COMPOSITIO MATHEMATICA

Singular units and isogenies between CM elliptic curves

Yingkun Li

Compositio Math. **157** (2021), 1022–1035.

doi:10.1112/S0010437X21007077

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY EST. 1865

https://doi.org/10.1112/S0010437X21007077 Published online by Cambridge University Press

Compositio Math. 157 (2021) 1022–1035
doi:10.1112/S0010437X21007077

Singular units and isogenies between CM elliptic curves

Yingkun Li

Abstract

In this note, we will apply the results of Gross–Zagier, Gross–Kohnen–Zagier and their generalizations to give a short proof that the differences of singular moduli are not units. As a consequence, we obtain a result on isogenies between reductions of CM elliptic curves.

1. Introduction

Let $\mathbb{H}$ be the complex upper half-plane, which is acted on discretely by the group $\Gamma := \mathrm{PSL}_2(\mathbb{Z})$. The modular curve $Y(1) := \Gamma\backslash\mathbb{H}$ is the coarse moduli space of isomorphism classes of elliptic curves over $\mathbb{C}$. The Klein-j invariant provides the uniformization

$$j : Y(1) \to \mathbb{C}.$$

Let $\mathrm{z} \in Y(1)$ be a CM point of discriminant $d < 0$, i.e. it corresponds to an elliptic curve E_{z} with complex multiplication by the order $\mathcal{O}_d := \mathbb{Z} + ((d + \sqrt{d})/2)\mathbb{Z}$ in the imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$. Then $j(\mathrm{z})$ is called a *singular modulus*. The classical theory of complex multiplication tells us that $j(\mathrm{z})$ is an algebraic integer generating the ring class field H_d of K, which corresponds to the order $\mathcal{O}_d$ via class field theory. The factorization of the difference of singular moduli was the subject of the seminal work by Gross and Zagier [GZ85] and has interesting implications for the CM elliptic curve E_{z}. For example, if $j(\mathrm{z})$ is divisible by a prime $\mathfrak{p}$ in H_d, then the reduction of E_{z} modulo $\mathfrak{p}$ is isomorphic to the reduction of the CM elliptic curve

$$E : y^2 = x^3 - 1,$$

whose corresponding j-invariant is zero.

More generally for $m \geq 1$, let $\varphi_m(X, Y) \in \mathbb{Z}[X, Y]$ be the modular polynomial defined by

$$\varphi_m(j(\mathrm{z}_1), j(\mathrm{z}_2)) := \prod_{\gamma \in \Gamma\backslash\Gamma_m} (j(\mathrm{z}_1) - j(\gamma \mathrm{z}_2)), \tag{1.1}$$

where Γ_m consists of matrices in $\mathrm{PGL}_2(\mathbb{Z})$ with determinant being $\pm m$ and is acted on by Γ via multiplication on the left. For example, $\varphi_1(X, Y)$ is simply $X - Y$. A prime that divides $\varphi_m(j(\mathrm{z}_1), j(\mathrm{z}_2))$ for two CM points $\mathrm{z}_1, \mathrm{z}_2$ then gives us a finite field, over which the reductions of E_{z_1} and E_{z_2} are m-isogenous.

Received 10 October 2019, accepted in final form 25 January 2021, published online 29 April 2021.
2020 Mathematics Subject Classification 11F30, 11G15, 14K02 (primary).
Keywords: singular moduli, higher Green's function.
The author is partially supported by the LOEWE research unit USAG.

In this note, we will apply the results by Gross and Zagier [GZ85], Gross, Kohnen and Zagier [GKZ87] and their generalizations by Schofer [Sch09], Bruinier, Kudla and Yang [BKY12] and Bruinier, Ehlen and Yang [BEY19] to prove the following result.

Theorem 1.1. *Let $m \in \mathbb{N}$ and $\mathrm{z}_1, \mathrm{z}_2 \in Y(1)$ be CM points of discriminants $d_1, d_2 < 0$, and H the composite of the ring class fields H_{d_i}. Then the norm of the algebraic integer $\varphi_m(j(\mathrm{z}_1), j(\mathrm{z}_2)) \in H$, if non-zero, satisfies the lower bound*

$$\log |\mathrm{Nm}_{H/\mathbb{Q}}\varphi_m(j(\mathrm{z}_1), j(\mathrm{z}_2))| \geq |Z(W) \cap T_{m,\epsilon}| \cdot Q_2(\cosh(\sqrt{2}\epsilon)) \tag{1.2}$$

for any $\epsilon > 0$. Here Q_{s-1} is the Legendre function of the second kind, $Z(W) \subset \mathcal{F}^2 \subset \mathbb{H}^2$ the set of Galois conjugates of $(\mathrm{z}_1, \mathrm{z}_2)$ (see (3.4) and (3.8)) with $\mathcal{F} \subset \mathbb{H}$ a fundamental domain of $Y(1)$ and $T_{m,\epsilon} \subset \mathbb{H}^2$ the ϵ-neighborhood (with respect to the Riemannian metric on $\mathbb{H}^2$) of the graph of the mth Hecke correspondence.

Remark 1.2. From the definition of Q_{s-1} in (2.1), it is clear that $Q_{s-1}(t)$ is positive and monotonic for $s, t > 1$. In fact, $Q_{s-1}(t) \gg_s t^{-s}\log((t+1)/(t-1))$. We can therefore make ϵ large enough such that $Z(W) \cap T_{m,\epsilon} \neq \emptyset$.

In [BHK20], it was shown that $j(\mathrm{z})$ is not a unit for any CM point $\mathrm{z} \in Y(1)$. This was an improvement over an earlier result for all (ineffectively) large discriminants in [Hab15]. By specializing Theorem 1.1 to $m = 1$ and $d_1 = -3$, we recover the main result of [BHK20]. By allowing both CM points to vary, we actually have the following more general result.

Corollary 1.3. *For any CM points $\mathrm{z}_1, \mathrm{z}_2 \in Y(1)$ and integer $m \geq 1$, the algebraic integer $\varphi_m(j(\mathrm{z}_1), j(\mathrm{z}_2))$ is never a unit.*

Remark 1.4. When one of the discriminants is fixed, the proof in [BHK20] can be adapted to prove the result above. However, this involves eliminating finitely many cases by computer calculation, and it is not clear if the same strategy works with both discriminants varying.

Remark 1.5. By definition, $\varphi_m(j(z_1, z_2)) = 0$ if and only if (z_1, z_2) lies on the graph of the mth Hecke correspondence. In particular, the corollary above implies that $\varphi_m(j(\mathrm{z}), j(\mathrm{z}))$ is never a unit for any CM point $\mathrm{z} \in Y(1)$. Note that this value is not zero as long as m is not a perfect square.

The results in [Hab15, BHK20] originated from a question of Masser, which was motivated by effective results of André–Oort type. As a generalization of Corollary 1.2 in [BHK20], we can deduce the following result from Theorem 1.1.

Corollary 1.6. *Let P be a polynomial in unknowns $X_1, \ldots, X_n$ with coefficients that are algebraic integers in $\mathbb{C}$. If P is divisible by the mth modular polynomial $\varphi_m(X_i, X_j)$ for some $m \geq 1$ and $1 \leq i \leq j \leq n$, then the subvariety in $\mathbb{C}^n$ defined by the equation $P(X_1, \ldots, X_n) = 1$ contains no special points.*

https://doi.org/10.1112/S0010437X21007077 Published online by Cambridge University Press

By the discussion concerning isogenies between elliptic curves, Theorem 1.1 also implies the following result.

