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The existence and asymptotic behaviour of the unique solution near the boundary to a singular Dirichlet problem with a convection term

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We show the existence and exact asymptotic behaviour of the unique solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ near the boundary to the singular nonlinear Dirichlet problem $-\Delta u = k(x)g(u) + \lambda|\nabla u|^q$, $u > 0$, $x \in \Omega$, $u|_{\partial\Omega} = 0$, where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $\lambda \in \mathbb{R}$, $q \in [0, 2]$, $g(s)$ is non-increasing and positive in $(0, \infty)$, $\lim_{s\to 0^+} g(s) = +\infty$, $k \in C^\alpha(\Omega)$ is non-negative non-trivial on Ω, which may be singular on the boundary.

1. Introduction and the main results

The purpose of this paper is to investigate the existence and exact asymptotic behaviour of the unique classical solution near the boundary of the following model problem:

$$-\Delta u = k(x)g(u) + \lambda|\nabla u|^q, \quad u > 0,\ x \in \Omega,\ u|_{\partial\Omega} = 0, \tag{1.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$ $(N \geqslant 1)$, $\lambda \in \mathbb{R}$, $q \in [0, 2]$, g satisfies the condition that

(g_1) $g \in C^1((0,\infty),(0,\infty))$, $g'(s) \leqslant 0$ for all $s > 0$, and $\lim_{s\to 0^+} g(s) = +\infty$;

and k satisfies the condition that

(k_1) $k \in C^\alpha(\Omega)$, for some $\alpha \in (0, 1)$, is non-negative non-trivial on Ω.

The problem arises in the study of non-Newtonian fluids, boundary-layer phenomena for viscous fluids, heterogeneous chemical catalysts, and in the theory of heat conduction in electric materials (see [4, 7, 9, 19, 21]).

The main feature of this paper is the presence of the three terms, the singular term $g(u)$, the convection term $\lambda|\nabla u|^q$ and the weight $k(x)$, which may be singular on the boundary.

The problem has been discussed in a number of works (see, for instance, [2–10, 12, 15, 16, 18–22, 24, 25]). For $\lambda = 0$, problem (1.1) becomes

$$-\Delta u = k(x)g(u), \quad u > 0,\ x \in \Omega,\ u|_{\partial\Omega} = 0, \tag{1.2}$$

for $k \equiv 1$ on Ω. Fulks and Maybee [7], Stuart [21] and Crandall *et al.* [4] showed that problem (1.2) has a unique solution $u \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$. Moreover, Crandall

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et al. [4, theorems 2.2 and 2.5] showed that, if $p \in C[0,a] \cap C^2(0,a]$ is the local solution of the problem

$$-p''(s) = g(p(s)), \quad p(s) > 0,\ 0 < s < a,\ p(0) = 0, \tag{1.3}$$

then there exist positive constants C_1 and C_2 such that

(I) $C_1 p(d(x)) \leqslant u(x) \leqslant C_2 p(d(x))$ near $\partial\Omega$, where $d(x) = \operatorname{dist}(x, \partial\Omega)$;

(II) $|\nabla u(x)| \leqslant C_2[d(x)g(C_1 p(d(x))) + p(d(x))/d(x)]$ near $\partial\Omega$.

In particular, when $g(u) = u^{-\gamma}$ and $\gamma > 1$, u has the following properties:

(I_1) $C_1[d(x)]^{2/(1+\gamma)} \leqslant u(x) \leqslant C_2[d(x)]^{2/(1+\gamma)}$ near $\partial\Omega$;

(I_2) $|\nabla u(x)| \leqslant C_2[d(x)]^{(1-\gamma)/(1+\gamma)}$ near $\partial\Omega$.

By constructing global subsolutions and supersolutions, Lazer and McKenna [16] showed that (I_1) continues to hold on $\bar{\Omega}$. Then $u \in H_0^1(\Omega)$ if and only if $\gamma < 3$. This is a basic characteristic of problem (1.2). Moreover, in [16, § 4, 'Remarks and generalizations'], there is the following additional information.

(I_3) If, instead of $k \equiv 1$ on Ω, we assume that $0 < c_1 \leqslant k(x)\varphi_1^{\sigma}(x) \leqslant c_2$ for $x \in \Omega$, where c_1 and c_2 are positive constants, $\sigma \in (0,2)$, and φ_1 is the first eigenfunction, corresponding to the first eigenvalue λ_1 of the Laplace operator with Dirichlet boundary conditions and $\gamma > 1$, then there exist positive constants C_3 and C_4 (C_3 is small and C_4 is large) such that

$$C_3[\varphi_1(x)]^{2/(1+\gamma)} \leqslant u(x) \leqslant C_4[\varphi_1(x)]^{(2-\sigma)/(1+\gamma)}, \quad \forall x \in \Omega.$$

Most recently, in [24], the authors showed the existence and global optimal estimate of the unique solution of problem (1.2) under

$$\int_1^\infty g(s)\,\mathrm{d}s < \infty.$$

Moreover, assume g satisfies (g_1) and the following conditions:

(g_2) there exist positive constants C_0, η_0 and $\gamma \in (0,1)$ such that $g(s) \leqslant C_0 s^{-\gamma}$, for all $s \in (0, \eta_0)$;

(g_3) there exist $\theta > 0$ and $t_0 \geqslant 1$ such that $g(\xi t) \geqslant \xi^{-\theta} g(t)$, for all $\xi \in (0,1)$ and all $t \in (0, t_0\xi]$;

(g_4) the function

$$\xi \in (0,\infty) \to T(\xi) = \lim_{t \to 0^+} \frac{g(\xi t)}{\xi g(t)}$$

is continuous;

and if k satisfies (k_1) and there exist positive constants δ_0, c_0 and a positive non-decreasing function $h \in C(0, \delta_0)$ such that the following are satisfied:

