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Minimal flows with arbitrary centralizer

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Abstract. Given a G-flow X, let $\mathrm{Aut}(G, X)$, or simply $\mathrm{Aut}(X)$, denote the group of homeomorphisms of X which commute with the G action. We show that for any pair of countable groups G and H with G infinite, there is a minimal, free, Cantor G-flow X so that H embeds into $\mathrm{Aut}(X)$. This generalizes results of [**2**, **7**].

Key words: minimal flow, centralizer, blueprint, strongly irreducible shift
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1. *Introduction*

Let G be an infinite countable group, and let X be a G-flow, that is, a compact Hausdorff space equipped with a continuous G-action $a: G \times X \to X$. When the action a is understood, we often write $g \cdot x$ or simply gx in place of $a(g, x)$. Given G-flows X and Y, a *G-map* $\varphi: X \to Y$ is a continuous map which respects the G-actions. A bijective G-map from X to itself is called an *automorphism* of the G-flow X, and we denote the group of automorphisms of X by $\mathrm{Aut}(X)$ when the G-action is understood. Sometimes, this group is called the *centralizer* of X. In this paper, we will be interested in the possible centralizers of *minimal* G-flows, that is, those G-flows with every orbit dense.

The study of the centralizers of G-flows has been an active area of research, especially in the case $G = \mathbb{Z}$. Usually some constraint is placed upon the flows X under consideration, for instance by demanding that X is a subshift over a finite alphabet (see, for instance, [**1**, **3**–**5**, **8**]). More recently, interest has turned to just considering minimality with no other constraints. Namely, does only the knowledge that X is a minimal G-flow place any algebraic constraints on the possible groups that can appear as $\mathrm{Aut}(X)$?

A natural constraint to place on X is that the underlying space of X be the Cantor space. We call G-flows with this property *Cantor* flows. This is not much of a constraint at all, since every countable group can act freely on Cantor space [**9**]. In [**1**], a construction is given of a minimal $\mathbb{Z}$-subshift whose automorphism group embeds $\mathbb{Q}$. Cortez and Petite in [**2**] construct for every residually finite countable group H a minimal Cantor $\mathbb{Z}$-flow X

such that H embeds into $\mathrm{Aut}(X)$. Independently and using different techniques, Glasner *et al* in [7] construct for any countable group G and any countable group H which embeds into a compact group a free, minimal, Cantor G-flow X for which H embeds into $\mathrm{Aut}(X)$. Recall that the G-flow X is *free* if for every $x \in X$, the *stabilizer* $G_x := \{g \in G : gx = x\}$ is trivial.

The goal of this paper is to prove the following theorem.

THEOREM 1.1. *Let G and H be any countable groups with G infinite. Then there is a minimal, free, Cantor G-flow X so that H embeds into* $\mathrm{Aut}(X)$.

We may assume without loss of generality that H is also infinite. We also note that it suffices to construct any minimal G-flow X, not necessarily free nor Cantor, so that H embeds into $\mathrm{Aut}(X)$. If X is a minimal G-flow such that H embeds into $\mathrm{Aut}(X)$, then by [7, Theorem 1.2], there is a minimal, free G-flow Y with $X \times Y$ also minimal. Then by arguing as in [7, Theorem 11.5], we can find Z a suitable highly proximal extension of $X \times Y$ which is homeomorphic to Cantor space and such that H still embeds into $\mathrm{Aut}(Z)$. However, it seems very likely that the construction given here gives an *essentially free* G-flow, in which case the appeal to [7, Theorem 1.2] is not needed.

We start with two preliminary sections. The first is on *blueprints*, a notion developed by Gao, Jackson and Seward in [6]. The second discusses strongly irreducible subshifts. The final section proves Theorem 1.1.

Notation. Our notation is mostly standard. The set $\mathbb{N} := \{0, 1, 2, \ldots\}$ of natural numbers contains zero. If $f : X \to Y$ is a function and $S \subseteq X$, we write $f[S] := \{f(s) : s \in S\}$, and we write $f|_S$ for the restriction of f to domain S.

2. *Blueprints*

The notion of a *blueprint* is developed by Gao, Jackson and Seward in [6] where, in particular, it is proven that every group carries a non-trivial blueprint. To keep this paper self-contained, we provide a proof of this. We delay the definition of a blueprint until we have actually constructed one. Throughout this section, we will use the group G; the group H will figure more heavily in the next section.

The idea behind a blueprint is that, starting with a rapidly growing sequence $A_0, A_1, \ldots$ of finite subsets of our group, we wish to pack translates of these into our group in such a way that for any $A_n \subseteq G$ in our sequence, translates of A_n appear 'syndetically often', that is, without arbitrarily large gaps. However, the packings for $A_m \subseteq A_n$ need to be coherent, so that a translate of A_m does not touch the boundary of any translate of A_n. The example of $G = \mathbb{Z}$ is perhaps too easy, since we can take the A_n to be intervals with lengths that divide one another. The notion of a blueprint is most useful when we do not have prior knowledge of the geometry of the group, or if the sets A_n we must use are constrained in some way. A surprisingly good example to keep in mind is that of $\mathbb{R}^2$; granted, $\mathbb{R}^2$ is not a countable group, but thinking of $A_n \subseteq \mathbb{R}^2$ as Euclidean balls of rapidly growing radius is still informative, and most of the difficulties that pop up in full generality already appear in this example.