Corollary 1.7. *For $i = 1, 2$, let E_i be an elliptic curve with CM by the order $\mathcal{O}_{d_i}$. For any $m \in \mathbb{N}$, there exists a prime $\mathfrak{p}$ of $H_{d_1}H_{d_2}$ such that the reductions of E_1 and E_2 modulo $\mathfrak{p}$ are m-isogenous.*

The idea of the proof of Theorem 1.1 is rather simple. In a nutshell, the result of Gross–Zagier expressed the left-hand side of (1.2) as a finite sum of non-negative quantities. Then a special case of the result of Gross–Kohnen–Zagier expressed the special value of a higher Green's function as a different linear combination of these non-negative quantities. From its definition as a Poincaré series, the higher Green's function clearly never vanishes. This then tells us that these non-negative quantities are not all zero. One can even obtain a bound as in (1.2).

In terms of Arakelov theory, the factorization of Gross–Zagier comes from explicitly calculating the archimedean and non-archimedean contributions to the self-intersection of Heegner points on the modular curve $Y(1)$, which add up to zero. The archimedean part is the negative of the norm of the difference of singular moduli, and the non-archimedean part gives the factorization. In the higher weight case, one would be calculating the self-intersection of Heegner cycles on Kuga–Sato varieties [Zha97]. This is still zero in some cases, and the local contributions are closely related to the case of the modular curve. The advantage though is that the archimedean contribution in the higher weight setting is visibly non-zero from definition. We then use the non-archimedean contributions as a bridge to pass this information to the modular curve case.

Despite its simplicity, this novel idea is rather robust and most of the tools used are available in more general settings. In particular, we hope to apply this idea to study the case of genus 2 and deduce analogues of results in [HP17].

In an earlier version, there were some conditions on the discriminants d_1, d_2, which are inherent in the results of Gross–Zagier and Gross–Kohnen–Zagier. These are now removed by the more general results by Schofer [Sch09], Bruinier, Kudla and Yang [BKY12] and Bruinier, Ehlen and Yang [BEY19]. To apply these more general results, one needs to identify certain toric orbits of CM points with suitable Galois orbits. In the case of singular moduli, this works out nicely when d_1, d_2 are coprime and fundamental (see e.g. § 3.2 of [YY19]). Otherwise, one can use the crucial fact that singular moduli generate ring class fields to still make suitable identifications. This is contained in Proposition 3.3, which is rather interesting and useful by itself, as one can use it to remove the conditions on the discriminants in the result of [GZ85] and prove Conjecture 1.7 in [LV15] (see § 4 of [YY19] for the general strategy).

Another essential ingredient is the non-negativity of Fourier coefficients of certain incoherent Eisenstein series. When d_1, d_2 are coprime and fundamental, these Fourier coefficients were explicitly computed in [GZ85], from which it is clear that they are non-negative. To compute these Fourier coefficients in general, one needs to evaluate certain local Whittaker integrals. Very general results in this regard have just become available in [YYY21], which we use here to deduce the non-negativity in Proposition 3.1.

Note that it is crucial that $\varphi_m(j(z_1), j(z_2))$ is the Borcherds product associated to a modular function whose principal part Fourier coefficients are all non-negative. For other modular functions where this is not satisfied, it is very much possible that their CM values are algebraic, integral units (even very often [YYY21]).

2. Higher Green's function

The function $\log|j(z_1)-j(z_2)|^2$ is the Green's function for the diagonal on two copies of the modular curve. In [GKZ87], higher Green's functions were studied. For $\Re(s)>1$, let

$$Q_{s-1}(t) := \int_0^\infty (t+\sqrt{t^2-1}\cosh v)^{-s}\,dv, \quad t>1 \tag{2.1}$$

be the Legendre function of the second kind, which satisfies the ordinary differential equation

$$(1-t^2)Q''(t) - 2tQ'(t) + s(s-1)Q(t) = 0.$$

From the definition above, we see that for any fixed $s\in(1,\infty)$, the function $Q_{s-1}(t)$ is positive and monotonically decreasing for $t\in(1,\infty)$. Define a function g_s on $\mathbb{H}^2$ by

$$g_s(z_1,z_2) := -2Q_{s-1}(\cosh d(z_1,z_2)) = -2Q_{s-1}\left(1+\frac{|z_1-z_2|^2}{2y_1y_2}\right) \tag{2.2}$$

for $(z_1,z_2)\in\mathbb{H}^2$ with $d(z_1,z_2)$ the hyperbolic distance between z_1 and z_2. By averaging over the $\Gamma(=\mathrm{PSL}_2(\mathbb{Z}))$-translates of the second variable, we obtain a function

$$G_s(z_1,z_2) := \sum_{\gamma\in\Gamma} g_s(z_1,\gamma z_2) \tag{2.3}$$

on $Y(1)^2$ symmetric in z_1 and z_2. Easy estimates show that the sum converges absolutely and uniformly on compact subsets of $\mathbb{H}^2$ when $\Re(s)>1$. In that case, $G_s(z_1,z_2)$ is an eigenfunction of the hyperbolic Laplacian $\Delta_{z_i,0}$ with eigenvalue $s(1-s)$, where

$$\begin{aligned} R_{z,\kappa} &:= 2i\partial_z + \frac{\kappa}{y}, \quad L_{z,\kappa} := -2iy^2\partial_{\bar z},\\ \Delta_{z,\kappa} &:= -R_{z,\kappa-2}L_{z,\kappa} = -L_{z,\kappa+2}R_{z,\kappa} - \kappa = -y^2(\partial_x^2+\partial_y^2) + i\kappa y(\partial_x + i\partial_y) \end{aligned} \tag{2.4}$$

for $\kappa\in\mathbb{Z}$. When $s=1$, the sum in (2.3) does not converge absolutely any more. One can however analytically continue $G_s(z_1,z_2)$ to $s=1$, where it will have a pole. After eliminating the pole using the real-analytic Eisenstein series, one obtains the function $2\log|j(z_1)-j(z_2)|$ (see [GZ85, Proposition 5.1]). Therefore, we will define

$$G_1(z_1,z_2) := -2\log|j(z_1)-j(z_2)| \tag{2.5}$$

for convenience later.

