Proceedings of the Edinburgh Mathematical Society (2017) **60**, 911–936
DOI:10.1017/S0013091516000481

THE HAUSDORFF DIMENSION IS CONVEX ON THE LEFT SIDE OF 1/4

LUDWIK JAKSZTAS

Faculty of Mathematics and Information Sciences, Warsaw University of Technology, ul. Koszykowa 75, 00-662 Warsaw, Poland (jaksztas@impan.gov.pl)

(Received 28 January 2014)

Abstract Let $d(c)$ denote the Hausdorff dimension of the Julia set J_c of the polynomial $f_c(z) = z^2 + c$. The function $c \mapsto d(c)$ is real-analytic on the interval $(-3/4, 1/4)$, which is in the domain bounded by the main cardioid of the Mandelbrot set. We prove that the function d is convex close to $1/4$ on the left side of it.

Keywords: Hausdorff dimension; Julia set; quadratic family

2010 *Mathematics subject classification:* Primary 37F45
Secondary 37F35

1. Introduction

Let us consider the family of quadratic polynomials of the form

$$f_c(z) = z^2 + c.$$

We define the filled-in Julia set K_c as the set of all points that do not escape to infinity under iteration of f_c. The Julia set J_c is defined as the boundary of K_c, i.e.

$$J_c = \partial K_c = \partial\{z \in \mathbb{C}\colon f_c^n(z) \nrightarrow \infty\}.$$

The Mandelbrot set $\mathcal{M}$ is the set of all parameters c for which the Julia set J_c is connected, or, equivalently,

$$\mathcal{M} = \{c \in \mathbb{C}\colon f_c^n(0) \nrightarrow \infty\}.$$

We are interested in the function $c \mapsto d(c)$, where $d(c) = \mathrm{HD}(J_c)$ denotes the Hausdorff dimension of the Julia set J_c.

Recall that a polynomial $f\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ (or more generally a rational function) is called hyperbolic (expanding) if there exists $n \in \mathbb{N}$ such that for all $z \in J(f)$, $|(f^n)'(z)| > 1$.

The function d is real-analytic on each hyperbolic component of $\mathrm{Int}\,\mathcal{M}$ (consisting of parameters related to hyperbolic maps) as well as on the exterior of $\mathcal{M}$ (see [**12**]). In particular, d is real-analytic on $\mathcal{M}^0$ and $\mathcal{M}^{1/2}$ (see the definitions below).

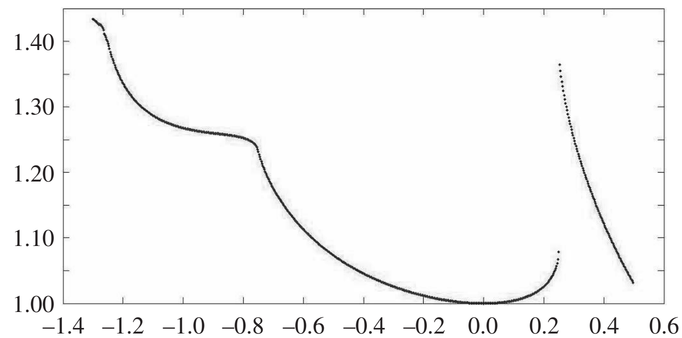

Figure 1. The Hausdorff dimension of J_c.

The component $\mathcal{M}^0$ (the largest component of $\operatorname{Int}\mathcal{M}$, bounded by the so-called main cardioid) consists of all parameters c for which the polynomial f_c has an attracting fixed point. Note that $\mathcal{M}^0 \cap \mathbb{R} = (-3/4, 1/4)$, and the polynomial $f_{1/4}$ has a parabolic fixed point with one petal: $f'_{1/4}(1/2) = 1$.

The component $\mathcal{M}^{1/2}$ (called the 1/2 bulb) consists of all parameters for which f_c has a minimal period 2 attracting periodic orbit. We have $\mathcal{M}^{1/2} \cap \mathbb{R} = (-5/4, -3/4)$, and the parameter $c = -3/4$ is a common point of the closures of $\mathcal{M}^0$ and $\mathcal{M}^{1/2}$. The polynomial $f_{-3/4}$ has a parabolic fixed point with two petals: $f'_{-3/4}(-1/2) = -1$.

Let us consider $c \in \mathbb{R}$. Bodart and Zinsmeister proved that the Hausdorff dimension is continuous when the parameter tends to 1/4 from the left (see [**1**]). It follows from [**9**] that the function $d|_{\mathbb{R}}$ is continuous on the interval $(c_{\text{feig}}, 1/4]$, where $c_{\text{feig}} \approx -1.401$ is the Feigenbaum parameter (in particular, $d|_{\mathbb{R}}$ is continuous on $[-5/4, 1/4] = (\overline{\mathcal{M}^{1/2} \cup \mathcal{M}^0}) \cap \mathbb{R}$). Note that $d|_{\mathbb{R}}$ is not right-continuous at 1/4, i.e. when c approaches 1/4 from outside of the Mandelbrot set (see [**2**]).

In [**4**] Havard and Zinsmeister studied more precisely the behaviour of d on the left side of 1/4. They proved the following theorem.

Theorem HZ. *There exist $c_1 < 1/4$ and $K > 1$ such that for every $c \in (c_1, 1/4)$,*

$$\frac{1}{K}\left(\frac{1}{4} - c\right)^{d(1/4)-3/2} \leqslant d'(c) \leqslant K\left(\frac{1}{4} - c\right)^{d(1/4)-3/2}.$$

We know from [**3**] that $d(1/4) < 3/2$. Thus, $d'(c) \to +\infty$ when $c \to 1/4^-$.

In § 7 we present a strategy for the proof of Theorem HZ. It can help the reader to understand results from §§ 8 and 9, which we need to prove the following theorem.

Theorem 1.1. *There exists $c_1 < 1/4$ such that*

$$d''(c) > 0,$$

where $c \in (c_1, 1/4)$ (i.e. d is a convex function on the interval $(c_1, 1/4)$). Moreover, $d''(c) \to \infty$ when $c \to 1/4^-$.

A motivation is the expectation that $d'(c)$ looks similar to the estimating function $(1/4-c)^{d(1/4)-3/2}$, whose derivative is positive. Theorem 1.1 is a step towards this.

It seems also plausible that the function d is convex on the interval $[-3/4, 1/4]$ (see Figure 1, which was made with the use of McMullen's program [**8**]). But until now it was only known that d was convex on a neighbourhood of 0 (see [**12**]); moreover,

$$d(c) = 1 + \frac{1}{4\log 2}|c|^2 + \text{higher-order terms.}$$

If c decreases to the left endpoint of $[-3/4, 1/4]$, then the derivative $d'(c)$ tends to $-\infty$ (see [**6**]). So presumably the function d is convex to the right of $-3/4$, but this cannot be proven in exactly the same way as Theorem 1.1.

