SUBRINGS OF THE MAXIMAL RING OF QUOTIENTS ASSOCIATED WITH CLOSURE OPERATIONS

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1. Introduction. This paper contains a number of results that grew out of an attempt to solve the following problem: Given a non-commutative ring R with suitable ascending chain condition, and a prime ideal P in R, to construct a corresponding local ring R_P in which the extension P' of P is a unique maximal prime, and to prove, if possible, that the intersection of the powers of P' is zero. The present investigation is at best a preliminary attack on this problem since the contribution to the complete solution is comparatively small and the central problem of the intersection of the powers of P' has not been touched. Nevertheless it is hoped that the method introduced may yet prove fruitful and that publication of this method and the results obtained so far may stimulate interest in non-commutative localization problems.

It is well known that in the commutative case a ring of quotients R_M can be constructed corresponding to any multiplicatively closed set M such that $0 \notin M$. If M contains no zero divisors, R_M is an extension of R. Otherwise it is an extension of a homomorphic image of R. The multiplicative system M also defines a closure operation $c(M)$ on the ideals of R, namely the mapping $A \to A_M$, where

$$A_M = \{x \in R \mid mx \in A \text{ for some } m \in M\}. \tag{1.1}$$

If $E(A)$ is the extension of the ideal A of R to R_M and if $C(A')$ is the contraction of the ideal A' of R_M to R (see **12**, p. 218), then the ring R_M and the closure $c(M)$ are related by

$$A^{c(M)} = C[E(A)].$$

If R is a Noetherian ring and M is the complement in R of a prime ideal P, then $R_M = R_P$ is a local ring in which $E(P)$ is the unique maximal prime.

In a non-commutative ring R the role of the multiplicative system M is usually taken by an m-system (**8**, **9**, **1**), since an ideal in R is prime if and only if its complement in R is an m-system. If M is any m-system in R a closure operation analogous to (1.1) can be defined on the ideals of R. (See **9**, Theorem 7.) This can be used to define a closure $c(M)$ on the lattice of right ideals of R such that all the closed right ideals are two-sided ideals. This we call a bilateral closure. By using the methods of Utumi (**11**) and Johnson (**4**) a subring R_c of the maximal ring of right quotients Q of R can be defined

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corresponding to any closure c on the right ideals of R. If c is a bilateral closure, $R \subseteq R_c \subseteq Q$. If $c = c(P)$ is the bilateral closure associated with the m-system $R - P$, one might hope, with suitable ascending chain conditions, that $R_{c(P)}$ would have the properties of a local ring. In fact we have been able to prove much less than this even when we restrict ourselves to prime or semi-prime rings. Many of the interesting questions are still open, however, as is pointed out in the last section.

2. Closure operations in a lattice. Let $\mathfrak{L}$ be a lattice of right or two-sided ideals in a ring R such that $R \in \mathfrak{L}$. A closure operation c is a mapping $A \rightarrow A^c$ of $\mathfrak{L}$ into itself that satisfies the three rules

$$\begin{aligned} &(1)\ A \subseteq A^c, \\ &(2)\ A \subseteq B \text{ implies } A^c \subseteq B^c, \\ &(3)\ (A^c)^c = A^c. \end{aligned}$$

Closure operations have been studied by R. E. Johnson **(5)** and others. We recapitulate their main properties. Proofs are immediate or can be found in **(5)**.

An element A of $\mathfrak{L}$ is c-closed if $A^c = A$. The intersection of any set of c-closed elements is c-closed and the closure A^c of A is the intersection of all c-closed elements that contain A. Thus the closure operation c is completely determined by the set $\mathfrak{L}^c$ of c-closed elements of $\mathfrak{L}$ and $\mathfrak{L}^c$ is an *inset* in the sense that it is a subset of $\mathfrak{L}$ containing R and closed under complete intersection. Conversely every inset $\mathfrak{I}$ of $\mathfrak{L}$ determines a unique closure operation c such that $\mathfrak{I} = \mathfrak{L}^c$. The closure operations can be partially ordered[1] according to the inclusion ordering of the corresponding insets. Thus if a, b are two closures we say $a \leqslant b$ if and only if $\mathfrak{L}^a \subseteq \mathfrak{L}^b$. This is equivalent to saying $a \leqslant b$ if and only if $A^b \subseteq A^a$ for all A in $\mathfrak{L}$. Since the partial ordering of the closure operations corresponds to the inclusion ordering of the insets of $\mathfrak{L}$, we can make the closure operations into a complete lattice by defining intersection and union of the corresponding insets. Thus if $\{c_\alpha\}$ is any set of closures and $\{\mathfrak{L}^{c_\alpha}\}$ are the corresponding insets of closed elements of $\mathfrak{L}$, we define $\bigcap \mathfrak{L}^{c_\alpha}$ as the set-theoretic intersection which is clearly an inset and therefore defines a unique closure which we take as $\bigcap c_\alpha$. Similarly $\bigcup \mathfrak{L}^{c_\alpha}$ is the inset consisting of all elements of $\mathfrak{L}$ of the form

$$\bigcap A_\sigma, \qquad A_\sigma \in \mathfrak{L}^{c_\alpha},$$

i.e. the smallest inset containing all the $\mathfrak{L}^{c_\alpha}$. The corresponding closure we take as $\bigcup c_\alpha$. It is now clear that the closure operations form a complete lattice under these definitions of intersection and union which is isomorphic to the lattice of insets of $\mathfrak{L}$. (In **(5)** this is a dual isomorphism owing to our reversal of the partial ordering.) We note also for future use that if $c = \bigcup c_\alpha$, then for every element A of $\mathfrak{L}$

[1]We have found it convenient to reverse Johnson's ordering.

(2.1) $$A^c = \bigcap A^{c_\alpha}$$

since this is the minimal element in $\mathfrak{L}^c$ that contains A. However, it is not true in general that $A^{\cap c_\alpha} = \bigcup A^{c_\alpha}$.

Let R be any ring. We shall use $\mathfrak{L}_r$, or $\mathfrak{L}_r(R)$, to denote the lattice of all right ideals of R and $\mathfrak{L}$, or $\mathfrak{L}(R)$ to denote the lattice of all two-sided ideals of R. We shall be mainly interested in closure operations c in $\mathfrak{L}_r$, having the property that $\mathfrak{L}_r{}^c \subseteq \mathfrak{L}$. Such a closure we shall call a *bilateral closure* in $\mathfrak{L}_r$ since the closed right ideals are two-sided ideals. A maximal bilateral closure m is obtained by choosing $\mathfrak{L}_r{}^m = \mathfrak{L}$ so that if $A \in \mathfrak{L}_r$, A^m is the minimal ideal containing A. If now b is any closure on $\mathfrak{L}$, then mb is a bilateral closure on $\mathfrak{L}_r$. It is clear from the definitions of union and intersection of insets that the bilateral closures in $\mathfrak{L}_r$ form a complete sublattice $\mathfrak{B}$ of the lattice of all closure operations in $\mathfrak{L}_r$.

3. The ring of quotients corresponding to a bilateral closure on $\mathfrak{L}_r(R)$. We shall assume throughout that R is an associative ring without total left zero divisors (i.e. $xR = 0$ implies $x = 0$). Utumi has shown that there is a unique maximal right quotient ring Q of R in the sense that $R \subseteq Q$ and if $\alpha, \beta \in Q$ and $\beta \neq 0$, then there exist elements a, b in R such that $\alpha a = b$ and $\beta a \neq 0$. There is no loss of generality in assuming $\beta a \in R$. We shall outline the construction of Q since we shall need the concepts used in this construction. The details may be found in (**11** and **4**).

Following Utumi we use the notation $Q \geqslant R$ or $R \leqslant Q$ to mean that Q is a right quotient ring of R in the above sense. We denote by $R^\blacktriangle$ the set of all right ideals I in R for which $R \geqslant I$. Since $xR = 0$ implies $x = 0$, it follows that $R \geqslant R$ and $R^\blacktriangle$ is not empty. Every right ideal in $R^\blacktriangle$ is an *essential* right ideal in the sense that it has non-zero intersection with every non-zero right-ideal of R. If $I, J \in R^\blacktriangle$, then $I \cap J$, IJ belong to $R^\blacktriangle$. We view R as a right R-module and consider the set $\mathfrak{R}$ of all R-homomorphisms α defined on a right ideal I_α in $R^\blacktriangle$ and having values in R. Such a mapping α is called a semi-R-endomorphism of R and I_α is its domain. If $I, J \in R^\blacktriangle$ and α is a semi-R-endomorphism defined on I, then the set $\{x \in I \mid \alpha x \in J\}$ is a right ideal in $R^\blacktriangle$. These facts enable us to define sums, products, and equivalence of semi-R-endomorphisms as follows:

1. If $I_\alpha = I_\beta$, $(\alpha + \beta)x = \alpha x + \beta x$ for $x \in I_\alpha$.
2. If $\beta I_\beta \subseteq I_\alpha$, $(\alpha\beta)x = \alpha(\beta x)$ and $I_{\alpha\beta} = \{x \in I_\beta \mid \beta x \in I_\alpha\} \in R^\blacktriangle$.
3. $\alpha \sim \beta$ if there exists a right ideal I in $R^\blacktriangle$ such that $I \subseteq I_\alpha \cap I_\beta$ and $\alpha x = \beta x$ for $x \in I$.