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For this section we fix an exhaustion $G = \bigcup_n A_n$, where each A_n is finite, symmetric, and contains the identity $1_G \in G$. We let $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ denote this exhaustion. We assume that each A_n is large enough to write $A_n = A_0^3 \cdot A_1^3 \cdots A_{n-1}^3 \cdot B_n$ for some finite set B_n containing 1_G which we now fix. In particular, notice that $A_0^3 \cdots A_{n-1}^3 \subseteq A_n$. Given $k < n$, we set $A_n(k) = A_k^3 \cdots A_{n-1}^3 \cdot B_n$. Notice that if $k' \leq k$, then $A_n(k) \subseteq A_n(k')$. Also notice that $A_k A_n(k) \supseteq A_n$.

If $F \subseteq G$ is finite, we say that $g, h \in G$ are *F-apart* if $Fg \cap Fh = \emptyset$. We say that $S \subseteq G$ is *F-spaced* if every $g \neq h \in S$ is F-apart.

Definition 2.1. An *$\mathcal{A}$-system of height n* is a collection $\mathcal{S} = \{\mathcal{S}(0), \ldots, \mathcal{S}(n)\}$ of subsets of A_n defined by reverse induction as follows.

- $\mathcal{S}(n) = \{1_G\}$.
- If $\mathcal{S}(k+1), \ldots, \mathcal{S}(n)$ have all been defined, we say that $g \in A_n$ is *k-admissible for $\mathcal{S}$* if, letting $\ell > k$ be least with $A_k \cdot g \cap A_\ell \cdot \mathcal{S}(\ell) \neq \emptyset$, then there is $h \in \mathcal{S}(\ell)$ with $A_k \cdot g \subseteq A_\ell(k) \cdot h$. Write $\mathrm{Ad}(k, \mathcal{S})$ for the set of $g \in A_n$ which are k-admissible for $\mathcal{S}$.
- $\mathcal{S}(k)$ is any maximal A_k-spaced subset of $\mathrm{Ad}(k, \mathcal{S})$ containing 1_G.

For the last item, notice by reverse induction that $1_G \in \mathrm{Ad}(k, \mathcal{S})$ for each $k < n$.

Let us immediately clarify an important point about the set $\mathrm{Ad}(k, \mathcal{S})$.

LEMMA 2.2. *Suppose $g \in \mathrm{Ad}(k, \mathcal{S})$. Then for any $m > k$ with $A_k \cdot g \cap A_m \cdot \mathcal{S}(m) \neq \emptyset$, there is $b \in \mathcal{S}(m)$ with $A_k \cdot g \subseteq A_m(k) \cdot b$.*

Proof. We induct on $m - k$. When $m - k = 1$, the lemma follows from the definitions. If $m - k > 1$, then consider the least $\ell > k$ with $A_k \cdot g \cap A_\ell \mathcal{S}(\ell) \neq \emptyset$. Using item (2) of the definition, there is $h \in \mathcal{S}(\ell)$ with $A_k \cdot g \subseteq A_\ell(k) \cdot h$. If $\ell = m$ we are done. If $\ell < m$, then $A_\ell \cdot h \cap A_m \cdot \mathcal{S}(m) \neq \emptyset$, so by induction we can find $b \in \mathcal{S}(m)$ with $A_\ell \cdot h \subseteq A_m(\ell) \cdot b$. Then $A_k \cdot g \subseteq A_\ell \cdot h \subseteq A_m(\ell) \cdot b \subseteq A_m(k) \cdot b$. □

For the moment, fix an $\mathcal{A}$-system $\mathcal{S}$ of height n. Our first main goal is Proposition 2.6, which shows that the sets $\mathcal{S}(k)$ are somewhat large.

LEMMA 2.3. *Suppose $g \in \mathrm{Ad}(k, \mathcal{S})$. Then we have $A_k^2 \cdot g \cap \mathcal{S}(k) \neq \emptyset$.*

Proof. If $A_k^2 \cdot g \cap \mathcal{S}(k)$ were empty, then $\mathcal{S}(k) \cup \{g\}$ would be a strictly larger A_k-spaced subset of $\mathrm{Ad}(k, \mathcal{S})$. □

LEMMA 2.4. *Suppose $\ell > k$ and $h \in \mathcal{S}(\ell)$. Then we have*

$$A_k^2 \cdot A_\ell(k+1) \cdot h \setminus A_k \cdot A_\ell(k+1) \cdot h \subseteq \mathrm{Ad}(k, \mathcal{S}).$$

Proof. Fix g in the left-hand side. Then $A_k \cdot g \subseteq A_\ell(k) \cdot h \setminus A_\ell(k+1) \cdot h$. Towards a contradiction, suppose there were some m, $k < m < \ell$, with $A_k \cdot g \cap A_m \cdot \mathcal{S}(m) \neq \emptyset$. Suppose $b \in \mathcal{S}(m)$ satisfies $A_k \cdot g \cap A_m \cdot b \neq \emptyset$. Then since $b \in \mathrm{Ad}(m, \mathcal{S})$, Lemma 2.2 implies that $A_m \cdot b \subseteq A_\ell(m) \cdot h$. But since we have $A_k \cdot g \cap A_\ell(k+1) \cdot h = \emptyset$, this is a contradiction. □

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Definition 2.5. Suppose $F \subseteq G$ is finite, $D \subseteq G$, and let $S \subseteq D$. We say that S is *F-syndetic in D* if for any $g \in G$ such that $Fg \subseteq D$, we have $Fg \cap S \neq \emptyset$. If $D = G$, we simply say that S is *F-syndetic*. We say that S is *syndetic* if there is a finite $F \subseteq G$ so that S is F-syndetic.