Moreover, $d'(c) \to -\infty$ when $c \to -3/4^-$ (see [**7**]). So this leads to the conjecture that d is concave on the left side of $-3/4$. Of course, this result also means that d cannot be convex on the interval $[-5/4, -3/4]$.

2. Thermodynamical formalism

We shall repeat after [**6**, § 2] the basic notions.

If $c \in \mathcal{M}$, then there exists the function $\Phi_c\colon \mathbb{C}\setminus\overline{\mathbb{D}} \to \mathbb{C}\setminus K_c$ (called the Böttcher coordinate), which is holomorphic, bijective, tangent to identity at infinity, and conjugating $T(s) = s^2$ to f_c (i.e. $\Phi_c \circ T = f_c \circ \Phi_c$). For $c \in \mathcal{M}^0 \cup \{1/4\}$ the Julia set J_c is a Jordan curve, and thus the function $\Phi_c\colon \mathbb{C}\setminus\overline{\mathbb{D}} \to \mathbb{C}\setminus K_c$ has homeomorphic extension to $\partial\mathbb{D}$ (Carathéodory's theorem), and Φ_c conjugates $T|_{\partial\mathbb{D}}$ to $f_c|_{J_c}$.

The map $(c, s) \mapsto \Phi_c(s)$ gives a holomorphic motion for $c \in \mathcal{M}^0$ (see [**5**]). Thus, the functions Φ_c are quasi-conformal, and then also Hölder continuous, whereas $c \mapsto \Phi_c(s)$ are holomorphic for every $s \in \mathbb{C}\setminus\mathbb{D}$ (in particular, for $s \in \partial\mathbb{D}$).

Now we use the thermodynamical formalism, which holds for hyperbolic rational maps. We will consider only such maps. Let $X = \partial\mathbb{D}$, $T(s) = s^2$, and let $\varphi\colon X \to \mathbb{R}$ be a potential function of the form $\varphi = -t\log|2\Phi_c|$ for $c \in (-3/4, 1/4)$, $t \in \mathbb{R}$. Note that $2\Phi_c(s) = f_c'(z)$, where $z = \Phi_c(s)$.

The *topological pressure* can be defined as

$$P(T,\varphi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\bar{x}\in T^{-n}(x)} \mathrm{e}^{S_n(\varphi(\bar{x}))},$$

where $S_n(\varphi) = \sum_{k=0}^{n-1} \varphi \circ T^k$, and the limit exists and does not depend on $x \in \partial\mathbb{D}$. If $\varphi = -t\log|2\Phi_c|$ and $\Phi_c(\bar{x}) = \bar{z}$, then $\mathrm{e}^{S_n(\varphi(\bar{x}))} = |(f_c^n)'(\bar{z})|^{-t}$, and hence

$$P(T, -t\log|2\Phi_c|) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\bar{z}\in f_c^{-n}(z)} |(f_c^n)'(\bar{z})|^{-t}.$$

The function $t \mapsto P(T, -t\log|2\Phi_c|)$ is strictly decreasing from $+\infty$ to $-\infty$. So, there exists a unique t_0 such that $P(T, -t_0\log|2\Phi_c|) = 0$. By Bowen's theorem (see [**10**, Corollary 9.1.7] or [**13**, Theorem 5.12]) we obtain

$$t_0 = \mathrm{HD}(J_c).$$

Thus, we have $P(T, -d(c)\log|2\Phi_c|) = 0$. Write $\varphi_c := -d(c)\log|2\Phi_c|$.

The *Ruelle operator*, or the *transfer operator*, $\mathcal{L}_\varphi\colon C^0(X) \to C^0(X)$ is defined as

$$\mathcal{L}_\varphi(u)(x) := \sum_{\bar{x}\in T^{-1}(x)} u(\bar{x})\mathrm{e}^{\varphi(\bar{x})}. \tag{2.1}$$

The Perron–Frobenius–Ruelle theorem [**13**, Theorem 4.1] asserts that $\beta = \mathrm{e}^{P(T,\varphi)}$ is a single eigenvalue of $\mathcal{L}_\varphi$ associated with an eigenfunction $\tilde{h}_\varphi > 0$. Moreover, there exists a unique probability measure $\tilde{\omega}_\varphi$ such that $\mathcal{L}^*_\varphi(\tilde{\omega}_\varphi) = \beta\tilde{\omega}_\varphi$, where $\mathcal{L}^*_\varphi$ is dual to $\mathcal{L}_\varphi$.

For $\varphi = \varphi_c$ we have $\beta = 1$, and then $\tilde{\mu}_{\varphi_c} := \tilde{h}_{\varphi_c}\tilde{\omega}_{\varphi_c}$ is a T-invariant measure called an equilibrium state (we assume that this measure is normalized). We denote by $\tilde{\omega}_c$ and $\tilde{\mu}_c$ the measures $\tilde{\omega}_{\varphi_c}$ and $\tilde{\mu}_{\varphi_c}$, respectively (measures supported on the unit circle). Next, we take $\mu_c := (\Phi_c)_*\tilde{\mu}_c$ and $\omega_c := (\Phi_c)_*\tilde{\omega}_c$ (measures supported on J_c).

So, the measure μ_c is f_c-invariant, whereas ω_c is called an *f_c-conformal measure with exponent d(c)*, i.e. ω_c is a Borel probability measure such that, for every Borel subset $A \subset J_c$,

$$\omega_c(f_c(A)) = \int_A |f_c'|^{d(c)}\,\mathrm{d}\omega_c, \tag{2.2}$$

provided that f_c is injective on A.

3. Variations of the Hausdorff dimension

Let μ be an ergodic f_c-invariant probability measure on J_c, where $c \in \mathcal{M}^0$ (so, c is a hyperbolic parameter). Then denote by $h_\mu(f_c)$ the measure-theoretic entropy of f_c with respect to μ, and let $\chi_\mu(f_c)$ be the Lyapunov characteristic exponent, i.e.

$$\chi_\mu(f_c) = \int_{J_c} \log|f_c'|\,\mathrm{d}\mu.$$

The Hausdorff dimension of any probability measure ν on J_c is defined as

$$\mathrm{HD}(\nu) := \inf\{\mathrm{HD}(Y)\colon \nu(Y) = 1\}.$$

It follows from [**10**, Theorem 11.4.1] that, due to hyperbolicity,

$$\mathrm{HD}(\mu) = \frac{h_\mu(f_c)}{\chi_\mu(f_c)}. \tag{3.1}$$

Obviously, $\mathrm{HD}(\mu) \leqslant \mathrm{HD}(J_c) = d(c)$. Note that the equality holds for the equilibrium state μ_c.