When $s=k\geq 1$ is an integer, the Legendre function $Q_{k-1}(t)$ has the form

$$Q_{k-1}(t) = \frac{P_{k-1}(t)}{2}\log\frac{t+1}{t-1} - R_{k-1}(t), \tag{2.6}$$

where $P_{k-1}(t)$ is the $(k-1)$st Legendre polynomial given by

$$P_{k-1}(t) = \frac{1}{2^{k-1}}\sum_{\ell=0}^{k-1}\binom{k-1}{\ell}^2 (t-1)^{k-1-\ell}(t+1)^\ell \tag{2.7}$$

https://doi.org/10.1112/S0010437X21007077 Published online by Cambridge University Press

and $R_{k-1}(t)$ is a unique polynomial. For $k = 1, 3, 5, 7$, they are given by

$$\begin{aligned}
P_0(t) &= 1, \quad R_0(t) = 0, \\
P_2(t) &= \frac{3t^2-1}{2}, \quad R_2(t) = \frac{3}{2}t, \\
P_4(t) &= \frac{35t^4-30t^2+3}{8}, \quad R_4(t) = \frac{35}{8}t^3 - \frac{55}{24}t, \\
P_6(t) &= \frac{231t^6-315t^4+105t^2-5}{16}, \quad R_6(t) = \frac{231}{16}t^5 - \frac{119}{8}t^3 + \frac{231}{80}.
\end{aligned} \tag{2.8}$$

For $m \in \mathbb{N}$, we can let the mth Hecke operator T_m act on one of z_1 and z_2 to define

$$G_k^m(z_1, z_2) := \sum_{\gamma \in \Gamma \backslash \Gamma_m} G_k(z_1, \gamma z_2). \tag{2.9}$$

It has a logarithmic singularity along the divisor

$$\mathcal{T}_m := \{(z, \gamma \cdot z) : z \in Y(1),\ \gamma \in \Gamma_m\} \subset Y(1)^2. \tag{2.10}$$

Given any weakly holomorphic modular form $f \in M^!_{2-2k}$ with the Fourier expansion $f(\tau) = \sum_{1 \le m \le m_0} c_f(-m)q^{-m} + O(1)$, we can define

$$G_f(z_1, z_2) := \sum_{1 \le m \le m_0} c_f(-m) m^{k-1} G_k^m(z_1, z_2). \tag{2.11}$$

Note that if $k = 1$ and $f(\tau) = J_m(\tau) := q^{-m} + O(q)$ is the unique modular function in $M^!_0$, then $G_f(z_1, z_2) = -2\log|\varphi_m(j(z_1), j(z_2))|$. In this case, Borcherds showed that $\log|G_f|$ is the regularized theta lift of f [Bor98]. The extension of this result to all $k \ge 1$ can be stated as follows (see [Bru02, Via11]).

Proposition 2.1 [Li18, Proposition 4.2]. *For an integer $r \ge 0$ and $\tau = u + iv \in \mathbb{H}$, denote $R^r_{\tau,\kappa} := R_{\tau,\kappa+2r-2} \circ R_{\tau,\kappa+2r-4} \circ \cdots \circ R_{\tau,\kappa}$. Then, for any $f \in M^!_{2-2k}$ and $z_1, z_2 \in \mathbb{H}$ with $k \ge 1$, we have*

$$G_f(z_1, z_2) = (4\pi)^{1-k} \lim_{T \to \infty} \int_{\mathcal{F}_T} f(\tau)(R^{k-1}_{\tau,0}\Theta_L)(\tau; z_1, z_2)\, d\mu(\tau), \tag{2.12}$$

where $\mathcal{F}_T := \mathcal{F} \cap \{\tau = u + iv \in \mathbb{H} : v \le T\}$ is the truncated fundamental domain and $\Theta_L(\tau; z_1, z_2)$ is the Siegel theta function associated to the unimodular lattice $L = M_2(\mathbb{Z})$. Furthermore, G_f has a logarithmic singularity along $\mathcal{T}_m$ for $m \ge 1$ if and only if $c_f(-m) \ne 0$.

To evaluate the theta integral above, one can try to find a preimage of $R^{k-1}_{\tau,2-2k}\Theta_L$ under the lowering operator L_τ. This is possible for $k = 1$ when one averages over a suitable toric orbit of CM points $(\mathrm{z}_1, \mathrm{z}_2)$, as we will see in the next section. For odd $k \ge 2$, one can apply the following operator $\mathcal{C}_{k-1}$ due to Cohen to obtain the desired preimage.

Proposition 2.2. *For a real-analytic function $F(\tau)$ on $\mathbb{H}^2$, suppose that there exists a real-analytic function $\tilde{F}(\tau_1, \tau_2)$ on $\mathbb{H}^2$ such that it is harmonic in τ_1, τ_2 and satisfies*

$$L_\tau \tilde{F}(\tau, \tau) = F(\tau).$$

https://doi.org/10.1112/S0010437X21007077 Published online by Cambridge University Press

Then, for $k \geq 1$ odd, we have $L_\tau(\mathcal{C}_{k-1}\tilde{F}) = (4\pi)^{1-k} R_{\tau,0}^{k-1}(F)$, where $\mathcal{C}_{k-1}$ is the Cohen operator defined by

$$(\mathcal{C}_{k-1}\tilde{F})(\tau) := (-2\pi i)^{1-k} \sum_{\ell=0}^{k-1} (-1)^\ell \binom{k-1}{\ell}^2 \partial_{\tau_1}^\ell \partial_{\tau_2}^{k-1-\ell} \tilde{F}(\tau_1, \tau_2)|_{\tau_1=\tau_2=\tau}$$

and satisfies

$$(\mathcal{C}_{k-1}\tilde{F})|_{2k}\gamma = \mathcal{C}_{k-1}(\tilde{F}|_{(1,1)}(\gamma, \gamma))$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{R})$.

Remark 2.3. The definition above is unchanged if one replaces $\partial_{\tau_i}^r$ with $(2i)^{-r} R_{\tau_i,1}^r$. The proposition can then be checked easily using (2.4).

With some technical conditions on the discriminants, the averaged value of $G_f(z_1, z_2)$ at CM points was studied in [GKZ87], extending results in [GZ85, GZ86]. As in the case $k = 1$, these averaged values are logarithms of rational numbers, which can be factored explicitly. In view of [BKY12], these results can be put in the framework of arithmetic intersection on a Hilbert modular surface and the technical conditions can be removed. We will recall these results in the next section.

3. Theorems of Gross–Zagier, Gross–Kohnen–Zagier and their generalizations

Let $\mathrm{z} \in Y(1)$ be a CM point of discriminant $d < 0$ and $\mathcal{O}_d, K, H_d$ as in the introduction. For an element t in the finite ideles $\mathbb{A}_{K,f}^\times$ of K, let $\sigma_t \in \mathrm{Gal}(H_d/K)$ be the associated element via the Artin map. Then σ_t acts naturally on z.

Given two CM points $\mathrm{z}_i \in Y(1)$ with discriminant d_i, one can realize $(\mathrm{z}_1, \mathrm{z}_2) \in Y(1)^2$ as small/big CM points (depending on whether $D := d_1 d_2$ is a perfect square or not) in the sense of [Sch09, BKY12]. The averaged values of G_f at these CM points can be expressed in terms of Fourier coefficients of incoherent Eisenstein series. In this section, we will recall these results.

3.1 Incoherent Eisenstein series

First, we quickly recall the incoherent Eisenstein series in the sense of Kudla [Kud97] (see e.g. § 4 of [BKY12]). Let F be a totally real field of degree n_0, E/F a quadratic, CM extension and $W = E$ an F-quadratic space with quadratic form $Q_F(x) = \alpha x \bar{x}$ for some $\alpha \in F^\times$. Denote by $\{\sigma_j : 1 \leq j \leq n_0\}$ the real embeddings of F and (V, Q) the restriction of scalar of (W, Q_F) to $\mathbb{Q}$. If α is chosen such that $\sigma_{n_0}(\alpha) < 0$ and $\sigma_j(\alpha) > 0$ for $1 \leq j \leq n_0 - 1$, then (V, Q) has signature $(n, 2)$.