PROPOSITION 2.6. *The set $\mathcal{S}(k) \subseteq A_n$ is A_k^5-syndetic in A_n.*

Proof. Suppose we have $g \in G$ with $A_k^5 \cdot g \subseteq A_n$. If $g \in \mathrm{Ad}(k, \mathcal{S})$, we are done by Lemma 2.3, so assume this is not the case. Let $\ell > k$ be least with $A_k \cdot g \cap A_\ell \cdot \mathcal{S}(\ell) \neq \emptyset$, and fix some $h \in \mathcal{S}(\ell)$ and $f \in A_k$ with $fg \in A_\ell \cdot h$. Notice that we cannot have $fg \in A_k \cdot A_\ell(k+1) \cdot h$, as this would imply that $g \in \mathrm{Ad}(k, \mathcal{S})$. In particular, for some $i \in \{2, 3, 4\}$, we have $fg \in A_k^i \cdot A_\ell(k+1) \cdot h \setminus A_k^{i-1} \cdot A_\ell(k+1) \cdot h$. In each case, we can find $f_0 \in A_k^2$ with $f_0 fg \in A_k^2 \cdot A_\ell(k+1) \cdot h \setminus A_k \cdot A_\ell(k+1) \cdot h$. By Lemma 2.4, we have $f_0 fg \in \mathrm{Ad}(k, \mathcal{S})$, so by Lemma 2.3, we have $A_k^2 \cdot f_0 fg \cap \mathcal{S}(k) \neq \emptyset$. We are done once we note that $A_k^2 \cdot f_0 f \subseteq A_k^5$. □

We now investigate how to modify $\mathcal{A}$-systems to create new ones. Definition 2.7 and Proposition 2.8 give a method to restrict to a smaller system, while Definition 2.9 and Proposition 2.10 allow us to print a smaller system inside a larger one.

Definition 2.7. Suppose $g \in \mathcal{S}(m)$. Then $(g \cdot \mathcal{S})|_m = \{(g \cdot \mathcal{S})|_m(0), \ldots, (g \cdot \mathcal{S})|_m(m)\}$ denotes the collection of subsets of A_m where for $k \leq m$ we set

$$(g \cdot \mathcal{S})|_m(k) = (\mathcal{S}(k) \cap (A_m \cdot g)) \cdot g^{-1}.$$

If $g = 1_G$, we simply write $\mathcal{S}|_m$.

PROPOSITION 2.8. *$(g \cdot \mathcal{S})|_m$ is an $\mathcal{A}$-system of height m.*

Proof. We first note that $(g \cdot \mathcal{S})|_m(m) = \{1_G\}$ since $\mathcal{S}(m)$ is A_m-spaced. Then we proceed by reverse induction on $k < m$. First we note that

$$\mathrm{Ad}(k, (g \cdot \mathcal{S})|_m) = (\mathrm{Ad}(k, \mathcal{S}) \cap (A_m \cdot g)) \cdot g^{-1}.$$

Then, if $b, h \in \mathrm{Ad}(k, \mathcal{S})$ with $A_k \cdot b \cap A_m \cdot g = \emptyset$ and $A_k \cdot h \subseteq A_m(k) \cdot g$, we have $A_k \cdot b \cap A_k \cdot h = \emptyset$. It follows that $(\mathcal{S}(k) \cap (A_k \cdot g)) \cdot g^{-1}$ is a maximal A_k-spaced subset of $\mathrm{Ad}(k, (g \cdot \mathcal{S})|_m)$. □

Definition 2.9. Let $\mathcal{S}$ be an $\mathcal{A}$-system of height n. Let $\mathcal{T}$ be an $\mathcal{A}$-system of height m for some $m < n$. Given $g \in \mathcal{S}(m)$, we let $(\mathcal{S}, \mathcal{T}, g) = \{(\mathcal{S}, \mathcal{T}, g)(0), \ldots, (\mathcal{S}, \mathcal{T}, g)(n)\}$ denote the collection of subsets of A_n where, for $k \leq n$, we have:

- $(\mathcal{S}, \mathcal{T}, g)(k) = \mathcal{S}(k)$ for $m \leq k \leq n$;
- $(\mathcal{S}, \mathcal{T}, g)(k) = (\mathcal{S}(k) \setminus A_m \cdot g) \cup (\mathcal{T}(m) \cdot g)$ for $k < m$.

PROPOSITION 2.10. *$(\mathcal{S}, \mathcal{T}, g)$ is an $\mathcal{A}$-system of height n.*

Proof. We proceed by reverse induction on $k \leq n$. For $k \geq m$ there is nothing to prove. For $k < m$, we observe that $\mathrm{Ad}(k, (\mathcal{S}, \mathcal{T}, g)) = (\mathrm{Ad}(k, \mathcal{S}) \setminus A_m \cdot g) \cup \mathrm{Ad}(k, \mathcal{T}) \cdot g$. Then we note that $\mathcal{S}(k) \setminus A_m \cdot g$ and $\mathcal{T}(m) \cdot g$ are A_k-apart. It follows that $(\mathcal{S}(k) \setminus A_m \cdot g) \cup \mathcal{T}(m) \cdot g$ is a maximal A_k-spaced subset of $\mathrm{Ad}(k, (\mathcal{S}, \mathcal{T}, g))$. □

We use Proposition 2.10 to construct particularly nice $\mathcal{A}$-systems.