In particular $\alpha \sim \beta$ if α extends β, i.e. if $I_\beta \subseteq I_\alpha$ and $\alpha x = \beta x$ for $x \in I_\beta$. Sums and products of equivalence classes can then be defined and it is easily verified that the equivalence classes of semi-R-endomorphisms form a ring $Q = \mathfrak{R}/\mathfrak{H}$, where $\mathfrak{H}$ consists of all α in $\mathfrak{R}$ for which $\alpha \sim 0$. Moreover Q contains a subring isomorphic to R. For if $a \in R$ left multiplication by a is a semi-R-endomorphism a' with domain R. If $a' \sim b'$, then $a'x = b'x$ or $ax = bx$ for all x in I where $I \in R^{\blacktriangle}$. This implies $(a - b)x = 0$, which contradicts $R \geqslant I$ if $a \neq b$. Hence $a' \sim b'$ implies $a = b$ and Q contains the subring isomorphic to R consisting of the equivalence classes $a' + \mathfrak{H}$ where $a \in R$. By identifying $a' + \mathfrak{H}$ with a we may assume $R \subseteq Q$. We shall refer to Q as the maximal ring of right quotients or maximal right quotient ring of R. It is not necessarily equal to the full ring of right quotients even if the latter exists. (See **2**, pp. 165–166.) It is clear that if $\alpha \in \mathfrak{K}$ and $\alpha a = b$, $a, b \in R$, then $\tilde{\alpha} a = b$ when a and b are viewed as elements of Q and $\tilde{\alpha}$ is the equivalence class to which α belongs.

Now let c be any bilateral closure on $\mathfrak{L}_r(R)$ and let $\mathfrak{K}_c$ denote the set of all $\alpha \in \mathfrak{K}$ such that for every right ideal $J \subseteq I_\alpha$,

$$\alpha J \subseteq J^c. \tag{3.1}$$

THEOREM 1. *If c is a bilateral closure on $\mathfrak{L}_r(R)$ and $\mathfrak{K}_c$ is defined as above, then*

$$R_c = (\mathfrak{K}_c + \mathfrak{H})/\mathfrak{H} \cong \mathfrak{K}_c/(\mathfrak{H} \cap \mathfrak{K}_c)$$

is a subring of Q and $R \subseteq R_c$.

Proof. If $\alpha, \beta \in \mathfrak{K}_c$ and $I_\alpha = I_\beta$, then clearly for $J \subseteq I_\alpha$,

$$(\alpha + \beta)J \subseteq \alpha J + \beta J \subseteq J^c$$

and $\alpha + \beta \in \mathfrak{K}_c$. Similarly if $\beta I_\beta \subseteq I_\alpha$, $I_{\alpha\beta} = \{x \in I_\beta \mid \beta x \in I_\alpha\}$, and if $J \subseteq I_{\alpha\beta}$ we have $(\alpha\beta)J = \alpha(\beta J) \subseteq \alpha(I_\alpha \cap J^c) \subseteq (I_\alpha \cap J^c)^c \subseteq I_\alpha{}^c \cap J^c \subseteq J^c$. Hence $\alpha\beta \in \mathfrak{K}_c$. Thus $\mathfrak{K}_c$ is closed under addition and multiplication as defined in (1) and (2) above, and $(\mathfrak{K}_c + \mathfrak{H})/\mathfrak{H}$ is a subring of Q. We denote this ring by R_c, and clearly $R_c \cong \mathfrak{K}_c/(\mathfrak{H} \cap \mathfrak{K}_c)$. Finally since J^c is a two-sided ideal, all the left multiplications by elements of R satisfy (3.1) and hence belong to $\mathfrak{K}_c$. It follows that $R \subseteq R_c \subseteq Q$. We call R_c the ring of quotients of R corresponding to the bilateral closure c.

4. Maximal closures and closure subrings of Q. To every bilateral closure c corresponds a ring R_c of Q such that $R \subseteq R_c \subseteq Q$. Different closures may well give rise to the same ring R_c. For example if R is the ring of integers let a be the closure for which $\mathfrak{L}^a$ is the set $\{(0), (6^r)\}$, $r = 0, 1, 2, 3, \ldots$, and let b be that for which $\mathfrak{L}^b = \{(0), (2^r 3^s)\}$, $r = 0, 1, 2, \ldots$, $s = 0, 1, 2, \ldots$. Since Q is the rational field, it is not hard to show that $R_a = R_b$, each being the ring of all rational numbers with denominator prime to 6.

THEOREM 2. *Let a be a bilateral closure in $\mathfrak{L}_r(R)$. There is a unique maximal bilateral closure $\bar{a}$ such that $R_a = R_{\bar{a}}$ and the mapping $a \to \bar{a}$ is a closure in the lattice $\mathfrak{B}$ of all bilateral closure operations in $\mathfrak{L}_r$.*

Proof. Let $\{c_i\}$ be any set of bilateral closures on $\mathfrak{L}_r$ and let $c = \cup c_i$. By (2.1) we have for any right ideal J of R

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$$J^c = \bigcap J^{c_i}.$$

From this and the definition of $\mathfrak{R}_c$ we see that

$$\mathfrak{R}_c = \bigcap \mathfrak{R}_{ci}$$

and hence that $R_c = \bigcap R_{c_i}$. It follows that the union c of all bilateral closures c_i for which $R_{ci} = R_a$ also has the property that $R_c = R_a$ and is the unique maximal bilateral closure having this property. The maximal closure c for which $R_c = R_a$ will be denoted by $\bar{a}$. It is clear that $a \leqslant \bar{a}$ and $\bar{\bar{a}} = \bar{a}$. Moreover if a, b are bilateral closures and $a \leqslant b$, then $J^b \subseteq J^a$ for all $J \in \mathfrak{L}_r$. It follows that $R_b \subseteq R_a$ and $R_{\bar{b}} \subseteq R_{\bar{a}}$. Now let $c = \bar{a} \cup \bar{b}$ and $R_c = R_{\bar{a}} \cap R_{\bar{b}} = R_{\bar{b}}$ and by the maximal property of $\bar{b}$, $c \leqslant \bar{b}$ and hence $\bar{a} \leqslant \bar{b}$. This shows that $a \to \bar{a}$ is a closure operation in the lattice of bilateral closures in $\mathfrak{L}_r$.

By a *maximal* bilateral closure we shall mean a closure $\bar{a}$ that is maximal in the above sense, i.e. $R_c = R_{\bar{a}}$ implies that $c \leqslant \bar{a}$. Since $a \to \bar{a}$ is a closure operation we have the following corollary.

COROLLARY 1. *The intersection of any set of maximal bilateral closures is a maximal bilateral closure.*

For future use we state also the following corollary, whose proof is contained in the proof of the theorem.

COROLLARY 2. *If a and b are maximal bilateral closures in $\mathfrak{L}_r(R)$, then $R_a \subseteq R_b$ if and only if $b \leqslant a$.*

By Corollary 1 above the set $\mathfrak{M}$ of all maximal bilateral closures on $\mathfrak{L}_r(R)$ becomes a complete lattice $(\mathfrak{M}; \cap, \cup^*, \leqslant)$ if we define a union operation $\cup^*$ in $\mathfrak{M}$ by

$$\cup^* c_i = \overline{\cup c_i} = \cap\, b, \qquad b \geqslant \cup c_i \text{ and } b \in \mathfrak{M}.$$

When we refer to the lattice $\mathfrak{M}$ of maximal bilateral closures the lattice operations will be understood to be $\cap$ and $\cup^*$.

A subring S of Q will be called a *closure subring* of Q if $S = R_c$ for some bilateral closure c. It may, of course, be assumed that $c \in \mathfrak{M}$. If $\{c_i\}$ is any set of bilateral closures in $\mathfrak{L}_r$ and if R_{c_i} are the corresponding rings, we have seen in the proof of Theorem 2 that $\cap R_{ci} = R_c$ where $c = \cup c_i$. Hence the set $\mathfrak{C}$ of closure subrings of Q is closed under complete intersection and becomes a lattice $(\mathfrak{C}; \cap, \cup^*, \subseteq)$ if we define the union operation $\cup^*$ by

$$\cup^* R_{ci} = \cap\, R_b, \qquad R_b \supseteq \cup R_{ci} \text{ and } b \in \mathfrak{M}.$$

The set $\mathfrak{C}$, of course, defines a closure operation on the lattice of subrings of Q that contain R.

