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VORONOI DOMAINS AND DUAL CELLS IN THE GENERALIZED KALEIDOSCOPE WITH APPLICATIONS TO ROOT AND WEIGHT LATTICES

Dedicated to H. S. M. Coxeter

R. V. MOODY AND J. PATERA

ABSTRACT. We give a uniform description, in terms of Coxeter diagrams, of the Voronoi domains of the root and weight lattices of any semisimple Lie algebra. This description provides a classification not only of all the facets of these Voronoi domains but simultaneously a classification of their dual or Delaunay cells and their facets. It is based on a much more general theory that we develop here providing the same sort of information in the setting of chamber geometries defined by arbitrary reflection groups. These generalized kaleidoscopes include the classical spherical, Euclidean, and hyperbolic kaleidoscopes as special cases. We prove that under certain conditions the Delaunay cells are Voronoi cells for the vertices of the Voronoi complex. This leads to the description in terms of Wythoff polytopes of the Voronoi cells of the weight lattices.

1. **Introduction.** Any discrete set of points Q in a metric space (X,d) decomposes X into regions, one around each point of Q, which are the Voronoi regions (or cells or domains) of Q. Specifically, for $q \in Q$,

$$\mathrm{Vor}(q) := \{x \in X \mid d(q,x) \leq d(p,x), \text{ for all } p \in Q\}.$$

The diversity of this concept in mathematics and physics is attested to by the variety of names attached it: Dirichlet cells, proximity cells, Wigner-Seitz cells, Brillouin cells, and so on. The case when the point set Q is a lattice in real n-space is by far the most important, not least because then the Voronoi cell around 0 has the property of being a fundamental region for the translation group and at the same time being invariant under the point group of the lattice. Yet it remains a difficult problem to describe the structure of the Voronoi cells of a lattice, and there are relatively few classes of lattices for which we have a complete description. A useful introduction to this subject is [Se]. Our own interest in the subject arose through the recent use of root and weight lattices in the theory of quasicrystals [KPZ, MP2].

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In this paper we describe Voronoi cells, and their facets, and at the same time the closely related dual cells called Delaunay cells, and also their facets. We do this in a uniform way for any root or weight lattice of finite type. There are two ingredients in this. The first is to realize that the Voronoi and Delaunay problems for the root lattices can be cast in the wider setting of solving the same problems for any set Q that is the orbit of a point under a reflection group W. The second is to understand that the natural duality between the Voronoi and Delaunay cells manifests itself in a particularly simple way in the reflection group context. The solution to the weight lattice problem is obtained by reversing the roles of the two types of cells.

It is remarkable that for such a well known and uniform class of lattices as the root and weight lattices that a complete study of their Voronoi and Delaunay cells was accomplished only very recently [CSl, CS2]. (In fact we learned of [CS2] only after the main body of this paper was complete.) In the second of these, Conway and Sloane give an explicit description of each type of cell for each indecomposable root and weight lattice of simply-laced type *i.e.* A_n, D_n, E_6, E_7, E_8. Their approach is through a detailed description of the lattices and involves a certain amount of computation which varies from case to case and is actually quite considerable for the weight lattices of the exceptional root systems. However their classification clearly reveals the uniformity of the description of the cells (although not so clearly their facets) in terms of Coxeter diagrams.

In this paper we obtain the same classification, but by a very different approach. By setting the Voronoi-Delaunay problem for root lattices in terms of a W-orbit Q, it acquires a combinatorial-geometrical interpretation that is no longer attached particularly to Euclidean spaces or Euclidean reflection groups but lends itself naturally to a uniform general solution in terms of the Coxeter diagrams.

The weight lattices are more difficult and our solution comes about through an interesting twist. The weight lattices are in general composed of several W-orbits and hence do not fall directly within the scope of our methods. However, the vertices of the Voronoi cells of a given W-orbit Q (also called the *holes* of Q) themselves form a set of points P which are centered in the Delaunay cells of Q. We give a necessary and sufficient condition for these Delaunay cells to be the Voronoi cells of P. Remarkably in the case of weight lattices (with one exception) it is easy to see that there is exactly one way in which to form a W-orbit (but not a lattice) so that the set P of holes is the weight lattice. This leads at once to a description of the Voronoi and Delaunay cells of the weight lattices but with the roles of polytopes and polyhedra reversed. The exception, perhaps not surprisingly, is E_8 where there is no such W-orbit. However in this case the root and weight lattices coincide.

The determinations of the Voronoi cells of the weight lattices of E_6 and E_7 were originally made by Worley [Wl, W2]. Each case required a detailed analysis of the weight lattice and was quite technical. Conway and Sloane base their analysis of these weight lattices on Worley's work. We are able to determine the structure of the Voronoi and Delaunay cell complexes of both root and weight lattices by inspection from Coxeter diagrams.

Our results apply to the non-simply-laced cases too, and although there is nothing new in the way of lattices to be obtained from this, the associated Coxeter groups are different and shed light on the symmetries involved.

The first part of the paper develops results that apply to any reflection group (Coxeter group), including not only the Euclidean groups that arise in connection with lattices, but also the spherical and hyperbolic groups, and even groups for which we do not know an invariant metric. We have to redefine the notions of Voronoi and Delaunay cells in a non-metrical way that can be reduced to the usual definition in the case that a metric is available.

An important result is Theorem 4.4. In Section 5 we reinterpret it as a simple algorithm played on the nodes of the Coxeter diagram of W that describes how to partition the diagram into two parts. One of these encodes a convex polyhedron, called a *reflection polyhedron*, that describes a facet (bounding figure of some lesser dimension) of the Voronoi cell. The other encodes a convex polytope, a Wythoff or reflection polytope, that describes the dual facet in the Delaunay cell. In keeping with this duality we use the term polyhedron for a convex figure described in terms of its faces (facets of maximum dimension) and polytope for a convex figure described by it vertices. The allowable partitions determine the facets and dual facets of various dimensions.

At this level of generality our results fit into the abstract theory of shadow geometries that were introduced by J. Tits [T] and subsequently developed by Scharlau [Sh] and G. Maxwell [Ma]. In one way or another most of our Theorem 4.4 can be found between these two papers, but not without requiring a considerable detour. The proof that we offer here is tailored to our needs and depends only on [B].

The generalized kaleidoscope is Coxeter's term for the simplicial complex that arises from a simplex of mirrors inclined to one another at various submultiples of π. The mirrors give rise to a discrete reflection group W (a Coxeter group) that classically was restricted to the cases in which the complex could be realized in either spherical, Euclidean, or hyperbolic space X [Cl, C2]. There is a well-known construction due to J. Tits which furnishes the setting of a similar geometrical realization with no restrictions on the Coxeter group serving as the reflection group. In Section 2 we recall this construction, that we call here the generalized kaleidoscope. It is the basis of everything to follow.

We begin, then, with a generalized kaleidoscope K based on the Coxeter group W acting on a Tits cone X. A point $* \in X$ is chosen arbitrarily and Q is the set of the W-translates of $* : Q = W(*)$. In Section 3 we relate this new picture, back to the classical spherical, Euclidean and hyperbolic pictures, and show how to give the Voronoi cells of Q a non-metrical description that we will use in the more general situation.

In Section 4 we prove Theorem 4.4 mentioned above that gives a complete description of the facets of the Voronoi cells. This leads in Section 5 to the algorithm that allows us to determine all this information at sight from the Coxeter diagram. In Section 6 we define the Delaunay complex in a combinatorial way and prove the important fact that the Delaunay cells meet face-to-face, *i.e.* the intersection of facets is again a facet. In

Section 7 we return to the classical kaleidoscopes and give the Delaunay cells their standard interpretation as convex hulls of certain subsets of Q. We also prove a key result of the paper: necessary and sufficient conditions for the Delaunay complex to be itself the Voronoi complex of the holes of the original Voronoi complex. It is this that allows us to determine the structure of the Voronoi cells of the weight lattices, as we show in Section 8. In Section 8 we also review the solution to the determination of the Voronoi and Delaunay complexes of all the indecomposable root lattices and extend the result to all root lattices. This classification in the indecomposable case appeared in our previous paper [MPl]. These results are summarized in Tables 2 and 3. Finally in Section 9, motivated by the discussion in [CS2] of contact cells (convex hulls of the points of contact of the spheres in the sphere packings of space determined by the root and weight lattices), we offer a description of these cells in the present context. These results are summarized in Table 4.

2. **Tits' construction of the generalized kaleidoscope.** In this section we review the construction due to J. Tits of the generalized kaleidoscope with an arbitrary Coxeter group. The details may be found in [B] or in [TBV]. Fix a nonnegative integer n and let $N := \{1,\ldots,n\}$.

We begin with a matrix $M = (m_{ij}), i,j \in N$, satisfying

(1) M is symmetric;

(2) $m_{ii} = 1$, for all i;

(3) $m_{ij} \in \{2,3,4,\ldots\} \cup \{\infty\}$, for all $i \neq j$.

Such a matrix is called a *Coxeter matrix*.

We set $E = \mathbb{R}^n$ with the standard basis $(e_1,\ldots,e_n)$ and define a symmetric bilinear form

$$B = B_M : B(e_i,e_j) = -\cos\Big(\frac{\pi}{m_{ij}}\Big)$$

(interpreted as -1 if $m_{ij} = \infty$) so

$$B(e_i,e_i) = 1,$$
$$B(e_i,e_j) \leq 0, \quad \text{if } i \neq j.$$

The Coxeter matrix M is *decomposable* if for some ordering of the basis the matrix $\big(B(e_i,e_j)\big)$ assumes the form

$$\left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right).$$

Otherwise M is *indecomposable*.

For each $i \in N$ define the involution

$$r_i\colon x \longmapsto x - 2B(x,e_i)e_i$$

on E and let W be the group generated by $r_1, \ldots, r_n$. Then W is a Coxeter group with presentation:

$$\text{generators:} \quad r_1, \ldots, r_n,$$
$$\text{relations:} \quad (r_i r_j)^{m_{ij}} = 1.$$

This is a nontrivial fact and is proved using the geometry induced on the dual space E° of E. Let $\langle \cdot, \cdot \rangle : E^\circ \times E \longrightarrow \mathbb{R}$ be the dual pairing and transfer the action of W to E° by defining

$$\langle wy, wx \rangle = \langle y, x \rangle,$$

for all $w \in W, y \in E^\circ, x \in E$

We need to know this action explicitly. Let $(e_1^\circ, \ldots, e_n^\circ)$ be the basis dual to $(e_1, \ldots, e_n)$:

$$(e_i^\circ, e_j) = \delta_{ij}.$$

Also let $\check{e}_i \in E^\circ, i \in N$, be defined by

$$\langle \check{e}_i, e_j \rangle = 2B(e_i, e_j), \quad j \in N$$

so

$$\check{e}_i = \sum_{j=1}^{n} 2B(e_i, e_j) e_j^\circ.$$

Then from

$$\langle r_i e_j^\circ, e_k \rangle = \langle e_j^\circ, r_i e_k \rangle = \langle e_j^\circ, e_k - 2B(e_k, e_i) e_i \rangle = \langle e_j^\circ - \delta_{ij} \check{e}_i, e_k \rangle$$

we see that

$$r_i e_j^\circ = e_j^\circ - \delta_{ij} \check{e}_i.$$

Now define for $i \in N$

$$A_i := \{x \in E^\circ \mid \langle x, e_i \rangle > 0\},$$
$$H_i := \{x \in E^\circ \mid \langle x, e_i \rangle = 0\} = \sum_{j \neq i} \mathbb{R} e_j^\circ,$$
$$\hat{A}_i := \{x \in E^\circ \mid \langle x, e_i \rangle \geq 0\} = A_i \cup H_i.$$

For each subset $S \subset N$ define

$$F_S^0 = \bigcap_{i \in S} H_i \cap \bigcap_{j \notin S} A_j$$

and

$$F_S := \bigcap_{i \in S} H_i \cap \bigcap_{j \notin S} \hat{A}_j.$$

Then

$$F_S = \coprod_{S \subset T \subset N} F_T^0$$

and

$$F := F_{\emptyset} = \{x \in E^{\circ} \mid \langle x, e_i \rangle \geq 0, \text{ for all } i \in N\}.$$

F is a simplicial cone with facets F_S that are also simplicial cones.