Fix an additive adelic character ψ of $\mathbb{Q}$ and denote $\psi_F = \psi \circ \mathrm{Tr}_{F/\mathbb{Q}}$. Associate to it is a Weil representation $\omega = \omega_{\psi_F}$ of $\mathrm{SL}_2(\mathbb{A}_F)$ on $S(W(\mathbb{A}_F)) = S(V(\mathbb{A}_\mathbb{Q}))$. Let χ be the quadratic Hecke character associated to E/F. For any element Φ in the principal series representation $I(s, \chi)$, one can define a Hilbert Eisenstein series

$$E(g, s, \Phi) := \sum_{\gamma \in B_F \backslash \mathrm{SL}_2(F)} \Phi(\gamma g, s), \quad g \in \mathrm{SL}_2(\mathbb{A}_F)$$

with $\Re(s) \gg 0$ and analytically continue it to $s \in \mathbb{C}$. At the infinite places, we choose Φ to be the unique eigenvector of $\mathrm{SL}_2(\mathbb{R})$ of weight 1.

https://doi.org/10.1112/S0010437X21007077 Published online by Cambridge University Press

At the finite places, one can use any $\phi \in S(V(\mathbb{A}_{\mathbb{Q},f}))$ to construct a section. Using this information, we can define a Hilbert Eisenstein series $E(\vec{\tau}, s, \phi)$, which is a real-analytic Hilbert modular form of parallel weight 1 in $\vec{\tau} = (\tau_1, \ldots, \tau_{n_0}) \in \mathbb{H}^{n_0}$ (see (4.4) in [BKY12]). We can further normalize it by

$$E^*(\vec{\tau}, s, \phi) = \Lambda(s+1, \chi) E(\vec{\tau}, s, \phi)$$

with $\Lambda(s, \chi)$ the completed L-function associated to χ. Usually, we will take $\phi = \phi_\mu$ the characteristic function of $(L+\mu) \otimes \hat{\mathbb{Z}}$ for $\mu \in L^\vee/L$ and $L \subset V$ some even integral lattice. In that case, we use $E^*(\vec{\tau}, s, L)$ to denote the vector-valued modular form $\sum_{\mu \in L^\vee/L} E^*(\vec{\tau}, s, \phi_\mu)\mathfrak{e}_\mu$.

Because of the choice of the section at the infinite places, this Eisenstein series is incoherent in the sense of Kudla [Kud97] and vanishes at $s = 0$. Its derivative at $s = 0$ is a real-analytic Hilbert modular form of parallel weight 1. For a totally positive $t \in F^\times$, denote its tth Fourier coefficient by $a(t, \phi)$. By Proposition 4.6 in [BKY12], one can write

$$a_m(\phi) := \sum_{t \in S_m} a(t, \phi) = \sum_p a_{m,p}(\phi) \log p \tag{3.1}$$

with $S_m := \{t \in F^\times : t \gg 0, \mathrm{Tr}(t) = m\}$ and $a_{m,p}(\phi)$ in the subfield of $\mathbb{C}$ generated by the values $\phi(x)$ for $x \in V(\mathbb{A}_{\mathbb{Q},f})$. When $\phi = \otimes \phi_{\mathfrak{p}}$ is factorizable, this is given by product of values of local Whittaker functions, which have been explicitly calculated in many cases (see e.g. [Yan05, KY10, YYY21]). Using these explicit formulas, we can say something more refined about $a(t, \phi)$ when $\phi = \phi_\mu$.

Proposition 3.1. *For any even, integral lattice $L \subset V$ and $\mu \in L^\vee/L$, we have*

$$-a(t, \phi_\mu) \geq 0 \tag{3.2}$$

whenever $t \in F^\times$ is totally positive. Furthermore, for any $m > 0$, the coefficient $a(t, \phi_\mu) = 0$ for all but finitely many $t \in S_m$.

Proof. Since any two lattices in V of full rank are commensurable, we can find a positive integer c such that $c\mathcal{O}_E$ is a sublattice of L and $c\mathcal{O}_E \subset L \subset L^\vee \subset (c\mathcal{O}_E)^\vee$. Let $\varpi : L^\vee/c\mathcal{O}_E \to L^\vee/L$ be the natural projection. We can then write

$$\phi_\mu = \sum_{\mu' \in L^\vee/c\mathcal{O}_E,\ \varpi(\mu') = \mu} \phi'_{\mu'}$$

with $\phi'_{\mu'}$ the characteristic function of $c\mathcal{O}_E + \mu'$. Therefore, it suffices to prove the claim for $L = c\mathcal{O}_E$. After scaling, we can then suppose that $L = \mathcal{O}_E$ and $0 \neq \alpha \in \mathcal{O}_F$, as L is even integral. Then the section $\phi = \phi_\mu = \otimes \phi_{\mathfrak{p}}$ is factorizable and the coefficient $a(t, \phi)$ is given by

$$a(t, \phi) = -2^{n_0} \frac{W^{*,\prime}_{t,\mathfrak{p}_0}(0, \phi_{\mathfrak{p}_0})}{\gamma(W_{\mathfrak{p}_0})} \prod_{\mathfrak{p} \nmid \mathfrak{p}_0 \infty} \frac{W^*_{t,\mathfrak{p}}(0, \phi_{\mathfrak{p}})}{\gamma(W_{\mathfrak{p}})}$$

if the 'Diff' set of Kudla $\mathrm{Diff}(W, t)$ consists of just a finite prime $\mathfrak{p}_0$ and vanishes otherwise (see Proposition 2.7 in [YY19]). Here $W^*_{t,\mathfrak{p}}(s, \phi)$ is the normalized local Whittaker function (see (2.25) in [YY19]) and $\gamma(W_{\mathfrak{p}})$ is the local Weil index. These local Whittaker functions have been

https://doi.org/10.1112/S0010437X21007077 Published online by Cambridge University Press

computed explicitly in all cases in the appendix of [YYY21], from which we know that

$$\frac{W^*_{t,\mathfrak{p}}(0,\phi_{\mathfrak{p}})}{\gamma(W_{\mathfrak{p}})} \geq 0, \quad \frac{W^{*,\prime}_{t,\mathfrak{p}}(0,\phi_{\mathfrak{p}})}{\gamma(W_{\mathfrak{p}})} \geq 0$$

for all totally positive $t \in F^\times$. Furthermore, there is a positive integer b depending on L, E, F such that $a(t,\phi) = 0$ for all $t \notin b^{-1}\mathcal{O}_F$. This proves the second claim. □