THEOREM 3. *The lattice $\mathfrak{M}$ of maximal bilateral closures on $\mathfrak{L}_r$ is dual isomorphic to the lattice $\mathfrak{C}$ of closure subrings of Q.*

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Proof. The correspondence $c \to R_c$ between the elements of $\mathfrak{M}$ and $\mathfrak{C}$ is clearly one to one, for if $R_b = R_c$, $R_{b \cup c} = R_b \cap R_c = R_b = R_c$ and hence if b and c are maximal $b = c = b \cup c$. By Corollary 2, Theorem 2, if $b, c \in \mathfrak{M}$, $R_b \subseteq R_c$ if and only if $b \geqslant c$. We have also $R_{b \cup^* c} = R_{\overline{b \cup c}} = R_{b \cup c} = R_b \cap R_c$, and $R_{b \cap c} = R_b \cup^* R_c$ follows in the usual way.

There is always a unique minimal closure subring R_m of Q, namely the intersection of all closure subrings. This ring must correspond to the unique maximal bilateral closure $m \in M$. Clearly m is the closure for which $\mathfrak{L}_r{}^m = \mathfrak{L}$, i.e. for which the closed right ideals are exactly the two-sided ideals. In many cases $R_m = R$. For example, if R is a (commutative) Noetherian semi-prime ring with unit element and P is a maximal prime ideal, we shall see later that the local ring R_P is a closure subring of Q. Since R_P consists of all quotients x/y, $x \in R$, $y \notin P$, and since every element of R other than 1 belongs to a maximal prime, $\cap R_P$, over all maximal primes P, is R, and hence $R_m = R$.

On the other hand it can happen that $R_m = Q$. For example, let R be a primitive ring with non-zero socle $\neq R$. Then R can be viewed as a dense ring of linear transformations of a vector space V over a division ring D. Utumi has shown (**11**, p. 11) that the maximal right quotient ring Q of R is the full ring of D-endomorphisms of V. Every two-sided ideal of R consists (**6**) of a set

$$\{\alpha \in R \mid \dim \alpha V < \sigma\}$$

for some transfinite cardinal σ less than or equal to $\dim V$. Now let J be any right ideal of R such that $J \in R^{\blacktriangle}$. Since J is essential it must contain the socle of R. Let τ be the least cardinal such that $\dim \alpha V < \tau$ for all $\alpha \in J$. The density of R plus the fact that J contains the socle implies that τ is not finite. The minimal two-sided ideal containing J is therefore

$$J^m = \{\alpha \in R \mid \dim \alpha V < \tau\},$$

since clearly $J \subseteq J^m$ and this would not be so if τ were replaced by a smaller cardinal. Now let $\beta \in Q$. If $J \subseteq I_\beta$ and $\alpha \in J$ we have $\beta\alpha \in R$ and

$$\dim(\beta\alpha V) \leqslant \dim(\alpha V) < \tau$$

and hence $\beta\alpha \in J^m$. Hence $\beta \in R_m$. Since β was any element of Q, we have $R_m = Q$.

5. Construction of the maximal closure $\bar{c}$. Let c be any bilateral closure in $\mathfrak{L}_r$ and let α be a semi-R-endomorphism of R that belongs to $\mathfrak{R}_c$. We shall denote by $\tilde{\alpha}$ the element of R_c to which α belongs, i.e. $\tilde{\alpha}$ is the equivalence class $\alpha + \mathfrak{H} \cap \mathfrak{R}_c$. Now if J is any right ideal of R we define

$$J^q = \sum \alpha(J \cap I_\alpha). \tag{5.1}$$

Here the summation extends over all α for which $\tilde{\alpha} \in R_c$. Since for all $a \in R$, $\tilde{a} \in R_c$ it follows that J^q is a two-sided ideal. Moreover $J \subseteq J^q$, and $J_1 \subseteq J_2$

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implies $J_1{}^q \subseteq J_2{}^q$. We denote by $\mathfrak{L}_r{}^q$ the set $\{J \in \mathfrak{L}_r \mid J^q = J\}$ and let $\{J_\sigma\}$ be any set of right ideals in $\mathfrak{L}_r{}^q$. By the above remarks

$$(\bigcap J_\sigma)^q \subseteq \bigcap J_\sigma{}^q = \bigcap J_\sigma.$$

Hence $\mathfrak{L}_r{}^q$ is closed under complete intersection and since it contains R, $\mathfrak{L}_r{}^q$ is an inset in $\mathfrak{L}_r$ and defines a closure q^* such that $\mathfrak{L}_r{}^{q^*} = \mathfrak{L}_r{}^q$. Moreover, since J^q is a two-sided ideal, $\mathfrak{L}_r{}^{q^*} \subseteq \mathfrak{L}$ and q^* is a bilateral closure.

THEOREM 4. *If c is any bilateral closure and q^* is the closure defined above, then $q^* = \bar{c}$.*

Proof. If $J^c = J$ and $\tilde{\alpha} \in R_c$, then $\alpha(J \cap I_\alpha) \subseteq J^c = J$. Hence $\mathfrak{L}^c \subseteq \mathfrak{L}^{q^*}$ and $c \leqslant q^*$. However, since the definition of q^* depends only on R_c and $R_{\bar{c}} = R_c$, we can also conclude that $\bar{c} \leqslant q^*$.

However, $c \leqslant q^*$ implies $R_{q^*} \subseteq R_c$, whereas if $\tilde{\alpha} \in R_c$ and $J \subseteq I_\alpha$, we have

$$\alpha J = \alpha(J \cap I_\alpha) \subseteq J^q \subseteq (J^{q^*})^q = J^{q^*}.$$

From this it follows that $\tilde{\alpha} \in R_{q^*}$ whence $R_c \subseteq R_{q^*}$. Hence $R_{q^*} = R_c$, which combined with $\bar{c} \leqslant q^*$ gives $q^* = \bar{c}$.

The closure q^* defined by means of (5.1) can be generalized somewhat. Let S be any set of elements of Q that contains the unit element of Q and all elements of R. If we define, for any right ideal J of R,

$$J^{q_S} = \sum_{\tilde{\alpha} \in S} \alpha(J \cap I_\alpha) \tag{5.2}$$

we see as before that J^{q_S} is a two-sided ideal, that $J \subseteq J^{q_S}$, and $J_1 \subseteq J_2$ implies $J_1{}^{q_S} \subseteq J_2{}^{q_S}$. We can then define a closure $q_S{}^*$ exactly as before and prove the following theorem.

THEOREM 5. *Let S be any subset of Q that contains R and the unit element of Q. Let $q_S{}^*$ be the closure defined above and let R_c be the minimal closure subring of Q containing S. Then $R_{q_S{}^*} = R_c$ and $q_S{}^* = q^* = \bar{c}$.*

Proof. If $\tilde{\alpha} \in S$ and $J \subseteq I_\alpha$, then $\alpha J \subseteq J^{q_S} \subseteq J^{q_S{}^*}$. Hence $\tilde{\alpha} \in R_{q_S{}^*}$ and $S \subseteq R_{q_S{}^*}$ and $R_c \subseteq R_{q_S{}^*}$. If on the other hand $S \subseteq R_b$ for some bilateral closure b, we have by Theorem 4, $J^{q_S{}^*} \subseteq J^b$ and hence $R_{q_S{}^*} \subseteq R_{\bar{b}} = R_b$. Thus

$$R_{q_S{}^*} \subseteq \bigcap_{S \leqslant R_b} R_b = R_c$$

and therefore $R_{q_S{}^*} = R_c$.

Now since $S \subseteq R_c$, $J^{q_S} \subseteq J^q$ and $q^* \leqslant q_S{}^*$. But since $R_c = R_{q_S{}^*}$, Theorem 4 gives $q^* = \bar{c} \geqslant q_S{}^*$ and hence $q_S{}^* = q^* = \bar{c}$.

If we define the cr-closure of a subset S of Q to be the minimal closure subring containing S, Theorem 5 shows that in constructing the maximal closure $\bar{c}$ corresponding to c, (5.1) may be replaced by (5.2), where S is any subset of Q containing 1 and R whose cr-closure is R_c.

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6. Bilateral closures induced by subrings of Q. Let R be any ring and let S be a ring containing R. If A is an ideal in R we denote by $E_S(A)$, or simply by $E(A)$ if no confusion is likely, the extension of A to S, i.e. the ideal in S generated by A. If we write

$$A^b = R \cap E_S(A),$$

it is well known that b is a closure operation on $\mathfrak{L}(R)$. If m is the maximal bilateral closure on $\mathfrak{L}_r$, then mb is a bilateral closure on $\mathfrak{L}_r$, which may also be written

$$J^{mb} = R \cap E_S(J),$$

where $E_S(J)$ is the *two-sided* ideal in S generated by the right ideal J of R. We shall call the closure mb so defined the *natural closure* induced in $\mathfrak{L}_r$ by S. If c is any bilateral closure in $\mathfrak{L}_r$ the natural closure induced in $\mathfrak{L}_r$ by the ring R_c will usually be denoted by c'. Thus

$$J^{c'} = R \cap E_c(J),$$

where $E_c(J)$ is the two-sided extension of J to R_c.