Let us fix $c_0 \in \mathcal{M}^0$ and let $c \in \mathcal{M}^0$. We have $\tilde{\mu}_{c_0} := (\Phi_{c_0}^{-1})_*\mu_{c_0}$, and next we take

$$\mu^c_{c_0} := (\Phi_c)_*\tilde{\mu}_{c_0} = (\Phi_c \circ \Phi_{c_0}^{-1})_*\mu_{c_0}.$$

So, this is the measure μ_{c_0} transported to changing J_c. Because $\Phi_c \circ \Phi_{c_0}^{-1}$ conjugates f_{c_0} to f_c, we conclude that $\mu_{c_0}^c$ is f_c-invariant. Moreover,

$$h_{\mu_{c_0}^c}(f_c) = h_{\mu_{c_0}}(f_{c_0}).$$

By (3.1) we have

$$\mathrm{HD}(\mu_{c_0}^c) = \frac{h_{\mu_{c_0}^c}(f_c)}{\chi_{\mu_{c_0}^c}(f_c)}. \tag{3.2}$$

So, we see that the numerator does not depend on c. Note that the equality $d(c_0) = \mathrm{HD}(\mu_{c_0}^c)$ holds for $c = c_0$ (i.e. for the unique equilibrium state $\mu_{c_0} = \mu_{c_0}^{c_0}$).

Next, the denominator can be rewritten as

$$\chi_{\mu_{c_0}^c}(f_c) = \int_{J_c} \log |f_c'| \, \mathrm{d}\mu_{c_0}^c = \int_{\partial\mathbb{D}} \log |2\Phi_c| \, \mathrm{d}\tilde{\mu}_{c_0}.$$

Since $c \mapsto \Phi_c(s)$ is a holomorphic function for every $s \in \partial\mathbb{D}$, we conclude that $c \mapsto \log|\Phi_c(s)|$ and

$$c \mapsto \int_{\partial\mathbb{D}} \log |2\Phi_c(s)| \, \mathrm{d}\tilde{\mu}_{c_0}(s) = \chi_{\mu_{c_0}^c}(f_c)$$

are harmonic functions on $\mathcal{M}^0$. Thus, we see from (3.2) that $c \mapsto \mathrm{HD}(\mu_{c_0}^c)$ is an analytic (and subharmonic) function. Note that the subharmonicity of $\mathrm{HD}(\mu_{c_0}^c)$ implies the subharmonicity of $d(c)$ (see [**11**, Chapter 6.5]).

Now we will assume that $c, c_0 \in \mathcal{M}^0 \cap \mathbb{R} = (-3/4, 1/4)$. Because c is a hyperbolic parameter, for every $i \in \mathbb{N}$ we have

$$\frac{\partial^i}{\partial c^i} \chi_{\mu_{c_0}^c}(f_c) = \frac{\partial^i}{\partial c^i} \int_{\partial\mathbb{D}} \log |2\Phi_c| \, \mathrm{d}\tilde{\mu}_{c_0} = \int_{\partial\mathbb{D}} \frac{\partial^i}{\partial c^i} \log |2\Phi_c| \, \mathrm{d}\tilde{\mu}_{c_0}. \tag{3.3}$$

Next, using the fact that $d(c) \geqslant \mathrm{HD}(\mu_{c_0}^c)$, where the equality holds for $c = c_0$, we obtain

$$d'(c_0) = \frac{\partial}{\partial c} \mathrm{HD}(\mu_{c_0}^c)\bigg|_{c=c_0} \quad \text{and} \quad d''(c_0) \geqslant \frac{\partial^2}{\partial c^2} \mathrm{HD}(\mu_{c_0}^c)\bigg|_{c=c_0}. \tag{3.4}$$

Thus, in order to prove Theorem 1.1, it is enough to estimate $(\partial^2/\partial c^2)\,\mathrm{HD}(\mu_{c_0}^c)$.

Differentiating both sides of (3.2) we obtain

$$\frac{\partial}{\partial c} \mathrm{HD}(\mu_{c_0}^c) = -h_{\mu_{c_0}}(f_{c_0}) \frac{(\partial/\partial c)\chi_{\mu_{c_0}^c}(f_c)}{(\chi_{\mu_{c_0}^c}(f_c))^2} = -\mathrm{HD}(\mu_{c_0}^c) \frac{(\partial/\partial c)\chi_{\mu_{c_0}^c}(f_c)}{\chi_{\mu_{c_0}^c}(f_c)}, \tag{3.5}$$

and then

$$\begin{aligned} \frac{\partial^2}{\partial c^2} \mathrm{HD}(\mu_{c_0}^c) &= -h_{\mu_{c_0}}(f_{c_0}) \frac{(\partial^2/\partial c^2)\chi_{\mu_{c_0}^c}(f_c)}{(\chi_{\mu_{c_0}^c}(f_c))^2} + 2h_{\mu_{c_0}}(f_{c_0}) \frac{((\partial/\partial c)\chi_{\mu_{c_0}^c}(f_c))^2}{(\chi_{\mu_{c_0}^c}(f_c))^3} \\ &= -\mathrm{HD}(\mu_{c_0}^c) \frac{(\partial^2/\partial c^2)\chi_{\mu_{c_0}^c}(f_c)}{\chi_{\mu_{c_0}^c}(f_c)} + 2\,\mathrm{HD}(\mu_{c_0}^c) \bigg(\frac{(\partial/\partial c)\chi_{\mu_{c_0}^c}(f_c)}{\chi_{\mu_{c_0}^c}(f_c)} \bigg)^2. \end{aligned} \tag{3.6}$$

Thus, (3.5) combined with (3.3) and (3.4) leads to the following proposition.

Proposition 3.1 (Havard and Zinsmeister [4, Proposition 2.1]). *If* $c_0 \in (-3/4, 1/4)$, *then*

$$d'(c_0) = \frac{-d(c_0)}{\int_{\partial\mathbb{D}} \log|2\Phi_{c_0}|\,\mathrm{d}\tilde{\mu}_{c_0}} \int_{\partial\mathbb{D}} \frac{\partial}{\partial c} \log|2\Phi_c|\bigg|_{c=c_0} \mathrm{d}\tilde{\mu}_{c_0}.$$

Next, (3.6) combined with (3.3) and (3.4) gives us the following lemma.

Lemma 3.2. *If* $c_0 \in (-3/4, 1/4)$, *then*

$$d''(c_0) \geqslant 2d(c_0)\biggl(\frac{(\partial/\partial c)\chi_{\mu^c_{c_0}}(f_c)|_{c=c_0}}{\chi_{\mu_{c_0}}(f_{c_0})}\biggr)^2 - \frac{d(c_0)}{\chi_{\mu_{c_0}}(f_{c_0})} \int_{\partial\mathbb{D}} \frac{\partial^2}{\partial c^2} \log|2\Phi_c|\bigg|_{c=c_0} \mathrm{d}\tilde{\mu}_{c_0}.$$

If $c \in (-3/4, 1/4)$, then the Lyapunov exponent $\chi_{\mu_c}(f_c)$ is positive. In fact, we know from [**4**, §§ 3.2 and 3.3] that

$$\lim_{c\to 1/4^-} \chi_{\mu_c}(f_c) = \lim_{c\to 1/4^-} \int_{\partial\mathbb{D}} \log|2\Phi_c|\,\mathrm{d}\tilde{\mu}_c = \int_{\partial\mathbb{D}} \log|2\Phi_{1/4}|\,\mathrm{d}\tilde{\mu}_{1/4} > 0. \tag{3.7}$$

Therefore, Theorem 1.1 follows from Lemma 3.2 and the following lemma.