3.2 Big CM points

When D is not a perfect square, $F = \mathbb{Q}(\sqrt{D})$ is a real quadratic field. For $i = 1, 2$, let $z_i \in \mathbb{H} \cap K_1K_2$ be a representative of z_i and $\mathfrak{a}_i := \mathbb{Z} + \mathbb{Z}z_i \subset K_i$ the corresponding $\mathbb{Z}$-module. Denote by $W = K_1K_2$ the F-quadratic space with the quadratic form $Q_F(x) = x\bar{x}/\sqrt{D}$. We can identify (V, Q) with the rational quadratic space $(M_2(\mathbb{Q}), \det)$ via the map

$$\sum_{i=1}^4 x_ie_i \in V \mapsto \begin{pmatrix} x_3 & x_1 \\ x_4 & x_2 \end{pmatrix} \in M_2(\mathbb{Q}),$$

$$e_1 := 1, \quad e_2 := -\overline{z_1}, \quad e_3 := z_2, \quad e_4 := -\overline{z_1}z_2,$$

under which $\overline{\mathfrak{a}_1}\mathfrak{a}_2$ is mapped to the unimodular lattice $L = M_2(\mathbb{Z})$. Define a torus

$$T(R) := \{(t_1, t_2) \in (K_1 \otimes_{\mathbb{Q}} R)^\times \times (K_2 \otimes_{\mathbb{Q}} R)^\times : t_1\overline{t_1} = t_2\overline{t_2}\}$$

for any $\mathbb{Q}$-algebra R. It embeds into the algebraic group

$$\mathrm{GSpin_V}(R) \cong \mathrm{H}(R) := \{(g_1, g_2) \in \mathrm{GL}_2(R) \times \mathrm{GL}_2(R) : \det(g_1) = \det(g_2)\} \tag{3.3}$$

via the map $\iota = (\iota_1, \iota_2) : T \to \mathrm{H}$, where

$$(e_1, e_2)\iota_1(r) = (re_1, re_2), \quad \iota_2(r)(e_3, e_1)^t = (\bar{r}e_3, \bar{r}e_1)^t.$$

Denote by $K_T := \iota^{-1}(\mathrm{H}(\hat{\mathbb{Z}})) \subset T(\mathbb{A}_f)$ a compact subgroup preserving the torus and $K_{T,i} := \iota_i^{-1}(\mathrm{H}(\hat{\mathbb{Z}})) \subset \mathbb{A}^\times_{K_i,f}$ for $i = 1, 2$. Then $K_{T,i} = \hat{\mathcal{O}}_{d_i}$ and $K_i^\times \backslash \mathbb{A}^\times_{K_i,f}/K_{T,i}$ is just the class group of the order $\mathcal{O}_{d_i}$, which we denote by $\mathrm{Cl}(d_i)$. As in Lemma 3.5 in [YY19], there is a natural injection

$$p' : T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_T \to \mathrm{Cl}(d_1) \times \mathrm{Cl}(d_2)$$

sending $[(t_1, t_2)]$ to $([t_1], [t_2])$. On the other hand, $H = H_1H_2$ implies that the natural map

$$p'' : \mathrm{Gal}(H/\mathbb{Q}) \to \mathrm{Gal}(H_1/\mathbb{Q}) \times \mathrm{Gal}(H_2/\mathbb{Q})$$

sending σ to $(\sigma|_{H_1}, \sigma|_{H_2})$ is injective. Since H_j/K_j is a ring class field, the Galois group $\mathrm{Gal}(H_j/\mathbb{Q})$ is a generalized dihedral group. This gives us the following result.

Lemma 3.2 [Coh85, Theorem 8.3.12]. *Let $H_0 := H_1 \cap H_2 \subset H$ be the intersection of ring class fields. Then $H_0/\mathbb{Q}$ is abelian.*

Proof. We collect the proof from [Coh85] here. By replacing H_0 with $H_0K_1K_2$, we can suppose that H_0 contains K_1K_2. Clearly, $H_0 \subset H_j$ is abelian over K_j for $j = 1, 2$. Therefore, it is also abelian over K_1K_2. Since $\mathrm{Gal}(H_0/\mathbb{Q})$ is a quotient of $\mathrm{Gal}(H_j/\mathbb{Q})$, which is generalized dihedral, we can find $\sigma_j \in \mathrm{Gal}(H_0/\mathbb{Q})$ of order 2 such that $\sigma_jh\sigma_j^{-1} = h^{-1}$ for all $h \in \mathrm{Gal}(H_0/K_j)$. That

https://doi.org/10.1112/S0010437X21007077 Published online by Cambridge University Press

means that $\mathrm{Gal}(H_0/\mathbb{Q})$ is generated by the abelian group $\mathrm{Gal}(H_0/K_1K_2)$ and the elements σ_1, σ_2. Since $\mathrm{Gal}(H_0/K_j)$ is abelian, we know that σ_j commutes with $\mathrm{Gal}(H_0/K_1K_2)$. But σ_1, σ_2 also commute since

$$\sigma_1\sigma_2\sigma_1^{-1} = \sigma_2^{-1} = \sigma_2.$$

This finishes the proof. □

After identifying $\mathrm{Cl}(d_i)$ with $\mathrm{Gal}(H_i/K_i) \subset \mathrm{Gal}(H_i/\mathbb{Q})$ via the Artin map, we can now state the following analogue of Lemma 3.8 of [YY19].

PROPOSITION 3.3. *Using the notation introduced above, the image of p' is contained in the image of p'', and $p''^{-1} \circ p' : T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_T \to \mathrm{Gal}(H/K_1K_2)$ is an isomorphism.*

Remark 3.4. When $(d_1, d_2) = 1$, the map p'' is an isomorphism and we recover Lemma 3.8 in [YY19].

Proof. The image of p'' is exactly given by

$$p''(\mathrm{Gal}(H/K_1K_2)) = \{(\sigma_1, \sigma_2) \in \mathrm{Gal}(H_1/K_1) \times \mathrm{Gal}(H_2/K_2) : \sigma_1|_{H_0} = \sigma_2|_{H_0}\}.$$

On the other hand, if $t_i \in K_i^\times\backslash\mathbb{A}_{K_i,f}^\times/K_{T,i}$ is associated to $\sigma_i \in \mathrm{Gal}(K_i^{\mathrm{ab}}/K_i)$ via the Artin map, then $t_i\overline{t_i} = \mathrm{Nm}_{K_i/\mathbb{Q}}(t_i)$ is associated to $\mathrm{res}(\sigma_i) \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ with $\mathrm{res} : \mathrm{Gal}(K_i^{\mathrm{ab}}/K_i) \to \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ the natural restriction map. So the image of p' is

$$\mathrm{Im}(p') = \{(\sigma_1, \sigma_2) \in \mathrm{Gal}(H_1/K_1) \times \mathrm{Gal}(H_2/K_2) : \sigma_1|_{\mathbb{Q}^{\mathrm{ab}}\cap H_0} = \sigma_2|_{\mathbb{Q}^{\mathrm{ab}}\cap H_0}\}.$$

Since $H_0/\mathbb{Q}$ is abelian by Lemma 3.2, this finishes the proof. □

Now let $Z(W)$ be the big CM point on $Y(1)^2$ associated to W as in [BKY12]. Then the argument in § 3 of [YY19] with the proposition above immediately implies that

$$Z(W) = \sum_{\sigma \in \mathrm{Gal}(H/K_1K_2)} (\mathfrak{z}_1, \mathfrak{z}_2)^\sigma + (-\overline{\mathfrak{z}_1}, \mathfrak{z}_2)^\sigma + (\mathfrak{z}_1, -\overline{\mathfrak{z}_2})^\sigma + (-\overline{\mathfrak{z}_1}, -\overline{\mathfrak{z}_2})^\sigma. \tag{3.4}$$

Now the value at $Z(W)$ of the higher Green's function G_f with $f \in M^!_{2-2k}$ and $k \geq 1$ odd can be explicitly given as a finite sum of Fourier coefficients of certain incoherent Eisenstein series. We state it as follows.