THEOREM 6. *If c is any bilateral closure in $\mathfrak{L}_r$, then $R_c \subseteq R_{c'}$, and $c' \leqslant \bar{c}$.*

Proof. If $\tilde{\alpha} \in R_c$ and $J \subseteq I_\alpha$ we have $\alpha J \subseteq R$ and hence

$$\tilde{\alpha} J \subseteq R \cap E_c(J) = J^{c'}$$

and therefore $\tilde{\alpha} \in R_{c'}$. Hence $R_c \subseteq R_{c'}$. Now by Theorem 2, Corollary 2, $\bar{c}' \leqslant \bar{c}$ and hence $c' \leqslant \bar{c}$.

Now let c be any closure in $\mathfrak{M}$. Denote c' by c_1 and c_i' by c_{i+1}, $i = 1, 2, 3, \ldots$. Since $c_1 \leqslant c$ implies $c_1' \leqslant c'$, we see that

$$c \geqslant c_1 \geqslant c_2 \geqslant c_3 \ldots$$

and hence

$$R_c \subseteq R_{c_1} \subseteq R_{c_2} \subseteq R_{c_3} \ldots .$$

If we let

$$c_\omega = \bigcap_{i=1}^{\infty} c_i \qquad \text{and} \qquad \bar{R} = \bigcup_{i=1}^{\infty} R_{c_j}$$

we can prove the following theorem.

THEOREM 7. (a) *The natural closure induced by $\bar{R}$ is equal to c_ω.*
(b) *If the* ACC *for right ideals holds in R, then $R_{c_\omega} = \bar{R}$ and hence $c_\omega' = c_\omega$.*
(c) *If* ACC *holds for right ideals in R, then*

$$\bar{c}_\omega = \bigcap_{i=1}^{\infty} \bar{c}_i.$$

Proof. (a) If A is any right ideal of R we use the notation $E_i(A)$ for the (two-sided) extension of A to the ring R_{c_i}. Now if $x \in E_{\bar{R}}(A)$,

$$x = \sum_{j=1}^{r} \alpha_j a_j \beta_j,$$

where $a_j \in A$, $\alpha_j, \beta_j \in \bar{R}$. Since $\bar{R} = \cup R_{ci}$ there is an integer s such that $\alpha_j, \beta_j \in R_{cs}$ for $j = 1, 2, \ldots, m$. Hence $x \in E_s(A)$ and

$$E_{\bar{R}}(A) \subseteq \bigcup_{i=1}^{\infty} E_i(A).$$

But since $E_i(A) \subseteq E_{\bar{R}}(A)$ we have

$$E_{\bar{R}}(A) = \bigcup_{i=1}^{\infty} E_i(A).$$

Now let r be the natural closure induced by $\bar{R}$ so that

$$A^r = R \cap E_{\bar{R}}(A) = R \cap \left[\bigcup_{i=1}^{\infty} E_i(A)\right] = \bigcup_{i=1}^{\infty} [R \cap E_i(A)] = \bigcup_{i=1}^{\infty} A^{ci'}$$

$$= \bigcup_{i=0}^{\infty} A^{ci+1} \subseteq A^{c_\omega}.$$

It follows that $A^{ci} \subseteq A^r \subseteq A^{c_\omega}$ for $i = 1, 2, 3, \ldots$ and hence $c_\omega \leqslant r \leqslant c_i$ and

$$c_\omega \leqslant r \leqslant \bigcap_{i=1}^{\infty} c_i = c_\omega$$

or $r = c_\omega$.

(b) Since $c_\omega \leqslant c_i$, $i = 1, 2, 3, \ldots$, $R_{ci} \subseteq R_{c_\omega}$ and $\bar{R} \subseteq R_{c_\omega}$. Now if $\tilde{\alpha} \in R_{c_\omega}$ and $J \subseteq I_\alpha$,

$$\alpha J \subseteq J^{c_\omega} = J^r = \bigcup_{i=1}^{\infty} J^{ci}.$$

Now the ACC implies that αJ is generated by a finite number of elements $a_1, a_2, \ldots, a_n$ and since each

$$a_j \in \bigcup_{i=1}^{\infty} J^{ci}$$

there exists an integer t such that each $a_j \in J^{ct}$ and hence $\alpha J \subseteq J^{ct}$ and $\tilde{\alpha} \in R_{ct} \subseteq \bar{R}$. Hence $R_{c_\omega} \subseteq \bar{R}$, which combined with the previous result gives $R_{c_\omega} = \bar{R}$. From this it follows that $c_\omega' = r = c_\omega$.

(c) Let

$$\dot{c} = \bigcap_{i=1}^{\infty} \bar{c}_i.$$

Then $\dot{c} \leqslant \bar{c}_i$, $R_{ci} \subseteq R_{\dot{c}}$ for all i and hence $\bar{R} \subseteq R_{\dot{c}}$. On the other hand

$$c_\omega = \bigcap_{i=1}^{\infty} c_i \leqslant \bigcap_{i=1}^{\infty} \bar{c}_i = \dot{c}.$$

Hence $R_{\dot{c}} \subseteq R_{c_\omega} = \bar{R}$ and hence $R_{\dot{c}} = \bar{R} = R_{c_\omega}$. Since $\dot{c}$ is a maximal closure, being the intersection of maximal closures, this implies $\bar{c}_\omega = \dot{c}$ as required.

7. Application to commutative rings. If R is commutative, in addition to the maximal quotient ring Q there is the full ring of quotients F consisting of the usual equivalence classes of formal quotients ab^{-1}, $a, b \in R$ and b regular. It is easy to see that $F \subseteq Q$. For if b is regular the mapping $br \to r$ is a semi-R-endomorphism β with $I_\beta = bR$. Now if $x, y \in R$, $y \neq 0$, $xa \in I_\beta$ for any $a \in I_\beta$. Moreover a can be chosen so that $ya \neq 0$ because $ybR = 0$ implies $yb = 0$ since R has no total zero-divisors and hence $y = 0$ since b is regular. Thus $I_\beta \in R^\blacktriangle$ and $\tilde{\beta} \in Q$. Clearly $\beta = b^{-1}$ and hence all regular elements of R have inverses in Q and $F \subseteq Q$. Findlay and Lambek have given an example (**2**, pp. 164–165) in which $F \neq Q$.

A semi-R-endomorphism is said to be irreducible if it cannot be extended to a larger domain. We can now prove the following theorem.

THEOREM 8. *If R is a commutative ring, a necessary and sufficient condition that $Q = F$ is that the domain of every irreducible semi-R-endomorphism in $\mathfrak{R}$ contains a regular element of R.*

Proof. The condition is obviously sufficient, for if $\alpha \in \mathfrak{R}$, $\alpha \sim \alpha'$ where α' is irreducible and $\alpha' a = b$ where a is a regular element in $I_{\alpha'}$ and $b \in R$. Hence $\tilde{\alpha} = \tilde{\alpha}' = ba^{-1} \in F$. Conversely, if $Q = F$ and $\tilde{\alpha} = ba^{-1}$ is any element of Q, we have $\tilde{\alpha}a = b \in R$. Now multiplication by $\tilde{\alpha}$ in Q defines a semi-R-endomorphism α^* of R whose domain,

$$I_{\alpha^*} = \{x \in R \mid \tilde{\alpha}x \in R\},$$

contains a. Since $\alpha x = y$ implies $\tilde{\alpha}x = y$, it is clear that $I_\alpha \subseteq I_{\alpha^*}$ and hence α^* extends α. Hence if α is irreducible, $I_{\alpha^*} = I_\alpha$ and $a \in I_\alpha$.

COROLLARY. *If $\tilde{\alpha} \in Q$ and $\tilde{\alpha}a \in R$, where $a \in R$, we may assume that $a \in I_\alpha$.*

For we have seen that $a \in I_{\alpha^*}$, where $\alpha^* \sim \alpha$.

THEOREM 9. *If R is commutative, c any closure in $\mathfrak{L}(R)$, and c' the natural closure induced by R_c, then $R_{c'} = R_c$ and $c' = \bar{c}$.*

Proof. Since R is commutative, it is known (**2**, p. 163) that Q and therefore R_c is commutative. Hence if A is any ideal of R, $E_c(A)$ is generated by elements of the form $\tilde{\alpha}a$, where $\tilde{\alpha} \in R_c$ and $a \in A$. Now if $\tilde{\alpha}a \in R$ we can assume by the above corollary that $a \in I_\alpha$ and hence $\alpha a \in A^c$ and $\tilde{\alpha}a \in A^c$. Hence we have

$$A^{c'} = R \cap E_c(A) \subseteq A^c,$$

whence $c \leqslant c'$ and $R_{c'} \subseteq R_c$. Hence, by Theorem 6, $R_{c'} = R_c$. Since, again by Theorem 6, $c' \leqslant \bar{c}$, we see that if c is maximal, $c = c'$. Hence for any c we have $\bar{c} = \bar{c}' = c'$.