Definition 2.11. Let $\mathcal{S}$ be an $\mathcal{A}$-system of height n. We call $\mathcal{S}$ *uniform* if $(g \cdot \mathcal{S})|_m = (h \cdot \mathcal{S})|_m$ for any $g, h \in \mathcal{S}(m)$ and any $m \leq n$.

PROPOSITION 2.12. *There is a sequence $\{\mathcal{S}_n : n \in \mathbb{N}\}$ of uniform $\mathcal{A}$-systems such that $\mathcal{S}_n$ has height n and $\mathcal{S}_n|_m = \mathcal{S}_m$ for any $m \leq n$.*

Proof. We proceed by (forward) induction. For $n = 0$ the unique $\mathcal{A}$-system of height zero is vacuously uniform. Suppose $\mathcal{S}_0, \ldots, \mathcal{S}_{n-1}$ have been constructed. Let $\mathcal{T} := \mathcal{T}_0$ be any $\mathcal{A}$-system of height n. For each $k < n$, we set

$$T(k) = \mathcal{T}(k) \setminus \bigg(\bigcup_{k<m<n} A_m \cdot \mathcal{T}(m) \bigg).$$

Note that the sets $T(0), \ldots, T(n-1)$ are pairwise disjoint. Fix some enumeration of $\bigcup_{k<n} T(k) = \{g_0, \ldots, g_{r-1}\}$, and for each $i < r$ let $\varphi(i) < n$ be the unique index with $g_i \in T(\varphi(i))$. We repeatedly use Proposition 2.10 to define $\mathcal{A}$-systems $\mathcal{T}_0, \ldots, \mathcal{T}_r$. If $\mathcal{T}_i$ has been built for some $i < r$, we set $\mathcal{T}_{i+1} = (\mathcal{T}_i, \mathcal{S}_{\varphi(i)}, g_i)$. We then set $\mathcal{S}_n = \mathcal{T}_r$. Then $\mathcal{S}_n$ is a uniform $\mathcal{A}$-system of height n as desired. □

Definition 2.13.

(1) A sequence $\vec{\mathcal{S}} := \{\mathcal{S}_n : n \in \mathbb{N}\}$ constructed as in Proposition 2.12 will be called a *coherent sequence*.

(2) Let $\vec{\mathcal{S}}$ be a coherent sequence. The *blueprint* of $\vec{\mathcal{S}}$ is the sequence $\{\vec{\mathcal{S}}(n) : n \in \mathbb{N}\}$, where $\vec{\mathcal{S}}(n) = \bigcup_{N \geq n} \mathcal{S}_N(n)$.

PROPOSITION 2.14. *Fix a coherent sequence $\vec{\mathcal{S}}$, and form its blueprint $\{\vec{\mathcal{S}}(n) : n \in \mathbb{N}\}$.*

(1) *$\vec{\mathcal{S}}(n) \supseteq \vec{\mathcal{S}}(n+1)$, and each $\vec{\mathcal{S}}(n)$ is A_n-spaced and A_n^5-syndetic.*

(2) *For any $k \leq n$, $g \in \vec{\mathcal{S}}(k)$, and $h \in \vec{\mathcal{S}}(n)$, we either have $A_k \cdot g \cap A_n \cdot h = \emptyset$ or $A_k \cdot g \subseteq A_n(k) \cdot h$.*

(3) *For any $k \leq n$ and $g, h \in \vec{\mathcal{S}}(n)$, we have*

$$(\vec{\mathcal{S}}(k) \cap (A_n \cdot g)) \cdot g^{-1} = (\vec{\mathcal{S}}(k) \cap (A_n \cdot h)) \cdot h^{-1}.$$

(4) *For each $n \in \mathbb{N}$, we have $|\vec{\mathcal{S}}(n) \cap A_{n+1}| \geq |A_n^2 \cdot B_{n+1}|/|A_n^2|$.*

Remark. Compare this to [6]. In fact, we have constructed what they call a *centered* blueprint.

Proof. (1) First, we note that for $n + 1 \leq N$ we have $\mathcal{S}_N(n+1) \subseteq \mathcal{S}_N(n)$. To see this, fix $g \in \mathcal{S}_N(n+1)$. Because $\mathcal{S}_N$ is uniform, we have that $(g \cdot \mathcal{S}_N)|_{n+1} = \mathcal{S}_N|_{n+1}$. In particular, since $1_G \in \mathcal{S}_N|_{n+1}(n)$, we have $1_G \in (g \cdot \mathcal{S}_N)|_{n+1}(n) = (\mathcal{S}_N(n) \cap (A_{n+1} \cdot g)) \cdot g^{-1}$, implying that $g \in \mathcal{S}_N(n)$ as desired. From this, it follows that $\vec{\mathcal{S}}(n) \supseteq \vec{\mathcal{S}}(n+1)$.

For the second claim, we note that for $n \leq N$ we have $\mathcal{S}_N(n) \subseteq \mathcal{S}_{N+1}(n)$ because $\vec{\mathcal{S}}$ is coherent. Each $\mathcal{S}_N(n)$ is A_n-spaced and A_n^5-syndetic in A_N. It follows that $\vec{\mathcal{S}}(n)$ is A_n-spaced and A_n^5-syndetic.