THEOREM 3.5. *Let $f \in M^!_{2-2k}$ with $k \geq 1$ odd and vanishing constant term if $k = 1$. Suppose that d_1d_2 is not a perfect square. Then*

$$\begin{aligned} G_f(Z(W)) &= \frac{|Z(W)|}{2\Lambda(0,\chi)} \sum_{m \geq 1} c_f(-m) a_m(\phi; k), \\ a_m(\phi; k) &:= m^{k-1} \sum_{t \in S_m} P_{k-1}\left(\frac{t - t'}{m}\right) a(t, \phi), \end{aligned} \tag{3.5}$$

where $a(t, \phi)$ is the tth Fourier coefficient of the holomorphic part of the incoherent Eisenstein series $E^{*,\prime}((\tau_1, \tau_2), 0, \phi)$ of parallel weight $(1, 1)$ with $\phi = \phi_{\mathfrak{a}_1,\mathfrak{a}_2} \in S(V \otimes \mathbb{A}_f) = S(W_\mathbb{Q} \otimes \mathbb{A}_f)$ the characteristic function of $\overline{\mathfrak{a}_1}\mathfrak{a}_2 \otimes \hat{\mathbb{Z}}$ and $\Lambda(s, \chi)$ the completed L-function.

https://doi.org/10.1112/S0010437X21007077 Published online by Cambridge University Press

Remark 3.6. The constant $|Z(W)|/2\Lambda(0,\chi)$ is explicitly given by $w_1w_2[H_1:K_1][H_2:K_2]/2h_1h_2[H_0:\mathbb{Q}]$ with $w_i := |\mathcal{O}_{K_i}^\times|$, h_i the class number of K_i and $H_0 := H_1\cap H_2$.

Remark 3.7. For any $\gamma\in\Gamma_m$ and CM point $\mathrm{z}\in Y(1)$ of discriminant d, the CM point $\gamma\cdot\mathrm{z}\in Y(1)$ has discriminant d' such that dd' is a perfect square. Since the singularity of G_f is supported on $\mathcal{T}_m\subset Y(1)^2$, it does not intersect the CM cycle $Z(W)$.

Proof. When $k=1$, this is just Theorem 5.2 in [BKY12]. When $k\geq 2$, this can be easily modified using the Cohen operator in Proposition 2.2 (see e.g. Theorem 5.10 in [BEY19]). For the convenience of the reader, we include some details here. By applying Lemma 4.3 and Proposition 4.5 in [BKY12] and Proposition 2.2, we obtain

$$(4\pi)^{1-k}\Theta_L(\tau;Z(W)) = -\frac{|Z(W)|}{2\Lambda(0,\chi)}L_\tau\mathcal{C}_{k-1}(E^{*,\prime}((\tau_1,\tau_2),0,\phi))$$

for any $k\geq 1$ odd. Substituting this into (2.12) and applying Stokes' theorem yields (cf. proof of Theorem 5.2 in [BKY12] and Theorem 5.10 in [BEY19])

$$G_f(Z(W)) = -\frac{|Z(W)|}{2\Lambda(0,\chi)}\lim_{T\to\infty}\int_{\mathcal{F}_T} d(f\cdot\mathcal{C}_{k-1}(E^{*,\prime})\,d\tau) = \frac{|Z(W)|}{2}\mathrm{CT}(f\mathcal{C}_{k-1}(\mathcal{E}_L^+)),$$

where

$$\mathcal{E}_L^+(\tau_1,\tau_2) = \frac{1}{\Lambda(0,\chi)}\sum_{m\in\mathbb{N},\ t\in S_m} a(t,\phi)\mathbf{e}(t\tau_1+t'\tau_2)$$

is the holomorphic part of $E'((\tau_1,\tau_2),0,\phi) = E^{*,\prime}((\tau_1,\tau_2),0,\phi)/\Lambda(0,\chi)$. We are now done after applying the identity

$$\begin{aligned}\mathcal{C}_{k-1}\mathbf{e}(t\tau_1+t'\tau_2) &= \left(\sum_{\ell=0}^{k-1}(-1)^\ell\binom{k-1}{\ell}^2 t^\ell(t')^{k-1-\ell}\right)\mathbf{e}((t+t')\tau)\\ &= P_{k-1}\left(\frac{t-t'}{t+t'}\right)(t+t')^{k-1}\mathbf{e}((t+t')\tau),\end{aligned}$$

which comes from the definitions of $\mathcal{C}_{k-1}$ in Proposition 2.2, P_{k-1} in (2.7) and the fact that $k-1$ is even. □

3.3 Small CM points

Suppose that $D=d_1d_2$ is a perfect square. Then K_1 and K_2 are the same quadratic field K of fundamental discriminant $d<0$. Let $z_i = x_i+iy_i\in\mathbb{H}\cap K$ be a representative of z_i. Then $y_1y_2\in\mathbb{Q}$ and the CM point (z_1,z_2) arises from a rational splitting of $(V,Q)=(M_2(\mathbb{Q}),\det)$. To be more precise, let $W\subset V$ be the rational, negative 2-plane spanned by the rational vectors $\Re Z(z_1,z_2), \sqrt{|d|}\Im Z(z_1,z_2)\in V$, where

$$Z(z_1,z_2) := \begin{pmatrix} z_1 & -z_1z_2\\ 1 & -z_2\end{pmatrix}\in V(\mathbb{C}). \tag{3.6}$$

Then the element $W\otimes\mathbb{R}$ in the Grassmannian of $V(\mathbb{R})$ corresponds to the points $z_0^+ := (z_1,z_2)\in\mathbb{H}^2$ and $z_0^- := (\overline{z_1},\overline{z_2})\in(\mathbb{H}^-)^2$.

On the level of lattice, denote $L = M_2(\mathbb{Z}) \subset V$ and consider a finite index sublattice

$$L_0 := L_+ \oplus L_- \subset L, \quad L_+ := L \cap W^\perp, \quad L_- := L \cap W.$$

The holomorphic theta function

$$\theta_{L_+}(\tau) := \sum_{\mu_1 \in L_+^\vee / L_+} \mathfrak{e}_{\mu_1} \theta_{L_+, \mu_1}(\tau), \quad \theta_{L_+,\mu_1}(\tau) := \sum_{\lambda \in L_+ + \mu_1} q^{Q(\lambda)}$$

is a vector-valued holomorphic modular form of weight 1 with respect to the Weil representation ρ_{L_+}. On the other hand, the incoherent Eisenstein series

$$E^{*,\prime}(\tau, 0, L_-) = \sum_{\mu_2 \in L_-^\vee / L_-} E^{*,\prime}(\tau, 0, \phi_{\mu_2}) \mathfrak{e}_{\mu_2}$$

is a real-analytic, elliptic modular form of weight 1 with respect to the Weil representation ρ_{L_-}. Their tensor product is a real-analytic modular form of weight 2 with respect to ρ_{L_0}. The following function on $\mathbb{H}^2$,

$$\tilde{F}(\tau_1, \tau_2; L_+, L_-) := \sum_{\mu = (\mu_1, \mu_2) \in L/L_0 \subset L_0^\vee / L_0} \theta_{L_+, \mu_1}(\tau) E^{*,\prime}(\tau, 0, \phi_{\mu_2}), \tag{3.7}$$

satisfies $\tilde{F}|_{1,1}(\gamma, \gamma) = \tilde{F}$ for all $\gamma \in \Gamma$.