(k_2) $\displaystyle\lim_{d(x)\to 0} \frac{k(x)}{h(d(x))} = c_0$;

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(k_3) $\lim_{t\to 0+} h(t)g(t) = \infty$;

Ghergu and Rădulescu [8] showed that the unique solution u of problem (1.2) satisfies

$$\lim_{d(x)\to 0} \frac{u(x)}{p(d(x))} = \xi_0, \tag{1.4}$$

where $T(\xi_0) = c_0^{-1}$ and $p \in C^1[0,a] \cap C^2(0,a] (a \in (0,\delta_0))$ is the local solution of the problem

$$-p''(s) = h(s)g(p(s)), \quad p(s) > 0, \quad 0 < s < a, \quad p(0) = 0. \tag{1.5}$$

For $\lambda > 0$ and $k(x) \equiv 1$ on Ω, there exist solutions to problem (1.1) (see [2,5,6,9, 10,25]). When $q = 2$ and

$$\lim_{s\to 0+} s^\gamma g(s) < \infty, \quad \gamma \in (0,1),$$

Ghergu and Rădulescu [9] showed that the unique classical solution u has the following properties:

(II$_1$) $C_1 d(x) \leqslant u_\lambda(x) \leqslant C_2 d(x)$, for all $x \in \Omega$, where C_1 and C_2 are positive constants depending on λ;

(II$_2$) $u_\lambda \in C^{1,1-\alpha}(\bar{\Omega})$.

Moreover, for $\lambda = \pm 1$, $0 < q < 2$, $k(x) \equiv 1$ on Ω and the function g $(0,\infty) \to (0,\infty)$ is locally Lipschitz continuous and decreasing, Giarrusso and Porru [10] showed that, if g satisfies the following conditions:

(g_5) $$\int_0^1 g(s)\,\mathrm{d}s = \infty, \qquad \int_1^\infty g(s)\,\mathrm{d}s < \infty;$$

(g_6) let

$$G(t) = \int_t^\infty g(s)\,\mathrm{d}s;$$

then there exist positive constants δ and M with $M > 1$ such that $G(t) < MG(2t)$, for all $t \in (0,\delta)$;

then the unique solution u has the properties:

(II$_3$) $|u(x) - p(d(x))| < \beta d(x)$, for all $x \in \Omega$ for $0 < q \leqslant 1$;

(II$_4$) $|u(x) - p(d(x))| < \beta d(x)[G(p(d(x)))]^{(q-1)/2}$, for all $x \in \Omega$ for $1 < q < 2$;

where β is a suitable positive constant and $p \in C[0,\infty) \cap C^2(0,\infty)$ is uniquely determined by

$$\int_0^{p(t)} \frac{\mathrm{d}s}{\sqrt{2G(s)}} = t, \quad \forall t \in (0,\infty). \tag{1.6}$$

These imply that

$$\lim_{d(x)\to 0} \frac{u(x)}{p(d(x))} = 1. \tag{1.7}$$

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In particular, if $g(u) = u^{-\gamma}$, $\gamma > 1$, then $p(s) = cs^{2/(1+\gamma)}$,

$$c = \left[\frac{(1+\gamma)^2}{2(\gamma-1)}\right]^{1/(1+\gamma)},$$

then u satisfies

$$\lim_{d(x)\to 0} \frac{u(x)}{[d(x)]^{2/(1+\gamma)}} = \left[\frac{(1+\gamma)^2}{2(\gamma-1)}\right]^{1/(1+\gamma)}. \tag{1.8}$$

In this paper, we first consider the asymptotic behaviour of the classical solution of problem (1.3) or (1.5) near to 0. Then, applying Ghergu and Rădulescu's argument [8], and constructing comparison functions, we show that (1.4) and (1.7) still hold for more general $g(u)$ and $k(x)$ in problem (1.1). We also show the existence of classical solutions to problem (1.1).

Our main results are summarized in the following theorems.

Theorem 1.1. *For $0 \leqslant q \leqslant 2$, $\lambda \in \mathbb{R}$, let g satisfy (g_1), (g_3), (g_4) and let $h \in C(0, \delta_0)$ satisfy the following condition:*

(h_0) *h is non-increasing on $(0, \delta_0)$ and $\lim_{t\to 0^+} h(t) = \infty$.*

If k satisfies (k_1), (k_2) and the following condition:

(k_4) *the linear problem*

$$-\Delta u = k(x), \quad u > 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = 0, \tag{1.9}$$

has a unique solution $v_0 \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$ (see [11, ch. 4, theorems 4.3 and 4.9, problems 4.3 and 4.6, pp. 70, 71]);

then the unique solution $u_\lambda \in C(\bar{\Omega}) \cap C^2(\Omega)$ to problem (1.1) satisfies (1.4), where $p \in C[0, a] \cap C^2(0, a]$ is the local solution of problem (1.5). In particular, if $g(u) = u^{-\gamma}$, $\gamma > 0$, $k(x) \cong c_0(d(x))^{-\beta}$ near the boundary and $\max\{0, 1-\gamma\} < \beta < 2$, then $p(s) = cs^{(2-\beta)/(1+\gamma)}$,

$$c = \left(\frac{(1+\gamma)^2}{(2-\beta)(\gamma+\beta-1)}\right)^{1/(1+\gamma)},$$

$\xi_0 = c_0^{1/(1+\gamma)}$, and u_λ satisfies

$$\lim_{d(x)\to 0} \frac{u_\lambda(x)}{[d(x)]^{(2-\beta)/(1+\gamma)}} = \left[\frac{c_0(1+\gamma)^2}{(2-\beta)(\gamma+\beta-1)}\right]^{1/(1+\gamma)}.$$

Remark 1.2. By (1.5), we see that the asymptotic behaviour (1.4) of u_λ is independent of $\lambda|\nabla u_\lambda|^q$.