Lemma 3.3. *There exists* $c_1 < 1/4$ *such that*

$$\int_{\partial\mathbb{D}} \frac{\partial^2}{\partial c^2} \log|2\Phi_c|\bigg|_{c=c_0} \mathrm{d}\tilde{\mu}_{c_0} < 0,$$

where $c_0 \in (c_1, 1/4)$. *Moreover, the above integral tends to* $-\infty$ *when* $c_0 \to 1/4^-$.

The rest of the paper is devoted to proving Lemma 3.3.

4. Cylinders

Now we define a partition of $\partial\mathbb{D} \setminus \{1\}$ onto cylinders C_n. Let

$$C_n^+ := \{\mathrm{e}^{2\pi\alpha} : \alpha \in (2^{-n-1}, 2^{-n}]\},$$

where $n \geqslant 1$. The sets C_n^+ form a partition of the upper half-circle. Write $C_n^- := \overline{C_n^+}$; then

$$C_n := C_n^+ \cup C_n^-.$$

We see that $\bigcup_{n\geqslant 1} C_n = \partial\mathbb{D} \setminus \{1\}$.

We will respectively denote by ζ_c and z_c the attracting and repelling fixed points, i.e.

$$\zeta_c = \frac{1-\sqrt{1-4c}}{2}, \qquad z_c = \frac{1+\sqrt{1-4c}}{2},$$

where $c \in (-3/4, 1/4)$. So, we see that $\zeta_c \leqslant 1/2 \leqslant z_c$ and for $c = 1/4$ the points ζ_c, z_c become the parabolic point $1/2$.

Since $\Phi_c(1) = z_c$, the function Φ_c allows us to define a corresponding partition of $J_c \setminus \{z_c\}$ onto cylinders $C_n(c)$.

The set of points that are 'near' the fixed point z_c is defined as

$$\mathcal{M}_N(c) := \overline{\bigcup_{n>N} C_n(c)} = \bigcup_{n>N} C_n(c) \cup \{z_c\},$$

where $N \in \mathbb{N}$. Write

$$\mathcal{B}_N(c) := J_c \setminus \mathcal{M}_N(c).$$

Hence, $\mathcal{B}_N(c)$ is the set of points that are 'far' from z_c. Related subsets of $\partial\mathbb{D}$ will be denoted by $\mathcal{M}_N$ and $\mathcal{B}_N$.

Now we give a few basic facts that we will need in the next section.

Lemma 4.1 (Jaksztas [6, Lemma A.1]). *For every $c \in [-3/4, 1/4]$ we have*

$$\bar{B}(0, 1/2) \subset K_c.$$

Moreover, if $c \in (-3/4, 1/4)$, then $\bar{B}(0, 1/2) \cap J_c = \emptyset$, whereas for $c \in \{-3/4, 1/4\}$ we have $\bar{B}(0, 1/2) \cap J_c = \{-1/2, 1/2\}$.

Lemma 4.2.

(1) *For every $\varepsilon_1 > 0$ there exist $N_1 \in \mathbb{N}$ and $c_1 < 1/4$ such that if $c \in (c_1, 1/4]$, then*

$$\mathcal{M}_{N_1}(c) \subset B(1/2, \varepsilon_1).$$

(2) *For every $N \in \mathbb{N}$ there exists $\varepsilon_2 > 0$ such that if $c \in [0, 1/4]$, then*

$$\mathcal{B}_N(c) \cap B(1/2, \varepsilon_2) = \emptyset.$$

Proof. The Julia set $J_{1/4}$ is a Jordan curve, so the function $\Phi_{1/4}$ has homeomorphic extension to $\partial\mathbb{D}$ (Carathéodory's theorem). Since $\Phi_{1/4}(1) = 1/2$, the statements hold for $c = 1/4$.

Next, because of the uniform convergence of Φ_c to $\Phi_{1/4}$ (see [**4**, § 3.2.]), the statements also hold for $c \in [c_1, 1/4]$ for some $c_1 < 1/4$.

Finally, since the Julia sets for $c \in [0, c_1]$ are uniformly separated from $\bar{B}(0, 1/2)$ (see Lemma 4.1), the second statement follows. □

Lemma 4.3. *There exists $K > 0$ such that, for every $n \geqslant 1$ and $c \in [0, 1/4]$,*

$$\operatorname{diam} C_n(c) \leqslant K \operatorname{dist}(C_n(c), [0, 1/2]).$$

Moreover, we can assume that the constant K is arbitrarily small if $c \leqslant 1/4$ is sufficiently close to $1/4$, and $n > N$ for N large enough.

Proof. We see from Lemma 4.1 and Lemma 4.2 (2) that for every $N \in \mathbb{N}$ and $c_1 < 1/4$ the Julia set J_c (for $c \in [0, c_1]$) and the sets $\mathcal{B}_N(c)$ (for $c \in [0, 1/2]$) are separated from $\bar{B}(0, 1/2)$. Since diameters of cylinders are bounded, the assertion obviously follows (possibly the constant K depends on c_1 and N).

https://doi.org/10.1017/S0013091516000481 Published online by Cambridge University Press

Now we can assume that $z \in \mathcal{M}_N$, where N is large enough, and c is close to $1/4$. Thus, if $z \in C_n(c)$ $(n > N)$, then it follows from Lemma 4.2 (1) that $f_c'(z)$ is close to 1. So $\operatorname{diam} C_{n+1}(c)/\operatorname{diam} C_n(c)$ is also close to 1, and the statement follows from the fact that cylinders $C_n(c)$ 'tend' to the fixed point $z_c \geqslant 1/2$ along the repelling direction (i.e. along the ray (z_c, ∞)). □

We end this section with important results from [**4**].

Proposition 4.4 (Havard and Zinsmeister [4, Proposition 3.2]). *There exist $c_1 < 1/4$ and $K > 0$ such that, for every $c \in (c_1, 1/4]$ and every sequence $z_n \in C_n(c)$,*

$$\sum_{n \geqslant 1} |\operatorname{Im} z_n| \leqslant K.$$

If $z_n \in C_n(c)$, then, for large n, $|\arg z_n|$ is close to $2|\operatorname{Im} z_n|$. Thus, convergence of the above series leads to the following corollary.

Corollary 4.5. *For every $\varepsilon > 0$ there exist $c_1 < 1/4$ and $N \in \mathbb{N}$ such that*

$$|\arg(f_c^k)'(z)| \leqslant \varepsilon,$$

where $z \in C_{N+n}(c)$, $1 \leqslant k \leqslant n$ and $c \in (c_1, 1/4]$.