(2) Find some $N \geq n$ with $g \in \mathcal{S}_N(k)$ and $h \in \mathcal{S}_N(n)$. If $k = n$, the claim holds since $\mathcal{S}_N(n)$ is A_n-spaced. If $k < n$, the claim holds since $\mathcal{S}_N$ is an $\mathcal{A}$-system of height N.

(3) Find some $N \geq n$ with $g, h \in \mathcal{S}_N(n)$. Then as $\mathcal{S}_N$ is uniform, we have $(g \cdot \mathcal{S}_N)|_n = (h \cdot \mathcal{S}_N)|_n$, so in particular $(g \cdot \mathcal{S}_N)|_n(k) = (h \cdot \mathcal{S}_N)|_n(k)$, that is,

$$(\mathcal{S}_N(k) \cap (A_n \cdot g)) \cdot g^{-1} = (\mathcal{S}_N(k) \cap (A_n \cdot h)) \cdot h^{-1}.$$

Since this is true for every large enough N, the result follows.

(4) The set $\mathcal{S}_{n+1}(n)$ is a maximal A_n-spaced subset of $\mathrm{Ad}(n, \mathcal{S}_{n+1}) = \{g \in G : A_n \cdot g \subseteq A_{n+1}\}$. Note that $A_n^2 \cdot B_{n+1} \subseteq \mathrm{Ad}(n, \mathcal{S}_{n+1})$. The result follows. □

3. *Strongly irreducible subshifts*

In this section we work with the group H. If M is a compact space, then H acts on the space M^H via the right shift, where, given $g \in H$ and $x \in M^H$, we define $g \cdot x \in M^H$ via $(g \cdot x)(h) = x(hg)$. A *subshift* is any non-empty closed $X \subseteq M^H$ which is H-invariant. Often, we will take M to be a finite set A; in this case we refer to A as an *alphabet*. Let $X \subseteq A^H$ be a subshift. If $C \subseteq H$ is finite, the set of C-*patterns* of X is given by $P_C(X) = \{x|_C : x \in X\} \subseteq A^C$. If $D \subseteq H$ is finite, sets $S_0, S_1 \subseteq H$ are called D-*apart* if $DS_0 \cap DS_1 = \emptyset$.

Definition 3.1. Let $D \subseteq H$ be finite. A subshift $X \subseteq A^H$ is D-*irreducible* if for any $S_0, S_1 \subseteq H$ which are D-apart and any $x_0, x_1 \in X$, there is $y \in X$ such that $y|_{S_i} = x_i|_{S_i}$ for each $i < 2$. We sometimes say that y *blends* $x_0|_{S_0}$ and $x_1|_{S_1}$. We say that X is *strongly irreducible* if X is D-irreducible for some finite $D \subseteq H$.

Fact 3.2. Let A and B be finite sets. If $X \subseteq A^H$ is D_X-irreducible and $Y \subseteq B^H$ is D_Y-irreducible, then $X \times Y \subseteq (A \times B)^H$ is $(D_X \cup D_Y)$-irreducible.

The remainder of this section discusses some examples of strongly irreducible flows that we will use in the construction of the next section. Given a finite $C \subseteq H$, any $S \subseteq H$, and $\gamma \colon S \to A$ for some finite alphabet A, we say that γ is C-*spaced* if $\gamma^{-1}(a) \subseteq H$ is C-spaced for every $a \in A$.

LEMMA 3.3. *Let $C \subseteq H$ be finite, and let $n = |C^{-1}C|$. Then, given any $S \subseteq H$ (possibly $S = \emptyset$) and C-spaced function $\delta \colon S \to \{0, \ldots, n-1\}$, there is a C-spaced function $\gamma \colon H \to \{0, \ldots, n-1\}$ with $\gamma|_S = \delta$.*

Proof. Enumerate $H = \{h_i : i \in \mathbb{N}\}$. Set $S_0 = S$ and $\gamma_0 = \delta$. If $S_i \subseteq H$ and C-spaced functions $\gamma_i \colon S_i \to \{0, \ldots, n-1\}$ have been determined, we set $S_{i+1} = S_i \cup \{h_i\}$. If $h_i \in S_i$, then we set $\gamma_{i+1} = \gamma_i$. If $h_i \notin S_i$, then we note that $|(C^{-1}Ch_i) \cap S_i| < n$. The inequality is strict since $|C^{-1}Ch_i| = n$ and $h_i \in (C^{-1}Ch_i) \setminus S_i$. In particular, we have

$$\{0, \ldots, n-1\} \setminus \gamma_i[(C^{-1}Ch_i) \cap S_i] \neq \emptyset.$$

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Choose $\gamma_{i+1}(h_i)$ to be anything from this non-empty set. We then set $\gamma = \bigcup_i \gamma_i$. $\square$

Definition 3.4. For any finite $C \subseteq H$ and any $n \geq |C^{-1}C|$, we set

$$\mathrm{Part}(C, n) := \{\gamma \in \{0, \ldots, n-1\}^H : \gamma^{-1}(k) \text{ is } C\text{-spaced for each } k < n\}.$$

The argument in Lemma 3.3 shows that $\mathrm{Part}(C, n)$ is C-irreducible.

As a warm-up for the next definition, first suppose that $C, D \subseteq H$ are finite. Let $X \subseteq A^H$ be D-irreducible, and fix $\alpha \in P_C(X)$. Suppose $S \subseteq H$ is DC-spaced. Then by repeatedly using D-irreducibility, we can find $x \in X$ such that $(h \cdot x)|_C = \alpha$ for each $h \in S$.