Remark 1.3. For a proof of the existence of solutions to problem (1.5) with $a \in (0, 1)$, see [1, corollary 2.1].

Remark 1.4. In addition to $h(t) = t^{-\beta}$, $\beta \in (0, 2)$, typical examples of singular weight functions satisfying the above assumptions are

(i) $h(t) = (t \ln^\beta(1+t))^{-1}$, $\beta \in [0, 1)$;

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(ii) $h(t) = (e^t - 1)^{-\beta}$, $\beta \in (0, 2)$.

REMARK 1.5. If $h(t) = t^{-\beta}$, $\beta \geqslant 2$, then problem (1.1) may not have any classical solution (see [24, theorem 1.2]).

REMARK 1.6. In addition to $g(u) = u^{-\gamma}$, $\gamma > 0$, typical examples of singularities satisfying (g_1), (g_3), (g_4) are

(i) $g(u) = (u \ln^{\gamma}(u + 1))^{-1}$, $\gamma \geqslant 0$, $T(\xi) = \xi^{-(2+\gamma)}$;

(ii) $g(u) = (e^u - 1)^{-\gamma}$, $\gamma > 0$, $T(\xi) = \xi^{-(1+\gamma)}$;

(iii) $$g(t) = t^{-\gamma} \exp\biggl(\int_t^a \frac{y(s)}{s}\,ds\biggr), \quad \gamma > 0, \quad T(\xi) = \xi^{-(1+\gamma)},$$

where $a > 0$, $y \in C[0, a]$, $y(0) = 0$.

REMARK 1.7. By (g_1) and the comparison principles [11, theorems 10.1 and 10.2], we see that problem (1.1) has at most one solution in $C^2(\Omega) \cap C(\bar{\Omega})$ (see the lemma 3.4).

THEOREM 1.8. *For $0 \leqslant q \leqslant 2$, $\lambda \in \mathbb{R}$, let g satisfy (g_1), (g_3), (g_4) and $k \equiv 1$ on Ω. If*

$$\int_1^{\infty} g(t)\,dt < \infty,$$

then the unique solution $u_\lambda \in C(\bar{\Omega}) \cap C^2(\Omega)$ to problem (1.1) satisfies (1.7), where $p \in C[0, \infty) \cap C^2(0, \infty)$ is the unique global solution of problem

$$\left.\begin{aligned} -p''(s) = g(p(s)), \quad p(s) > 0, \quad s \in (0, \infty), \\ p(0) = 0, \quad \lim_{s\to\infty} p'(s) = b \geqslant 0. \end{aligned}\right\} \tag{1.10}$$

In particular, if $g(u) = u^{-\gamma}$, $\gamma > 1$, then u_λ satisfies (1.8).

THEOREM 1.9. *Let g satisfy (g_1) and $\lim_{s\to\infty} g(s) = 0$.*

(I) *If $q = 2$, k satisfies (k_1) and (k_4), then problem (1.1) has a unique solution $u_\lambda \in C(\bar{\Omega}) \cap C^2(\Omega)$ for every $\lambda \geqslant 0$.*

(II) *If $k \equiv 1$ on Ω, then problem (1.1) has a unique solution $u_\lambda \in C(\bar{\Omega}) \cap C^2(\Omega)$ in one of the following cases:*

(i) $q \in [0, 2]$, $\lambda \leqslant 0$;

(ii) $q \in [0, 1)$, $\lambda \geqslant 0$;

(iii) $q = 1$, $0 \leqslant \lambda < \sqrt{\lambda_1}$.

REMARK 1.10. It is not known in theorem 1.9(I) whether or not problem (1.1) has a solution if $q \in (0, 2)$ and, in theorem 1.9(II)(iii), whether or not $\sqrt{\lambda_1}$ is exact.

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2. The exact asymptotic behaviour

First we give some preliminary considerations.

LEMMA 2.1 (Lazer and McKenna [17, lemma 2.1]). *If g satisfies (g_1), $b > 0$, then*

$$\lim_{s\to 0+} \frac{\int_s^b g(t)\,\mathrm{d}t}{g(s)} = 0. \tag{2.1}$$

LEMMA 2.2. *Let $q \in [0, 2]$ and $g \in C^1((0,\infty),(0,\infty))$. If*

$$\int_1^\infty g(t)\,\mathrm{d}t < \infty,$$

then problem (1.10) has a unique global solution $p \in C[0,\infty) \cap C^2(0,\infty)$. Moreover, if $\lim_{t\to 0^+} g(t) = \infty$, then p has the following properties:

$$\lim_{s\to 0+} \frac{p'(s)}{p''(s)} = 0, \qquad \lim_{s\to 0+} \frac{(p'(s))^q}{p''(s)} = 0. \tag{2.2}$$

Proof. Note that, if $p(s)$ is a positive global classical solution of problem (1.10) on $(0,\infty)$, $p(0) = 0$ and $p''(s) < 0$, then $p'(s)$ is decreasing on $(0,\infty)$ and $p'(s) > 0$, $p(s)$ is increasing. Let $\lim_{s\to\infty} p(s) = p_0 \in (0,\infty]$. Multiplying equation (1.10) by $p'(s)$ and integrating on $[s,t]$, we get

$$[p'(s)]^2 = [p'(t)]^2 + 2\int_{p(s)}^{p(t)} g(z)\,\mathrm{d}z, \quad s > 0. \tag{2.3}$$