COROLLARY. *If c is maximal, the c-closed ideals of R are exactly the ideals of R that are contractions of ideals of R_c.*

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Proof. If $A^c = A$, then $A = A^{c'}$ = the contraction to R of $E_c(A)$. Conversely if $A = R \cap A'$, where A' is an ideal of R_c, it is well known that $A^{c'} = A$ (**12**, p. 219) and hence $A^c = A$.

Now let M be any multiplicative system of elements of R. If A is any ideal of R its M-component A_M is defined by

$$A_M = \{x \in R \mid mx \in A \text{ for some } m \in M\}.$$

The mapping $A \to A_M$ is a closure operation in $\mathfrak{L}(R)$ which we shall denote by $c(M)$. If M contains only regular elements of R we shall denote by R_M the usual quotient ring constructed from formal quotients am^{-1}, $a \in R$, $m \in M$.

THEOREM 10. *If R is commutative, $Q = F$, and M is a multiplicative system that contains only regular elements of R, then $R_{c(M)} = R_M$.*

Proof. If $m \in M$, since m is regular, it has an inverse m^{-1} in Q whose domain is mR. If $A \subseteq mR$, $A = mB$, where B is an ideal of R. Clearly $m^{-1}A \subseteq B \subseteq A^{c(M)}$ and hence $m^{-1} \in R_{c(M)}$. It follows that $R_M \subseteq R_{c(M)}$. On the other hand if $Q = F$ every element $\tilde{\alpha}$ of Q has the form $\tilde{\alpha} = ba^{-1}$, b, $a \in R$ and a regular. Hence $\tilde{\alpha}(aR) \subseteq R$ and, by the corollary to Theorem 8, $aR \subseteq I_\alpha$. Now if $\tilde{\alpha} \in R_{c(M)}$, $\alpha(aR) = bR \subseteq (aR)^{c(M)}$. Hence for every $r \in R$ there exists $m_1 \in M$ such that $m_1 br \in aR$. Choose $r = m_2 \in M$ and let $m = m_1 m_2$. Then $mb \in aR$ where $m \in M$ and is therefore regular. Hence $mb = ax$ and $\tilde{\alpha} = ba^{-1} = xm^{-1} \in R_M$. Thus $R_{c(M)} \subseteq R_M$ and the theorem follows.

THEOREM 11. *Let R be a commutative ring in which the* ACC *holds, and let M be any multiplicative system in R. The closure $c(M)$ is maximal if and only if the zero ideal is $c(M)$-closed, and this occurs if and only if M contains only regular elements of R.*

Proof. Since $0^{c(M)} = \{x \in r \mid mx = 0, m \in M\}$ it is clear that $0^{c(M)} = 0$ if and only if M contains only regular elements. It is also clear that if $c(M)$ is maximal, since then $c(M) = c(M)'$, the zero ideal is $c(M)$-closed. (Alternatively, if the zero ideal is not $c(M)$-closed it can clearly be added to the inset of $c(M)$-closed ideals without changing the ring $R_{c(M)}$.)

Conversely suppose the zero ideal is $c(M)$-closed so that every element of M has an inverse m^{-1} in Q. The domain of m^{-1} is mR. If $A \subseteq mR$, $A = mB$, where B is an ideal of R. Clearly $B \subseteq A^{c(M)}$ and $m^{-1}A \subseteq B \subseteq A^{c(M)}$. Hence $m^{-1} \in R_{c(M)}$ and every element of M has an inverse in $R_{c(M)}$.

Now the ACC implies that $A^{c(M)} = (x_1, x_2, \ldots, x_n)$, where $m_i x_i \in A$, $m_i \in M$. Hence if $m = m_1 m_2 \ldots m_n$, $m \in M$ and $mA^{c(M)} \subseteq A$, whence $A^{c(M)} \subseteq m^{-1}A$. Now let $X = A \cap mR$ and we have

$$X^{c(M)} = A^{c(M)} \cap (mR)^{c(M)} = A^{c(M)}$$

since $(mR)^{c(M)} = R$. Hence

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$$m^{-1}X = m^{-1}(A \cap mR) \supseteq m^{-1}(mA^{c(M)}) = A^{c(M)} = X^{c(M)}.$$

But since $m^{-1} \in R_{c(M)}$, and $X \subseteq mR$, $m^{-1}X \subseteq X^{c(M)}$. Hence $m^{-1}X = X^{c(M)}$.

Now if $c(M) < \overline{c(M)}$ we can choose the ideal A above so that $A^{\overline{c(M)}} \subset A^{c(M)}$ and then $X^{\overline{c(M)}} \subseteq A^{\overline{c(M)}} \subset A^{c(M)} = X^{c(M)}$. Since $m^{-1}X = X^{c(M)} \not\subseteq X^{\overline{c(M)}}$ this means that $m^{-1} \notin R_{\overline{c(M)}}$, which is a contradiction since $R_{c(M)} = R_{\overline{c(M)}}$. Hence $c(M)$ is maximal.

Corollary. *If M contains only regular elements, $c(M) = c(M)'$ and for any ideal A of R,*

$$R \cap E_{c(M)}(A) = A_M.$$

8. Bilateral closures associated with an arbitrary m-system. In a non-commutative ring R the role of the multiplicatively closed set is, for many purposes, filled by an m-system **(8)**. A set M of elements of R is an m-system if when $x, y \in M$ there exists an element r in R such that $xry \in M$. The null set is also, by definition, an m-system. An ideal P is prime if and only if its complementary set $R - P$ is an m-system. We have seen that in the commutative case a closure $c(M)$, and hence a ring of quotients $R_{c(M)}$, can be associated with any multiplicative system M. In the non-commutative case, if M is an m-system, an analogous closure $c(M)$ is provided by the upper M-component $u(A, M)$ defined in **(9)**. The original definition of $u(A, M)$ involved considerations that are not needed here. We therefore use the definition given by Barnes **(1)** suitably adapted to the present context.

Let R be any ring and A any ideal in R. An element x of R is right prime[2] (rp) to A if $xRy \subseteq A$ implies $y \in A$. Otherwise x is not right prime (nrp) to A. An ideal is nrp to A if all its elements are nrp to A. Otherwise it is rp to A. Now let M be any set of elements in R. If $M = \emptyset$ we define the right upper M-component of A to be A itself. If $M \neq \emptyset$ let $\mathfrak{S}$ be the set of all ideals B in R such that all elements of M are right prime to B. Since $R \in \mathfrak{S}$, $\mathfrak{S}$ is not empty. If $\{A_\sigma\}$ is any set of ideals each of which belongs to $\mathfrak{S}$ it is clear that $\cap A_\sigma \in \mathfrak{S}$. Hence $\mathfrak{S}$ is an inset in $\mathfrak{L}(R)$, and defines a closure $c(M)$. We call $A^{c(M)}$ the right upper M-component of A and denote it by $u_r(A, M)$. It is clear that if $M \cap A$ is non-null, then R is the only element of $\mathfrak{S}$ that contains A and hence $u_r(A, M) = R$. But if $M \cap A = \emptyset$, $u_r(A, M)$ is the intersection of all ideals X containing A such that all elements of M are rp to X. By the intersection property all elements of M are rp to $u_r(A, M)$.

Although in the above there is no restriction on the set M we are interested primarily in the case in which M is an m-system. If R is commutative and M is multiplicatively closed (and hence an m-system) we know from **(9**, Theorem **7)** that $u(A, M) = A_M$, and thus use of the same notation $c(M)$ for the closure

[2]The left-right terminology of **(9)** and **(1)** has been reversed in order to conform with that used in **(7)** and at the same time to fit better the requirements of the present paper.

$A \to u(A, M)$ is justified. If M is a non-null m-system we define, as in **(9)**, the right lower M-component $l_r(A, M)$ to be the ideal

$$\{x \in R \mid mRx \subseteq A \text{ for some } m \in M\}.$$

If $M = \emptyset$ we define $l_r(A, M)$ to be A itself. We know from **(9)** that

$$A \subseteq l_r(A, M) \subseteq u_r(A, M).$$

The other properties of these components derived in **(9)** will be assumed. When no confusion will result we omit the subscripts r, although $c_r(M)$ and $c_l(M)$ will be used when necessary to distinguish the closures defined by $u_r(A, M)$ and $u_l(A, M)$.

