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Exponentials of de Branges–Rovnyak kernels

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Abstract. In this note, we give a new property of de Branges–Rovnyak kernels. As the main theorem, it is shown that the exponential of de Branges–Rovnyak kernel is strictly positive definite if the corresponding Schur class function is nontrivial.

1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, and let H^∞ be the Banach algebra consisting of all bounded analytic functions on $\mathbb{D}$. Then, we set

$$\mathcal{S} = \{\varphi \in H^\infty : |\varphi(\lambda)| \leq 1\ (\lambda \in \mathbb{D})\},$$

which is called the Schur class. For any function φ in H^∞, it is well known that φ belongs to $\mathcal{S}$ if and only if

$$\frac{1 - \overline{\varphi(\lambda)}\varphi(z)}{1 - \overline{\lambda}z} \tag{1.1}$$

is positive semi-definite. This equivalence relation based on the properties of the Szegö kernel is crucial in the operator theory on the Hardy space over $\mathbb{D}$, in particular, theories of Pick interpolation, de Branges–Rovnyak spaces and sub-Hardy Hilbert spaces (see Agler–McCarthy [2], Ball–Bolotnikov [4], Fricain–Mashreghi [6], and Sarason [14]). The kernel (1.1) is called the de Branges–Rovnyak kernel.

Before introducing our study, we should mention that not only the original de Branges–Rovnyak kernel but also its variants have been studied by a number of authors. For example, Zhu [16, 17] initiated the study on the kernel

$$\frac{1 - \overline{\varphi(\lambda)}\varphi(z)}{(1 - \overline{\lambda}z)^2} \tag{1.2}$$

in the Bergman space over $\mathbb{D}$. The reproducing kernel Hilbert space induced by the kernel (1.2) is called a sub-Bergman Hilbert space (see also Abkar–Jafarzadeh [1], Ball–Bolotnikov [3], Chu [5], Nowak–Rososzczuk [11], and Sultanic [15]). Further, powers

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of the de Branges–Rovnyak kernel

$$\left(\frac{1-\overline{\varphi(\lambda)}\varphi(z)}{1-\overline{\lambda}z}\right)^n \quad (n \in \mathbb{N}) \tag{1.3}$$

are naturally obtained from the theory of hereditary functional calculus for weighted Bergman spaces on $\mathbb{D}$ (see Example 14.48 in [2] for the case where $n = 2$) and have appeared also in p. 3672 of Jury [8].

Now, the purpose of this paper is to study the structure of the kernel

$$\exp\left(t\frac{1-\overline{\varphi(\lambda)}\varphi(z)}{1-\overline{\lambda}z}\right) \quad (t > 0). \tag{1.4}$$

Note that our kernel (1.4) is obtained by binding all kernels in (1.3) together. Thus, we expect that new properties of the de Branges–Rovnyak kernel (1.1) are drawn out from our kernel (1.4). In fact, as the main theorem, we will show that the exponential of the de Branges–Rovnyak kernel is strictly positive definite if φ is nontrivial.

Here, we shall give some remarks on strictly positive definite kernels. In general, it is not difficult to construct positive semi-definite kernels. On the other hand, for strictly positive definite kernels, nontrivial methods depending on each case are often needed (for example, see Micchelli [9]). Moreover, it might be worth while mentioning that strictly positive definite kernels have received attention in machine learning (see Rasmussen–Williams [13]).

This paper is organized as follows. In Section 2, basic properties of the reproducing kernel Hilbert space $\exp \mathcal{H}_t(\varphi)$ constructed from our kernel (1.4) are given. In Section 3, unbounded multipliers on $\exp \mathcal{H}_t(\varphi)$ are introduced and studied. In Section 4, we prove the main theorem.