Lemma 4.6 (Havard and Zinsmeister [4, § 4]). *There exists $c_1 < 1/4$ and for every $N \in \mathbb{N}$ there is a constant $\lambda(N) > 0$ such that if $z \in J_c$ and $c \in (c_1, 1/4]$, then*

$$f_c^n(z) \in \mathcal{B}_N(c) \implies \frac{1}{|(f_c^n)'(z)|} \leqslant \frac{\lambda(N)}{n^2}.$$

5. Partition of $\mathcal{B}_N$

For every $N, N_0 \in \mathbb{N}$ we define a family of sets $\{A_{N,n}^{N_0}\}_{n \geqslant 0}$, which forms a partition of $\mathcal{B}_N$ (cf. [**4**, Proof of Proposition 4.1] and [**6**, § 12]). Let

$$A_{N,n}^{N_0} = T^{-N_0}(C_{N+n}) \cap \mathcal{B}_N \quad \text{for } n \geqslant 1$$

and

$$A_{N,0}^{N_0} = T^{-N_0}(\mathcal{B}_N) \cap \mathcal{B}_N.$$

The sets $A_{N,n}^{N_0}(c)$ are defined as the images of $A_{N,n}^{N_0}$ under Φ_c.

Let us recall from [**4**, § 3.3] (cf. [**6**, Lemma 7.1]) that there exists $\lambda > 1$ such that, for every $n \geqslant 1$,

$$\lambda^{-1} \sum_{k=n}^{\infty} \tilde{\omega}_c(C_k) \leqslant \tilde{\mu}_c(C_n) \leqslant \lambda \sum_{k=n}^{\infty} \tilde{\omega}_c(C_k), \tag{5.1}$$

and also [**4**, § 4] (cf. [**6**, Lemma 7.3])

$$\left.\frac{\mathrm{d}\tilde{\mu}_c}{\mathrm{d}\tilde{\omega}_c}\right|_{\mathcal{B}_N} \leqslant \lambda(N), \tag{5.2}$$

where $c < 1/4$ is close to $1/4$ and $\lambda(N)$ depends on N.

Now we will give two estimates of $\tilde{\mu_c}(A_{N,n}^{N_0})$. First, in Lemma 5.1, is a stronger version of the estimate from [**4**, proof of Proposition 4.1].

Lemma 5.1. *There exist $K > 0$ and $c_1 < 1/4$ such that for every $N \in \mathbb{N}$, $N_0, n \geqslant 1$ and $c \in (c_1, 1/4]$ we have*

$$\tilde{\mu_c}(A_{N,n}^{N_0}) \leqslant K N_0 \tilde{\omega_c}(C_{N+n}).$$

Proof. The preimage of $C_n(c)$ under f_c consists of $C_{n+1}(c)$ and the set $C'_{n+1}(c)$, which is placed symmetrically to $C_{n+1}(c)$ with respect to 0.

Since $C'_n(c) \subset \mathcal{B}_0(c)$, (5.2) gives us

$$\mu_c(C'_n(c)) \leqslant \lambda(0)\omega_c(C'_n(c)).$$

By the uniqueness of ω_c we get $\omega_c(C'_n(c)) = \omega_c(C_n(c))$. Thus, the fact that μ_c is f_c-invariant leads to

$$\mu_c(f_c^{-k}(C'_n(c))) \leqslant \lambda(0)\omega_c(C_n(c)). \tag{5.3}$$

The set $f_c^{-N_0}(C_{N+n}(c))$ can be written as

$$f_c^{-N_0}(C_{N+n}(c)) = C_{N+n+N_0}(c) \cup \bigcup_{k=1}^{N_0} f_c^{k-N_0}(C'_{N+n+k}(c)).$$

Thus, the above combined with (5.3) leads to

$$\begin{aligned}\tilde{\mu_c}(A_{N,n}^{N_0}) = \mu_c(f_c^{-N_0}(C_{N+n}(c)) \cap \mathcal{B}_N(c)) &\leqslant \sum_{k=1}^{N_0} \mu_c(f_c^{k-N_0}(C'_{N+n+k}(c))) \\ &\leqslant \lambda(0) \sum_{k=1}^{N_0} \omega_c(C_{N+n+k}(c)) \\ &\leqslant \lambda(0) N_0 \omega_c(C_{N+n}(c)),\end{aligned}$$

and the lemma follows. □

Lemma 5.2. *For every $\varepsilon > 0$ there exist $\tilde{n} \in \mathbb{N}$, $c_1 < 1/4$ such that*

$$\tilde{\omega_c}(C_n) \leqslant \varepsilon \tilde{\mu_c}(C_n),$$

where $n > \tilde{n}$ and $c \in (c_1, 1/4]$.

Proof. Using Lemma 4.2 (1) we can assume that f'_c is as close to 1 as necessary on the set $\mathcal{M}_N(c)$, where N is large enough and $c < 1/4$ close to 1/4. Thus, it follows from (2.2) that $\omega_c(C_{n+1}(c))/\omega_c(C_n(c))$ is close to 1, and the assertion follows from (5.1). □

Lemmas 5.1 and 5.2 lead to the following corollary.

Corollary 5.3. *For every $N_0 \in \mathbb{N}$ and $\varepsilon > 0$ there exist $\tilde{n} \in \mathbb{N}$, $c_1 < 1/4$ such that*

$$\tilde{\mu_c}(A_{N,n}^{N_0}) \leqslant \varepsilon \tilde{\mu_c}(C_{N+n}),$$

where $N + n > \tilde{n}$ and $c \in (c_1, 1/4]$.

Let $f_{c,\nu}^{-k}$ be an inverse branch of f_c^k defined on $\mathbb{C}\setminus(-\infty,1/2]$, where $k\in\mathbb{N}$. Since the trajectory of the critical point 0 is included in the interval $[0,\zeta_c)\subset[0,1/2]$, Lemma 4.3 and the Koebe distortion theorem lead to the following lemma.

