposed to the previous corollaries, Theorem 2.1 is not always the most convenient tool to prove the TImC property. For example, every variety defined by a set of balanced identities satisfies TImC by Chajda, *et al.* [5]. In particular, so do the variety of all semigroups and that of all commutative semigroups. We wonder what the situation is with other important varieties of semigroups.

5. Natural Maltsev conditions and TImC

We have shown in Corollary 4.1 that every variety of lattices with additional monotone operations satisfies TImC. A natural generalisation would be to consider varieties with a *majority term* (a ternary term m such that the identities $m(x, x, y) \approx x$, $m(x, y, x) \approx x$, and $m(y, x, x) \approx x$ hold in $\mathcal{V}$). Lattices have such a term. Also, they constitute an *idempotent* variety. (A variety $\mathcal{V}$ is idempotent if $t(x, \ldots, x) \approx x$ is an identity of $\mathcal{V}$ for every basic operation t.) The following example shows that these conditions together are still not sufficient to establish TImC.

PROPOSITION 5.1. *There exists an idempotent variety $\mathcal{V}$ with a majority term such that TImC fails in $\mathcal{V}$ and $\mathcal{V}$ is generated by a three-element algebra.*

PROOF. Let $A = \{0, 1, 2\}$, and define an algebra $\mathbf{A} = (A; f, g, m)$, where f and g are idempotent quaternary operations and m is a ternary majority operation defined as follows:

$$f(x_i: i < 4) = \begin{cases} 1 & \text{if } (x_i: i < 4) \in \{(1, 1, 1, 1), (1, 0, 0, 2)\}, \\ 2 & \text{if } (x_i: i < 4) = (2, 2, 2, 2), \\ 0 & \text{otherwise;} \end{cases}$$

$$g(x_i: i < 4) = \begin{cases} 2 & \text{if } (x_i: i < 4) \in \{(2, 2, 2, 2), (1, 0, 0, 2)\}, \\ 1 & \text{if } (x_i: i < 4) = (1, 1, 1, 1), \\ 0 & \text{otherwise;} \end{cases}$$

$$m(x_0, x_1, x_2) = \begin{cases} 0 & \text{if } |\{x_0, x_1, x_2\}| = 3, \\ j & \text{if } |\{i : x_i = j\}| \geq 2. \end{cases}$$

Then $\mathcal{V}$, the variety generated by $\mathbf{A}$, is an idempotent variety with a majority term. Consider the relation $\boldsymbol{\tau} = A^2 \setminus \{(1, 2), (2, 1)\}$. We show that $\boldsymbol{\tau}$ is a tolerance of $\mathbf{A}$.

Suppose for a contradiction that f does not preserve $\boldsymbol{\tau}$. Then there are $(a_i, b_i) \in \boldsymbol{\tau}$ such that $(f(a_i: i < 4), f(b_i: i < 4)) \notin \boldsymbol{\tau}$. By symmetry, we can assume that $(f(a_i: i < 4), f(b_i: i < 4)) = (1, 2)$. However, then $(a_0, b_0) = (1, 2) \notin \boldsymbol{\tau}$ is a contradiction. Hence f preserves $\boldsymbol{\tau}$. So does $g(x_i: i < 4)$ since it is the '1-2 dual' of $f(x_{3-i}: i < 4)$. Next, suppose for a contradiction that $(a_i, b_i) \in \boldsymbol{\tau}$ for $i < 3$ but, say, $(m(a_i: i < 3), m(b_i: i < 3)) = (1, 2)$. Then at least two of the a_i equal 1 and at least two of the b_i equal 2. Thus there is an $i \in \{0, 1, 2\}$ such that $(a_i, b_i) = (1, 2) \notin \boldsymbol{\tau}$, which is a contradiction. Therefore, $\boldsymbol{\tau}$ is indeed a tolerance of $\mathbf{A}$.