2 Preliminaries

For $t > 0$, let $\mathcal{H}_t(\varphi)$ denote the reproducing kernel Hilbert space with kernel

$$tk^{\varphi}(z,\lambda) = t\frac{1-\overline{\varphi(\lambda)}\varphi(z)}{1-\overline{\lambda}z} \quad (\varphi \in \mathcal{S}),$$

and we will use notations $tk^{\varphi}_{\lambda}(z) = tk^{\varphi}(z,\lambda)$ and $\mathcal{H}(\varphi) = \mathcal{H}_1(\varphi)$. Then, since

$$\langle tk^{\varphi}_{\lambda}, tk^{\varphi}_{z}\rangle_{\mathcal{H}_t(\varphi)} = tk^{\varphi}(z,\lambda) = t^{-1}\langle tk^{\varphi}_{\lambda}, tk^{\varphi}_{z}\rangle_{\mathcal{H}(\varphi)},$$

the trivial linear mapping $f \mapsto f$ from $\mathcal{H}(\varphi)$ onto $\mathcal{H}_t(\varphi)$ is bounded and invertible. Particularly, $\mathcal{H}_t(\varphi) = \mathcal{H}(\varphi)$ as vector spaces. In this section, we construct the exponential of $\mathcal{H}_t(\varphi)$ and give its basic properties. The contents of this section are well known to specialists. For example, see Exercise (k) in p. 320 of Nikolski [10] and Chapter 7 in Paulsen–Raghupathi [12]. However, we give the details for the sake of readers.

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2.1 Construction of $\exp \mathcal{H}_t(\varphi)$

Let $\mathcal{H}_t(\varphi)^n$ be the reproducing kernel Hilbert space obtained by the pull-back construction with the n-fold tensor product space

$$\mathcal{H}_t(\varphi)^{\otimes n} = \mathcal{H}_t(\varphi) \otimes \cdots \otimes \mathcal{H}_t(\varphi)$$

and the n-dimensional diagonal map

$$\Delta_n : \mathbb{D} \to \mathbb{D}^n, \ \lambda \to (\lambda, \ldots, \lambda)$$

(for the pull-back construction, see Theorem 5.7 in [12]). We note that $(tk_\lambda^\varphi)^{\otimes n} \circ \Delta_n = (tk_\lambda^\varphi)^n$ is the reproducing kernel of $\mathcal{H}_t(\varphi)^n$. Let $\oplus_{n=0}^\infty \mathcal{H}_t(\varphi)^n$ denote the Hilbert space with the inner product

$$\langle (f_0, f_1, \ldots)^\top, (g_0, g_1, \ldots)^\top \rangle_{\oplus_{n=0}^\infty \mathcal{H}_t(\varphi)^n} = \sum_{n=0}^\infty \frac{1}{n!} \langle f_n, g_n \rangle_{\mathcal{H}_t(\varphi)^n},$$

where we set $\mathcal{H}_t(\varphi)^0 = \mathbb{C}$. Moreover, we define the linear map Γ as follows:

$$\Gamma : \begin{pmatrix} f_0 \\ f_1 \\ \vdots \end{pmatrix} \mapsto \sum_{n=0}^\infty \frac{1}{n!} f_n \quad \left(\begin{pmatrix} f_0 \\ f_1 \\ \vdots \end{pmatrix} \in \oplus_{n=0}^\infty \mathcal{H}_t(\varphi)^n \right).$$

Proposition 2.1 *The following statements hold:*

(1) *Γ is a map from $\oplus_{n=0}^\infty \mathcal{H}_t(\varphi)^n$ to* $\mathrm{Hol}(\mathbb{D})$.
(2) $\ker \Gamma$ *is closed.*

Proof For any $F = (f_0, f_1, \ldots)^\top$ in $\oplus_{n=0}^\infty \mathcal{H}_t(\varphi)^n$, we have

$$\begin{aligned} \left| \sum_{\ell=n+1}^m \frac{1}{\ell!} f_\ell(\lambda) \right| &\le \sum_{\ell=n+1}^m \left| \frac{1}{\ell!} f_\ell(\lambda) \right| \\ &\le \sum_{\ell=n+1}^m \frac{1}{\ell!} \| f_\ell \|_{\mathcal{H}_t(\varphi)^\ell} \| (tk_\lambda^\varphi)^\ell \|_{\mathcal{H}_t(\varphi)^\ell} \\ &\le \left(\sum_{\ell=n+1}^m \frac{1}{\ell!} \| f_\ell \|^2_{\mathcal{H}_t(\varphi)^\ell} \right)^{1/2} \left(\sum_{\ell=n+1}^m \frac{1}{\ell!} \| (tk_\lambda^\varphi)^\ell \|^2_{\mathcal{H}_t(\varphi)^\ell} \right)^{1/2} \\ &= \left(\sum_{\ell=n+1}^m \frac{1}{\ell!} \| f_\ell \|^2_{\mathcal{H}_t(\varphi)^\ell} \right)^{1/2} \left(\sum_{\ell=n+1}^m \frac{1}{\ell!} \| tk_\lambda^\varphi \|^{2\ell}_{\mathcal{H}_t(\varphi)} \right)^{1/2}. \end{aligned} \tag{2.1}$$