It is important to note that F_S^0 is open in the linear space that it spans (namely $\bigcap_{i \in S} H_i$) and that F_S is the closure of F_S^0 in $\mathbb{R}^n (= E^{\circ})$.

Set

$$X = X_M = \bigcup_{w \in W} wF,$$
$$\mathcal{F} = \mathcal{F}_M = \{wF_S^0 \mid w \in W, S \subset N\}.$$

THEOREM 2.1 (J. TITS [TBV,B]). *(i) X is a convex cone,*

(ii) for all $x, y \in X$, the line segment $[x, y]$ meets only finitely many elements of $\mathcal{F}$,

(iii) the cone F is a fundamental domain for W on X. Specifically for $S, S' \subset N$, $w, w' \in W$, one has $wF_S^0 \bigcap wF_{S'}^0 \neq \emptyset \Rightarrow$

(a) $S = S'$,

(b) $wF_S^0 = w'F_{S'}^0$,

(c) $w^{-1}w' \in W_S := \langle r_i \mid i \in S \rangle$.

(iv) $X = \coprod_{G \in \mathcal{F}} G$. ■

COROLLARY 2.2. *If $x \in F_S^0$ is fixed by some $w \in W$ then $w \in W_S$ and w fixes every point of F_S.* ■

We refer to $\mathcal{K}(M) := (X_M, \mathcal{F}_M)$ as the generalized kaleidoscope of the matrix M. We will need the Coxeter diagram Γ_M of M which is the graph with n nodes and an edge $\bigcirc\!\overset{m_{ij}}{\text{—}}\!\bigcirc$ between each pair of nodes i and j for which $m_{ij} > 2$. It is customary to omit the label 3 when $m_{ij} = 3$.

The bilinear form B_M is indecomposable if and only if the diagram Γ_M is connected.

3. **The classical cases.** There are three cases of the generalized kaleidoscope that are particularly important geometrically. Suppose that M is indecomposable.

(i) SPHERICAL CASE. If B_M is positive definite then W is a finite subgroup of the orthogonal group of the Euclidean space $(\mathbb{R}^n, B_M)$. Using B_M we identify E° and E. Let $\bar{X}$ be the unit sphere $\{x \in E^{\circ} \mid B_M(x, x) = 1\}$ and note that W acts faithfully on $\bar{X}$. We find that $X = E = E^{\circ}$ and $\mathcal{F} \cap \bar{X}$ covers $\bar{X}$ and determines a simplicial complex (with spherical simplices) on $\bar{X}$.

(ii) EUCLIDEAN CASE. If B_M is positive semidefinite but not positive definite then $\operatorname{rad} E = \mathbb{R}u$ for some $u \in \mathbb{R}^n$ (one dimensionality follows from the assumption that B_M is indecomposable). Set $\bar{X} := \{x \in E^{\circ} \mid \langle x, u \rangle = 1\}$ and set $V = \{x \in E^{\circ} \mid \langle x, u \rangle = 0\}$. Then V is isomorphic to the dual space of $E / \operatorname{rad} E$ and hence carries a positive definite form derived from B_M. We have a natural simply-transitive action

$$V \times \bar{X} \longrightarrow \bar{X}$$

of V on $\bar{X}$ defined by addition and in this way $\bar{X}$ may be viewed as Euclidean space. We find that $\mathcal{F}\cap\bar{X}$ covers $\bar{X}$ and determines a simplicial complex on $\bar{X}$ and W acts as a group of affine symmetries of this complex.

(iii) HYPERBOLIC CASE. We assume that B_M is indefinite but for all principal $(n-1)\times(n-1)$ submatrices M' of M that $B_{M'}$ is positive definite. Then relative to some suitable basis of $\mathbb{R}^n$ (not $(e_1,\ldots,e_n)$!), we have

$$B_M(x,x) = x_1^2 + \cdots + x_{n-1}^2 - x_n^2.$$

Since B_M is non degenerate we may identify E and E° using it. Let Γ be the cone $\{x \in E \mid B_M(x,x) < 0\}$. The X is precisely one of the two nappes, say Γ_+, of this cone. Set

$$\bar{X} := \{x \in \Gamma_+ \mid B_M(x,x) = -1\}$$

which is W-stable.

Then $\bar{X}$ is a model of hyperbolic space with the metric $ds^2 = dx_1^2 + \cdots + dx_{n-1}^2 - dx_n^2$ and we find that $\mathcal{F}\cap\bar{X}$ is a simplicial complex (with hyperbolic simplices) on $\bar{X}$ with W acting as a group of symmetries.

In each of these three cases we have determined a metric space $(\bar{X},d)$ of dimension $(n-1)$ and a simplicial complex $\bar{\mathcal{F}}\cap\bar{X}$ on which W acts faithfully as a group of symmetries. If we define $w\bar{F}_S := wF_S\cap\bar{X}$ and $\bar{F} := F\cap\bar{X}$ then $\bar{F}$ is a fundamental region for action of W on $\bar{X}$ and the conclusions of Theorem 2.1(iii) and its Corollary are true when applied to the facets $w\bar{F}_S$.

The pair $\bar{\mathcal{K}}(M) := (\bar{X},\bar{\mathcal{F}})$ is called a *classical kaleidoscope* and $\mathcal{K}(M) = (X,\mathcal{F})$ is called its *linear cover*.

Let $\mathcal{K}(M) = (X,\mathcal{F})$ be the linear cover of a classical kaleidoscope and let $*\in\bar{F}$ be chosen arbitrarily. Set $Q = W(*)$ and let $\overline{V(*)}$ denote the Voronoi region of $\bar{X}$ determined by $*$ and the set Q. Let

$$W_* = \mathrm{Stab}_W(*).$$

According to Corollary 2.2, W_* is generated by the reflections r_i for which the corresponding reflection hyperplane passes through $*$. Let $K' := \{i \mid r_i(*) = *\}$ so $W_* = W_{K'}$.

The next result is due to Conway and Sloane [CS1] in the case of root lattices.

PROPOSITION 3.1.
$$\overline{V(*)} = \bigcup_{w\in W_{K'}} w\bar{F}.$$

PROOF. Set $V_0 = \bigcup_{w\in W_{K'}} w\,\mathrm{Int}(\bar{F})$. Let $x\in V_0$ and let p be a point in Q that minimizes $d(x,p)$. We show that $p = *$. Since V_0, Q, and $*$ are stabilized by W_*, we may assume that $x\in\mathrm{Int}(\bar{F})$.

Suppose that some hyperplane H_k, $k\in N$, separates p and x. Then using the well-known fact that

$$|r_k p - x| < |p - x|,$$

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we obtain a contradiction to the definition of p. Thus no hyperplane H_k separates p and x and $p \in \bar{F}$. But $p \in W(*)$ and $\bar{F}$ is a fundamental region for W. Thus $p = *$. This proves that $V_0 \subset \overline{V(*)}$ are hence, since $\overline{V(*)}$ is closed, $\bigcup_{w \in W_{K'}} w\bar{F} \subset \overline{V(*)}$.

Conversely let $y \in \overline{V(*)}$. Suppose that some hyperplane H_k, $k = 1, \ldots, n$, separates y and $*$. Then $|r_k y - *| \leq |y - *| \Rightarrow |y - r_k *| \leq |y - *|$. Since $r_k * \in Q$ this can happen only if these are equalities and so $|r_k y - *| = |y - *|$ and $* \in H_k$. Thus $k \in K'$, so the only hyperplanes separating y and $*$ are those passing through $*$. The group W_* is finite and $\bigcup_{w \in W_*} w\bar{F}$ is a neighbourhood of $*$ on which W_* acts with fundamental region $\bar{F}$. Thus using some $w \in W_*$ we obtain wy not separated by any hyperplane from $\bar{F}$. Then $wy \in \bar{F}$ and $y \in W_* \bar{F}$. ∎

4. **Voronoi cells in the generalized kaleidoscope.** With $\bar{X}, \bar{F}, *$ as above, we have determined that $\overline{V(*)} = W_* \bar{F}$, where W_* is the stabilizer of $*$ in W. There are two reasons why it is convenient to lift this picture back to the original Tits cone X and its fundamental region F. The point $*$ lies in $F \setminus \{0\}$ and we may look at $V(*) := W_*(F)$. The facet structure of $\overline{V(*)}$ lifts canonically to the facet structure of $V(*)$, each facet of $\overline{V(*)}$ being replaced by the union of rays through it (for the definitions of faces and facets see below). Inclusions and multiplicities of facets containing others, *etc.* are all preserved by lifting, the only difference being that $V(*)$ has a facet $\{0\}$ that corresponds to no facet of $\overline{V(*)}$.

The first advantage of lifting back is that it is technically easier to work in the vector space setting. The more important advantage is that $W_*(F)$ is defined for any point $* \in F \setminus \{0\}$ in *any* generalized kaleidoscope $K(M)$ for any Coxeter matrix M and so we can study the problem of the structure of $W_*(F)$ even in the non-classical setting.

Henceforth M is a Coxeter matrix, *not necessarily indecomposable*, with diagram Γ, generalized kaleidoscope $\mathcal{K}(M) = (X_M, \mathcal{F}_M)$, and a fundamental region F.

We fix a point $* \in F \setminus \{0\}$ and write

$$* = \sum_{j \in K} c_j e_j^\circ, \quad c_j > 0, \text{ for all } j \in K.$$

Set $K' := N \setminus K$. The stabilizer of $*$ in W is

$$W_K' = \langle r_i \mid i \in K' \rangle.$$

Define

$$V(*) := W_{K'} F = \bigcup_{w \in W_{K'}} wF,$$
$$Q := W(*) \subset X.$$

For $w \in W$ define $V(w*) = wV(*) = wW_{K'}F$. We call the sets $V(q), q \in Q$, the *Voronoi cells* of Q in X. Clearly $X = \bigcup_{q \in Q} V(q)$.

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LEMMA 4.1. *Let $S \subset N$, $i \in N \backslash S$, $w \in W_S$, and let $\ell \colon W \longrightarrow \mathbb{Z}_+$ be the length function on W. Then*

$$\ell(r_i w) > \ell(w).$$

PROOF. The alternative is $\ell(r_i w) < \ell(w)$. Then if $r_{i_1} \cdots r_{i_k}$ is a reduced expression for w we have by the exchange condition [B]

$$r_i w = r_{i_1} \cdots \hat{r}_{i_j} \cdots r_{i_k},$$

where the hat indicates a deleted entry, and then we find $r_i \in W_S$ which is false. ∎

LEMMA 4.2. *Let $D = \bigcup_{G \in \mathcal{G}} G$ be any union of facets of the form $G = wF_S$, $w \in W$, $S \subset N$. Then D is closed in the Tits cone X.*

PROOF. Let $x \in X$ be in the closure of D. Suppose if possible that $x \notin D$. Since any open ball $\mathcal{B}$ about x meets D we can choose $y \in D$ so that $[x,y] \subset \mathcal{B}$. Since $[x,y]$ is covered by finitely many facets of the form $G^0 = wF_S^0$, $w \in W$, $S \subset N$, and these are convex, there is a G^0 so that $]x,z[\subset G^0$ for some point $z \in]x,y[$. Now $]x,z[\cap D \neq \phi \Rightarrow G^0 \cap D \neq \emptyset \Rightarrow G^0 \subset G_1$ for some $G_1 \in \mathcal{G}$. Since G_1 is closed in $\mathbb{R}^n$, $x \in G_1 \subset D$, and we obtain a contradiction. ∎

PROPOSITION 4.3. *(i) $V(*) \subset \hat{A}_i$, for all $i \in K$,*
(ii) $f := V(*) \cap r_i V(*) = V(*) \cap H_i$, for all $i \in K$,
(iii) $V(*)$ and f are convex and closed in X.

PROOF. $V(*) = W_{K'}(F)$. Let $w \in W_{K'}$ and let $i \in K$. Then $\ell(r_i w) > \ell(w)$ by Lemma 4.1 and so by [TBW] or [B, Chapter V, Section 4.4, (P_n)], we have $wF \subset \hat{A}_i$. Then $V(*) \subset \hat{A}_i$ and $f \subset \hat{A}_i \cap r_i \hat{A}_i = H_i$.

The proof that $V(*)$ is convex is a simple variation of the proof [B] that X is convex. Let $x, y \in V(*)$. We prove that the line segment $[x,y]$ is covered by a finite number of sets wF_J^0, $w \in W_{K'}$. We may assume that $x \in F$ and $y = wv$, where $v \in F$, $w \in W_{K'}$. If $y \in F$ the result is trivial, so we assume that $y \notin F$.