Lemma 5.4. *There exists $K>0$ such that for every $c\in[0,1/4]$, $n\geqslant 1$, $x,y\in C_n(c)$, and an inverse branch $f_{c,\nu}^{-k}$, we have*

$$K^{-1}<\left|\frac{(f_{c,\nu}^{-k})'(x)}{(f_{c,\nu}^{-k})'(y)}\right|<K.$$

Lemma 5.5. *There exists $K>0$ such that for every $c\in[0,1/4]$, $n\geqslant 1$, and an inverse branch $f_{c,\nu}^{-k}$, where $k\in\mathbb{N}$, if $x,y\in B=f_{c,\nu}^{-k}(C_n(c))$, then*

$$K^{-1}<\frac{\mathrm{d}\mu_c}{\mathrm{d}\omega_c}(x)\Bigg/\frac{\mathrm{d}\mu_c}{\mathrm{d}\omega_c}(y)<K.$$

Proof. The density $\mathrm{d}\mu_c/\mathrm{d}\omega_c$ is the limit of $\mathcal{L}_{\varphi_c}^n(\mathbf{1})$ (see [**13**, Theorem 4.1]), where $\varphi_c=-d(c)\log|2\Phi_c|$. Since $2\Phi_c=f_c'(\Phi_c)$, (2.1) leads to

$$\mathcal{L}_{\varphi_c}^n(\mathbf{1})(s):=\sum_{\bar{s}\in T^{-n}(s)}|(f_c^n)'(\Phi_c(\bar{s}))|^{-d(c)}.$$

Thus, the assertion follows from Lemma 5.4. □

Lemma 5.6. *There exists $K>0$ such that for every $c\in[0,1/4]$, $m\geqslant 1$, an inverse branch $f_{c,\nu}^{-k}$, and a continuous function $F\colon C_m(c)\to\mathbb{R}^+$, we have*

$$K^{-1}\leqslant\left(\frac{1}{\mu_c(B)}\int_B F(T^k)\,\mathrm{d}\mu_c\right)\Bigg/\left(\frac{1}{\mu_c(C_m(c))}\int_{C_m(c)}F\,\mathrm{d}\mu_c\right)\leqslant K,$$

where $B=f_{c,\nu}^{-k}(C_m(c))$. Moreover, the statement remains valid if we replace B by $A_{N,n}^{N_0}(c)$, which is a union of images of $C_m(c)$ under $f_{c,\nu}^{-k}$, where $m=N+n$ and $k=N_0$.

Proof. Let $C\subset C_m(c)$ and let $A=f_{c,\nu}^{-k}(C)$ (i.e. $A\subset B$). Because ω_c is a conformal measure (see (2.2)), Lemma 5.4 leads to

$$K_1^{-1}\leqslant\frac{\tilde{\omega}_c(A)}{\tilde{\omega}_c(B)}\Bigg/\frac{\tilde{\omega}_c(C)}{\tilde{\omega}_c(C_m)}\leqslant K_1,$$

Next, Lemma 5.5 gives us

$$K_2^{-1}\leqslant\frac{\tilde{\mu}_c(A)}{\tilde{\mu}_c(B)}\Bigg/\frac{\tilde{\mu}_c(C)}{\tilde{\mu}_c(C_m)}\leqslant K_2,$$

and the lemma follows. □

6. The functions $\dot{\varPhi}_c$ and $\ddot{\varPhi}_c$

Let $\varPhi_c = u + \mathrm{i}v$, and let $\dot{\varPhi}_c := (\partial/\partial c)\varPhi_c$, $\ddot{\varPhi}_c := (\partial^2/\partial c^2)\varPhi_c$. Then we have

$$\frac{\partial}{\partial c}\log|2\varPhi_c| = \frac{u\dot{u} + v\dot{v}}{u^2 + v^2} = \mathrm{Re}\left(\frac{\dot{\varPhi}_c}{\varPhi_c}\right). \tag{6.1}$$

Moreover, the key expression whose integrals are to be estimated (see Lemma 3.3) can be rewritten as

$$\begin{aligned}\frac{\partial^2}{\partial c^2}\log|2\varPhi_c| &= \frac{(u\ddot{u} + v\ddot{v} + \dot{u}\dot{u} + \dot{v}\dot{v})(u^2+v^2) - 2(u\dot{u} + v\dot{v})^2}{(u^2+v^2)^2} \\ &= \frac{u\ddot{u} + v\ddot{v}}{u^2+v^2} - \left(\frac{u\dot{u} + v\dot{v}}{u^2+v^2}\right)^2 + \left(\frac{u\dot{v} - v\dot{u}}{u^2+v^2}\right)^2 \\ &= \mathrm{Re}\left(\frac{\ddot{\varPhi}_c}{\varPhi_c}\right) - \mathrm{Re}^2\left(\frac{\dot{\varPhi}_c}{\varPhi_c}\right) + \mathrm{Im}^2\left(\frac{\dot{\varPhi}_c}{\varPhi_c}\right) \\ &= \mathrm{Re}\left(\frac{\ddot{\varPhi}_c}{\varPhi_c}\right) - \mathrm{Re}\left(\frac{\dot{\varPhi}_c}{\varPhi_c}\right)^2. \end{aligned} \tag{6.2}$$

Now we will deal with $\dot{\varPhi}_c$ and $\ddot{\varPhi}_c$. The function $\varPhi_c$ conjugates $T(s) = s^2$ to $f_c(z) = z^2 + c$, so

$$\varPhi_c(s^2) = \varPhi_c^2(s) + c.$$

Differentiating both sides with respect to c, we have

$$\dot{\varPhi}_c(s^2) = 2\varPhi_c(s)\dot{\varPhi}_c(s) + 1, \tag{6.3}$$

and therefore

$$\dot{\varPhi}_c(s) = -\frac{1}{2\varPhi_c(s)} + \frac{1}{2\varPhi_c(s)}\dot{\varPhi}_c(s^2).$$

Next, replacing s by $s^2, s^4, \dots, s^{2^{m-1}}$, we obtain

$$\dot{\varPhi}_c(s) = -\sum_{k=0}^{m-1}\frac{1}{2\varPhi_c(s)\cdot 2\varPhi_c(s^2)\cdot\dots\cdot 2\varPhi_c(s^{2^k})} + \frac{1}{2\varPhi_c(s)\cdot 2\varPhi_c(s^2)\cdot\dots\cdot 2\varPhi_c(s^{2^{m-1}})}\dot{\varPhi}_c(s^{2^m}).$$

We have $f_c'(z) = 2z$, and then $2\varPhi_c(s) = f_c'(\varPhi_c(s))$. Thus,

$$\dot{\varPhi}_c(s) = -\sum_{k=1}^{m}\frac{1}{(f_c^k)'(\varPhi_c(s))} + \frac{1}{(f_c^m)'(\varPhi_c(s))}\dot{\varPhi}_c(T^m(s)). \tag{6.4}$$

Let $N \in \mathbb{N}$. If $s \in C_{N+n}$, then define

$$\dot{\varPsi}_{N,c}(s) := -\sum_{k=1}^{n}\frac{1}{(f_c^k)'(\varPhi_c(s))}. \tag{6.5}$$

Differentiating (6.3), we get

$$\ddot{\Phi}_c(s^2) = 2\dot{\Phi}_c^2(s) + 2\Phi_c(s)\ddot{\Phi}_c(s).$$

Thus,

$$\ddot{\Phi}_c(s) = -\frac{2\dot{\Phi}_c^2(s)}{2\Phi_c(s)} + \frac{1}{2\Phi_c(s)}\ddot{\Phi}_c(s^2).$$