Hence, $\sum_{n=0}^\infty \frac{1}{n!} f_n(\lambda)$ converges uniformly on any compact subset of $\mathbb{D}$. This concludes (1). Next, let K be a compact set in $\mathbb{D}$. Then, since

$$\| tk_\lambda^\varphi \|^2_{\mathcal{H}_t(\varphi)} = t \frac{1 - |\varphi(\lambda)|^2}{1 - |\lambda|^2},$$

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there exists a constant $C_K > 0$ such that

$$\sup_{\lambda\in K} \|tk_\lambda^\varphi\|^{2n}_{\mathcal{H}_t(\varphi)} \leq C_K^n.$$

Moreover, in (2.1), we essentially showed that

$$|(\Gamma F)(\lambda)| \leq \|F\|_{\oplus_{n=0}^\infty \mathcal{H}_t(\varphi)^n} \exp \frac{\|tk_\lambda^\varphi\|^2_{\mathcal{H}_t(\varphi)}}{2}. \tag{2.2}$$

Hence, we have

$$\sup_{\lambda\in K} |(\Gamma F)(\lambda)| \leq \|F\|_{\oplus_{n=0}^\infty \mathcal{H}_t(\varphi)^n} \exp \frac{C_K}{2}.$$

Therefore, Γ is continuous. This concludes (2).

By Proposition 2.1, the pull-back construction can be applied to Γ.

Definition 2.1 We define $\exp \mathcal{H}_t(\varphi)$ as the reproducing kernel Hilbert space obtained by the pull-back construction with the linear map

$$\Gamma : \oplus_{n=0}^\infty \mathcal{H}_t(\varphi)^n \to \mathrm{Hol}(\mathbb{D}).$$

2.2 Basic properties of $\exp \mathcal{H}_t(\varphi)$

We summarize basic properties of $\exp \mathcal{H}_t(\varphi)$.

Proposition 2.2 *$\exp \mathcal{H}_t(\varphi)$ is a reproducing kernel Hilbert space consisting of holomorphic functions on $\mathbb{D}$. More precisely, for any f in $\exp \mathcal{H}_t(\varphi)$, there exists a vector $(f_0, f_1, \dots,)^\top$ in $\oplus_{n=0}^\infty \mathcal{H}_t(\varphi)^n$ such that*

$$f = \sum_{n=0}^\infty \frac{1}{n!} f_n$$

converges uniformly on any compact subset of $\mathbb{D}$. Moreover,

(1) *the following norm estimate holds:*

$$\|f\|^2_{\exp \mathcal{H}_t(\varphi)} \leq \sum_{n=0}^\infty \frac{1}{n!} \|f_n\|^2_{\mathcal{H}_t(\varphi)^n},$$

(2) *the reproducing kernel of $\exp \mathcal{H}_t(\varphi)$ is*

$$\sum_{n=0}^\infty \frac{1}{n!} (tk_\lambda^\varphi)^n = \exp tk_\lambda^\varphi,$$

that is,

$$f(\lambda) = \langle f, \exp tk_\lambda^\varphi \rangle_{\exp \mathcal{H}_t(\varphi)}$$

for any λ in $\mathbb{D}$,

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(3) *the following growth condition holds:*

$$|f(\lambda)|^2 \leq \|f\|^2_{\exp \mathcal{H}_t(\varphi)} \exp\left(t\frac{1-|\varphi(\lambda)|^2}{1-|\lambda|^2}\right)$$

for any λ in $\mathbb{D}$.

Proof By the definition of the norm and the inner product of $\exp \mathcal{H}_t(\varphi)$, we have conclusions.

3 Unbounded multipliers

We shall investigate into unbounded multipliers of $\exp \mathcal{H}_t(\varphi)$.