Let $[x,y]$ pass through the face $F_j := F_{\{j\}}$ of F at the point z. Then $x \in \hat{A}_j$, $y \in r_j A_j$. Since $V(*) \subset \hat{A}_i$ whenever $i \in K$, we see that $j \in K'$. Thus $wF \subset r_j \hat{A}_j$ and by [B, loc.cit.] $\ell(r_j w) < \ell(w)$.

Now $[z,y] = r_j[z, r_j wv]$, $r_j w \in W_{K'}$, and we may assume by induction on length that $[z, r_j wv]$ is covered by finitely many facets of the required type. The same then applies to $[z,y]$ and hence also to $[x,y] = [x,z] \cup [z,y]$.

This proves that $V(*)$ is convex, hence also f. By Lemma 4.2, $V(*)$, and then f, is closed in X. ∎

Let $A, B \subset \mathbb{R}^n$. We say that *$A$ supports B* if the linear spans $[A]$ and $[B]$ of A and B are equal. Thus A supports B if and only if B supports A.

Let C be a convex set in $\mathbb{R}^n$. A convex subset $f \subset C$ is a *facet* of C if whenever $x, y \in C$ and $]x,y[\cap f \neq \emptyset$ then $[x,y] \subset f$. It is *proper* if it is nonempty and is not equal to C.

Clearly if f and g are facets of C then $f \cap g$ is also a facet of C.

Let H be an affine hyperplane of the affine span of C and suppose that

(i) $H \cap C$ supports H,

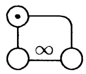
(ii) C lies in one of the two closed halfspaces determined by H.

Then $f := H \cap C$ is a facet of C. Indeed, if $x, y \in C$ and $(]x,y[) \cap H \cap C \neq \emptyset$ then by (ii) $x, y \in H$ and $[x,y] \in H \cap C$.

The facets of the form just described are called the *faces* of C. The faces have codimension 1 in C. See [Br] for more details and definitions on faces and facets. Note that we use the words faces and facets where he uses the words facets and faces respectively.

REMARK 1. We do not insist that C be closed and so, under this definition, an open simplex has no faces! These peculiarities are necessitated because when W is infinite the Tits cone is not in general closed in E°. For $V(*)$, the only points where caution is required are those lying in the boundary of X.

REMARK 2. A second peculiarity of dealing with convex figures with potentially infinitely many faces is that not every proper facet need lie in a face. However this situation can occur only if $*$ lies on the boundary of X, and in particular cannot happen for a kaleidoscope for which F lies in the interior of X (or equivalently for which $\mathrm{Stab}_W(e_j^{\circ}) = W_{N\setminus j}$ is finite for all $j = 1, \ldots, n$). A simple example is povided by the diagram

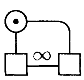


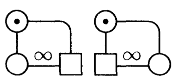
which is most easily visualized as a kaleidoscope in the hyperbolic plane. The fundamental domain is a triangle with one vertex at infinity. The Voronoi region $V(*)$ is has infinitely many sides, all on one W_*-orbit, and three orbits of vertices. Two of the orbits make up the natural end points of the edges. The point $*$ is the third orbit. It is a facet of $V(*)$ but is not on any edge. See Section 5 for the notation.

A facet is *extraordinary* if it is proper and does not lie in a face, otherwise it is *ordinary*.

In this paper, the word facet, unmodified, will be taken to mean an ordinary facet, and no other type of facet will occur in the discussion. Our classification of facets by diagrams applies only to these ordinary facets.

For example, as we will see in the sequel, the classification of the facets of the above kaleidosope procedes by the diagrams


edges


vertices

and does not include the extraordinary vertex $*$.

THEOREM 4.4. *Let $* = \sum_{j \in K} c_j e_j^0$, $c_j > 0$, be a point of the fundamental region F of the kaleidoscope $\mathcal{K}(M)$. Let $V(*) = W_{K'}F$, where $K' = N \setminus K$. Then*

(i) Every facet of $V(*)$ is convex and has a $W_{K'}$-translate f for which $f \cap F = F_S$ (for some $S \subset N$) supports f. (We say then that f is in special position*).*

(ii) *If f is in special position with $f \cap F = F_S$ then S satisfies the condition*

(**C**) *Every connected component of S (viewing S as a set of nodes of the Coxeter diagram Γ) contains points of K.*

(iii) *For f as in (ii),*

$$f = f_S := \bigcap_{w \in W_S} V(w*) = W_{K' \cap S^\perp} F_S.$$

(iv) *With f as in (ii)*

$$\mathrm{Stab}_W(f_S) = W_S \times W_{K' \cap S^\perp}$$

and W_S is the pointwise fixer of f_S in W.

(v) *Let*

$$\mathcal{C} := \{S \subset N \mid S \text{ satisfies } (\mathbf{C})\}.$$

Then the mapping $f_S \longmapsto S$ from the set of facets of $V(*)$ that are in special position into $\mathcal{C}$ is a bijection.

We will prove Theorem 4.4 in the following sequence of results. The main ingredient is an induction on the number card(N) of Coxeter generators of W.

PROPOSITION 4.5. *The faces of $V(*) = W_{K'}F$ are supported by sets of the form wF_i, $w \in W_{K'}$, $i \in K$. Every such set supports some face of $V(*)$.*

PROOF. Let the hyperplane H support a face f of $V(*)$. Then f is the union of sets $wF_J \cap H$, $w \in W_{K'}$, $J \subset N$. Now since f is convex and spans H it contains an open subset of H. Since $\{wF_J \mid w \in W_{K'}, J \in N\}$ is countable, H must intersect wF_i in an open set for some $i \in N$. But wF_i supports wH_i and so $H = wH_i$. If $i \notin K$ then $r_i \in W_{K'}$ and $V(*) = r_i V(*)$ so wF and $r_i wF$ both lie in $V(*)$, contrary to $V(*)$ lying on one side of H. Thus $i \in K$.

Conversely consider wH_i, $w \in W_{K'}$, $i \in K$. We know $V(*) \subset \hat{A}_i$ (Proposition 4.3(i)) whence $V(*) = wV(*)$ lies on one side of wH_i. Also $wF_i \subset V(*)$, whence $wH_i \cap V(*)$ is a face supported by wF_i. ∎

In order to study the facet structure of $V(*)$ we will study the facet structure of its faces. Since these are all of the form $wH_i \cap V(*)$, where $w \in W_{K'}$, and $i \in K$, it is sufficient to study the faces f of the form $f = H_i \cap V(*)$, $i \in K$.

Let $i \in N$. We define

$$\mathrm{conn}(i) := \{j \in N \setminus \{i\} \mid B(e_j, e_i) \neq 0\}.$$

Let $S \subset N$. We define

$$S^\perp := \{j \in N \mid B(e_j, e_i) = 0, \text{for all } i \in S\}.$$

If $S = \{i\}$ we write $i^\perp$ for $\{i\}^\perp$.

LEMMA 4.6. *Let $S \subset N$, $i \in N \setminus S$, $w \in W_S$ and suppose that $r_i w = w r_i$. Then $w \in W_{S \cap i^\perp}$.*

PROOF. By Theorem 2.1, W_S is the pointwise stabilizer of $\{e_j^\circ \mid j \in N \setminus S\}$. Since $i \notin S$

$$w r_i e_i^\circ = r_i w e_i^\circ = r_i e_i^\circ$$

so w fixes $r_i e_i^\circ = e_i^\circ - 2\sum_{m \in \mathrm{conn}(i) \cup \{i\}} B(e_i, e_m) e_m^\circ$. Since w fixes e_i°, w fixes $-2\sum_{m \in \mathrm{conn}(i)} B(e_i, e_m) e_m^\circ \in F$ and hence by Theorem 2.1,

$$w \in \left\langle r_p \;\middle|\; \sum_{m \in \mathrm{conn}(i)} B(e_i, e_m) e_m^\circ \subset H_p \right\rangle = \langle r_p \mid p \in i^\perp \cup \{i\} \rangle = W_{i^\perp \cup \{i\}}.$$

Finally $w \in W_S \cap W_{i^\perp \cup \{i\}} = W_{S \cap i^\perp}$. ∎

PROPOSITION 4.7. *Let $i \in K$. Then $V(*) \cap H_i$ is a face of $V(*)$ and*

$$V(*) \cap V(r_i *) = V(*) \cap H_i = W_{K' \cap i^\perp} F_i.$$

PROOF. We already know by Propositions 4.5 and 4.3 that $f := V(*) \cap H_i$ is a face of $V(*)$ and the first equality holds. Since $F_i \subset V(*) \cap V(r_i *) = f \subset H_i$ and F_i contains a nonempty open subset of H_i and since f is convex, we see that every point of f lies in the closure of an open subset of H_i lying in f. Now $V(*) \cap V(r_i *)$ is the disjoint union of at most countably many sets of the form wF_J^0, $J \subset N$, $J \neq \emptyset$ and since wF_J lies in a subspace of dimension $k < n$ if $\operatorname{card} J > 1$, we see that in fact every point of f lies in the closure of the union of subsets $wF_j \subset V(*) \cap V(r_i *)$. However, by Lemma 4.2, the union of such sets is closed in X and hence we have proved that $f = \bigcup_p w_p F_{j_p}$ for some $w_p \in W_K, j_p \in N$.

Now $V(*) \cap r_i V(*)$ must be a union of sets $wF_j = r_i w' F_k$, where $w, w' \in W_{K'}, j, k \in N$. Then by Theorem 2.1, $j = k$ and $w^{-1} r_i w' \in \langle r_k \rangle$, whence $r_i \in w \langle r_k \rangle w'^{-1} \subset W_{K' \cup \{k\}}$. Since $i \in K$, we have $i = k$ and we have proved that

$$f = \bigcup_{w \in U} wF_i$$

for some subset $U \subset W_{K'}$. However $f \subset H_i$ and

$$wF_i \subset H_i \Rightarrow r_i w F_i = wF_i \Rightarrow w^{-1} r_i w \in \langle r_i \rangle \Rightarrow w^{-1} r_i w = r_i \Rightarrow r_i w = w r_i.$$

Then by Lemma 4.6, $w \in W_{K' \cap i^\perp}$. Conversely $w \in W_{K' \cap i^\perp} \Rightarrow wF_i \subset H_i$ and hence $f = W_{K' \cap [i]^\perp} F_i$. ∎

We now prepare the way for the induction step.

Fix $i \in K$. Define $*_i$ to be the unique point on the line segment $[*, r_i *]$ that lies on H_i. Since

$$\begin{aligned} r_i * &= * - \langle *, e_i \rangle \check{e}_i \\ &= \sum_{j \in K} c_j e_j^\circ - c_i \Big\{ 2 \sum_m B(e_i, e_m) e_m^\circ \Big\} \end{aligned}$$

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we see that $*_i$ has the form

$$*_i = \sum_{j\in K_i} d_j e_j^\circ,$$

where

$$K_i := K \setminus \{i\} \cup \mathrm{conn}(i),\ d_j > 0, \quad \text{for all } j \in K_i.$$

For future reference we define

$$K_i' := (N \setminus \{i\}) \setminus K_i$$

and note that

$$K_i' = K' \cap i^\perp.$$

Set

$$E_i := \sum_{p\neq i} \mathbb{R}e_p,$$

$$E_i^\circ := E^\circ / \mathbb{R}e_i^\circ,$$

$$M^i := (m_{pq})_{(p,q)\in N\setminus\{i\}\times N\setminus\{i\}},$$

$$\langle\cdot,\cdot\rangle: E_i^\circ \times E_i \longrightarrow \mathbb{R}, \text{ the natural pairing induced from } \langle\cdot,\cdot\rangle \text{ on } E^\circ \times E,$$

$$\pi = \pi_i: E^\circ \longrightarrow E_i^\circ, \text{ the natural map (also denoted by an over-tilde)},$$

$$B^i = B_M|_{E_i\times E_i}.$$

Then M^i defines through B^i and $\{e_j \mid j \in N \setminus \{i\}\}$ a generalized kaleidoscope $\mathcal{K}^i = \mathcal{K}(M^i)$ on the pair (E_i°, E_i°). The basis of E_i° dual to $\{e_j \mid j \neq i\}$ is $\{\tilde{e}_j^\circ \mid j \neq i\}$. The fundamental region is by definition

$$F^i := \{\tilde{x} \in E_i^\circ \mid \langle\tilde{x}, e_j\rangle \geq 0, j \in N \setminus \{i\}\}.$$

LEMMA 4.8. *(i) $\pi|_{H_i}$ determines an isomorphism of H_i and E_i°. In particular each $\tilde{x} \in E_i^\circ$ has a unique preimage under π lying in H_i,*
(ii) $F^i = \pi(F) = \pi(F_i)$.