Definition 3.5. Let $C, D \subseteq H$ be finite, let $E \subseteq H$ be finite with $DC \subseteq E$, and let $N \geq |E^{-1}E|$. We define

$$\begin{aligned}\mathrm{Print}(X, \alpha, E, N) := \{(x_0, \ldots, x_{N-1}) \in X^N &: \text{there exists } \gamma \in \mathrm{Part}(E, N) \text{ for all } h \in H \\ &\quad \text{such that } (h \cdot x_{\gamma(h)})|_C = \alpha\} \\ \subseteq \{(x_0, \ldots, x_{N-1}) \in X^N &: \text{for all } h \in H \text{ there exists } i < N \\ &\quad \text{such that } (h \cdot x_i)|_C = \alpha\}.\end{aligned}$$

In the notation, notice that C is implicit, as $C = \mathrm{dom}(\alpha)$. Although D is implicit as well via the assumption that X is D-irreducible, this is less important as E is presumed to be suitably large. Notice that if $(x_0, \ldots, x_{N-1}) \in \mathrm{Print}(X, \alpha, E, N)$ as witnessed by $\gamma \in \mathrm{Part}(E, N)$, then for any $g \in H$, we have that $g \cdot (x_0, \ldots, x_{N-1}) = (g \cdot x_0, \ldots, g \cdot x_{N-1})$ is in $\mathrm{Print}(X, \alpha, E, N)$ as witnessed by $g \cdot \gamma \in \mathrm{Part}(E, N)$.

PROPOSITION 3.6. *$\mathrm{Print}(X, \alpha, E, N)$ is $EE^{-1}D$-irreducible.*

Proof. Let $(x_0, \ldots, x_{N-1}), (y_0, \ldots, y_{N-1}) \in \mathrm{Print}(X, \alpha, E, N)$ as witnessed by γ_x, $\gamma_y \in \mathrm{Part}(E, N)$. Let $S_x, S_y \subseteq H$ be $EE^{-1}D$-apart. For each $i < N$, we define

$$S_x(i) = S_x \cup \Big(\bigcup\{Ch : h \in H \text{ with } \gamma_x(h) = i \text{ and } h \in E^{-1}DS_x\}\Big),$$

and we define $S_y(i)$ similarly. Note that $S_x(i) \subseteq CE^{-1}DS_x$, and similarly for $S_y(i)$; in particular, $S_x(i)$ and $S_y(i)$ are D-apart since $DC \subseteq E$.

Since $\mathrm{Part}(E, N)$ is E-irreducible, we can find $\gamma \in \mathrm{Part}(DC, N)$ blending $\gamma_x|_{E^{-1}DS_x}$ and $\gamma_y|_{E^{-1}DS_y}$. Notice that if $h \notin E^{-1}D(S_x \cup S_y)$, then Ch and $S_x \cup S_y$ are D-apart; if we also have $\gamma(h) = i$, then Ch and $S_x(i) \cup S_y(i)$ are D-apart. Now for each $i < N$, find $z_i \in X$ which blends $x_i|_{S_x(i)}$ and $y_i|_{S_y(i)}$ and satisfies $(h \cdot z_i)|_C = \alpha$ whenever $\gamma(h) = i$. Then $(z_0, \ldots, z_{N-1}) \in \mathrm{Print}(X, \alpha, N)$ is as desired. $\square$

4. *The construction*

In this section we construct a $(G \times H)$-subshift $X \subseteq 2^{G\times H}$ which is essentially free (in fact free) as an H-flow and minimal as a G-flow. This will prove Theorem 1.1. We will often think of $2^{G\times H}$ as either the G-flow $(2^H)^G$ or as the H-flow $(2^G)^H$ as needed.

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However, to keep the roles of G and H clear, we use different notation. Given $g \in G$ and $x \in 2^{G\times H}$, we will write $\langle g, x\rangle$ instead of $(g, 1_H) \cdot x$, and if $h \in H$, we write $h \cdot x$ instead of $(1_G, h) \cdot X$. We will first construct an H-flow $Y = \varprojlim Y_n \subseteq 2^{G\times H}$. Then we will set $X = \overline{\langle G, Y\rangle}$, where $\langle G, Y\rangle := \{\langle g, y\rangle : g \in G, y \in Y\}$. The main work in this section is the construction of Y.

We start by fixing both an exhaustion $G = \bigcup_n A_n$ as in §2 and a coherent sequence $\vec{\mathcal{S}}$ on G. We will adhere to the notation developed in §2 as much as possible. We will often assume that each A_{n+1} is suitably large compared to A_n to proceed as we need, especially in regard to item 4 of Proposition 2.14. For each $n \in \mathbb{N}$, the H-flow Y_n will be a subshift of $(2^{A_n})^H$, and for $m < n$, the projection $\pi^n_m \colon Y_n \to Y_m$ will be the one induced by the restriction map from 2^{A_n} to 2^{A_m}. We also fix an exhaustion $H = \bigcup_n C_n$ with each C_n finite, symmetric, and containing the identity $1_H \in H$.

It will be helpful to 'finitize' the G-action as follows.

Definition 4.1.

(1) Suppose $A \subseteq G$ and $\alpha \in 2^A$. Given $g \in G$, we let $\langle g \mid \alpha\rangle \in 2^{Ag^{-1}}$ be defined by $\langle g \mid \alpha\rangle(ag^{-1}) = \alpha(a)$ for $a \in A$. Note that $\langle g_0g_1 \mid \alpha\rangle = \langle g_0 \mid \langle g_1 \mid \alpha\rangle\rangle$.