Lemma 3.1 *Let ψ be a function in $\mathcal{H}_t(\varphi)$. Then, for any function f in $\mathcal{H}_t(\varphi)^n$, ψf belongs to $\mathcal{H}_t(\varphi)^{n+1}$.*

Proof We define the bounded linear operator τ_ψ as follows:

$$\tau_\psi : \mathcal{H}_t(\varphi)^{\otimes n} \to \mathcal{H}_t(\varphi)^{\otimes n+1}, \quad F \mapsto \psi \otimes F.$$

Then, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}_t(\varphi)^{\otimes n} & \xrightarrow{\tau_\psi} & \mathcal{H}_t(\varphi)^{\otimes n+1} \\ {\scriptstyle \Delta_n}\big\downarrow & & \big\downarrow{\scriptstyle \Delta_{n+1}} \\ \mathcal{H}_t(\varphi)^n & \xrightarrow[M_\psi|_{\mathcal{H}_t(\varphi)^n}]{} & \mathcal{H}_t(\varphi)^{n+1}, \end{array}$$

where M_ψ denotes the multiplication operator with symbol ψ. This concludes the proof.

Theorem 3.2 *Let ψ be a function in $\mathcal{H}_t(\varphi)$. Then, the multiplication operator M_ψ is a densely defined closable linear operator in $\exp \mathcal{H}_t(\varphi)$.*

Proof Let $F = (f_0, f_1, \dots, f_N, 0 \dots)^\top$ be a vector having finite support in $\oplus_{n=0}^\infty \mathcal{H}_t(\varphi)^n$. We set $\Gamma F = f$. Then,

$$\psi f = \psi \sum_{n=0}^{N} \frac{1}{n!} f_n = \sum_{n=0}^{N} \frac{1}{n!} \psi f_n = \sum_{n=0}^{N} \frac{1}{(n+1)!}(n+1)\psi f_n = \sum_{n=1}^{N+1} \frac{1}{n!} n\psi f_{n-1},$$

where we note that $n\psi f_{n-1}$ belongs to $\mathcal{H}_t(\varphi)^n$ by Lemma 3.1. Hence, setting

$$G = (0, \psi f_0, 2\psi f_1, \dots, (N+1)\psi f_N, 0, \dots)^\top,$$

G belongs to $\oplus_{n=0}^\infty \mathcal{H}_t(\varphi)^n$ and $\Gamma G = \psi f$, that is, ψf belongs to $\exp \mathcal{H}_t(\varphi)$. Therefore, M_ψ is a densely defined linear operator in $\exp \mathcal{H}_t(\varphi)$. Moreover, it is easy to see that M_ψ is closable.

Corollary 3.3 *Let ψ be a function in $\mathcal{H}_t(\varphi)$. Then the adjoint operator M_ψ^* of M_ψ is a densely defined closed linear operator in $\exp \mathcal{H}_t(\varphi)$, and every $\exp tk_\lambda^\varphi$ is an eigenfunction of M_ψ^*. More precisely,*

$$M_\psi^* \exp tk_\lambda^\varphi = \overline{\psi(\lambda)} \exp tk_\lambda^\varphi.$$

4 Main results

Let X be a set. A function k on $X \times X$ is called a strictly positive definite kernel on X if $k(x, y) = \overline{k(y, x)}$ for any x and y in X and

$$\sum_{i,j=1}^{n} c_i \overline{c_j} k(x_j, x_i) > 0$$

for any n in $\mathbb{N}$, any $(c_1, \ldots, c_n)^\top$ in $\mathbb{C}^n \setminus \{\mathbf{0}\}$ and any n distinct points $x_1, \ldots, x_n$ in X. For example, it is well known that

$$k(z, \lambda) = \exp(\overline{\lambda} z)$$

is a strictly positive definite kernel on $\mathbb{C}$. In fact, this is the reproducing kernel of the Segal–Bargmann space. Now, we note that if $\varphi = z^2$ then

$$e^{-1} \exp\left(\frac{1 - \overline{\varphi(\lambda)}\varphi(z)}{1 - \overline{\lambda} z}\right) = \exp(\overline{\lambda} z).$$

Motivated by this observation, we shall give new examples of strictly positive definite kernels. We consider the following three conditions: (C1) $\varphi(0) = \varphi'(0) = 0$, (C2) $\varphi(\mu) = 0$ for some μ in $\mathbb{D} \setminus \{0\}$, (C3) the dimension of $\mathcal{H}(\varphi)$ is infinite.