PROOF. (i) is obvious. For (ii) we clearly have $\pi(F) \subset F^i$. If $\tilde{x} \in F^i$ and x is its unique preimage in H_i then $\langle x, e_j\rangle \geq 0$, for all $j \in N \setminus \{i\}$ and $\langle x, e_i\rangle = 0$ so that $x \in F_i$. ∎

Define W^i to be the Weyl group of the new kaleidoscope. Then W^i is generated by $n-1$ reflections $\tilde{r}_j, j \in N \setminus \{i\}$, and these form a system of Coxeter generators of W^i with Coxeter matrix M^i. We have a natural embedding $W^i \hookrightarrow W, \tilde{r}_j \longmapsto r_j$, by which we identify W^i as a subgroup of W.

LEMMA 4.9. *$\pi: E^\circ \longrightarrow E_i^\circ$ is a W^i-equivariant map.*

PROOF. Let $j \in N \setminus \{i\}$ and let $x \in E^\circ$. Then

$$\begin{aligned} r_j x &= x - \langle x, e_j\rangle \check{e}_j \\ &= x - 2\langle x, e_j\rangle \sum B(e_j, e_k) e_k^\circ \overset{\pi}{\longmapsto} \tilde{x} - 2\langle\tilde{x}, e_j\rangle \sum_{k\neq i} B(e_j, e_k)\tilde{e}_k^\circ \\ &= \tilde{r}_j \tilde{x}. \end{aligned}$$

∎

REMARK. $W^i e_i^\circ = e_i^\circ$ so $W^i \subset W$ acts naturally on $E^\circ / \mathbb{R}e_i^\circ$. This action is identical to that obtained from the kaleidoscope $\mathcal{K}^i$.

Let f be the face $V(*) \cap r_i V(*) = V(*) \cap H_i$ of $V(*)$ (Proposition 4.7) and let $V(*_i)$ be the Voronoi region for $*_i$ in the kaleidoscope $\mathcal{K}^i$.

LEMMA 4.10. *$\pi_i(f) = V(*_i)$ and $f = \pi_i^{-1} V(*) \cap H_i$.*

PROOF.

$$\begin{aligned} \pi_i(f) &= \pi(W_{K' \cap i^\perp} F) \quad \text{(Proposition 4.7)} \\ &= W^i_{K' \cap i^\perp}(\pi F) \quad \text{(Lemma 4.9)} \\ &= W^i_{K_i'}(F^i) \quad \text{(Lemma 4.8)} \\ &= V(*_i) \quad \text{(definition).} \end{aligned}$$

The second part follows from Lemma 4.8. ∎

Assuming that Theorem 4.4 holds for all kaleidoscopes $\mathcal{K}(M')$, where M' is a $k \times k$ Coxeter matrix and $k < n = \operatorname{card} N$, we may apply it to $*_i = \sum_{j \in K_i} d_j e_j^\circ$ and $V(*_i) = W_{K_i'} F^i$ to conclude:

LEMMA 4.11. *(i) Every facet of $V(*_i)$ is convex and has a $W_{K_i'}$-translate f^i for which $f^i \cap F^i$ supports f^i (f^i is in special position).*

(ii) If f^i is in special position and $f^i \cap F^i = F^i_{S^i}$ then S^i satisfies

($\mathbf{C}^i$) *Every connected component of S^i in $\Gamma(M^i)$ contains points of K_i.*

(iii) With f^i as in (ii),

$$f^i = f^i_{S^i} := \bigcap_{w \in W^i_{S^i}} V(w *_i) = W_{K_i' \cap [S^i]^\perp} F^i_{S^i}.$$

(iv) $\operatorname{Stab}_{W^i}(f^i_{S^i}) = W^i_{S^i} \times W^i_{K' \cap [S^i]^\perp}$ *and $W^i_{S^i}$ is the pointwise fixer of $f^i_{S^i}$ in W^i.*

(v) Let

$$C^i := \{ S^i \subset N \setminus \{i\} \mid S^i \text{ satisfies } (\mathbf{C}^i) \}.$$

Then $f^i_{S^i} \longmapsto S^i$ is a bijection between the set of facets of $V(*_i)$ in special position and C^i. ∎

Let $C(i) := \{ S \in C \mid i \in S \}$. For each set $S \subset N$ for which $i \in S$ we define $S^i := S \setminus \{i\}$.

LEMMA 4.12. *The mapping $S \longmapsto S^i$ determines a bijection $C(i) \longrightarrow C^i$.*

PROOF. $S \in C(i) \Leftrightarrow i \in S$ and each connected component of S contains a point of $K \Leftrightarrow S = S^i \amalg \{i\}$ and each connected component of S^i contains a point of $K \setminus \{i\} \cup \operatorname{conn}(i) = K_i \Leftrightarrow S^i \in C^i$. ∎

Let $S \in C(i)$ and let $S^i := S \setminus \{i\}$. Define

$$f_S := \bigcap_{w \in W_S} V(w*).$$

LEMMA 4.13. *(i)* $f_S = \pi_i^{-1}(f^i_{S^i}) \cap H_i = \pi_i^{-1}(f^i_{S^i}) \cap f$,
(ii) f_S *is pointwise fixed by* W_S.

PROOF. Let

$$h := \bigcap_{w \in W_{S^i}} V(w*) \cap \bigcap_{w \in W_{S^i}} V(wr_i*).$$

Then from

$$\pi\big(V(w*) \cap V(wr_i*)\big) = \pi w\big(V(*) \cap V(r_i*)\big)$$
$$= w\pi\big(V(*) \cap V(r_i*)\big) = wV(*_i) = V(w*_i),$$

we see by Lemma 4.11(iii) that

$$\pi(h) \subset f^i_{S^i}.$$

To see that $\pi(h) = f^i_{S^i}$, let $\bar{x} \in f^i_{S^i}$ and let $x \in V(*) \cap V(r_i*)$ be a preimage. Then for all $w \in W_{S^i}$, $wx \in V(w*) \cap V(wr_i*)$ and $\pi(wx) = w\pi(x) = w\bar{x} = \bar{x}$, by Lemma 4.11(iv). Thus $wx = x + a(w)e^\circ_i$ for some $a(w) \in \mathbb{R}$. Since W_{S^i} fixes e°_i, $a\colon W_{S^i} \longrightarrow (\mathbb{R}, +)$ is a homomorphism, and since W_{S^i} is generated by involutions, $a \equiv 0$. This shows that $x \in h$ and hence $\bar{x} \in \pi(h)$.

In fact this argument shows that h is pointwise fixed by W_{S^i} and since also $h \subset V(*) \cap V(r_i*) \subset H_i$, it is pointwise fixed by r_i. Thus h is pointwise fixed by W_S. In particular $h \subset \bigcap_{w \in W_S} wV(*) = f_S$ and we have $h = f_S$. ∎

LEMMA 4.14. *Let* $S \in \mathcal{C}(i)$. *Then* f_S *is in special position and* $f_S \cap F = F_S$.

PROOF. We have $F_S \subset F \subset V(*)$ and F_S is pointwise fixed by W_S, whence $F_S \subset \bigcap_{w \in W_S} V(w*) = f_S$. Thus $F_S \subset f_S \cap F$.

Now f_S is the union of elements of $\mathcal{F}$ and in particular, from Theorem 2.1(iii)(b), $f_S \cap F$ is a union of sets F_T, $T \subset N$. Suppose that $F_T \subset f_S \cap F$. Then $F_T \subset H_i$ and so $i \in T$ and $F^i_{T^i} = \pi(F_T) \subset \pi f_S = f^i_{S^i}$. Now $f^i_{S^i}$ is in special position and $f^i_{S^i} \cap F^i = F^i_{S^i}$. Thus $F^i_{T^i} \subset F^i_{S^i}$ and so $T^i \supset S^i$, $T \supset S$, and $F_T \subset F_S$.

Since f_S is pointwise fixed by W_S, $f_S \subset \sum_{j \in N \setminus S} \mathbb{R} e^\circ_j$. Since $\sum_{j \in N \setminus S} \mathbb{R}_{\geq 0} e^\circ_j \subset F_S$ we see that F_S supports f_S and so f_S is in special position. ∎

LEMMA 4.15. *Set* $S \in \mathcal{C}(i)$. *Then* $f_S = W_{K' \cap S^\perp} F_S$.

PROOF. $f_S \overset{\pi_i}{\twoheadrightarrow} f^i_{S^i} = W_{K'_i \cap [S^i]^\perp} F^i_{S^i}$. But $K'_i \cap [S^i]^\perp = K' \cap i^\perp \cap [S^i]^\perp = K' \cap S^\perp$. Now $W_{K' \cap S^\perp} F_S \overset{\pi}{\twoheadrightarrow} W_{K'_i \cap [S^i]^\perp} F^i_{S^i}$, and since $W_{K' \cap S^\perp}$ stabilizes H_i $(i \in S)$, $W_{K' \cap S^\perp} F_S \subset H_i$. Together with $f_S \subset H_i$ (Lemma 4.13) and Lemma 4.8 we see that $f_S = W_{K' \cap S^\perp} F_S$. ∎

LEMMA 4.16. *Let* $S \in \mathcal{C}(i)$. *Then* $\mathrm{Stab}_W f_S = W_S \times W_{K' \cap S^\perp}$ *and* W_S *is the pointwise stabilizer of* f_S.

PROOF. The second statement is a consequence of Lemma 4.13 and Lemma 4.14.

Let $w \in \mathrm{Stab}_W f_S$. Then $V(*) \supset f_S \Rightarrow V(w*) \supset f_S \supset F_S$. Now $V(w*)$ is a union of sets $ww'F^0_T$, where $w' \in W_{K'}$ and $T \subset N$ (see the definition of $V(*)$) and hence for some $w' \in W_{K'}$ and some $T \subset N$, $ww'F^0_T = F^0_S$. Then $T = S$ and $ww' \in W_S$. Replacing

w by some other element w_1 of the coset $W_S w \subset \mathrm{Stab}_W f_S$, we may assume that $w_1 \in W_{K'} \subset W^i$. Then $w_1 f^i_{S^i} = w_1 \pi_i f_S = \pi_i w_1 f_S = \pi_i f_S = f^i_{S^i}$ and so by Lemma 4.11(iv), $w_1 \in W_{S^i} \times W_{K'_i \cap [S^i]^\perp}$. Finally $w \in W_S w_1 \subset W_S \times W_{K' \cap S^\perp}$. ∎

We now prove Theorem 4.4.

PROOF (THEOREM 4.4). Let g be any proper facet of $V(*)$. Then g lies in some face of $V(*)$ (see Remark 2), which, by Proposition 4.5, is supported by a set of the form wF_i, $w \in W_{K'}$, $i \in K$. Since $w^{-1}V(*) = V(*)$, we may replace g by $w^{-1}g$ and supppose that g is a facet of a face $f = V(*) \cap H_i$ supported by F_i for some $i \in K$. Using π_i we have a bijective linear map

$$\pi_i|_f : f \longrightarrow V(*_i)$$

and this necessarily determines a bijection between their facets. Thus some $W_{K'_i}$-translate $\pi_i(vg)$ of $\pi_i(g)$ is $f^i_{S^i}$, where $f^i_{S^i} \cap F^i = F^i_{S^i}$ and $S^i \in \mathcal{C}^i$ (see Lemma 4.11). Then $vg \subset vf = f$ (since $K'_i \subset i^\perp$) and we may thus assume that at the outset $g \subset f$, f is supported by F_i, $\pi f = V(*_i)$, and $\pi g = f^i_{S^i}$. In particular, g is convex.

Now $f \subset H_i$ and by Lemma 4.13, $g = \pi^{-1} f^i_{S^i} \cap H_i = f_S$, where $S = S^i \cup \{i\}$ and $S \in \mathcal{C}(i)$. Using Lemma 4.14 we obtain Theorem 4.4(i).