(2) Suppose $A \subseteq G$ is finite and $z \in (2^A)^H$. Then, for any $g \in G$, we define $\langle g, z\rangle \in (2^{Ag^{-1}})^H$ where, for $z \in (2^A)^H$ and $h \in H$, we have $\langle g, z\rangle(h) = \langle g \mid z(h)\rangle$. Again, note that $\langle (g_0g_1), z\rangle = \langle g_0, \langle g_1, z\rangle\rangle$.

(3) Note that if $Z \subseteq (2^A)^H$ is an H-subshift, then $\langle g, Z\rangle := \{\langle g, z\rangle : z \in Z\} \subseteq (2^{Ag^{-1}})^H$ is also an H-subshift.

Example 4.2. Suppose $G = \mathbb{Z}$, and that $A = \{-10, \ldots, 10\}$. Then, if $\alpha \in 2^A$ and $g = 5$, the domain of $\langle 5, \alpha\rangle$ is $\{-15, \ldots, 5\}$. This might seem a little counterintuitive, but this definition agrees with how we defined our shift action.

We build the flows Y_n by induction, and we set $Y_0 = (2^{A_0})^H$. Trivially, Y_0 is $\{1_H\} := D_0$-irreducible. Suppose $Y_0, \ldots, Y_{n-1}$ have been constructed, where each Y_k is an H-subflow of $(2^{A_k})^H$, and are all D_{n-1}-irreducible for some finite symmetric $D_{n-1} \subseteq H$. Fix some finite $E_{n-1} \subseteq H$ with $D_{n-1}C_{n-1} \subseteq E_{n-1}$. For each $k < n$, set

$$S_n(k) = \mathcal{S}_n(k) \setminus \bigg(\bigcup_{k<m<n} A_m \cdot \mathcal{S}_n(m)\bigg).$$

Notice that $S_n(n-1) = \mathcal{S}_n(n-1) = \vec{\mathcal{S}}(n-1) \cap A_n$. For $k < n$, set $T_n(k) = A_k \cdot S_n(k)$. We also set $T_n(n) := A_n \setminus \bigcup_{k<n} T_n(k)$. To define Y_n, we will first define a subshift $Z_n \subseteq (2^{T_n(n-1)})^H$. We will then put

$$Y_n := (2^{T_n(n)})^H \times Z_n \times \prod_{k=0}^{n-2} \prod_{g\in S_n(k)} \langle g^{-1}, Y_k\rangle.$$

Since $A_n = T_n(n) \cup T_n(n-1) \cup \bigcup_{0\le k\le n-2} T_n(k)$, we see that $Y_n \subseteq (2^{A_n})^H$ as desired. We note that Y_n will be strongly irreducible as long as Z_n is.

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Let $r := |S_n(n-1)| = |\vec{\mathcal{S}}(n-1) \cap A_n|$. How large does r need to be? Consider the set $P_{C_{n-1}}(Y_{n-1}) := \{\alpha_0, \ldots, \alpha_{\ell-1}\} \subseteq (2^{A_{n-1}})^{C_{n-1}}$. We will want to ensure that

$$r > |E_{n-1}^{-1}E_{n-1}| \cdot 2^{|A_{n-1}\times C_{n-1}|}.$$

Using part (4) of Proposition 2.14, we see that as long as A_n is suitably large compared to A_{n-1}, r will satisfy this inequality.

Having fixed $\ell = |P_{C_{n-1}}(Y_{n-1})|$, the size of r allows us to find disjoint sets $F_i \subseteq S_n(n-1)$ for each $i < \ell$ with $|F_i| = |E_{n-1}^{-1}E_{n-1}| := q$, while ensuring that $F_\ell := S_n(n-1) \setminus \bigcup_{i<\ell} F_i \neq \emptyset$. We also demand that $1_G \in F_\ell$. For $i < \ell$, write $F_i = \{g_0^i, \ldots, g_{q-1}^i\}$.

Recall the flow Print from the previous section. We define a map

$$\Phi_i\colon \mathrm{Print}(Y_{n-1}, \alpha_i, E_{n-1}, q) \to (2^{A_{n-1}\cdot F_i})^H = (2^{A_{n-1}\cdot g_0^i})^H \times \cdots \times (2^{A_{n-1}\cdot g_{q-1}^i})^H$$

via $\Phi_i((x_0, \ldots, x_{q-1})) = (\langle (g_0^i)^{-1}, x_0\rangle, \ldots, \langle (g_{q-1}^i)^{-1}, x_{q-1}\rangle)$. Note that Φ_i is injective. We set $Q_i = \mathrm{Im}(\Phi_i)$.

We then set

$$Z_n = \prod_{g\in F_\ell} \langle g^{-1}, Y_{n-1}\rangle \times \prod_{i<\ell} Q_i.$$

Note that Z_n is strongly irreducible; hence Y_n is as well.

We will need the following lemma. For any $B \subseteq A_n$, we let $\pi_B^n\colon (2^{A_n})^H \to (2^B)^H$ denote the restriction map. If $B = A_m$, we simply write π_m^n instead of $\pi_{A_m}^n$.