Finally we show that TImC fails in $\mathcal{V}$. Suppose for a contradiction that $\mathbf{B} \in \mathcal{V}$, $\boldsymbol{\vartheta}$ is a congruence of $\mathbf{B}$, $\varphi \colon \mathbf{B} \to \mathbf{A}$ is a homomorphism, and $\varphi(\boldsymbol{\vartheta}) = \boldsymbol{\tau}$. Pick $(a, b), (c, d) \in \boldsymbol{\vartheta}$ such that $((\varphi(a), \varphi(b)) = (1, 0) \in \boldsymbol{\tau}$ and $((\varphi(c), \varphi(d)) = (0, 2) \in \boldsymbol{\tau}$. Observe that the identity $f(x, x, y, y) \approx g(x, x, y, y)$ holds in $\mathcal{V}$ since it holds in $\mathbf{A}$. Therefore

$$f(a, b, c, d) \, \boldsymbol{\vartheta} \, f(a, a, d, d) = g(a, a, d, d) \, \boldsymbol{\vartheta} \, g(a, b, c, d),$$

and we obtain the following contradiction:

$$
\begin{aligned}
(1, 2) &= (f(1, 0, 0, 2), \ g(1, 0, 0, 2)) \\
&= (f(\varphi(a), \varphi(b), \varphi(c), \varphi(d)), \ g(\varphi(a), \varphi(b), \varphi(c), \varphi(d))) \\
&= (\varphi(f(a, b, c, d)), \ \varphi(g(a, b, c, d))) \in \varphi(\boldsymbol{\vartheta}) = \boldsymbol{\tau}. \qquad \square
\end{aligned}
$$

Another frequently considered Maltsev condition is congruence permutability. Each congruence permutable variety $\mathcal{V}$ satisfies TImC since every tolerance of an algebra in $\mathcal{V}$ is known to be a congruence; see Smith [20] (explicitly) or Werner [22] (implicitly). As an illustration, we give a new proof, based on Theorem 2.1.

COROLLARY 5.2. *Every congruence permutable variety satisfies TImC.*

PROOF. Assume that $\mathcal{V}$ is a congruence permutable variety. By a classical result of Mal'cev [17], $\mathcal{V}$ has a Maltsev term p, that is, a ternary term p such that $p(x, x, y) \approx y \approx p(y, x, x)$ holds in $\mathcal{V}$. Assume that f and g satisfy (3.1) in $\mathcal{V}$. Let

$$
h(u_i, v_i : i < n \, ; \, x_i, y_i : i < n) = p(f(x_i, y_i : i < n), f(x_i, u_i : i < n), g(x_i, u_i : i < n)).
$$

Obviously, this h witnesses that $M(n)$ holds in $\mathcal{V}$, and Theorem 2.1 applies. $\qquad \square$

The strength of the property TImC is very well shown by the following theorem, which refutes a possible generalisation.

THEOREM 5.3. *A congruence n-permutable variety has TImC if and only if it is congruence permutable.*

PROOF. The previous corollary shows one direction, so suppose that a variety $\mathcal{V}$ is n-permutable and satisfies TImC. By the results of Hagemann and Mitschke [13], there exist ternary terms $p_1, \ldots, p_n$ such that the following are identities of $\mathcal{V}$:

$$
\begin{aligned}
x &\approx p_1(x, y, y), \\
p_i(x, x, y) &\approx p_{i+1}(x, y, y) \quad \text{for } 1 \leq i \leq n - 1, \\
p_n(x, x, y) &\approx y.
\end{aligned}
$$

We shall construct a term p such that $\mathcal{V}$ satisfies the identities $x \approx p(x, y, y)$ and $p(x, x, y) \approx p_3(x, y, y)$. This replaces p_1 and p_2 above, implying that the variety $\mathcal{V}$ is actually $(n - 1)$-permutable. Then we shall be done by induction on n.

Define

$$
f(x, u, v, y) = p_1(x, u, y) \quad \text{and} \quad g(x, u, v, y) = p_2(x, v, y).
$$

Then $f(x, x, y, y) \approx g(x, x, y, y)$ is an identity of $\mathcal{V}$, since this reduces to the identity $p_1(x, x, y) \approx p_2(x, y, y)$. Thus $M(2)$ implies the existence of an 8-ary term h satisfying the following identities:

$$
\begin{aligned}
h(x, u, v, y, x, u, v, y) &\approx f(x, u, v, y) \approx p_1(x, u, y), \\
h(u, x, y, v, x, u, v, y) &\approx g(x, u, v, y) \approx p_2(x, v, y).
\end{aligned}
$$

Since p_1 does not depend on v, we can substitute $v \to y$ in the first identity, and similarly, $u \to x$ in the second identity, so

$$h(x, u, y, y, x, u, y, y) \approx p_1(x, u, y) , \tag{5.1}$$

$$h(x, x, y, v, x, x, v, y) \approx p_2(x, v, y) \tag{5.2}$$

still hold in $\mathcal{V}$. Finally, let

$$p(a, b, c) = h(a, b, c, b, a, b, b, c).$$

Then the substitution $u \to y$ in (5.1) gives

$$p(x, y, y) = h(x, y, y, y, x, y, y, y) \approx p_1(x, y, y) \approx x,$$

and the substitution $v \to x$ in (5.2) yields

$$p(x, x, y) = h(x, x, y, x, x, x, x, y) \approx p_2(x, x, y) \approx p_3(x, y, y),$$

proving the theorem.   □

Theorem 5.3 leads to further examples of varieties without TImC. For example, the variety of implication algebras is 3-permutable (see Mitschke [18]), while that of n-Boolean algebras (see Schmidt [19] and [13]) is $(n + 1)$-permutable. Hence it follows from Theorem 5.3 that these (nonidempotent) varieties do not satisfy TImC since they are not congruence permutable.