Now suppose that g above is already in special position and $g \cap F = F_T$. Our argument shows that for some $w \in W_{K'}$, and $i \in S$, $wg = f_S \subset f = V(*) \cap H_i$, where $wg \cap F = F_S$. Then from $f_S = W_{K' \cap S^\perp} F_S$ (Lemma 4.15), g is the union of sets $w^{-1}uF_S$, where $u \in W_{K' \cap S^\perp}$. It follows that $F_T = w^{-1}uF_S$ for some u and so $T = S$ and $w^{-1}u \in W_S$. Thus $g = w^{-1}f_S = w^{-1}uf_S = f_S$ by Lemma 4.16. This proves that every facet in general position is of the form f_S, where $S \in \mathcal{C}$. This proves Theorem 4.4(ii). Parts (iii) and (iv) have been proved in Lemmas 4.15 and 4.16.

Finally for (v), if $S \in \mathcal{C}$ then $S \in \mathcal{C}(i)$ for some $i \in K$ and by Lemma 4.13 and Lemma 4.11, $f^i_{S^i}$ is a facet of $V(*_i)$ and f_S is a facet of $V(*) \cap H_i$ in special position with $f_S \cap F = F_S$. This proves that the mapping of (v) is surjective. Since f_S is entirely determined by S (and K) (see Lemma 4.15) the mapping is injective.

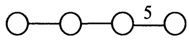
This completes the proof of the Theorem. ∎

5. **Classification of faces, facets, and dual cells by Coxeter diagrams.** In this section we explain how the meaning of Theorem 4.4 can be seen through decorations of the Coxeter diagram and how this leads to a simple algorithm for determining the facet structure of Voronoi cells $V(*)$. At the same time we will see that Theorem 4.4 contains all the relevant information about the facet dual to each facet and that the decorated Coxeter diagram simultaneously displays the Wythoff polytopes of these dual cells.

We assume then that $M = (M_{ij})_{(i,j)\in N\times N}$ is a Coxeter matrix and $\mathcal{K}(M)$ the corresponding kaleidoscope based on vector spaces E° and E with dual bases $\{e^\circ_i \mid i \in N\}$ and $\{e_i \mid i \in N\}$ respectively. We let

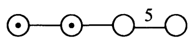
$$* := \sum_{j\in K} c_j e^\circ_j, \quad c_j > 0,$$

be a fixed point, $* \neq 0$, in the fundamental region F of E°. To illustrate the procedure that we will describe, we will use the kaleidoscope whose Coxeter diagram is



This is a spherical kaleidoscope and so may be viewed either as a linear kaleidoscope in four dimensional space or as a tesselation by spherical simplicies of the three dimensional surface of a sphere in 4-space. Our description below is taken in terms of the linear model. Thus we take the fundamental region as a simplicial cone bounded by rays. In the spherical model the fundamental region becomes a simplex bounded by vertices. It is merely a matter of convenience which description we use. In the case of root lattices (Section 8) we will use the classical models.

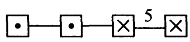
The first step is to indicate the nodes appearing in the support of $*$, *i.e.* those belonging to K, by marking each with a dot. In our example we will choose K as indicated here.



As we will see (and it is already clear from Theorem 4.4) the actual coefficients c_j of $*$ play no role in the description of $V(*)$ and of its facets. It is only the set K which is relevant. (In Section 8, where metrical considerations appear, the precise position of $*$ will be important).

R1. Replace each node by a square box and put an $\times$ in each box that is not indexed by K.

In our example we have



The box in position i stands for the ray $\mathbb{R}_{\geq 0}e_i{}^\circ$ and the entire collection of boxes stands for the convex hull C of the corresponding rays. The $\times$ in a box i indicates that we should allow the corresponding reflection r_i to operate on C and the entire collection of $\times$'s indicates the reflection group that is allowed to act on C.

Thus, in the diagram above, C is the convex hull of $\{\mathbb{R}_{\geq 0}e_i^\circ \mid i = 1, \ldots, 4\}$, which is in fact the fundamental chamber F, and the reflection group indicated is $\langle r_j \mid j \notin K\rangle = W_{K'}$. The diagram stands for

$$W_{K'}F = \bigcup_{w \in W_{K'}} wF$$

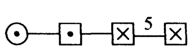
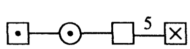
and this is precisely the definition of the Voronoi cell $V(*)$ (see Section 4).

To determine the faces of $V(*)$ we use two rules:

R2. Replace exactly one box, not marked by an $\times$, by a circle.

R3. Re-mark all the remaining boxes with $\times$'s according to the restrictions that no marked box may be joined by an edge to a circle node and no dotted node may be marked by an $\times$.

We obtain in our example two faces:

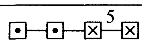
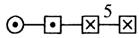
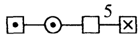
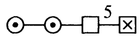
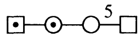
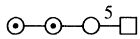
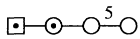
Face	Diagram	#
$V(*)$		1
2-face		1
2-face		5
1-face		5
1-face		5
0-face		5
0-face		1

TABLE 1. THE EXAMPLE OF SECTION 5. CLASSIFICATION OF FACES OF TESSELATION OF THE 3-DIMENSIONAL SURFACE OF A SPHERE IN 4-SPACE. DOTS INDICATE THE SUPPORT OF THE CHOSEN FIXED POINT $*$ OF THE FUNDAMENTAL REGION. THE SURFACE IS FORMED BY 1440 COPIES OF $V(*)$. THE NUMBER OF FACES OF $V(*)$ OF EACH TYPE IS SHOWN IN THE LAST COLUMN.

The interpretation of these is the same as above. The boxed nodes describe the rays generating a convex figure F_S and the $\times$'s indicate a reflection group acting on it.

Now this tallies precisely with Theorem 4.4. Every face of $V(*)$ has a unique $W_{K'}$-translate in standard position. If f is such a face then $f = W_{K'\cup i^\perp}F_i$, where $i \in K$ (see also Proposition 4.7). Our rules above force us to replace by a circle a box from K and to mark with $\times$'s those boxes in $K' \cap i^\perp$.

To obtain lower dimensional facets we continue in the same way, using rules R2 and R3. Since the boxes marked with $\times$'s are always the set of nodes orthogonal to the circled nodes, R2 guarantees that the circled nodes always form a set S whose components lead back to K. Rule R3 then provides the reflection group $W_{K'\cup S^\perp}$.

The entire procedure then gives rise to Table 1. It is not hard to determine more detailed information about $V(*)$. The group W of the kaleidoscope is of order 14400 [C] and the set $Q = W(*)$ has $14400/|W_{K'}| = 14400/10 = 1440$ points. There are then 1440 copies of $V(*)$ that tile the space. $V(*)$ has the dihedral group D_5 of order 10 as its symmetry group. In terms of the more easily visualized classical model it has $10/10 = 1$ pentagonal face and $10/2 = 5$ faces that are isoceles triangles.

At each stage of the determination of the facets f of the Voronoi cell we have a subdiagram of boxed nodes and a complementary diagram of circled nodes that has the familiar form of a Wythoff diagram [C]. This complementary diagram is exactly the Wythoff diagram that describes the convex hull of the set $W_S(*)$, which in turn is the facet f° of the Delaunay complex dual to the facet f. This is how the duality of the Voronoi and Delaunay cells is manifested in the partition of the Coxeter diagram. The Delaunay complex is described in more detail in Section 6 and Section 7.

6. **The Voronoi and Delaunay complexes.** Let

$$\mathcal{K} = \mathcal{K}(M) = (X_M, \mathcal{F}_M)$$

be a generalized kaleidoscope. Let $* \in F \setminus \{0\}$, $Q = W(*)$, and let

$$\mathcal{V} := \{wf_S \mid w \in W, S \in \mathcal{C}\}$$

be the corresponding set of all the reflection polyhedra and partially order it by inclusion. $\mathcal{V}$ with the partial ordering is the *Voronoi complex* of $(\mathcal{K}, *)$.

In $\mathcal{V}$ we find the extreme cases $wf_N = \{0\}$ and $wf_\emptyset = wW_{K'}F_\emptyset = wV(*)$.

LEMMA 6.1. *(i) Let $w \in W, S \in \mathcal{C}$. Then $wF_S \subset V(*)$ if and only if $wf_S = w'f_S$ for some $w' \in W_{K'}$ and $w \in W_{K'}W_S$. In either case wf_S is a facet of $V(*)$.*

(ii) Let $q \in Q, S \in \mathcal{C}$. Then $V(q) \supset f_S$ if and only if $q \in W_S(*)$. In either case f_S is in a facet of $V(q)$.

PROOF. (i) Suppose that $wF_S \subset V(*)$. Then $F_S \subset f_S \subset w^{-1}V(*) = w^{-1}W_{K'}F$ and hence $F_S = w^{-1}w'F_S$ for some $w' \in W_{K'}$ and $w^{-1}w' \in W_S$. Thus $wf_S = w'f_S$ and $w \in W_{K'}W_S$. We know that $w'f_S$ is a facet of $V(*) = W_{K'}F$. The converse is obvious.

(ii) Write $q = w^{-1}*$. Then $V(q) \supset f_S \Leftrightarrow wf_S \subset V(*) \Leftrightarrow w \in W_{K'}W_S \Leftrightarrow q = w^{-1}* \in W_SW_{K'}* = W_S(*)$. ■

LEMMA 6.2. *Let $w \in W$, $T \in \mathcal{C}$. Then $wf_T \cap V(*)$ is a facet of $V(*)$.*

PROOF. We will use induction on $\ell(w)$. If $\ell(w) = 1$ it is obvious. Suppose that $\ell(w) > 1$ and choose i so that $\ell(r_iw) < \ell(w)$. Set $g := wf_T \cap V(*)$. If $i \in K'$ then $r_iV(*) = V(*)$ and we have $r_ig = r_iwf_T \cap V(*)$. This is a facet of $V(*)$.

Suppose that $i \in K$. Then F and wF lie on opposite sides of H_i [B, Chapter V, Section 4.4,(P_n)] and hence $wF_T \cap V(*) \subset H_i$. It follows that $g \subset H_i$. Thus

$$\begin{aligned} g = r_ig &= r_iwf_T \cap V(r_i*) \cap H_i \\ &= r_iwf_T \cap V(*) \cap H_i \subset r_iwf_T \cap V(*) =: g'. \end{aligned}$$

By the induction assumption g' is a facet of $V(*)$. Finally

$$g' \cap f_i = g' \cap V(*) \cap H_i = g' \cap H_i = g' \cap r_ig' = g$$

is a facet of $V(*)$ since the intersection of any two facets of a convex set is again a facet (see the definition of facets in Section 4). ■

LEMMA 6.3. *(i) $\bigcup_{g \in \mathcal{V}} g = X$,*
(ii) if $g, h \in \mathcal{V}$ then $g \cap h \in \mathcal{V}$.

PROOF. (i) is obvious from Theorem 2.1. For (ii) we can suppose that h is a facet of $V(*)$. Now

$$g = uf_S = u \bigcap_{w \in W_S} V(w*)$$

for some $S \in \mathcal{C}, u \in W$.

It will suffice to show that $h \cap uwV(*)$ is a facet of $V(*)$ for each $w \in W_S$, for we know that the intersection of facets of $V(*)$ is a facet of $V(*)$. But $h \cap uwV(*) = uw\big((uw)^{-1}h \cap V(*)\big)$ is a facet of $uwV(*)$ lying in $V(*)$. By Lemma 6.2 it is a facet of $V(*)$. ■

COROLLARY 6.4. *Let $R \subset Q$ be any subset, $R \neq \emptyset$. Then there exists $S \in \mathcal{C}, w \in W$ so that*

(i) $\bigcap_{r \in R} V(r) = w \bigcap_{u \in W_S} V(u*)$,

(ii) $R \subset wW_S(*)$.

PROOF. (i) $\bigcap_{r \in R} V(r)$ is a facet and we know that every facet has the form of the right hand side of (i).

(ii)

$$r \in R \Rightarrow V(r) \supset wf_S \Rightarrow V(w^{-1}r) \supset f_S \Rightarrow w^{-1}r \in W_S(*)$$

by Lemma 6.1(ii). ∎

We are now ready to define a second complex called the *Delaunay complex*. In the setting here it is a combinatorial object made of subsets of the set $Q = W(*)$ or equivalently from cosets of $W/W_{K'}$. In the classical cases we will provide a more geometric interpretation for it.