We need the following lemma.

Lemma 4.1 *Let $\lambda_1, \ldots, \lambda_n$ be n distinct points in $\mathbb{D}$. Suppose one of (C1), (C2) and (C3). Then there exists a function ψ in $\mathcal{H}_t(\varphi)$ such that $\psi(\lambda_i) \neq \psi(\lambda_j)(i \neq j)$.*

Proof Since $\mathcal{H}_t(\varphi) = \mathcal{H}(\varphi)$ as vector spaces, it suffices to show the statement for $\mathcal{H}(\varphi)$. First, we assume (C1). Then, since φ/z is in $\mathcal{S}$ by the Schwarz lemma and $(\varphi/z)(0) = 0$, we have

$$(I - T_\varphi T_\varphi^*)z = z - T_\varphi T_{\varphi/z}^* T_z^* z = z - T_\varphi T_{\varphi/z}^* 1 = z,$$

where T_φ denotes the Toeplitz operator with symbol φ on the Hardy space H^2 over $\mathbb{D}$. Hence z belongs to $\mathcal{H}(\varphi)$, and we may take $\psi = z$.

Secondly, we assume (C2). Let μ be a nonzero zero point of φ. Then, we have

$$k_\mu^\varphi = (I - T_\varphi T_\varphi^*)k_\mu = k_\mu.$$

Hence, $(1 - \overline{\mu} z)^{-1}$ belongs to $\mathcal{H}(\varphi)$, and we may take $\psi = (1 - \overline{\mu} z)^{-1}$.

Thirdly, we assume (C3). Then, by Lemma 31.2 in [6], the family $\{k_{\lambda_j}^\varphi : 1 \leq j \leq n\}$ is minimal. Hence, we have that $\dim \operatorname{span}\{k_{\lambda_j}^\varphi : 1 \leq j \leq n\} = n$. Let T be the linear map defined as follows:

$$T : \operatorname{span}\{k^{\varphi}_{\lambda_j} : 1 \le j \le n\} \to \mathbb{C}^n, \quad f \mapsto (f(\lambda_1), \dots, f(\lambda_n)).$$

Then, it is easy to see that $\ker T = \{0\}$. Hence, there exists a function ψ in $\mathcal{H}(\varphi)$ such that $\psi(\lambda_i) \ne \psi(\lambda_j)(i \ne j)$.

Theorem 4.2 *Let φ be a function in $\mathcal{S}$. If φ satisfies one of* (C1), (C2) *and* (C3), *then the kernel*

$$k_t(z, \lambda) = \exp\left(t \frac{1 - \overline{\varphi(\lambda)}\varphi(z)}{1 - \overline{\lambda} z}\right) \quad (t > 0)$$

is strictly positive definite.

Proof It suffices to show that $\{\exp tk^{\varphi}_{\lambda_j}\}_{j=1}^n$ is linearly independent for any n in $\mathbb{N}$ and any n distinct points $\lambda_1, \dots, \lambda_n$ in $\mathbb{D}$. Suppose that

$$\sum_{j=1}^{n} c_j \exp tk^{\varphi}_{\lambda_j} = 0$$

for some n in $\mathbb{N}$, some n distinct points $\lambda_1, \dots, \lambda_n$ in $\mathbb{D}$, and some $c_1, \dots, c_n$ in $\mathbb{C}$. Then, for any function ψ in $\mathcal{H}_t(\varphi)$, by Corollary 3.3 and the assumption, we have