THEOREM 12.[3] *If the* ACC *holds for two-sided ideals in R, then for every ideal A and every m-system M, $u(A, M) = l(A, M)$.*

Proof. If $M = \emptyset$, $u(A, M) = l(A, M) = A$ by definition. Also if $M \cap A \neq 0$, $u(A, M) = l(A, M) = R$. Assume $M \neq \emptyset$ but $M \cap A = \emptyset$. Since the ACC holds we can write

$$l(A, M) = L = (c_1, c_2, \ldots, c_n).$$

For each c_i there exists an element $m_i \in M$ such that $m_iRc_i \subseteq A$. Since M is an m-system there is an element $m = m_1x_1m_2x_2m_3 \ldots x_{n-1}m_n \in M$ and clearly $mRL \subseteq A$.

Now suppose $m'Rx \subseteq L$, $m' \in M$. Then $mRm'Rx \subseteq A$ and if we choose r so that $m'' = mrm' \in M$ we have $m''Rx \subseteq A$ and $x \in L$. Hence every element m' of M is rp to L and $u(A, M) \subseteq L$. But since $L \subseteq u(A, M)$, equality follows.

THEOREM 13. *If M is any m-system and $c(M)$ the corresponding closure in $\mathfrak{L}(R)$, then $c(M)$ is an m-closure in the sense of Johnson* **(5)**, *that is it is a $\cap$-endomorphism of $\mathfrak{L}(R)$.*

Proof. We must show that for any ideals A and B,

$$u(A \cap B, M) = u(A, M) \cap u(B, M). \tag{8.1}$$

This is trivially true if $M = \emptyset$. If $M \neq \emptyset$ we first prove (8.1) for the lower components. Clearly $l(A \cap B, M) \subseteq l(A, M) \cap l(B, M)$. If

$$x \in l(A, M) \cap l(B, M)$$

choose $m_1, m_2 \in M$ so that $m_1Rx \subseteq A$, $m_2Rx \subseteq B$. Then if $m = m_1rm_2 \in M$, $mRx \subseteq A \cap B$ and $x \in l(A \cap B, M)$. Hence we have

$$l(A \cap B, M) = l(A, M) \cap l(B, M). \tag{8.2}$$

From (8.2) we see that

[3]This theorem was first proved by W. E. Barnes but has not been published previously.

$$l^2(A \cap B, M) = l[l(A, M) \cap l(B, M), M]$$
$$= l^2(A, M) \cap l^2(B, M)$$

and by transfinite induction, for any ordinal σ,

$$l^\sigma(A \cap B, M) = l^\sigma(A, M) \cap l^\sigma(B, M).$$

Now (8.1) follows from **(9**, Theorem 5, Corollary 1**)**.

If $M = R - P$, where P is a prime ideal, we write $c(P)$ for $c(M)$, $u(A, P)$, $l(A, P)$ for $u(A, M)$, $l(A, M)$ and we refer to the latter as the (right) upper and lower P-components of A.

For the remainder of this section we shall assume the ACC for the ideals of R. In this case another characterization of the closure $c(P)$ can be obtained by using concepts introduced by Lesieur and Croisot. For completeness we shall include the relevant definitions from **(7)**. We use the notation xR^*y, introduced in **(7)**, for the set $\{xRy, xy\}$, and prove the following lemma.

LEMMA 1. *An element a is* nrp *to an ideal A if and only if there exists an element $x \notin A$ such that $aR^*x \subseteq A$.*

Proof. The "if" part is obvious. Conversely, if a is nrp to A there exists an $x \notin A$ such that $aRx \subseteq A$. If $ax \in A$, then $aR^*x \subseteq A$. If $ax \notin A$, let $ax = x_1$ and clearly $aR^*x_1 \subseteq A$, $x_1 \notin A$.

DEFINITION. *The (right) tertiary radical of an ideal A is the set $t(A)$ which consists of all elements a of R such that in every principal ideal generated by an element not in A there exists an element $x \notin A$ such that $aR^*x \subseteq A$.*

It is shown in **(7)** that $t(A)$ is an ideal and it is clear from the definition that $t(A)$ is nrp to A.

DEFINITION. *An ideal T is (right) tertiary if $xR^*y \subseteq T$ implies that either $y \in T$ or $x \in t(T)$.*

By **(7**, Theorem 5.3**)**, the tertiary radical of a tertiary ideal is prime. Let T be P-tertiary, that is, tertiary with tertiary radical P. It follows from the above lemma and the definition of a tertiary ideal that every element that is nrp to T is contained in P. But we have seen that P is nrp to T and hence the elements that are nrp to T are exactly the elements of P. It follows **(1,** p. 2**)** that T is (right) primal with adjoint P in the sense of the following definition.

DEFINITION. *An ideal A is (right) primal if the elements* nrp *to A form an ideal Q. The ideal Q is called the adjoint of A and A is then said to be Q-primal.*

In the presence of the ACC for ideals the adjoint of a primal ideal is prime **(1**, p. 7, Cor. 2.**)**

The above definition of a primal ideal is that of Barnes **(1)**. It follows from **(1**, Theorem 10**)** that it is equivalent to that of Lesieur and Croisot when the ACC holds. Although every P-tertiary ideal is P-primal the converse is false.

Lesieur and Croisit **(7)** have shown that if the ACC holds in R for two-sided ideals, then every ideal A has a short representation

$$A = T_1 \cap T_2 \cap \ldots \cap T_n, \tag{8.3}$$

where T_i is a (right) tertiary ideal whose tertiary radical is a prime P_i. In any such short representation the tertiary radicals $P_1, P_2, \ldots, P_n$ are distinct prime ideals and are uniquely determined by A. We shall call them the (right) *associated primes* of A. The following theorem can be used to provide a characterization of the closure $c(P)$ associated with a prime ideal P.

THEOREM 14. *Let A be an ideal in a ring R which satisfies the* ACC *for ideals. Every associated prime of A is* nrp *to A and every prime ideal P that is* nrp *to A is contained in one of the associated primes of A.*

Proof. Let (8.3) be a short representation of A by (right) tertiary ideals and let T_1, with tertiary radical P_1, be any one of the tertiary components. Since (8.3) is irredundant we can choose $b \in T_2 \cap \ldots \cap T_n$ such that $b \notin T_1$. By the definition of tertiary radical if $a \in P_1$, there exists an element $x \in (b)$ such that $x \notin T_1$ and $aR^*x \subseteq T_1$. Since $x \in (b) \subseteq T_2 \cap \ldots \cap T_n$, we have

$$aRx \subseteq aR^*x \subseteq T_1 \cap T_2 \cap \ldots \cap T_n = A$$

and $x \notin A$ since $x \notin T_1$. Hence a is nrp to A and since a was arbitrary, P_1 is nrp to A, proving the first part of the theorem.

Now if a is nrp to A by Lemma 1 we can choose $x \notin A$ such that $aR^*x \subseteq A$. Hence $aR^*x \subseteq T_i$ for all i and $x \notin T_j$ for a least one j. Since T_j is P_j-tertiary it follows that $a \in P_j$. Hence if P is nrp to A, $P \subseteq \cup P_i$, the set-theoretic union of the associated primes of A. It now follows by a familiar argument (**1**, Lemma 5, p. 4) that $P \subseteq P_i$ for some i.

COROLLARY 1. *If R satisfies the* ACC *for ideals, the maximal* nrp *to A primes of A* **(1)** *are among the associated primes of A.*

COROLLARY 2. *If R satisfies the* ACC *for ideals and P is any prime ideal of R, an ideal A is $c(P)$-closed if and only if its associated primes are all contained in P.*

Proof. A is $c(P)$-closed if and only if $u(A, P) = A$, i.e. if and only if every element m not in P is rp to A. This is equivalent to saying that every element that is nrp to A belongs to P, i.e. all the associated primes of A are contained in P.

COROLLARY 3. *If R satisfies the* ACC *for ideals, P is any prime ideal, and A any ideal of R, then $u(A, P)$ is the intersection of all ideals containing A whose associated primes are all contained in P.*

In **(1)** Barnes has defined an upper B-component $u(A, B)$ of A where B

is any ideal. This is precisely the upper M-component $u(A, M)$ where M is the set of elements that are rp to B. Writing $c(B) = c(M)$ we have a closure in $\mathfrak{L}(R)$ corresponding to any ideal B. If the ACC holds in R we see from Theorem 14 that A is $c(B)$-closed if and only if every associated prime of A is contained in an associated prime of B. Thus $u(A, B)$ is the intersection of all ideals containing A each of whose associated primes is contained in an associated prime of B.