For each $S \in \mathcal{C}$ we define

$$f_S^\circ := W_S(*)$$

with the extreme cases $f_N^\circ = W(*) = Q, f_\emptyset^\circ = \{*\}$. The Delaunay complex is the set of subsets

$$\mathcal{D} := \{wf_S^\circ \mid S \in \mathcal{C}, w \in W\}$$

together with the partial ordering by inclusion. The set wf_S° is called the *facet* of $\mathcal{D}$ dual to wf_S. Of special importance are the cells wf_S°, where $S = N \setminus \{j\}$ for some j. In this case wf_S is the ray $w\mathbb{R}_{\geq 0}e_j^\circ$ and the dual facet is $D(w\mathbb{R}_{\geq 0}e_j^\circ) = wW_{N\setminus\{j\}}(*)$. By abuse of language we call the sets $w\mathbb{R}_{\geq 0}e_j^\circ, N \setminus \{j\} \in \mathcal{C}$, the *vertices* of $\mathcal{V}$.

We could have also defined the Delaunay complex using rays $W_S(\mathbb{R}_{\geq 0}*)$. However there is nothing to be gained from this.

PROPOSITION 6.5. *For $S, T \in \mathcal{C}$, $w, w' \in W$,*

$$wf_S^\circ \subset w'f_T^\circ \Leftrightarrow wf_S \supset w'f_T.$$

PROOF. $wW_S(*) \subset w'W_T(*) \Rightarrow w\bigcap_{v \in W_S} V(v*) \supset w'\bigcap_{v \in W_T} V(v*) \Rightarrow wf_S \supset w'f_T$. Conversely

$$\begin{aligned} wf_S \supset w'f_T &\Rightarrow w \bigcap_{v \in W_S} V(v*) \supset w' \bigcap_{u \in W_T} V(u*) \\ &\Rightarrow w'^{-1}wV(v*) \supset \bigcap_{u \in W_T} V(u*), \quad \text{for all } v \in W_S, \\ &\Rightarrow w'^{-1}wv \in W_T W_{K'}, \quad \text{for all } v \in W_S \text{ by Lemma 6.1}, \\ &\Rightarrow w'^{-1}wW_S \subset W_T W_{K'} \\ &\Rightarrow wf_S^\circ \subset w'f_T^\circ. \end{aligned}$$

∎

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PROPOSITION 6.6. *(i) Any properly ascending chain of facets of $\mathcal{D}$ has at most $n+1$ elements.*

(ii) The intersection of any collection of facets of $\mathcal{D}$ is again a facet.

PROOF. (i) A properly ascending chain of dual facets implies a properly descending chain of facets in $\mathcal{V}$ and since dimensions drop, the result follows.

(ii) From (i) it will suffice to show that the intersection of any two facets of $\mathcal{D}$ is a facet of $\mathcal{D}$. We will show that $f_S^\circ \cap wf_T^\circ$ is an facet of $\mathcal{D}$.

By Lemma 6.1, $r \in f_S^\circ \Leftrightarrow r \in W_S(*) \Leftrightarrow V(r) \supset f_S$ and similarly $r \in wf_T^\circ \Leftrightarrow V(r) \supset wf_T$.

Thus

$$R := f_S^\circ \cap wf_T^\circ = \{r \in Q \mid V(r) \supset f_S \cap wf_T\}.$$

But $f_S \cap wf_T$ is some facet vf_P of $V(*)$ and

$$\{r \in Q \mid V(r) \supset vf_P\} = vf_P^\circ. \qquad \blacksquare$$

7. **The classical kaleidoscopes.** In order to make use of Delaunay complex, we return to the case of a classical kaleidoscope $\bar{\mathcal{K}}(M)$ (M indecomposable) with its fundamental region $\bar{F}$ and Voronoi complex $\bar{\mathcal{V}}$ determined from $\mathcal{V}$ by intersection with $\bar{X}$. Each ray $w(\mathbb{R}_{\geq 0}e_j^\circ)$ contains exactly one point on $\bar{X}$ and these points are the *vertices* of $\bar{\mathcal{V}}$. They are also called the *holes* of $\mathcal{V}$ or Q and are the points of $\bar{X}$ that locally are most distant from the points of Q. The set of holes is denoted by Ω.

We will assume that our base point $*$ lies in $\bar{F}$, whence the entire Delaunay complex $\mathcal{D}$ consists of subsets of $\bar{X}$.

For any set A of $\bar{X}$, $\langle A\rangle_{\mathrm{conv}}$ will denote its convex hull. Convexity for hyperbolic and spherical spaces is defined, as for Euclidean spaces, by declaring that for each two points x and y of the set the geodesic $[x,y]$ also lies in the set.

For $S \in \mathcal{C}$ we define

$$\overline{wf_S^\circ} = w\bar{f}_S^\circ = \langle wf_S^\circ\rangle_{\mathrm{conv}} = \langle wW_S(*)\rangle_{\mathrm{conv}}.$$

For each $S \subset N, S \neq N$, W_S is a finite group and hence $w\bar{f}_S^\circ$ is a closed bounded convex set in $\bar{X}$.

The set of convex sets $w\bar{f}_S^\circ$, $w \in W$, $S \in \mathcal{C}$, partially ordered by inclusion, is called the *geometric* Delaunay complex for $\left(\bar{\mathcal{K}}(M), *\right)$. Its maximal elements, dual to the vertices, are called the *Delaunay cells* and are denoted by $D(\alpha)$, $\alpha \in \Omega$.

PROPOSITION 7.1. *$W_S(*)$ is the set of extreme points of $\bar{f}_S^\circ$.*

PROOF. Certainly the extreme points of $\bar{f}_S^\circ$ lie in the set $W_S(*)$. But $W_S(*)$ is a group of symmetries of $\bar{f}_S^\circ$ and is transitive on $W_S(*)$. Thus all the points of $W_S(*)$ are extreme. $\blacksquare$

PROPOSITION 7.2. *Let $\bar{X}$ be as above. For each facet $\bar{f}_S$ of $\overline{V(*)}$, $W_S(*)$ is precisely the set of points of Q that are closest to $\bar{f}_S$ (each point $x \in \bar{f}_S$ is equidistant from the points of $W_S(*)$ and more distant from every other point of Q).*

PROOF. Since $\bar{f}_S$ is a facet of the Voronoi region $\overline{V(*)}$, each point x of $\bar{f}_S$ is as close to $*$ as to any other point of Q. Now the distance from x to $q \in Q$ is minimized if and only if $x \in V(q)$, and hence $\bar{f}_S$ is as close to q as to $*$ if and only if $\bar{f}_S \subset V(q)$. But $\bar{f}_S \subset V(q) \Leftrightarrow q \in W_S(*)$ by Lemma 6.1(ii). ∎

The classical definition [CS] of Delaunay facet $\bar{f}^\circ$ dual to a facet $\bar{f}$ of the Voronoi complex $\bar{\mathcal{V}}$ is precisely

$$\begin{aligned}\bar{f}^\circ &= \langle q \in Q \mid q \text{ as close as possible to } f\rangle_{\text{conv}} \\ &= \langle q \in Q \mid f \text{ is a facet of } V(q)\rangle_{\text{conv}}.\end{aligned}$$

This explains our definition of the Delaunay complex of the generalized kaleidoscope. For $S \in \mathcal{C}, f_S^\circ$, defined above, coincides with the classical definition.

It also explains why the part of diagram made up of the circle-nodes determines, as a Wythoff polytope, the facet dual to the facet described by the complementary part of the diagram. A Wythoff polytope is by definition the convex hull of an orbit of points defined by a point $*$ that is in some facet of the fundamental region. The facet in which $*$ lies is indicated by dotting the nodes corresponding to the extreme points of this facet. The entire set of nodes of the Wythoff diagram describes the Coxeter group that is to be used to construct the orbit. In our case the orbits are the sets $W_S(*)$. For more on Wythoff polytopes consult [C] and [CS1].

We now ask under what conditions the geometric Delaunay complex is itself the Voronoi complex of the set Ω of holes of Q. This has a remarkably simple answer that leads us in Section 8 to the determination of the Voronoi complex of any weight lattice.

LEMMA 7.3. *A necessary and sufficient condition for the cells $D(\alpha), \alpha \in \Omega$, to be Voronoi cells for Ω is*

(Vor 1) *for all pairs $\alpha, \beta \in \Omega$, and for all $x \in D(\alpha) \cap D(\beta)$, $d(x, \alpha) = d(x, \beta)$.*

PROOF. Suppose that the condition (Vor 1) holds. $D(\alpha)$ is a closed convex body. Its faces are sets $[\alpha, \beta]^\circ$, where $\beta \in \Omega$ and $[\alpha, \beta]$ is an edge of the Voronoi complex $\mathcal{V}$. For each $\gamma \in \Omega$ let $H_{\alpha,\gamma}$ denote the perpendicular bisector of $[\alpha, \gamma]$ and let $A_{\alpha,\gamma}$ denote the corresponding closed halfspace that contains α. Then by (Vor 1) the face corresponding to $[\alpha, \beta]$ is $D(\alpha) \cap H_{\alpha,\beta}$ and

$$D(\alpha) = \bigcap_{[\alpha,\beta] \text{ an edge in } \mathcal{V}} A_{\alpha,\beta}$$

(see [Br], Corollary 9.4).

On the other hand the Voronoi cell $V_\Omega(\alpha)$ for α in the set Ω is $\bigcap_{\gamma\in\Omega} A_{\alpha,\gamma}$ by definition. Thus $V_\Omega(\alpha) \subset D(\alpha)$. Now both $\{V_\Omega(\alpha) \mid \alpha \in \Omega\}$ and $\{D(\alpha) \mid \alpha \in \Omega\}$ consist of compact polyhedral sets that cover $\bar{X}$ with overlaps of measure 0 (they tile $\bar{X}$). If $V_\Omega(\alpha) \neq D(\alpha)$ then $D(\alpha) \setminus V_\Omega(\alpha)$ contains nonempty open set and it is impossible for $\{V_\Omega(\alpha) \mid \alpha \in \Omega\}$ to tile $\bar{X}$. Thus $D(\alpha) = V_\Omega(\alpha)$ is a Voronoi cell.

Conversely if, for all $\alpha \in \Omega, D(\alpha)$ is a Voronoi cell then (Vor 1) is obvious. ∎

Let $\alpha, \beta \in \Omega$ and suppose that $D(\alpha) \cap D(\beta) \neq \emptyset$. Then $D(\alpha) \cap D(\beta) = \bar{f}^\circ$ for some facet $\bar{f}$ of $\mathcal{V}$. As far as condition (Vor 1) is concerned, it will be equivalent to consider those cases in which $\bar{f} = \bar{f}_S$ is in standard form. Then $\bar{f}_S^\circ = \langle (W_S(*)\rangle_{\text{conv}}$ and (Vor 1) can be rewritten as

(Vor 2) for all facets $\bar{f}_S$ in standard form, for all $\alpha, \beta \in \Omega$ with $D(\alpha) \cap D(\beta) = \bar{f}_S^\circ$, and for all $w \in W_S$, $d(w*, \alpha) = d(w*, \beta)$.

Now for $\alpha \in \Omega$ and $\bar{f}_S^\circ \subset D(\alpha)$ we have α is a vertex of $\bar{f}_S = W_{K' \cap S^\perp} \bar{F}_S$, whence α is expressible as $\alpha = u\alpha_\circ$, where $u \in W_{K' \cap S^\perp}, \alpha_\circ \in \Omega \cap \bar{F}_S$.

We have

$$\begin{aligned} d(w*, \alpha) &= d(w*, u\alpha_\circ) = d(*, w^{-1}u\alpha_\circ) \\ &= d(*, uw^{-1}\alpha_\circ) \quad \{\text{since } w \in W_S, u \in W_{S^\perp}\} \\ &= d(u^{-1}*, \alpha_\circ) \quad \{\text{since } \alpha_\circ \in \bar{F}_S\} \\ &= d(*, \alpha_\circ) \quad \{\text{since } u \in W_{K'}\}. \end{aligned}$$

Similar considerations with β lead to the new sufficient condition for the cells $D(\alpha)$ to be Voronoi cells:

(Vor 3) for all $S \in \mathcal{C}$, for all $\alpha_\circ, \beta_\circ \in \Omega \cap \bar{F}_S$, $d(*, \alpha_\circ) = d(*, \beta_\circ)$.