The torus T_W, whose R-points are $(R \otimes K)^\times$ for any $\mathbb{Q}$-algebra R, is embedded into the algebraic group H defined in (3.3) through the map $\iota = (\iota_1, \iota_2) : T_W \hookrightarrow \mathrm{H}$ defined by

$$\iota_i\left(a + b\frac{y_1 y_2}{\sqrt{d}}\right) := a \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + (-1)^{i+1} b \frac{y_{3-i}}{\sqrt{|d|}} \begin{pmatrix} x_i & -|z_i|^2 \\ 1 & -x_i \end{pmatrix}$$

for $i = 1, 2$. Simple routine calculations then show that $\iota^{-1}(\mathrm{H}(\mathbb{Z})) = \mathcal{O}_{d'} \subset K$ with $d' := -d_1 d_2/(d_1, d_2)^2 < 0$ the largest negative discriminant divisible by d_1 and d_2. The class group $T_W(\mathbb{Q}) \backslash T_W(\mathbb{A}_f)/\hat{\mathcal{O}}_{d'}$ is just the class group of the order $\mathcal{O}_{d'}$. Therefore, the small CM 0-cycle $Z(W) := T_W(\mathbb{Q}) \backslash (\{z_0^\pm\} \times T_W(\mathbb{A}_f)/\hat{\mathcal{O}}_{d'})$ on $Y(1)^2$ becomes

$$Z(W) = \sum_{\sigma \in \mathrm{Cl}(d')} (\mathrm{z}_1, \mathrm{z}_2)^\sigma + (-\overline{\mathrm{z}_1}, -\overline{\mathrm{z}_2})^\sigma. \tag{3.8}$$

We can now apply Theorem 1.1 in [Sch09] to give a formula for $G_f(Z(W))$ when $f \in M_0^!$ and generalize it to a higher Green's function as in Theorem 3.5.

Theorem 3.8. *Let $f \in M_{2-2k}^!$ with $k \geq 1$ odd and vanishing constant term if $k = 1$. Suppose that the singularity of G_f does not intersect $Z(W)$. Then*

$$G_f(Z(W)) = \frac{|Z(W)|}{\Lambda(0, \chi)} \sum_{m \geq 1} c_f(-m) \kappa(m; k), \tag{3.9}$$

where $\kappa(m; k)$ is the mth Fourier coefficient of $\mathcal{C}_{k-1} \tilde{F}(\tau_1, \tau_2; L_+, L_-)$.

By Remark 3.7, it is possible for $Z(W)$ to intersect $\mathcal{T}_m$. We will give a simple criterion to see when this happens.

https://doi.org/10.1112/S0010437X21007077 Published online by Cambridge University Press

Lemma 3.9. *For $m \geq 1$, the CM cycle $Z(W)$ intersects the divisor $\mathcal{T}_m$ if and only if there exists $\lambda \in L_+$ such that $Q(\lambda) = m$, i.e. the mth Fourier coefficient of $\theta_{L_+,0}(\tau)$ is positive.*

Proof. The divisor $\mathcal{T}_m \subset Y(1)^2$ is an example of a special cycle. In particular, its preimage in $\mathbb{H}^2$ is the following $\Gamma \times \Gamma$-invariant set:

$$\left\{(z_1', z_2') \in \mathbb{H}^2 : \text{ there is } \lambda \in L \text{ such that } Z(z_1', z_2') \perp \lambda \text{ and } \det(\lambda) = m\right\}.$$

Note that if $Y(1)^2$ is replaced by a Hilbert modular surface, then the analogue of $\mathcal{T}_m$ is the Hirzebruch–Zagier divisor. From this description, we know that $Z(W)$ intersects $\mathcal{T}_m$ if and only if there exists $\lambda \in L$ satisfying $Q(\lambda) = m$ and $\lambda \perp Z(z_1, z_2)$, i.e. $\lambda \in W^\perp \cap L = L_+$. The lemma is now clear. □

We can now give an explicit expression for $\kappa(m; k)$.

Proposition 3.10. *Suppose that the (n_1)th Fourier coefficient of $\theta_{L_+,\mu_1}(\tau)$ is $b(n_1, \mu_1)$ for $n_1 \geq 0, \mu_1 \in L_+^\vee/L_+$. Then $\kappa(m; k)$ is given by*

$$\kappa(m;k) = m^{k-1}\Bigg(b(m,0)a(0,\phi_0)P_{k-1}(1) + \sum_{\substack{\mu=(\mu_1,\mu_2)\in L/L_0 \\ n_1\geq 0,\ n_2>0 \\ n_1+n_2=m}} b(n_1,\mu_1)a(n_2,\phi_{\mu_2})P_{k-1}\left(\frac{n_1-n_2}{m}\right)\Bigg),$$

where $a(n_2, \phi_{\mu_2})$ is the (n_2)th Fourier coefficient of the incoherent Eisenstein series $E^{*,\prime}(\tau, 0, \phi_{\mu_2})$. Furthermore, $c_f(-m)b(m, 0) = 0$ for all $m \geq 1$ if and only if the singularity of G_f does not intersect $Z(W)$.

Proof. From the definition of $\mathcal{C}_{k-1}$ and (3.7), one can derive the formula for κ with sum over $n_i \geq 0$. If $(n_1, n_2) = (m, 0)$, then we know from Definition 2.16 in [Sch09] that $a(0, \phi_{\mu_2}) = 0$ for all $\mu_2 \neq 0 \in L_-^\vee/L_-$. This proves the first claim. The second part follows from Proposition 2.1 and Lemma 3.9. □

4. Proof of Theorem 1.1

When d_1d_2 is not a perfect square, Proposition 3.3 and Theorem 3.5 imply that

$$\begin{aligned}
&2\log \mathrm{Nm}_{H/\mathbb{Q}}\varphi_m(j(\mathrm{z}_1) - j(\mathrm{z}_2)) \\
&\quad = 2\log \mathrm{Nm}_{H/K}(\varphi_m(j(\mathrm{z}_1) - j(\mathrm{z}_2))\varphi_m(j(-\overline{\mathrm{z}_1}) - j(\mathrm{z}_2))\varphi_m(j(\mathrm{z}_1) - j(-\overline{\mathrm{z}_2}))\varphi_m(j(\mathrm{z}_1) - j(-\overline{\mathrm{z}_2}))) \\
&\quad = -G_{J_m}(Z(W)) = -G_1^m(Z(W)) = \frac{|Z(W)|}{2\Lambda(0,\chi)}\sum_{t\in S_m} -a(t,\phi).
\end{aligned}$$