Let $t \to \infty$ in (2.3). We then see that

$$[p'(s)]^2 = b^2 + 2\int_{p(s)}^{p_0} g(z)\,\mathrm{d}z, \quad s > 0, \tag{2.4}$$

which can be integrated on $[0,t]$, yielding

$$\int_0^t \frac{p'(s)\,\mathrm{d}s}{\sqrt{b^2 + 2\int_{p(s)}^{p_0} g(z)\,\mathrm{d}z}} = \int_0^{p(t)} \frac{\mathrm{d}v}{\sqrt{b^2 + 2\int_v^{p_0} g(z)\,\mathrm{d}z}} = t, \quad t > 0.$$

Let $t \to +\infty$. We then see that

$$\int_0^{p_0} \frac{\mathrm{d}v}{\sqrt{b^2 + 2\int_v^{p_0} g(z)\,\mathrm{d}z}} = +\infty.$$

It follows, by $p(s) > 0$ on $(0,\infty)$ and

$$\int_1^\infty g(t)\,\mathrm{d}t < \infty,$$

that $p_0 = +\infty$. Thus,

$$\int_0^{p(t)} \frac{\mathrm{d}v}{\sqrt{b^2 + 2\int_v^{+\infty} g(z)\,\mathrm{d}z}} = t, \quad t \geqslant 0. \tag{2.5}$$

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Hence, defining $p(t)$ on $[0,\infty)$ by (2.5), we see that $p \in C[0,\infty) \cap C^2(0,\infty)$ is the unique global solution of problem (1.10) with

$$\lim_{s\to+\infty} p(s) = +\infty \quad \text{and} \quad \lim_{s\to+\infty} p'(s) = b.$$

Moreover, for $c > 0$, by lemma 2.1 and (g_1), we have

$$\begin{aligned}\lim_{t\to0^+} \frac{(p'(t))^2}{p''(t)} &= -\lim_{t\to0^+} \frac{b^2 + 2\int_{p(t)}^c g(s)\,\mathrm{d}s + 2\int_c^\infty g(s)\,\mathrm{d}s}{g(p(t))} \\ &= -\lim_{u\to0^+} \frac{b^2 + 2\int_u^c g(s)\,\mathrm{d}s + 2\int_c^\infty g(s)\,\mathrm{d}s}{g(u)} \\ &= 0.\end{aligned}$$

Since

$$0 < [p'(0)]^2 = b^2 + 2\int_0^\infty g(z)\,\mathrm{d}z \leqslant \infty,$$

we see that

$$\lim_{t\to0^+} \frac{(p'(t))^q}{p''(t)} = -\lim_{t\to0^+} \frac{(p'(t))^2}{p''(t)} \lim_{t\to0^+} \frac{1}{(p'(t))^{2-q}} = 0, \quad \text{for } 0 \leqslant q < 2;$$

and

$$\lim_{t\to0^+} \frac{p'(t)}{p''(t)} = \lim_{t\to0^+} \frac{(p'(t))^2}{p''(t)} \lim_{t\to0^+} \frac{1}{p'(t)} = 0.$$

This completes the proof. □

LEMMA 2.3. *Let g satisfy (g_1), and h satisfy (h_0). If $p \in C[0,a] \cap C^2(0,a]$ is the local solution of problem (1.5), then (2.2) holds.*

Proof. Since $p(s)$ is a positive concave on $(0,a]$ and $p(0) = 0$, then $p'(0) \in (0,\infty]$ and we can choose $a \in (0,1)$ such that $p'(s) > 0$ on $(0,a]$. It follows by $p''(s) < 0$ on $(0,a]$ that $p'(s)$ is decreasing on $(0,a]$. By ensuring that h is non-increasing, multiplying equation (1.5) by $p'(s)$ and integrating on $[t,a]$, $0 < t < a$, we get

$$\begin{aligned}(p'(a))^2 + 2h(a)\int_{p(t)}^{p(a)} g(y)\,\mathrm{d}y &\leqslant (p'(a))^2 + 2\int_t^a h(s)g(p(s))p'(s)\,\mathrm{d}s \\ &= (p'(t))^2 \\ &\leqslant (p'(a))^2 + 2h(t)\int_{p(t)}^{p(a)} g(y)\,\mathrm{d}y,\end{aligned}$$

It follows by lemma 2.1 and (g_1), (h_0) that

$$\begin{aligned}&\lim_{t\to0^+} \frac{(p'(a))^2 + 2h(t)\int_{p(t)}^{p(a)} g(y)\,\mathrm{d}y}{h(t)g(p(t))} \\ &\quad = (p'(a))^2 \lim_{t\to0^+} \frac{1}{h(t)} \lim_{u\to0^+} \frac{1}{g(u)} + 2\lim_{u\to0^+} \frac{\int_u^{p(a)} g(y)\,\mathrm{d}y}{g(u)} \\ &\quad = 0,\end{aligned}$$

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i.e.

$$\lim_{t\to 0^+}\frac{(p'(t))^2}{p''(t)}=0.$$

The other results follow by the same proof of lemma 2.2. □

LEMMA 2.4 (Ghergu and Rădulescu [8, lemma 2.1]). *If g satisfies (g_3) and (g_4), then the function*

$$T(\xi)=\lim_{t\to 0^+}\frac{g(\xi t)}{\xi g(t)}$$

is decreasing on $(0,\infty)$ and $\lim_{\xi\to 0^+}T(\xi)=+\infty$.