LEMMA 4.3. *If $g \in \mathcal{S}_n(n-1)$, then $\pi_{A_{n-1}\cdot g}^n[Y_n] \subseteq \langle g^{-1}, Y_{n-1}\rangle$.*

Proof. To see this, first note that if $g \in F_\ell$, we have $\pi_{A_{n-1}\cdot g}^n[Y_n] = \langle g^{-1}, Y_{n-1}\rangle$ straight from the definition of Z_n. If $g \in F_i = \{g_0^i, \ldots, g_{q-1}^i\}$ for some $i < \ell$, say that $g = g_j^i$ for some $j < q$. Suppose $y \in Y_n$. Then, setting $z = \pi_{A_{n-1}\cdot F_i}^n(y)$, we have $z \in Q_i = \mathrm{Im}(\Phi_i)$. If $(x_0, \ldots, x_{q-1}) \in \mathrm{Print}(Y_{n-1}, \alpha_i, E_{n-1}, q)$ is such that $\Phi_i(x_0, \ldots, x_{q-1}) = z$, then we have that $\pi_{A_{n-1}\cdot g}^n(z) = \langle g^{-1}, x_j\rangle$. Since $x_j \in Y_{n-1}$, we have the result. □

COROLLARY 4.4. *If $g \in \mathcal{S}_n(k)$ for $k < n$, then $\pi_{A_k\cdot g}^n[Y_n] \subseteq \langle g^{-1}, Y_k\rangle$.*

Proof. We induct on $n-k$ for every n simultaneously. If $n-k = 1$, the result follows from Lemma 4.3. If $g \in S_n(k)$, then $\pi_{A_k\cdot g}^n[Y_n] = \langle g^{-1}, Y_k\rangle$ straight from the definition of Y_n. If $g \in \mathcal{S}_n(k) \setminus S_n(k)$, then there is m with $k < m < n$ so that $g \in A_m \cdot \mathcal{S}_n(m)$. In particular, let $h \in \mathcal{S}_n(m)$ be such that $g \in A_m \cdot h$. By the induction hypothesis, we have $\pi_{A_m\cdot h}^n[Y_n] \subseteq \langle h^{-1}, Y_m\rangle$. Now notice that since $\vec{\mathcal{S}}$ is a coherent sequence, we have $gh^{-1} \in \mathcal{S}_m(k)$. By the induction hypothesis, we have $\pi_{A_k\cdot gh^{-1}}^m[Y_m] \subseteq \langle hg^{-1}, Y_k\rangle$. Putting everything together, we have $\pi_{A_k\cdot g}^n[Y_n] = \pi_{A_k\cdot gh^{-1}}^m[\langle h, \pi_{A_m\cdot h}^n[Y_n]\rangle] \subseteq Y_k$ as desired. □

We now set $Y = \varprojlim Y_n \subseteq 2^{(\bigcup_n A_n)\times H} = 2^{G\times H}$, where the inverse limit is taken along the maps π_m^n, and we set $X = \overline{\langle G, Y\rangle}$. If $B \subseteq G$, we let $\pi_B\colon 2^{G\times H} \to (2^B)^H$ denote the restriction map. If $B = A_n$, we simply write π_n instead of π_{A_n}.

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PROPOSITION 4.5. *X is essentially free as an H-flow and minimal as a G-flow.*

Remark. Note that this immediately implies that X is in fact free as an H-flow, since each $h \in H$ acts as an automorphism of the minimal G-flow X.

Proof. We note that each Y_n is essentially free, since $\pi_0^n[Y_n] = Y_0 = (2^{A_0})^H$. Hence Y is essentially free, from which it follows that $\overline{\langle G, Y\rangle}$ is essentially free as an H-flow.

To show that X is G-minimal, fix $x, y \in Y$, and fix an open $V \ni y$. We need to show that the *visiting set* $\mathrm{Vis}(x, V) := \{g \in G : \langle g, x\rangle \in V\}$ is syndetic. We may assume that $V = \{z \in Y : z|_{A_{n-1}\times C_{n-1}} = y|_{A_{n-1}\times C_{n-1}} = \alpha_i\}$, where we use notation (α_i, Q_i, etc.) defined the construction of Y_n from Y_{n-1}.

Pick any $g \in \vec{\mathcal{S}}(n)$. Fix some $N \geq n$ so that $g \in \mathcal{S}_N(n)$. Then since $\pi_N(x) \in Y_N$, we have

$$\langle g, x\rangle|_{A_n\times H} = \langle g, \pi_{A_n\cdot g}(x)\rangle = \langle g, \pi^N_{A_n\cdot g}(\pi_N(x))\rangle \in Y_n$$

by Corollary 4.4. It follows that $\langle g, x\rangle|_{(A_{n-1}\cdot F_i)\times H} \in Q_i$. By the definition of Print, there is $j < q$ with

$$\langle g^i_j, \langle g, x\rangle\rangle|_{A_{n-1}\times C_{n-1}} = \langle g^i_j \cdot g, x\rangle = \alpha_i.$$

It follows that $g^i_j \cdot g \in \mathrm{Vis}(x, V)$. Since g was an arbitrary element of $\vec{\mathcal{S}}(n)$, an A_n^5-syndetic set, and since $F_i \subseteq \mathcal{S}_n(n-1) \subseteq A_n$, we see that $\mathrm{Vis}(x, V)$ is A_n^6-syndetic as desired. □

One drawback of the techniques used in this paper is the asymmetry between the roles of G and H. For example, the following 'symmetric' version of the result remains open.

Question 4.6. Let G and H be countable infinite groups. Is there a free $(G \times H)$-flow which is simultaneously a minimal G-flow and a minimal H-flow?

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