For $k = 3, 5, 7$, the space $M^!_{2-2k}$ has a basis $\{f_{k,m}(\tau) = q^{-m} + O(1) : m \geq 1\}$ since the space of cusp forms of weight $2k$ is trivial. We can specialize Theorem 3.5 to $f = f_{k,m}$ to obtain

$$-m^{1-k}G_{f_{k,m}}(Z(W)) = -G_k^m(Z(W)) = \frac{|Z(W)|}{2\Lambda(0,\chi)}\sum_{t\in S_m} P_{k-1}\left(\frac{t-t'}{m}\right)(-a(t,\phi))$$

for $k = 3, 5, 7$ and any $m \geq 1$.

https://doi.org/10.1112/S0010437X21007077 Published online by Cambridge University Press

By Proposition 3.1, we can easily deduce that

$$2\log|\mathrm{Nm}_{H/\mathbb{Q}}\varphi_m(j(\mathrm{z}_1),j(\mathrm{z}_2))| \geq \frac{|Z(W)|}{2\Lambda(0,\chi)}\sum_{t\in S_m}P_{k-1}\left(\frac{t-t'}{m}\right)(-a(t,\phi))$$
$$= -G_k^m(Z(W)) \tag{4.1}$$

since $P_{k-1}(r)\leq 1$ for all $r\in[-1,1]$. To bound the term $-G_k^m(Z(W))$ from below, we can first apply its definition to write

$$-G_k^m(Z(W)) = \sum_{(z_1,z_2)\in Z(W)}\sum_{\gamma\in\Gamma\backslash\Gamma_m} -G_k(z_1,\gamma z_2) = \sum_{(z_1,z_2)\in Z(W)\cap\mathcal{F}^2,\ \gamma\in\Gamma_m} 2Q_{k-1}(\cosh d(z_1,\gamma z_2)).$$

If we denote by d_2 the distance on $\mathbb{H}^2$ associated to the product Riemannian metric, then

$$2d_2((z_1,z_2),(z,\gamma z))^2 = 2(d(z_1,z)^2+d(z_2,\gamma z)^2) = 2(d(\gamma z_1,\gamma z)^2+d(z_2,\gamma z)^2)$$
$$\geq (d(\gamma z_1,\gamma z)+d(z_2,\gamma z))^2 \geq d(\gamma z_1,z_2)^2$$

for any $\gamma\in\mathrm{PSL}_2(\mathbb{R})$ by the triangle inequality. Therefore, for any $\epsilon>0$, $(z_1,z_2)\in Z(W)\cap\mathcal{F}^2\cap T_{m,\epsilon}\subset\mathbb{H}^2$ implies that there exists $\gamma\in\Gamma_m$ such that $d(z_1,\gamma z_2)<\sqrt{2}\epsilon$, i.e. $Q_{k-1}(\cosh d(z_1,\gamma z_2)) > Q_{k-1}(\cosh(\sqrt{2}\epsilon))$. Combining this with (4.1) and setting $k=3$ finishes the proof.

If d_1d_2 is a perfect square, the same argument goes through with Theorem 3.5 replaced by Theorem 3.8 and Proposition 3.10. Note that $\varphi_m(j(\mathrm{z}_1),j(\mathrm{z}_2))\neq 0$ is equivalent to that the cycle $Z(W)$ does not intersect the singularity of G_{J_m}, which has the same support as that of $G_{f_{k,m}}$. This finishes the proof of Theorem 1.1.

Acknowledgements

The author would like to thank Philipp Habegger for helpful comments. He would also like to thank Jan Bruinier for providing him with a preliminary draft of [BEY19] and Tonghai Yang for helpful discussions about [BKY12, YY19, YYY21]. Finally, he would like to thank the anonymous referee for helpful comments.

References

BHK20 Y. Bilu, P. Habegger and L. Kühne, *No singular modulus is a unit*, Int. Math. Res. Not. IMRN **2020** (2020), 10005–10041; MR 4190395.

Bor98 R. E. Borcherds, *Automorphic forms with singularities on Grassmannians*, Invent. Math. **132** (1998), 491–562; MR 1625724 (99c:11049).

Bru02 J. H. Bruinier, *Borcherds products on O(2, l) and Chern classes of Heegner divisors*, Lecture Notes in Mathematics, vol. 1780 (Springer, Berlin, 2002); MR 1903920 (2003h:11052).

BEY19 J. H. Bruinier, S. Ehlen and T. Yang, *CM values of higher automorphic Green functions for orthogonal groups*, Invent. Math., to appear. Preprint (2019), arXiv:1912.12084.

BKY12 J. H. Bruinier, S. S. Kudla and T. Yang, *Special values of Green functions at big CM points*, Int. Math. Res. Not. IMRN **2012** (2012), 1917–1967; MR 2920820.

Coh85 H. Cohn, *Introduction to the construction of class fields*, Cambridge Studies in Advanced Mathematics, vol. 6 (Cambridge University Press, Cambridge, 1985); MR 812270.

GKZ87 B. Gross, W. Kohnen and D. Zagier, *Heegner points and derivatives of L-series. II*, Math. Ann. **278** (1987), 497–562; MR 909238 (89i:11069).

https://doi.org/10.1112/S0010437X21007077 Published online by Cambridge University Press

GZ85 B. H. Gross and D. B. Zagier, *On singular moduli*, J. Reine Angew. Math. **355** (1985), 191–220; MR 772491 (86j:11041).

GZ86 B. H. Gross and D. B. Zagier, *Heegner points and derivatives of L-series*, Invent. Math. **84** (1986), 225–320.

Hab15 P. Habegger, *Singular moduli that are algebraic units*, Algebra Number Theory **9** (2015), 1515–1524; MR 3404647.

HP17 P. Habegger and F. Pazuki, *Bad reduction of genus 2 curves with CM jacobian varieties*, Compos. Math. **153** (2017), 2534–2576; MR 3705297.

Kud97 S. S. Kudla, *Central derivatives of Eisenstein series and height pairings*, Ann. of Math. (2) **146** (1997), 545–646; MR 1491448.

KY10 S. S. Kudla and T. Yang, *Eisenstein series for SL(2)*, Sci. China Math. **53** (2010), 2275–2316; MR 2718827.

LV15 K. Lauter and B. Viray, *On singular moduli for arbitrary discriminants*, Int. Math. Res. Not. IMRN **2015** (2015), 9206–9250; MR 3431591.

Li18 Y. Li, *Average CM-values of higher Green's function and factorization*, Preprint (2018), arXiv:1812.08523.

Sch09 J. Schofer, *Borcherds forms and generalizations of singular moduli*, J. Reine Angew. Math. **629** (2009), 1–36; MR 2527412.

Via11 M. Viazovska, *CM values of higher Green's functions*, Preprint (2011), arXiv:1110.4654.

Yan05 T. Yang, *CM number fields and modular forms*, Pure Appl. Math. Q. **1** (2005), 305–340.

YY19 T. Yang and H. Yin, *Difference of modular functions and their CM value factorization*, Trans. Amer. Math. Soc. **371** (2019), 3451–3482; MR 3896118.

YYY21 T. Yang, H. Yin and P. Yu, *The lambda invariants at CM points*, Int. Math. Res. Not. (IMRN), **2021** (2021), 5542–5603.

Zha97 S. Zhang, *Heights of Heegner cycles and derivatives of L-series*, Invent. Math. **130** (1997), 99–152; MR 1471887.

Yingkun Li li@mathematik.tu-darmstadt.de
Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstrasse 7,
D-64289 Darmstadt, Germany