Proof of theorems 1.1 and 1.8. Let $u_\lambda\in C^{2+\alpha}(\Omega)\cap C(\bar{\Omega})$ be the unique solution of problem (1.1) for every $\lambda\in\mathbb{R}$. In the following, when $k\equiv 1$ on Ω, we set $h\equiv 1$ on $(0,\delta_0)$. Define $\psi:(0,\infty)\to(0,\infty)$ by

$$\psi(\xi)=\lim_{d(x)\to 0}\frac{k(x)g(\xi p(d(x)))}{\xi h(d(x))g(p(d(x)))},\quad \forall\xi>0.$$

The definition implies that $\psi(\xi)=c_0T(\xi)$, for all $\xi>0$. Set $\xi_0=\psi^{-1}(1)$, i.e. $T(\xi_0)=c_0^{-1}$. Let $\varepsilon\in(0,\frac{1}{2})$, $\xi_{1\varepsilon}$ and $\xi_{2\varepsilon}$ be such that $\psi(\xi_{1\varepsilon})=1-2\varepsilon$ and $\psi(\xi_{2\varepsilon})=1+2\varepsilon$. Lemma 2.4 implies that $\xi_{1\varepsilon}>\xi_{2\varepsilon}>0$. We can further choose ε small enough that $\xi_0/2<\xi_{2\varepsilon}<\xi_{1\varepsilon}<2\xi_0$. Fix ε for any $\delta>0$, where we define $\Omega_\delta=\{x\in\Omega: d(x)\leqslant\delta\}$. For every $\lambda\in\mathbb{R}$, by the regularity of $\partial\Omega$ and lemmas 2.2 and 2.3, we can choose δ sufficiently small such that

(i) $d(x)\in C^2(\Omega_\delta)$;

(ii)

$$\left|\frac{p'(s)}{p''(s)}\Delta d(x)+\lambda\xi_{i\varepsilon}^{q-1}\frac{(p'(s))^q}{p''(s)}\right|<\varepsilon,\quad \forall(x,s)\in\Omega_\delta\times(0,\delta),\quad i=1,2;$$

(iii)

$$\frac{\xi_{2\varepsilon}h(d(x))g(p(d(x)))}{g(\xi_{2\varepsilon}p(d(x)))}(\psi(\xi_{2\varepsilon})-\varepsilon)<k(x)<\frac{\xi_{1\varepsilon}h(d(x))g(p(d(x)))}{g(\xi_{1\varepsilon}p(d(x)))}(\psi(\xi_{1\varepsilon})+\varepsilon),$$

for all $x\in\Omega_\delta$, $i=1,2$.

For any $x\in\Omega_\delta$, we define $\bar{u}_\varepsilon=\xi_{1\varepsilon}p(d(x))$ and $\underline{u}_\varepsilon=\xi_{2\varepsilon}p(d(x))$. It follows by $|\nabla d(x)|=1$ that

$$\begin{aligned}
&\Delta\bar{u}_\varepsilon(x)+k(x)g(\bar{u}_\varepsilon(x))+\lambda|\nabla\bar{u}_\varepsilon(x)|^q\\
&\quad=k(x)g(\xi_{1\varepsilon}p(d(x)))+\xi_{1\varepsilon}p'(d(x))\Delta d(x)+\xi_{1\varepsilon}p''(d(x))+\lambda\xi_{1\varepsilon}^q(p'(d(x))^q\\
&\quad=\xi_{1\varepsilon}h(d(x))g(p(d(x)))\\
&\qquad\times\left[\frac{k(x)g(\xi_{1\varepsilon}p(d(x)))}{\xi_{1\varepsilon}h(d(x))g(p(d(x)))}-1-\frac{p'(d(x))}{p''(d(x))}\Delta d(x)-\lambda\xi_{1\varepsilon}^{q-1}\frac{(p'(d(x)))^q}{p''(d(x))}\right]\\
&\quad\leqslant\xi_{1\varepsilon}h(d(x))g(p(d(x)))\\
&\qquad\times\left[(\psi(\xi_{1\varepsilon})+\varepsilon)-1-\frac{p'(d(x))}{p''(d(x))}\Delta d(x)-\lambda\xi_{1\varepsilon}^{q-1}\frac{(p'(d(x)))^q}{p''(d(x))}\right]\\
&\quad\leqslant 0
\end{aligned}$$

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and

$$\begin{aligned}
&\Delta \underline{u}_\varepsilon(x) + k(x)g(\underline{u}_\varepsilon(x)) + \lambda \nabla |\underline{u}_\varepsilon(x)|^q \\
&\quad = k(x)g(\xi_{2\varepsilon}p(d(x))) + \xi_{2\varepsilon}p'(d(x))\Delta d(x) + \xi_{2\varepsilon}p''(d(x)) + \lambda \xi_{2\varepsilon}^q (p'(d(x))^q \\
&\quad = \xi_{2\varepsilon}h(d(x))g(p(d(x))) \\
&\qquad \times \left[\frac{k(x)g(\xi_{2\varepsilon}p(d(x)))}{\xi_{2\varepsilon}h(d(x))g(p(d(x)))} - 1 - \frac{p'(d(x))}{p''(d(x))}\Delta d(x) - \lambda \xi_{2\varepsilon}^{q-1}\frac{(p'(d(x)))^q}{p''(d(x))} \right] \\
&\quad \geqslant \xi_{2\varepsilon}h(d(x))g(p(d(x))) \\
&\qquad \times \left[(\psi(\xi_{2\varepsilon}) - \varepsilon) - 1 - \frac{p'(d(x))}{p''(d(x))}\Delta d(x) - \lambda \xi_{2\varepsilon}^{q-1}\frac{(p'(d(x)))^q}{p''(d(x))} \right] \\
&\quad \geqslant 0.
\end{aligned}$$

It follows by (g_1) and the maximum principle [11, theorem 10.1] that

$$\xi_{2\varepsilon}p(d(x)) = \underline{u}_\varepsilon(x) \leqslant u_\lambda(x) \leqslant \bar{u}_\varepsilon(x) = \xi_{1\varepsilon}p(d(x)), \quad \forall x \in \Omega_\delta.$$

Let $\varepsilon \to 0$. We then see that

$$\lim_{d(x)\to 0} \frac{u_\lambda(x)}{p(d(x))} = \xi_0.$$

This completes the proof. □

3. Existence of solutions

First we introduce a sub–supersolution method with a boundary restriction (see [6, 25]).