Taking $S = \emptyset$, we obtain the sufficiency of

PROPOSITION 7.4. *A necessary and sufficient condition for the set of Delaunay cells $D(\lambda), \lambda \in \Omega$, to be the Voronoi cells for Ω is*

(Vor) *for all $\alpha, \beta \in \Omega \cap \bar{F}$, $d(*, \alpha) = d(*, \beta)$.*

PROOF. It remains to see that the condition is necessary. If $\alpha, \beta \in \Omega \cap \bar{F}$ then $* \in D(\alpha) \cap D(\beta)$ and so for $D(\alpha)$ and $D(\beta)$ to be Voronoi domains we need $d(*, \alpha) = d(*, \beta)$. ∎

The discussions of Voronoi and Delaunay cells in the vector space set-up do not require any particular consideration for decomposable Coxeter matrices, and indeed no such assumption was made in Section 4, Section 5, Section 6. However for the classical kaleidoscopes our discussion so far has been restricted to indecomposable Coxeter matrices.

To consider the decomposable case we begin with some generalities. If (X_i, d_i), $i = 1, \ldots, k$, are metric spaces and Q_i is a discrete set of points in X_i for each i, then in $X := X_1 \times \cdots \times X_k$ with the metric

$$d\big((x_1, \ldots, x_k), (y_1, \ldots, y_k)\big)^2 := d(x_1, y_1)^2 + \cdots + d(x_k, y_k)^2$$

and the discrete set

$$Q := Q_1 \times \cdots \times Q_k$$

we have for all $q = (q_1, \ldots, q_k) \in Q$

$$\mathrm{Vor}(q) = \mathrm{Vor}(q_1) \times \cdots \times \mathrm{Vor}(q_k)$$

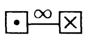
with facets $f = f_1 \times \cdots \times f_k$ and dual facets $f^\circ = f_1^\circ \times \cdots \times f_k^\circ$.

Thus the entire business of Voronoi and Delaunay cells extends in a straighforward way to the product of spaces.

Now if M is a Coxeter matrix with indecomposable components $M_1, \ldots, M_k$ then the kaleidoscope $\mathcal{K}(M)$ can be indentified naturally with the product $\mathcal{K}(M_1) \times \cdots \times \mathcal{K}(M_k)$ with Tits cone $X := X_1 \times \cdots \times X_k$. As we have just indicated, the results of Section 4, Section 5, Section 6 apply to M and are simply the results that one would obtain by glueing together the corresponding results for the individual components.

Now suppose that each indecomposable component is of classical type. Then we may form the classical kaleidoscopes $\bar{\mathcal{K}}(M_i)$. We define the kaleidoscope $\bar{\mathcal{K}}(M)$ to be the product of the spaces $\bar{X}_i$ with simplicial complex $\bar{\mathcal{F}} := \bar{\mathcal{F}}_1 \times \cdots \times \bar{\mathcal{F}}_k$. Now we already know that $\mathcal{F}_i$ and $\bar{\mathcal{F}}_i$ differ only in that there is no facet of $\bar{\mathcal{F}}_i$ corresponding to $\{0\}$ in $\mathcal{F}_i$. Thus the facets of Vor($*$) and the corresponding dual facets can be read directly from the Coxeter diagrams as before with the single change that no component diagram can consist entirely of circle-nodes (for then the corresponding Voronoi facet is $\{0\}$).

A simple example will illustrate the point. We consider the classical kaleidoscope $\bar{\mathcal{K}}$ whose Coxeter graph is



This is a straight line tesselated by reflected images of a line segment. The Voronoi and Delaunay facets for the product $\bar{\mathcal{K}} \times \bar{\mathcal{K}}$ are given by the following set of diagrams:

Note that

do not appear in this.

8. **The Voronoi and Delaunay complexes for root and weight lattices.** We now apply the results of the previous sections to the problem of determining the Voronoi and Delaunay cells and their facets for the root and weight lattices of the simple and semisimple Lie groups. We treat only the indecomposable root lattices and their weight lattices (the simple group case). The semisimple case can be handled by using the discussion above.

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The Voronoi and Delaunay cells for indecomposable root lattices have been described in detail in [MP], so we will simply recall here how their structure is related to the kaleidoscope picture.

Let Q be an indecomposable root lattice and let Γ denote the Coxeter-Dynkin diagram of the underlying root system Δ. Let $\hat{\Gamma}$ denote the extension of Γ to an affine or Euclidean Coxeter-Dynkin diagram by using the lowest *short* root. This diagram forms the basic diagram for the decoration process. Let $\hat{\Gamma}^\circ$ denote the diagram obtained from $\hat{\Gamma}$ by reversing the arrows and let $m_i, i = 0, \dots, n$, be the *marks* of $\hat{\Gamma}^\circ$. For $i = 1, \dots, n$ these are the coefficients of the highest root of $\check{\Delta}$, the root system dual to Δ. We have $m_0 := 1$. These numbers m_i are used to describe the fundamental region.

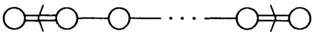
Corresponding to $\hat{\Gamma}$ there is an affine chamber complex $\mathcal{A}$ in a Euclidean space $\mathbb{R}^n$ (the real span of Δ) whose Weyl group W is a semidirect product of the finite Weyl group W_{fin} of Δ (isotropy group of 0) and the group Q acting on $\mathbb{R}^n$ by translations. If $\{\omega_1, \dots, \omega_n\}$ is the corresponding set of fundamental weights then the chamber complex has fundamental region $\bar{F}$ with vertices 0 and $\{\omega_i/m_i \mid i = 1, \dots, n\}$. Then $\mathcal{A}$ is a model of the Euclidean kaleidoscope $\bar{\mathcal{K}}(M)$, where M is the Coxeter matrix with $m_{ij} = 2, 3, 4, 6, \infty$ according as the Cartan matrix entries A_{ij} satisfy $A_{ij}A_{ji} = 0, 1, 2, 3, 4$.

The root lattice Q is $W(0)$ and hence, by choosing $* = 0$, we may determine the Voronoi and Delaunay cell complexes directly from the kaleidoscope. We illustrate this with the examples of the root lattice of type B_n.

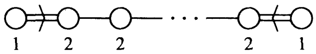
The Coxeter-Dynkin diagram is

Γ: 

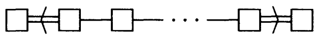
the extended diagram (by the lowest short root) is

$\hat{\Gamma}$: 

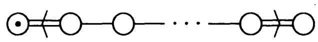
and the marked dual diagram is

$\hat{\Gamma}^\circ$: 

The fundamental simplex for the chamber geometry associated with $\hat{\Gamma}$ is the convex hull of $\{0\}$ and $\{\omega_1/1, \omega_2/2, \dots, \omega_{n-1}/2, \omega_n/1\}$. In our notation this set of vertices is indicated by



The kaleidoscope with $* = 0$ chosen is described as



In the context of root lattices we prefer to use the Coxeter-Dynkin diagrams (*i.e.* diagrams with arrows that indicate the relative lengths of the basic root vectors) as we have been doing here, rather than the straight Coxeter diagrams. As far as classification of facets

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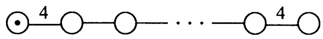
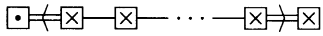
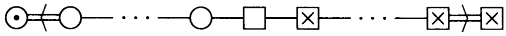
goes there is no difference between them. Thus the diagram above has the same meaning as



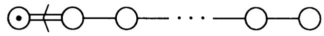
The Voronoi cell is given according to Section 5 as



with facets and dual facets given by the diagrams



and Delaunay cells

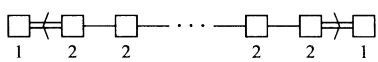


The determination of the Voronoi cells of weight lattices has to be made in a somewhat different way. The weight lattice P contains the root lattice Q with index $c = [P : Q]$ equal to the order of the centre of the corresponding simply connected simple Lie group. The different classes of weights modulo Q are called *congruence classes*. We know from above that the fundamental simplex $\bar{F}$ for the affine kaleidoscope, used to determine the Voronoi complex of Q, has vertices ω_i/m_i, $i = 1, \ldots, n$ and $\{0\}$. It is well known, and in any case easy to check case by case, that there are exactly c values of i for which $m_i = 1$ (including $m_0 = 1$). The vertex $\{0\}$ and the vertices are ω_i/m_i for which $m_i = 1$ occur at the *tips* of the extended Coxeter-Dynkin diagram and $\{0, \omega_i/m_i \mid m_i = 1\}$ is a complete set of representatives for the congruence classes. Since the congruence classes are W-invariant and W contains the translations by Q we obtain P as the disjoint union of $Q = W(0)$ and the sets $W(\omega_i)$, $m_i = 1$. The first thing then, is that since there is in general more than one congruence class, P is not a single W-orbit, and hence P cannot be written as $W*$ for some $* \in \hat{F}$. Thus we cannot use the kaleidoscope in the usual way to obtain the Voronoi cells of P.

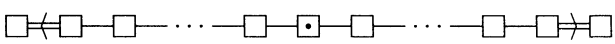
However according to Proposition 7.4, under appropriate conditions we may also find Voronoi cells for the holes Ω of $W*$. Thus in our affine kaleidoscope we want $\{0, \omega_i \mid m_i = 1\}$ to be a full set of representatives for the holes of some set $\underset{\sim}{Q} = W*$, or equivalently, we want the tips of the diagram to classify the vertices of the Voronoi cells of $\underset{\sim}{Q}$.

Using Proposition 7.4 we can see easily that in all cases except one there is exactly one solution to this problem. The exception is E_8, where there is no description of the Voronoi cells of P by Delaunay cells. However, since in this case $P = Q$ we already have a description of the Voronoi cells of Q.

As an illustration consider the weight lattices B_n^* of types B_n. The kaleidoscope is determined by the Coxeter-Dynkin diagram and values $\{m_i\}$:



(with $(n+1)$ nodes). The diagrams (where the dots are centrally located)

n even: 

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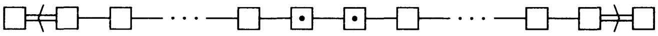
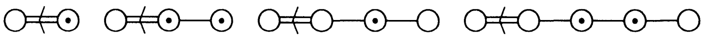
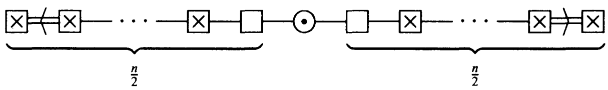
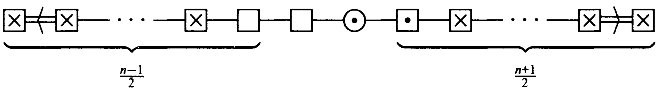
n odd: 

determine sets Q for which the holes are represented by the two tip boxes. Choosing $* \in \bar{F}$ equidistant from the two holes lying in $\bar{F}$, which is clearly possible by the symmetry of the diagram, we find from Proposition 7.4 the Wythoff constructions



and so on for entire series of weight lattices B_n^*. These diagrams appear in [CS1] for the D-type weight lattices. But the weight lattices B_n^* and D_n^* are isomorphic. If we go through the same procedure for the weight lattice D_n^*, we obtain the same Wythoff polytopes but now in terms of D type diagrams. This is also how they appear in [CS2]. The Delaunay cells (up to symmetries) are

n even: 

n odd: 

and the facets are obtained by the usual convention of removing boxes.

The weight lattices for the remaining cases $A_n, B_n, C_n, D_n, E_6, E_7, F_4$, and G_2 are equally easy to work out. We have listed this information in Table 2.

9. **Contact polytopes.** Let Q be a discrete set of points in the metric space (X, d). For each $q \in Q$ we may form the inscribed sphere $S(q)$ of $V(q)$—the largest sphere centered on q that lies entirely inside $V(q)$. The spheres $S(q)$ as q runs through Q then produce a sphere packing of the space X. Sphere packings for Euclidean spaces have a long and important history and the subject is still replete with unanswered questions [CS1].

Each sphere $S(q)$ touches several neighbouring spheres of the sphere packing, the points of contact lying on various faces of $V(q)$. The convex hull of the contact points of $S(q)$ form the *contact polytope* about q. In the case of lattices or in the case that Q is a single orbit under a group of isometries of X, all the contact polytopes are of course isometric.