In view of Theorem 5.3, it would be interesting to see if there is a connection between TImC and other famous Maltsev conditions.

References

[1]   I. Chajda, 'Partitions, coverings and blocks of compatible relations', *Glas. Mat.* **14**(34) (1979), 21–26.
[2]   I. Chajda, *Algebraic Theory of Tolerance Relations* (Palacký University Olomouc, Olomouc, 1991).
[3]   I. Chajda and G. Czédli, 'How to generate the involution lattice of quasiorders?', *Stud. Sci. Math. Hungar.* **32** (1996), 415–427.
[4]   I. Chajda, G. Czédli and R. Halaš, 'Independent joins of tolerance factorable varieties'. *Algebra Universalis*, to appear.
[5]   I. Chajda, G. Czédli, R. Halaš and P. Lipparini, 'Tolerances as images of congruences in varieties defined by linear identities', *Algebra Universalis*, to appear.
[6]   I. Chajda, J. Niederle and B. Zelinka, 'On existence conditions for compatible tolerances', *Czech. Math. J.* **26** (1976), 304–311.
[7]   G. Czédli, 'Factor lattices by tolerances', *Acta Sci. Math. (Szeged)* **44** (1982), 35–42.
[8]   G Czédli and G. Grätzer, 'Lattice tolerances and congruences', *Algebra Universalis* **66** (2011), 5–6.
[9]   G. Czédli, E. K. Horváth and P. Lipparini, 'Optimal Mal'tsev conditions for congruence modular varieties', *Algebra Universalis* **53** (2005), 267–279.
[10] G. Czédli and L. Klukovits, 'A note on tolerances of idempotent algebras', *Glas. Mat.* **18**(38) (1983), 35–38.

[11] A. Day and Ch. Herrmann, 'Gluings of modular lattices', *Order* **5** (1988), 85–101.

[12] E. Fried and G. Grätzer, 'Notes on tolerance relations of lattices: a conjecture of R. N. McKenzie', *J. Pure Appl. Algebra* **68** (1990), 127–134.

[13] J. Hagemann and A. Mitschke, 'On n-permutable congruences', *Algebra Universalis* **3** (1973), 8–12.

[14] Ch. Herrmann, 'S-verklebte Summen von Verbänden', *Math. Z.* **130** (1973), 255–274.

[15] B. Jónsson, 'Algebras whose congruence lattices are distributive', *Math. Scand.* **21** (1967), 110–121.

[16] M. Kindermann, 'Über die Äquivalenz von Ordnungspolynomvollständigkeit und Toleranzein-fachheit endlicher Verbände', *Proc. Klagenfurt Conf.* **1978** (1979), 145–149.

[17] A. I. Mal'cev, 'On the general theory of algebraic systems', *Mat. Sb. N. Ser.* **35(77)** (1954), 3–20 (in Russian).

[18] A. Mitschke, 'Implication algebras are 3-permutable and 3-distributive', *Algebra Universalis* **1** (1971), 182–186.

[19] E. T. Schmidt, 'Kongruenzrelationen algebraischer Strukturen', *Math. Forsch.* **25** (1969), Berlin.

[20] J. D. H. Smith, *Mal'cev Varieties* (Springer, Berlin, 1976).

[21] W. Taylor, 'Characterizing Mal'cev conditions', *Algebra Universalis* **3** (1973), 351–397.

[22] H. Werner, 'A Mal'cev condition for admissible relations', *Algebra Universalis* **3** (1973), 263.

GÁBOR CZÉDLI, Bolyai Institute, University of Szeged,
Aradi vértanúk tere 1, 6720 Szeged, Hungary
e-mail: czedli@math.u-szeged.hu

EMIL W. KISS, Department of Algebra and Number Theory,
Eötvös University, Pázmány Péter sétány 1/c, 1117 Budapest, Hungary
e-mail: ewkiss@cs.elte.hu