Similarly to before we obtain

$$\ddot{\Phi}_c(s) = -\sum_{k=1}^{m} \frac{2\dot{\Phi}_c^2(T^{k-1}(s))}{(f_c^k)'(\Phi_c(s))} + \frac{1}{(f_c^m)'(\Phi_c(s))}\ddot{\Phi}_c(T^m(s)). \tag{6.6}$$

If $N \in \mathbb{N}$ and $s \in C_{N+n}$, then we define

$$\ddot{\Theta}_{N,c}(s) := -\sum_{k=1}^{n} \frac{2\dot{\Phi}_c^2(T^{k-1}(s))}{(f_c^k)'(\Phi_c(s))}$$

and

$$\ddot{A}_{N,c}(s) := -\sum_{k=1}^{n} \frac{2\dot{\Psi}_{N,c}^2(T^{k-1}(s))}{(f_c^k)'(\Phi_c(s))}. \tag{6.7}$$

7. Comments and Theorem HZ

From (6.2) we have

$$\int_{\partial\mathbb{D}} \frac{\partial^2}{\partial c^2} \log|2\Phi_c| \,\mathrm{d}\tilde{\mu}_c = \int_{\partial\mathbb{D}} \mathrm{Re}\left(\frac{\ddot{\Phi}_c}{\Phi_c}\right) \mathrm{d}\tilde{\mu}_c - \int_{\partial\mathbb{D}} \mathrm{Re}\left(\frac{\dot{\Phi}_c}{\Phi_c}\right)^2 \mathrm{d}\tilde{\mu}_c. \tag{7.1}$$

Thus, in order to prove Lemma 3.3 (and consequently Theorem 1.1), it is enough to show that

$$\int_{\partial\mathbb{D}} \mathrm{Re}\left(\frac{\dot{\Phi}_c}{\Phi_c}\right)^2 \mathrm{d}\tilde{\mu}_c \to \infty \quad \text{and} \quad \int_{\partial\mathbb{D}} \mathrm{Re}\left(\frac{\ddot{\Phi}_c}{\Phi_c}\right) \mathrm{d}\tilde{\mu}_c \to -\infty, \tag{7.2}$$

where $c \to 1/4^-$. Sections 8 and 9 are devoted to proving these two facts (see Propositions 8.1 and 9.1).

Note that the right-hand side integral of (7.2) diverges faster, and therefore it has a decisive influence on (7.1). But it is convenient to prove divergence of both the integrals.

The schemes of these two proofs, and some ideas, are very similar to that of the proof that

$$\int_{\partial\mathbb{D}} \mathrm{Re}\left(\frac{\dot{\Phi}_c}{\Phi_c}\right) \mathrm{d}\tilde{\mu}_c \to -\infty. \tag{7.3}$$

This is the main ingredient in the proof of Theorem HZ (i.e. [**4**, Theorem 1.1]). Although, in order to obtain Theorem HZ, the following more precise result is needed.

Proposition 7.1. *For every $N \in \mathbb{N}$ there exist $K > 1$ and $c_1 < 1/4$ such that*

$$-K\left(\frac{1}{4}-c\right)^{d(1/4)-3/2} \leqslant \int_{\partial\mathbb{D}} \operatorname{Re}\left(\frac{\dot{\Phi}_c}{\Phi_c}\right) \mathrm{d}\tilde{\mu_c} \leqslant -\frac{1}{K}\left(\frac{1}{4}-c\right)^{d(1/4)-3/2}.$$

Indeed, we know that $d(1/4) < 3/2$, and thus the integral tends to $-\infty$, and Theorem HZ is a consequence of the formula from Proposition 3.1 and (3.7).

The rest of this section is devoted to a presentation of the scheme of the proof of Proposition 7.1.

We will need three lemmas. Note that Proposition 8.1 as well as Proposition 9.1 will be obtained in a similar way as a consequence of related lemmas. Moreover, we will use some ideas from the proofs of Lemmas 7.3 and 7.4 (especially Lemma 7.3), and we will draw a conclusion from Lemma 7.2.

Let us divide $\partial\mathbb{D}$ into two sets, $\mathcal{B}_N$ and $\mathcal{M}_N$.

Lemma 7.2. *For every $N \in \mathbb{N}$ there exist $K > 1$ and $c_1 < 1/4$ such that*

$$\frac{1}{K}\left(\frac{1}{4}-c\right)^{d(1/4)-3/2} \leqslant \int_{\mathcal{M}_N} |\dot{\Psi}_{N,c}| \,\mathrm{d}\tilde{\mu_c} \leqslant K\left(\frac{1}{4}-c\right)^{d(1/4)-3/2}.$$

In particular, $\lim_{c\to 1/4^-} \int_{\mathcal{M}_N} |\dot{\Psi}_{N,c}| \,\mathrm{d}\tilde{\mu_c} = \infty$.

Lemma 7.3. *There exists $c_1 < 1/4$ such that for every $N \in \mathbb{N}$ there is a constant $\lambda(N) > 0$ such that*

$$\int_{\mathcal{B}_N} |\dot{\Phi}_c| \,\mathrm{d}\tilde{\mu_c} \leqslant \lambda(N),$$

provided that $c \in (c_1, 1/4)$.

Lemma 7.4. *There exists $c_1 < 1/4$ such that for every $N \in \mathbb{N}$ there is a constant $\lambda(N) > 0$ such that*

$$\int_{\mathcal{M}_N} |\dot{\Phi}_c - \dot{\Psi}_{N,c}| \,\mathrm{d}\tilde{\mu_c} \leqslant \lambda(N),$$

provided that $c \in (c_1, 1/4)$.

By using Corollary 4.5, we can assume that $\dot{\Psi}_{N,c}$ is close to $-|\dot{\Psi}_{N,c}|$ on the set $\mathcal{M}_N$ (where N is large enough). So, the above lemmas and the fact that the function Φ_c is separated from zero on $\mathcal{B}_N$, whereas Φ_c is close to $1/2$ on $\mathcal{M}_N$, lead to

$$\int_{\partial\mathbb{D}} \operatorname{Re}\left(\frac{\dot{\Phi}_c}{\Phi_c}\right) \mathrm{d}\tilde{\mu_c} \asymp \int_{\mathcal{M}_N} \operatorname{Re}\left(\frac{\dot{\Phi}_c}{\Phi_c}\right) \mathrm{d}\tilde{\mu_c} \asymp \int_{\mathcal{M}_N} \operatorname{Re}\left(\frac{\dot{\Psi}_{N,c}}{\Phi_c}\right) \mathrm{d}\tilde{\mu_c} \asymp -\int_{\mathcal{M}_N} |\dot{\Psi}_{N,c}| \,\mathrm{d}\tilde{\mu_c}.$$

Thus, Proposition 7.1 follows, and we see that the integral of $-|\dot{\Psi}_{N,c}|$ over $\mathcal{M}_N$ has a decisive influence on $\int_{\partial\mathbb{D}} \operatorname{Re}(\dot{\Phi}_c/\Phi_c)\,\mathrm{d}\tilde{\mu_c}$.