We consider the following more general problem:

$$-\Delta u = f(x, u, \nabla u), \quad u > 0, \ x \in \Omega, \ u|_{\partial\Omega} = 0, \tag{3.1}$$

where $f(x, s, \eta)$ satisfies the following two conditions:

(D_1) $f(x, s, \eta)$ is locally Hölder continuous in $\Omega \times (0, \infty) \times \mathbb{R}^N$ and continuously differentiable with respect to the variables s and η;

(D_2) for any $\Omega_1 \subset\subset \Omega$ and any $a, b \in (0, \infty)$ with $a < b$, there exists a corresponding constant $C = C(\Omega_1, a, b) > 0$ such that

$$|f(x, s, \eta)| \leqslant C(1 + |\eta|^2), \quad \forall x \in \bar{\Omega}_1, \ \forall u \in [a, b], \ \forall \eta \in \mathbb{R}^N.$$

Definition 3.1. A function $\underline{u} \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$ is called a subsolution of problem (3.1) if

$$-\Delta \underline{u} \leqslant f(x, \underline{u}, \nabla \underline{u}), \quad \underline{u} > 0, \ x \in \Omega, \ \underline{u}|_{\partial\Omega} = 0. \tag{3.2}$$

Definition 3.2. A function $\bar{u} \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$ is called a supersolution of problem (3.1) if

$$-\Delta \bar{u} \geqslant f(x, \bar{u}, \nabla \bar{u}), \quad \bar{u} > 0, \ x \in \Omega, \ \bar{u}|_{\partial\Omega} = 0. \tag{3.3}$$

We have the following basic existence result.

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Lemma 3.3 (Cui [6, lemma 3]). *Suppose problem (3.1) has a supersolution* $\bar{u}$ *and a subsolution* $\underline{u}$ *such that* $\underline{u} \leqslant \bar{u}$ *on* Ω*. Then problem (3.1) has at least one solution* $u \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$ *in the ordered interval* $[\underline{u}, \bar{u}]$*.*

At the same time, we use the following basic unique result.

Lemma 3.4 (Theorems 10.1 and 10.2 in [11]). *If* $f(x, s, \eta)$ *satisfies the two conditions that*

(D_3) f *is non-increasing in* s *for each* $(x, \eta) \in \Omega \times \mathbb{R}^N$*;*

(D_4) f *is continuously differentiable with respect to the* η *variables in* $\Omega \times (0, \infty) \times \mathbb{R}^N$*;*

then problem (3.1) has at most one solution in $C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$*.*

For the following convenience, we denote

$$|u|_\infty = \max_{x \in \bar{\Omega}} u(x), \quad u \in C(\bar{\Omega}).$$

Now we apply the lemmas to consider the existence and uniqueness of solutions to problem (1.2), which is the corresponding to the result in [23] for $\Omega = \mathbb{R}^N$.

Lemma 3.5. *Let* $k \in C^\alpha(\Omega)$ *be non-negative and non-trivial on* Ω*. If* g *satisfies the condition that*

(g_{11}) $g \in C^1((0, \infty), (0, \infty))$, $g'(s) \leqslant 0$ *for all* $s > 0$,

then problem (1.2) has a unique solution $u \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$ *if and only if problem (1.9) is solvable.*

Proof. Let

$$H(u) = \int_0^u \frac{1}{g(s)}\,\mathrm{d}s \quad \text{for } u \geqslant 0.$$

It follows that $H : [0, \infty) \to [0, \infty)$ is strictly increasing and

$$H'(u) = \frac{1}{g(u)} \quad \text{for } u > 0.$$

Let $\bar{u}(x) = H^{-1}(v_0(x))$, $x \in \Omega$, where H^{-1} denotes the inverse function of H, and v_0 is the unique classical solution of problem (1.9). We see that $\bar{u}|_{\partial\Omega} = 0$ and

$$-\Delta\bar{u} + \frac{g'(\bar{u})|\nabla\bar{u}|^2}{g(\bar{u})} = k(x)g(\bar{u}), \quad x \in \Omega.$$

It follows by (g_{11}) that

$$-\Delta\bar{u} \geqslant k(x)g(\bar{u}), \quad x \in \Omega,$$

i.e. $\bar{u} = H^{-1}(v_0)$ is a supersolution of problem (1.2).

On the other hand, hypothesis (g_{11}) implies that $\lim_{s\to 0+} g(s) \in (0, \infty]$, so

$$\lim_{s\to 0+} \frac{g(s)}{s} = +\infty.$$

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There then exists $c_0 \in (0,1)$ such that

$$\frac{g(c_0|v_0|_\infty)}{c_0} \geqslant 1.$$

Let $\underline{u} = c_0 v_0$. We see that

$$-\Delta \underline{u} = c_0 k(x) \leqslant k(x) g(c_0|v_0|_\infty) \leqslant k(x) g(\underline{u}), \quad x \in \Omega,$$

i.e. $\underline{u} = c_0 v_0$ is a subsolution of problem (1.2). Moreover, we see that

$$H(c_0 v_0(x)) = \int_0^{c_0 v_0(x)} \frac{1}{g(s)}\,\mathrm{d}s \leqslant \frac{c_0 v_0(x)}{g(c_0|v_0|_\infty)} \leqslant v_0(x), \quad x \in \Omega,$$

i.e. $\underline{u} \leqslant \bar{u}$ on Ω. By lemma 3.3, we see that problem (1.2) has at least one solution $u \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$ in ordered interval $[\underline{u}, \bar{u}]$. And the uniqueness follows by lemma 3.4.