In the subsequent discussion we will assume that we are in the setting of a generalized kaleidoscope $\mathcal{K} = (X, \mathcal{F})$ with a W-invariant metric or in one of the classical kaleidoscopes.

The contact point c determined by the two spheres $S(q)$ and $S(p)$ is the midpoint of the line segment $[q,p]$ and hence lies in a face f of $V(q)$ whose dual facet is precisely the edge $[q,p]$. Thus we need to determine the midpoints of the edges of the Delaunay cells. If there is more than one orbit of edges we also need to determine which of the midpoints are actually closest to q, for only these can be contact points.

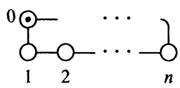
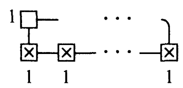
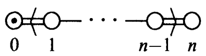
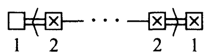
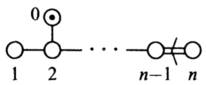
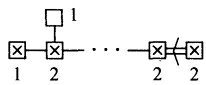
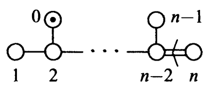
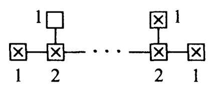
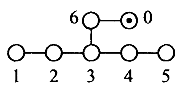
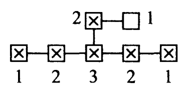
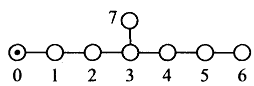
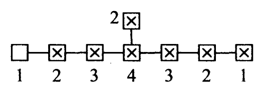
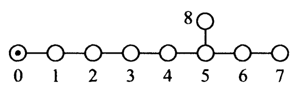
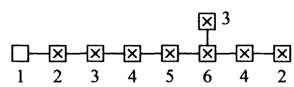
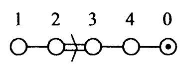
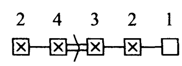
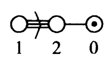
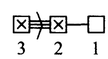
Type	Root lattice diagram with numbering of nodes	$V(0)$ diagram with marks	Weyl group order
A_n, $n \geq 2$			$(n+1)!$
B_n, $n \geq 2$			$2^n n!$
C_n, $n \geq 3$			$2^n n!$
D_n, $n \geq 4$			$2^{n-1} n!$
E_6			$2^7 3^4 5$
E_7			$2^{10} 3^4 5 7$
E_8			$2^{14} 3^5 5^2 7$
F_4			$2^7 3^2$
G_2			$2^2 3$

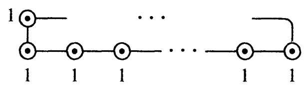
TABLE 2. LATTICE DIAGRAMS WITH NUMBERING OF NODES, DECORATED DIAGRAMS OF THE VORONOI CELL $V(0)$ WITH MARKS, AND ORDERS OF THE WEYL GROUP.

PROPOSITION 9.1. *The contact polytope around the point q of the generalized kaleidoscope $\mathcal{K}$ is the convex hull of those midpoints of the edges of the Delaunay cells that are closest to q.*

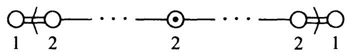
Now we consider the indecomposable root lattice $Q = W(0)$ in the classical Euclidean kaleidoscope. There is only one orbit of edges for the Delaunay cells and the midpoint of the edges is represented by the midpoint of the edge defined by 0 and the highest short root ξ_s, ie. by $\xi_s/2$. See also [MP1] for more explanation of this. The contact polytope

Kaleidoscope

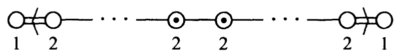
A_n



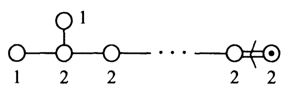
B_n n even



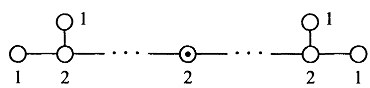
B_n n odd



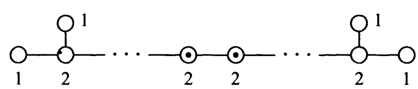
C_n



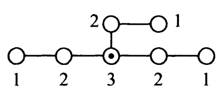
D_n n even



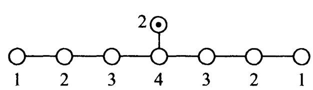
D_n n odd



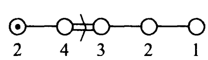
E_6



E_7



F_4



G_2

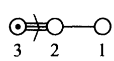


TABLE 3A. DIAGRAMS OF THE KALEIDOSCOPES WITH MARKS FOR THE WEIGHT LATTICES A_n, B_n, C_n, D_n, E_6, E_7, F_4, G_2.

around 0 is the convex hull of the orbit of this point under $\mathrm{Stab}_W(0)$ and hence is a Wythoff polytope. The extension node of the affine diagram gives the necessary information about which facet of the fundamental region the point $\xi_s/2$ lies. This leads at once to the results in the first part of Table 4.

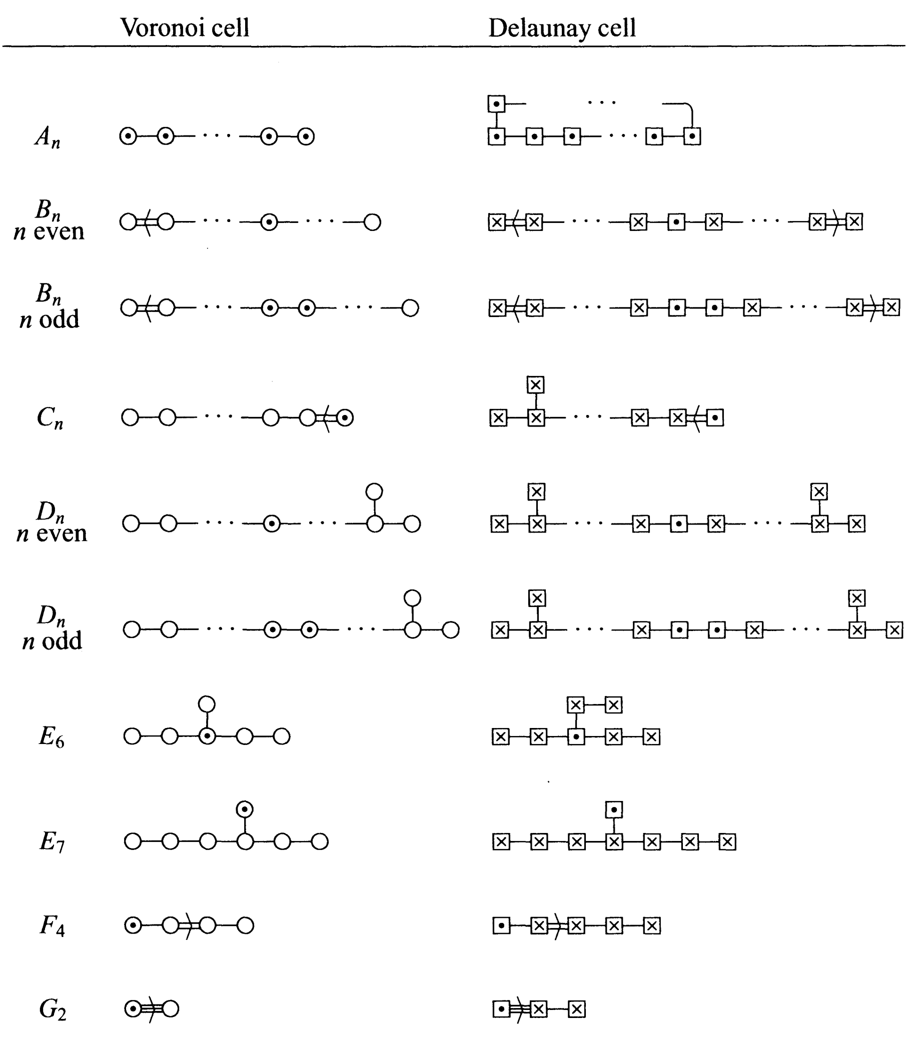
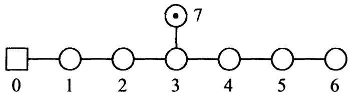


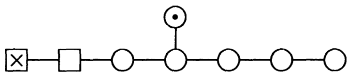
TABLE 3B. DIAGRAMS OF THE VORONOI AND DELAUNAY CELLS FOR THE WEIGHT LATTICES A_n, B_n, C_n, D_n, E_6, E_7, F_4, G_2.

The determination of the contact polytopes for the weight lattices is slightly more complicated because there are several orbits of midpoints to deal with and they are not in general equally close to the origin. We have to determine these distances in order to select the correct orbit(s). The illustration of the argument in the cases of E_7^* and A_n^* will serve to show what is involved. The tables of 'quadratic form matrices' in [BMP] are useful for looking up the appropriate square lengths. The results for all the weight lattices are shown in the second part of Table 4.

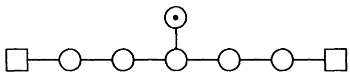
The Voronoi cell of the weight lattice E_7^* together with its corresponding dual (a vertex of the Delaunay cell) is given by the diagram



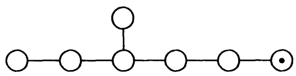
Here the dual is a single point that is the centre of the Voronoi cell and is the vertex 0 of the fundamental region $\bar{F}$. The edges of the Delaunay cells are of two types derived from the diagrams



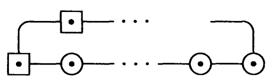
and



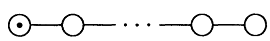
The midpoints of these edges are ω_1 and $\omega_6/2$. Their square lengths [BMP] are respectively $(\omega_1, \omega_1) = 2$ and $(\omega_6/2, \omega_6/2) = 3/8$ respectively. Thus the contact points are given by the orbit under $W(E_7)$ of $\omega_6/2$:



In the case of the weight lattice A_n^* there are two orbits of edges,

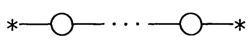


and its mirror image that are equivalent under the full symmetry group of the lattice. These lead to two orbits of contact points determined by $\omega_1/2$ and $\omega_n/2$. Each orbit gives rise to a Wythoff polytope:



and its mirror image.

The contact polytope is the convex hull of these two simplices that are in fact oppositely opposed (-1 interchanges the two orbits). In the notation of Conway and Sloane this is indicated as



where the stars indicate that the polytope is the convex hull of the union of the Wythoff orbits determined by the individually starred nodes.

They call it a diplo-simplex. Diplo-polytopes also occur in the case of E_6.

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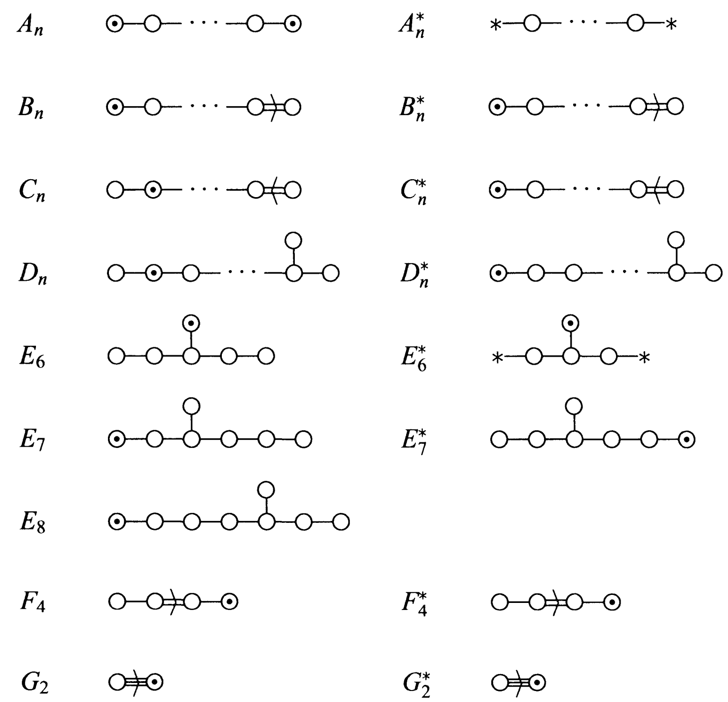

TABLE 4. CONTACT POLYTOPES FOR THE ROOT LATTICES AND THEIR DUALS.

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