THEOREM 15. *If R satisfies the* ACC *for ideals and B is an ideal of R with associated primes $P_1, P_2, \ldots, P_n$, then $u(A, B) = \cap u(A, P_i)$, whence $c(B) = \cup c(P_i)$ and if $c(B)$, $c(P_i)$ are used to define bilateral closures in $\mathfrak{L}_r(R)$ we have $R_{c(B)} = \cap R_{c(P_i)}$.*

Proof. Since $u(A, P_i)$ is the intersection of all ideals containing A whose associated primes are contained in P_i it is clear that $u(A, B) \subseteq \cap u(A, P_i)$. On the other hand $u(A, B) = \cap X$ over all $X \supseteq A$ such that each associated prime of X is contained in one of the P_i. Hence each X is the intersection of tertiary ideals each of whose radicals is contained in a P_i. Grouping together those tertiary components of the various X's whose radicals are contained in P_1, in P_2, etc. we find

$$u(A, B) = \bigcap_{i=1}^{n} \left(\bigcap_{\sigma} X_{\sigma}^{(i)} \right),$$

where for a fixed i the associated primes of each $X_\sigma{}^{(i)}$ are contained in P_i. It follows that

$$u(A, P_i) \subseteq \bigcap_{\sigma} X_{\sigma}^{(i)} \quad \text{and} \quad u(A, B) \supseteq \bigcap_{i=1}^{n} u(A, P_i).$$

This proves the first statement of the theorem and the other two statements are immediate consequences of it.

The remaining theorems of this section give additional properties of the associated primes of an ideal and the corresponding components in a ring with ACC for ideals.

THEOREM 16. *If R satisfies the* ACC *for ideals and P is an associated prime of an ideal A, then P is* nrp *to $u(A, P)$. It follows that in this case $u(A, P)$ is P-primal.*

Proof. Let (8.3), where T_i is P_i-tertiary, be a short representation of A. Choose any associated prime of A, say P_1. By Theorem 13, $u(A, P_1) = \cap u(T_i, P_1)$. By the definition of a P_1-tertiary ideal it follows that $u(T_1, P_1) = T_1$ and hence

$$u(A, P_1) = T_1 \cap u(T_2, P_1) \cap \ldots \cap u(T_n, P_1). \tag{8.4}$$

If the component T_1 were redundant in (8.4) we would have

$$T_1 \supseteq \bigcap_{i=2}^{n} u(T_i, P_1) \supseteq \bigcap_{i=2}^{n} T_i$$

contrary to the irredundance of the original short representation of A. Thus (8.4) leads to a short representation of $u(A, P_1)$ that contains a P_1-tertiary component. Hence P_1 is an associated prime of $u(A, P_1)$ and, by Theorem 14, P_1 is nrp to $u(A, P_1)$. Since every element not in P_1 is rp to $u(A, P_1)$ this is equivalent to saying that $u(A, P_1)$ is (right) P_1-primal.

COROLLARY. *If $P_1, \ldots, P_r$ are the maximal* nrp *to A primes, then*

$$A = u(A, P_1) \cap \ldots \cap u(A, P_r) \tag{8.5}$$

is a short representation of A as an intersection of (right) primal ideals.

Proof. The validity of (8.5) follows from Theorem 12 and (**1**, Theorem 19). The maximal primes $P_1, \ldots, P_n$ are associated primes of A by Cor. 1, Theorem 14, and hence each $u(A, P_1)$ is P_i-primal. Since $P_i \not\subseteq P_j$ for $i \neq j$ the representation (8.5) is short.

It is known from (**9**, Theorem 18) that if A has a right primary decomposition (which is, of course, also a right tertiary decomposition) the associated primes of A are exactly those primes P for which P is nrp to $u(A, P)$. Theorem 16 shows that even when a primary decomposition fails to exist the associated primes of A still have this property. That not every prime P for which P is nrp to $u(A,P)$ is an associated prime of A may be deduced from the following.

THEOREM 17. *If P is a minimal prime of A, $u(A, P)$ is P-tertiary (and therefore P-primal).*

Proof. Since P is a minimal prime of A it is also a minimal prime of $u(A, P)$. The ACC implies (**9**, p. 51) that $u(A, P)$ has a minimal prime P' that is nrp to $u(A, P)$. Since every element not in P is rp to $u(A, P)$, we have $P' \subseteq P$ and since P is minimal, $P' = P$. Hence P is nrp to $u(A, P)$ and $u(A, P)$ is P-primal.

Now suppose $u(A, P) = T_1 \cap \ldots \cap T_s$, where T_i is P_i-tertiary. By Theorem 14 each P_i is nrp to $u(A, P)$ and hence $P_i \subseteq P$ for all i. Hence

$$P \supseteq P_i \supseteq T_i \supseteq u(A, P) \supseteq A \qquad (i = 1, 2, \ldots, s)$$

and since P is a minimal prime of A, $P_i = P$ for all i, and $u(A, P)$ is P-tertiary.

Theorem 17 shows that every minimal prime P of A has the property that P is nrp to $u(A, P)$. But not all minimal primes of A need occur among the associated primes of A, as is shown by taking A to be any P-tertiary ideal T which is not P-primary. Thus the condition that P be nrp to $u(A, P)$ is necessary but not sufficient for P to be an associated prime of A. These results, together with (**9**, Theorem 18), suggest the conjecture that if every prime P for which P is nrp to $u(A, P)$ occurs as an associated prime of A, then A has a right primary decomposition. I have not been able to prove or disprove this conjecture.

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9. Extended and contracted ideals. If R and S are rings and $R \subseteq S$, the extended ideals of S are those that are extensions to S of ideals of R. The contracted ideals of R are those that are contractions of ideals of S, i.e. of the form $R \cap A'$, where A' is an ideal of S. The one-to-one correspondence between these two sets of ideals and the various properties of it are well known and may be found for the commutative case in **(12)**. We list below the main results that hold in the general case. There are important differences (to be discussed later) from the corresponding list for the commutative case **(12**, p. 219**)**.

We denote by c' the natural closure induced in $\mathfrak{L}(R)$ by S, and by c'' the corresponding mapping in $\mathfrak{L}(S)$ defined by

$$X^{c''} = E_S(R \cap X),$$

where $X \in \mathfrak{L}(S)$. The mapping c'' is an anti-closure in the sense of Querré **(10)**, that is, it satisfies the rules (a) $X^{c''} \subseteq X$, (b) $X \subseteq Y$ implies $X^{c''} \subseteq Y^{c''}$, and (c) $(X^{c''})^{c''} = X^{c''}$. We denote by $\mathfrak{L}^{c'}(R)$ the set of c'-closed ideals in R and by $\mathfrak{L}^{c''}(S)$ the set of c''-closed ideals in S and easily derive the following properties.

1. The contracted ideals of R are exactly those in $\mathfrak{L}^{c'}(R)$ and the extended ideals of S are exactly those of $\mathfrak{L}^{c''}(S)$.

2. The mapping $A \to E_S(A)$ is a one-to-one mapping of $\mathfrak{L}^{c'}(R)$ onto $\mathfrak{L}^{c''}(S)$ whose inverse mapping is $X \to R \cap X$.

3. $\mathfrak{L}^{c'}(R)$ is closed under complete intersection and hence becomes a lattice $(\mathfrak{L}^{c'}, \cap, \cup^*, \subseteq)$ when a union $\cup^*$ is defined by $\cup^* A_\sigma = (\sum A_\sigma)^{c'}$.

4. $\mathfrak{L}^{c''}(S)$ is closed under complete sums and hence becomes a lattice $(\mathfrak{L}^{c''}, \cap^*, +, \subseteq)$ when an intersection $\cap^*$ is defined by $\cap^* X_\sigma = (\cap X_\sigma)^{c''}$.

5. The two lattices defined in (3) and (4) are isomorphic under the mapping defined in (2).

It is important to note that whereas in the commutative case the mapping $A \leftrightarrow E_S(A)$ is also an isomorphism with respect to multiplication of ideals, this is not so in general for the non-commutative case.

The standard theorems **(12**, Ch. IV, Section 10**)** about contracted and extended ideals in R and R_M hold also, in the non-commutative case, for R and R_c provided the following three conditions are satisfied:

(a) $\mathfrak{L}^{c''}(R_c) = \mathfrak{L}(R_c)$, i.e. every ideal of R_c is an extended ideal.

(b) $E_c(AB) = E_c(A)E_c(B)$ for all ideals A, B of R.

(c) $c = c'$.

In every commutative ring (b) holds, and (c) is true for all maximal closures c. Condition (a) presents more difficulty and will be discussed further below.

THEOREM 18. *In a non-commutative ring R which satisfies the* ACC *for ideals, conditions* (a), (b), *and* (c) *imply*:

(1) *The contracted prime ideals P of R are mapped one-to-one onto the prime ideals P' of R_c by the mapping $P \to E_c(P)$, and $P' \to R \cap P'$ is the inverse mapping.*

(2) *If P and $P' = E_c(P)$ are corresponding primes as in* (1), *then $A \to E_c(A)$ is a one-to-one mapping of the* (*right*) *P-primary ideals A of R onto the* (*right*) *P'-primary ideals A' of R_c and $A' \to R \cap A'$ is the inverse mapping.*

Here an ideal A in R is said to be right primary if for any ideals B and C in R, $BC \subseteq A$ implies that either $C \subseteq A$ or $B^n \subseteq A$ for some integer n. If R contains a unit element and satisfies the ACC for ideals, this is equivalent to the definition $xRy \subseteq A$ implies either $y \in A$ or $x \in$ the radical of A. The proof of (1) and (2) is the standard application of (a), (b), and (c) together with the lattice isomorphism between $\mathfrak{L}^{c'}(R)$ and $\mathfrak{L}^{c''}(R_c)$.