Inversely, let problem (1.2) have a solution $u \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$, and let $c_1 = g(|u|_\infty)$. It follows by (g_{11}) that

$$-\Delta u = k(x) g(u) \geqslant c_1 k(x), \quad x \in \Omega,$$

i.e. u/c_1 is a supersolution of problem (1.9) and 0 is a subsolution of problem (1.9). It follows by lemma 2.1 and the Höpf strong maximum principle (see [11, theorem 3.5,p. 35]) that the linear problem (1.9) is solvable. The proof is complete. □

Proof of theorem 1.9. We first consider the following problem:

$$-\Delta u = k(x) g(u) + \lambda |\nabla u|^2 + \sigma, \quad u > 0,\ x \in \Omega,\ u|_{\partial\Omega} = 0, \tag{3.4}$$

where σ is a non-negative constant.

The change of variable $v = \mathrm{e}^{\lambda u} - 1$ transforms problem (3.4) into the equivalent one:

$$-\Delta v = k(x)(v+1) h_\lambda(v) + \lambda\sigma(v+1), \quad v > 0,\ x \in \Omega,\ v|_{\partial\Omega} = 0, \tag{3.5}$$

where $h_\lambda(s) = \lambda g(\lambda^{-1} \ln(s+1))$ and $s > 0$.

Fix $\lambda > 0$. We then see by (g_1) and $\lim_{s\to\infty} g(s) = 0$ that h_λ is non-increasing on $(0,\infty)$ and that

$$\lim_{s\to 0^+} (s+1) h_\lambda(s) = +\infty, \qquad \lim_{s\to\infty} \frac{(1+s) h_\lambda(s)}{s} = 0.$$

So there exist positive constants C_λ and c_λ with $c_\lambda < |v_0|_\infty^{-1}$ such that

$$(s+1) h_\lambda(s) \leqslant 2 h_\lambda(s) + c_\lambda s + C_\lambda, \quad \forall s > 0. \tag{3.6}$$

Let $v_{\lambda 1} \in C(\bar{\Omega}) \cap C^2(\Omega)$ be the solution of the following problem (see lemma 3.5):

$$-\Delta v = 2k(x) h_\lambda(v), \quad v > 0,\ x \in \Omega,\ v|_{\partial\Omega} = 0. \tag{3.7}$$

Let

$$M_\lambda = \frac{c_\lambda |v_1|_\infty + C_\lambda}{1 - c_\lambda |v_0|_\infty} \quad \text{and} \quad v_{\lambda 2} = M_\lambda v_0.$$

We then have that

$$\begin{aligned}-\Delta v_{\lambda 2} &= M_\lambda k(x)\\ &\geqslant k(x)(C_\lambda + c_\lambda |v_{\lambda 1}|_\infty + c_\lambda M_\lambda |v_0|_\infty)\\ &\geqslant k(x)[C_\lambda + c_\lambda(v_{\lambda 1} + v_{\lambda 2})], \quad x \in \Omega.\end{aligned}$$

(I) $\sigma = 0$. We see by (3.6) that $\bar{v}_\lambda = v_{\lambda 1} + v_{\lambda 2}$ satisfies

$$\begin{aligned}-\Delta \bar{v}_\lambda &= 2k(x)h_\lambda(v_{\lambda 1}) + M_\lambda k(x)\\ &\geqslant k(x)[2h_\lambda(v_{\lambda 1} + v_{\lambda 2}) + c_\lambda(v_{\lambda 1} + v_{\lambda 2}) + C_\lambda]\\ &\geqslant k(x)(\bar{v}_\lambda + 1)h_\lambda(\bar{v}_\lambda), \quad x \in \Omega,\end{aligned}$$

i.e. $\bar{v}_\lambda = v_{\lambda 1} + v_{\lambda 2}$ is a supersolution of problem (3.5).

Obviously, the solution $\underline{v}_\lambda$ of the problem (lemma 3.5)

$$-\Delta v = k(x)h_\lambda(v), \quad v > 0,\ x \in \Omega,\ v|_{\partial\Omega} = 0, \tag{3.8}$$

is a subsolution of problem (3.5) and $\underline{v}_\lambda \leqslant v_{\lambda 1} \leqslant \bar{v}_\lambda$ on Ω. Thus, problem (3.5) has at least one solution $v_\lambda \in C(\bar{\Omega}) \cap C^2(\Omega)$ in ordered interval $[\underline{v}_\lambda, \bar{v}_\lambda]$ for each fixed $\lambda > 0$, i.e. problem (1.1) has a solution $u_\lambda \in C(\bar{\Omega}) \cap C^2(\Omega)$ for each $\lambda > 0$ and $q = 2$.