$$\begin{aligned}
\begin{pmatrix} 1 & \cdots & 1 \\ \overline{\psi(\lambda_1)} & \cdots & \overline{\psi(\lambda_n)} \\ \vdots & \vdots & \vdots \\ \overline{\psi(\lambda_1)}^{n-1} & \cdots & \overline{\psi(\lambda_n)}^{n-1} \end{pmatrix}
\begin{pmatrix} c_1 \exp tk^{\varphi}_{\lambda_1} \\ c_2 \exp tk^{\varphi}_{\lambda_2} \\ \vdots \\ c_n \exp tk^{\varphi}_{\lambda_n} \end{pmatrix}
&= \begin{pmatrix} \sum_{j=1}^n c_j \exp tk^{\varphi}_{\lambda_j} \\ \sum_{j=1}^n \overline{\psi(\lambda_j)} c_j \exp tk^{\varphi}_{\lambda_j} \\ \vdots \\ \sum_{j=1}^n c_j \overline{\psi(\lambda_j)}^{n-1} \exp tk^{\varphi}_{\lambda_j} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{j=1}^n c_j \exp tk^{\varphi}_{\lambda_j} \\ M_\psi^* \sum_{j=1}^n c_j \exp tk^{\varphi}_{\lambda_j} \\ \vdots \\ (M_\psi^*)^{n-1} \sum_{j=1}^n c_j \exp tk^{\varphi}_{\lambda_j} \end{pmatrix} \\
&= \mathbf{0}.
\end{aligned}$$

Further, by Lemma 4.1, there exists a function ψ in $\mathcal{H}_t(\varphi)$ such that

$$\prod_{1 \le i < j \le n} (\psi(\lambda_i) - \psi(\lambda_j)) \ne 0.$$

Then, the Vandermonde matrix

$$\begin{pmatrix} 1 & \cdots & 1 \\ \overline{\psi(\lambda_1)} & \cdots & \overline{\psi(\lambda_n)} \\ \vdots & \vdots & \vdots \\ \overline{\psi(\lambda_1)}^{n-1} & \cdots & \overline{\psi(\lambda_n)}^{n-1} \end{pmatrix}$$

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is nonsingular. Therefore, we have that

$$\begin{pmatrix} c_1 \exp tk^{\varphi}_{\lambda_1} \\ c_2 \exp tk^{\varphi}_{\lambda_2} \\ \vdots \\ c_n \exp tk^{\varphi}_{\lambda_n} \end{pmatrix} = \mathbf{0}.$$

This concludes that $c_1 = \cdots = c_n = 0$.

The well-known fact mentioned at the beginning of this section is included in Theorem 4.2.

Corollary 4.3 *The kernel function*

$$k(z,\lambda) = \exp(\overline{\lambda} z)$$

is strictly positive definite on $\mathbb{C}$.

Proof For any n distinct points $\lambda_1, \dots \lambda_n$ in $\mathbb{C}$, we set $R = \max_{1 \le j \le n} |\lambda_j| + 1$. Then $\lambda_1/R, \dots \lambda_n/R$ are in $\mathbb{D}$. Hence, by Theorem 4.2 in the case where $\varphi = z^2$ and $t = R^2$, we have

$$\begin{aligned} \sum_{i,j=1}^{n} c_i \overline{c_j} \exp(\overline{\lambda_i}\lambda_j) &= e^{-R^2} \sum_{i,j=1}^{n} c_i \overline{c_j} \exp(R^2 + \overline{\lambda_i}\lambda_j) \\ &= e^{-R^2} \sum_{i,j=1}^{n} c_i \overline{c_j} \exp(R^2(1 + \overline{(\lambda_i/R)}(\lambda_j/R))) \\ &= e^{-R^2} \sum_{i,j=1}^{n} c_i \overline{c_j} \exp\left(R^2 \frac{1 - \overline{\varphi(\lambda_i/R)}\varphi(\lambda_j/R)}{1 - \overline{(\lambda_i/R)}(\lambda_j/R)}\right) > 0 \end{aligned}$$

for any $(c_1, \dots, c_n)^\top$ in $\mathbb{C}^n \setminus \{\mathbf{0}\}$.

Although the next result is just a simple consequence of Theorem 4.2, from the viewpoint of the theory of model spaces (see Garcia–Mashreghi–Ross [7]), it will be worth while mentioning it as a theorem.

Theorem 4.4 *Let φ be an inner function. If φ is neither a constant nor $e^{i\theta}z$, then the kernel*

$$k_t(z,\lambda) = \exp\left(t \frac{1 - \overline{\varphi(\lambda)}\varphi(z)}{1 - \overline{\lambda} z}\right) \quad (t > 0)$$

is strictly positive definite.

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