Of course this is highly unsatisfactory since it is far from clear in what non-commutative rings and for what closures c conditions (a), (b), and (c) hold. Condition (b) especially suggests a high degree of commutativity. By Theorem 7 every maximal closure c contains a closure c_ω for which (c) holds, but it is not clear, for example, if $c = c(P)$, which of the $c(P)$-closed ideals are also c_ω-closed and hence are contracted ideals. In the next section we discuss conditions that ensure (a) and a weakened form of (b) but which nevertheless yield both results of Theorem 18 provided we use left primary ideals and $c = c_l(P)$.

10. Semi-prime rings with right quotient conditions. Let R be a semi-prime ring that satisfies A. W. Goldie's right quotient conditions, namely, (a) R has finite dimension as a right R-module and (b) R satisfies the ACC for annihilator right ideals. Goldie has shown (**3**, Theorem 5) that R has a full ring of right quotients F and (**3**, Lemma 4.2) that the right singular ideal[4] of R is zero. It follows from (**11**, Theorem 3) that $R^\blacktriangle$ consists precisely of the essential right ideals of R. Since by (**3**, Theorem 4.8), a right ideal is essential if and only if it contains a regular element, it follows that $Q = F$.

THEOREM 19. *Let R be a semi-prime ring that satisfies Goldie's right quotient conditions. Let c be a bilateral closure in $\mathfrak{L}_r(R)$ such that every element $\tilde{\alpha}$ of R_c can be written in the form $\tilde{\alpha} = ba^{-1}$, where $a, b \in R$, a is regular, and $a^{-1} \in R_c$. Then* (a) $\mathfrak{L}^{c''}(R_c) = \mathfrak{L}(R_c)$ *and* (b) *if $A, B \in \mathfrak{L}(R)$ and B is c-closed,*

$$E_c(AB) = E_c(A)E_c(B).$$

Proof. (a) Let A' be any ideal of R_c and let $a \in A'$. Then $\tilde{\alpha} = ba^{-1}$, where $a^{-1} \in R_c$. Since $\tilde{\alpha}a = b \in R \cap A'$, $\tilde{\alpha} = ba^{-1} \in E_c(R \cap A')$. Hence

$$A' = E_c(R \cap A')$$

and every ideal of R_c is an extended ideal.

(b) Clearly $E_c(AB) \subseteq E_c(A)E_c(B)$. Now $E_c(A)E_c(B)$ is generated by elements of the form $\tilde{\alpha}a\tilde{\beta}b\tilde{\gamma}$ where $a \in A$, $b \in B$ and $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \in R_c$, α β, γ

[4]The right singular ideal of R (**4**, p. 894) is the two-sided ideal consisting of all elements x such that the right annihilator of x is an essential right ideal.

being semi-R-endomorphisms in $\mathfrak{R}_c$. The domain of the semi-R-endomorphism $\alpha a\beta b\gamma$ is the essential right ideal

$$I = \{x \in I_\gamma \mid b\gamma x \in I_\beta \text{ and } a\beta b\gamma x \in I_\alpha\},$$

which, by Goldie's result, contains a regular element x. Since $x \in I_\gamma$, $\gamma x \in R$ and $b\gamma x \in I_\beta \cap B$. Hence since $\beta \in \mathfrak{R}_c$ and B is c-closed, $\beta b\gamma x \in B$ and $\alpha a\beta b\gamma x = \alpha ab'$, $b' \in B$. Under the assumption of the theorem x can be so chosen that $x^{-1} \in R_c$ and hence $\tilde{\alpha}a\tilde{\beta}b\tilde{\gamma} = \tilde{\alpha}ab'x^{-1} \in E_c(AB)$. Hence

$$E_c(A)E_c(B) = E_c(AB).$$

From this we have the following theorem.

THEOREM 20. *Let R be a ring satisfying the hypotheses of Theorem 19 and the ACC for two-sided ideals. Let c be a bilateral closure in $\mathfrak{L}_r(R)$ satisfying $c' = c$ and the hypothesis of Theorem 19. Then*

(1) *the mapping $P \to E_c(P)$ maps the set of contracted prime ideals of R into the set of prime ideals of R_c,*

(2) *if P and $P' = E_c(P)$ are corresponding primes as in (1) and A is a right P-primary ideal of R, then $E_c(A)$ is a right P'-primary ideal of R_c.*

Proof. The proof is immediate. The only effect of the weakening of Condition (b) is to invalidate the proof in Theorem 18 that if P' is prime in R_c then $R \cap P'$ is prime in R and similarly for the primary ideals. However, this feature can be restored for the closures $c_l(P)$ defined by the *left* upper P-components. We have in fact the following theorem.

THEOREM 21. *Let R be a ring, P a prime ideal in R, and let c be the bilateral closure defined by*

$$J^c = u_l(J^m, P).$$

If R and c satisfy the hypotheses of Theorem 20, then Theorem 18(1) follows and also Theorem 18(2) for left primary ideals. Moreover $E_c(P)$ is a unique maximal prime of R_c and $[E_c(P)]^n = E_c(P^n)$.

Proof. The conclusions of Theorem 20 follow as before. Suppose P' is a prime ideal in R_c. We wish to show that $R \cap P'$ is prime in R. Since $R \cap P'$ is a contracted ideal it is c'-closed and hence c-closed since we are assuming $c = c'$. Let A, B be ideals in R. Since $B^c = u_l(B, P)$ by Theorem 12, $B^cRm \subseteq B$ for some element $m \notin P$ and hence $AB^cRm \subseteq AB$ and $AB^c \subseteq (AB)^c$. Now if $AB \subseteq R \cap P'$, since $R \cap P'$ is c-closed, $(AB)^c \subseteq R \cap P'$ and hence $AB^c \subseteq R \cap P'$. Since B^c is c-closed Theorem 19 now yields $E_c(A)E_c(B^c) = E_c(AB^c) \subseteq E_c(R \cap P') = P'$. Hence $E_c(A) \subseteq P'$ or $E_c(B^c) \subseteq P'$ and, by contracting, either $A^c \subseteq R \cap P'$ or $B^c \subseteq R \cap P'$ and a fortiori either A or B is contained in $R \cap P'$.

Now suppose P' is a prime ideal in R_c and A' is left P'-primary. Suppose B, C are ideals of R and $BC \subseteq R \cap A'$. The same argument as above yields $E_c(B)E_c(C^c) \subseteq A'$ and since A' is left P'-primary $E_c(B) \subseteq A'$ or $[E_c(C^c)]^n \subseteq A'$.

Since C^c is c-closed the latter gives $E_c[(C^c)^n] \subseteq A'$ and by contracting we get either $B \subseteq R \cap A'$ or $C^n \subseteq (C^c)^n \subseteq R \cap A'$. Hence $R \cap A'$ is left primary. Now suppose the radical of $R \cap A'$ is P_1. Since P' is nilpotent mod A', $R \cap P'$ is nilpotent mod $R \cap A'$. Hence $R \cap A' \subseteq R \cap P' \subseteq P_1$. But $R \cap P'$ is prime and P_1 is a minimal prime of $R \cap A'$. Hence $P_1 = R \cap P'$ and $R \cap A'$ is left $(R \cap P')$-primary.

Finally if P' is any prime of R_c, $R \cap P'$ is c-closed and hence either $R \cap P' = R$ or $R \cap P' \subseteq P$. It follows that $P' = E_c(R \cap P')$ is either R_c itself or is contained in $E(P)$. Hence $E(P)$ is a unique maximal prime of R_c and $[E_c(P)]^n = E_c(P^n)$.

If $c = c(P)$ when P is a prime ideal in R, a necessary condition for $c = c'$ is clearly that the 0 ideal is c-closed, that is $u(0, P) = 0$. This means that no two-sided ideal can be annihilated by multiplying on the left (right) by an element not in P. If $u(0, P) = N \neq 0$ we can of course work in the ring R/N, as is done in the commutative case, since c induces a closure in R/N for which the zero ideal is closed.

It is clear that many fundamental questions are left unanswered by the above theorems. For example if $c = c(P)$ is c maximal if $u(0, P) = 0$? For what primes in what rings is $c = c'$? Are P and the P-primary ideals of R c_ω-closed, where c_ω is the closure constructed in Theorem 7? When does the ring $R_{c(P)}$ have a unique maximal prime $E_c(P)$ and if it does is $\bigcap[E_c(\mathrm{P})]^n = 0$? For what rings and what closures is the hypothesis of Theorem 19 satisfied?

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