(II) $\sigma > 0$ and $k \equiv 1$ on Ω. In this case we see that $v_0 \in C^1(\bar{\Omega}) \cap C^{2+\alpha}(\Omega)$ is the unique solution of the problem

$$-\Delta v = 1, \quad v > 0, \quad x \in \Omega, \quad v|_{\partial\Omega} = 0. \tag{3.9}$$

Let $0 < \lambda\sigma < \lambda_1$. We can then choose c_λ in (3.6) such that $c_\lambda < \lambda_1 - \lambda\sigma$. It follows by [14, theorem 3.2, p. 128] and [13, p. 1363] that the problem

$$-\Delta v = (\lambda\sigma + c_\lambda)v + \lambda\sigma + C_\lambda + (\lambda\sigma + c_\lambda)|v_{\lambda 1}|_\infty, \quad v > 0, \quad x \in \Omega, \quad v|_{\partial\Omega} = 0, \tag{3.10}$$

has a unique solution $v_{\lambda 3}$, where $v_{\lambda 1}$ is the unique solution of problem (3.7) with $k \equiv 1$ on Ω. Then $\bar{v}_\lambda = v_{\lambda 1} + v_{\lambda 3}$ satisfies

$$\begin{aligned}-\Delta \bar{v}_\lambda &= 2h_\lambda(v_{\lambda 1}) + (\lambda\sigma + c_\lambda)v_{\lambda 3} + \lambda\sigma + C_\lambda + (\lambda\sigma + c_\lambda)|v_{\lambda 1}|_\infty\\ &\geqslant 2h(v_{\lambda 1} + v_{\lambda 3}) + (\lambda\sigma + c_\lambda)(v_{\lambda 3} + v_{\lambda 1}) + C_\lambda + \lambda\sigma\\ &\geqslant (\bar{v}_\lambda + 1)h_\lambda(\bar{v}_\lambda) + \lambda\sigma(\bar{v}_\lambda + 1), \quad x \in \Omega,\end{aligned}$$

i.e. $\bar{v}_\lambda = v_{\lambda 1} + v_{\lambda 3}$ is a supersolution of problem (3.5). Obviously, the unique solution $\underline{v}_\lambda$ of the problem (3.8) with $k \equiv 1$ on Ω is a subsolution of problem (3.5) and $\underline{v}_\lambda \leqslant v_{\lambda 1} \leqslant \bar{v}_\lambda$ on Ω. Thus, problem (3.5) has at least one solution $v \in C(\bar{\Omega}) \cap C^2(\Omega)$ in the ordered interval $[\underline{v}_\lambda, \bar{v}_\lambda]$, i.e. problem (3.4) has a solution $u_\lambda \in C(\bar{\Omega}) \cap C^2(\Omega)$ for $k \equiv 1$ on Ω, $q = 2$ and $\sigma > 0$ with $0 < \lambda\sigma < \lambda_1$.

When $0 < q < 2$, for an arbitrary positive constant C, we have the basic inequality [25, proof of theorem 1.2, p. 922]

$$s^q \leqslant s^2 C^{1-q/2} + C^{q/2}, \quad \forall s \geqslant 0. \tag{3.11}$$

We now consider the following problem:

$$-\Delta v = g(u) + \lambda C^{q/2-1}|\nabla u|^2 + \lambda C^{q/2}, \quad u > 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = 0. \tag{3.12}$$

Let

$$\lambda C^{q/2}(\lambda C^{q/2-1}) = \lambda^2 C^{q-1} < \lambda_1. \tag{3.13}$$

By the above proof, we see that problem (3.12) has a unique solution $\bar{u}_\lambda$, which is a supersolution of problem (1.1) and that the solution u_0 to problem (1.2) with $k \equiv 1$ on Ω is a subsolution of problem (1.1). It follows by the maximum principle that $u_0 \leqslant \bar{u}_\lambda$ on Ω. Problem (1.1) then has at least one solution $u_\lambda \in C(\bar{\Omega}) \cap C^2(\Omega)$ in the ordered interval $[u_0, \bar{u}_\lambda]$.

Now we analyse the inequality (3.13). Given every $\lambda > 0$, for $q = 1$, we can choose $\lambda < \sqrt{\lambda_1}$ such that (3.13) holds and, for $q \in (0,1) \cup (1,2)$, we can choose $C < (\lambda_1/\lambda^2)^{1/(q-1)}$ such that (3.13) holds, i.e. problem (1.1) has one solution $u_\lambda \in C(\bar{\Omega}) \cap C^2(\Omega)$ for $k \equiv 1$ on Ω and for one of the following cases:

(ii) $q \in [0,1)$, $\lambda \geqslant 0$;

(iii) $q = 1$, $0 \leqslant \lambda < \sqrt{\lambda_1}$.

When $\lambda < 0$, it is obvious that u_0 is a supersolution of problem (1.1), where u_0 is the solution of problem (1.2). Besides, for fixed $\lambda < 0$, we can see, by the fact that

$$\lim_{s \to 0^+} \frac{s - \lambda s^q |\nabla v_0|_\infty^q}{g(s|v_0|_\infty)} = 0,$$

that there exists a positive constant b_λ such that $b_\lambda < c_0$ and

$$s - \lambda s^q |\nabla v_0|_\infty^q \leqslant g(s|v_0|_\infty), \forall s \in (0, b_\lambda),$$

where v_0 is the solution of problem (3.9).

It follows that $\underline{u}_\lambda = c_{\lambda 1} v_0$ satisfies

$$\begin{aligned} -\Delta \underline{u}_\lambda(x) = c_{\lambda 1} &\leqslant g(c_{\lambda 1}|v_0|_\infty) + \lambda c_{\lambda 1}^q |\nabla v_0(x)|^q \\ &\leqslant g(\underline{u}_\lambda(x)) + \lambda |\nabla \underline{u}_\lambda(x)|^q, \quad x \in \Omega, \end{aligned}$$

where $c_{\lambda 1} \in (0, b_\lambda)$, i.e. $\underline{u}_\lambda = c_{\lambda 1} v_0$ is a subsolution of problem (1.1). By the proof of lemma 3.5, we see that $\underline{u}_\lambda \leqslant c_0 v_0 \leqslant u_0$ on Ω. Thus, problem (1.1) has at least one solution $u_\lambda \in C(\bar{\Omega}) \cap C^2(\Omega)$ for every $\lambda < 0$. The uniqueness follows by lemma 3.4. This completes the proof. □

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