We will not prove Lemma 7.2. But let us note that the proof relies on estimates of diameters of cylinders $C_n(c)$. Indeed, if $z \in C_n(c)$, then $1/(f_c^k)'(z)$ can be estimated by $\operatorname{diam} C_n(c)/\operatorname{diam} C_{n-k}(c)$, whereas $\tilde{\mu_c}(C_n)$ can be estimated by $(\operatorname{diam} C_n(c))^{d(c)}$.

https://doi.org/10.1017/S0013091516000481 Published online by Cambridge University Press

Proof of Lemma 7.3. Let us fix $N \in \mathbb{N}$. For every $N_0 \geqslant 1$ and $s \in A_{N,n}^{N_0}$, using (6.4), we can get

$$\dot{\Phi}_c(s) = -\sum_{k=1}^{N_0+n} \frac{1}{(f_c^k)'(\Phi_c(s))} + \frac{1}{(f_c^{N_0+n})'(\Phi_c(s))}\dot{\Phi}_c(T^{N_0+n}(s)).$$

So, we have divided $\dot{\Phi}_c$ into two parts, the finite sum and the 'tail'.

First, in Step 1, we will prove that the integral of the 'tail' is less than $\frac{1}{2}\int_{\mathcal{B}_N} |\dot{\Phi}_c| \, \mathrm{d}\tilde{\mu_c}$ (for N_0 large enough, depending on N). Next, in Step 2, we will see that the integral of the finite sum is bounded by a constant $K(N, N_0)$ (depending on N and N_0), which means that

$$\int_{\mathcal{B}_N} |\dot{\Phi}_c| \, \mathrm{d}\tilde{\mu_c} \leqslant K(N, N_0) + \frac{1}{2}\int_{\mathcal{B}_N} |\dot{\Phi}_c| \, \mathrm{d}\tilde{\mu_c}.$$

Since N_0 depends only on N, the assertion follows.

Step 1. The measure $\tilde{\mu_c}$ is T-invariant, and $T^{N_0+n}(A_{N,n}^{N_0}) \subset \mathcal{B}_N$, so

$$\int_{A_{N,n}^{N_0}} |\dot{\Phi}_c(T^{N_0+n})| \, \mathrm{d}\tilde{\mu_c} \leqslant \int_{\mathcal{B}_N} |\dot{\Phi}_c| \, \mathrm{d}\tilde{\mu_c}.$$

Next, if $s \in A_{N,n}^{N_0}$, then $f_c^{N_0+n}(\Phi_c(s)) \in \mathcal{B}_N(c)$. Thus, Lemma 4.6 leads to

$$\int_{A_{N,n}^{N_0}} \left|\frac{\dot{\Phi}_c(T^{N_0+n}(s))}{(f_c^{N_0+n})'(\Phi_c(s))}\right| \mathrm{d}\tilde{\mu_c}(s) \leqslant \frac{\lambda_1(N)}{(N_0+n)^2}\int_{\mathcal{B}_N} |\dot{\Phi}_c(s)| \, \mathrm{d}\tilde{\mu_c}(s).$$

So, we get

$$\sum_{n=0}^{\infty}\int_{A_{N,n}^{N_0}} \left|\frac{\dot{\Phi}_c(T^{N_0+n})}{(f_c^{N_0+n})'(\Phi_c)}\right| \mathrm{d}\tilde{\mu_c} \leqslant \left(\sum_{n=0}^{\infty}\frac{\lambda_1(N)}{(N_0+n)^2}\right)\int_{\mathcal{B}_N} |\dot{\Phi}_c| \, \mathrm{d}\tilde{\mu_c}.$$

For N_0 large enough (depending on N), we obtain

$$\sum_{n=0}^{\infty}\frac{\lambda_1(N)}{(N_0+n)^2} < \tfrac{1}{2}.$$

Step 2. Lemma 5.1 and [**4**, Remark 3.5] lead to

$$\tilde{\mu_c}(A_{N,n}^{N_0}) \leqslant K_1 N_0 \tilde{\omega_c}(C_{N+n}) \leqslant K_2 N_0 (\operatorname{diam} C_{N+n}(c))^{d(c)} \leqslant K_3 N_0 (N+n)^{-2d(c)},$$

where $n > 1$. Since $|f_c'(z)| > 1$, we obtain

$$\begin{aligned}\int_{A_{N,0}^{N_0}} \left|\sum_{k=1}^{N_0}\frac{1}{(f_c^k)'(\Phi_c(s))}\right| \mathrm{d}\tilde{\mu_c} &+ \sum_{n=1}^{\infty}\int_{A_{N,n}^{N_0}} \left|\sum_{k=1}^{N_0+n}\frac{1}{(f_c^k)'(\Phi_c(s))}\right| \mathrm{d}\tilde{\mu_c} \\ &\leqslant N_0\tilde{\mu_c}(A_{N,0}^{N_0}) + \sum_{n=1}^{\infty}(N_0+n)K_3N_0(N+n)^{-2d(c)} \\ &\leqslant K(N, N_0),\end{aligned}$$

where the constant $K(N, N_0)$ depends on N and N_0. □

Proof of Lemma 7.4. The definition of $\dot{\Psi}_{N,c}$ and (6.4) give us

$$\int_{C_{N+n}} |\dot{\Phi}_c - \dot{\Psi}_{N,c}| \,\mathrm{d}\tilde{\mu_c} = \int_{C_{N+n}} \left| \frac{1}{(f_c^n)'(\Phi_c)} \dot{\Phi}_c(T^n) \right| \mathrm{d}\tilde{\mu_c}.$$

Thus, similar to in the proof of Lemma 7.3 (Step 1), Lemma 4.6 and the fact that $T^n(C_{N+n}) \subset \mathcal{B}_N$ lead to

$$\sum_{n=1}^{\infty} \int_{C_{N+n}} |\dot{\Phi}_c - \dot{\Psi}_{N,c}| \,\mathrm{d}\tilde{\mu_c} = \sum_{n=1}^{\infty} \frac{\lambda(N)}{n^2} \int_{\mathcal{B}_N} |\dot{\Phi}_c| \,\mathrm{d}\tilde{\mu_c}.$$

Since $\sum_{n=1}^{\infty} \lambda(N)/n^2$ is finite, the assertion follows from Lemma 7.3. □

8. The integral of $\mathrm{Re}(\dot{\Phi}_c/\Phi_c)^2$ is positive

Proposition 8.1. *There exists* $c_1 < 1/4$ *such that*

$$\int_{\partial\mathbb{D}} \mathrm{Re}\left(\frac{\dot{\Phi}_c}{\Phi_c}\right)^2 \mathrm{d}\tilde{\mu_c} > 0,$$

provided that $c \in (c_1, 1/4)$. *Moreover,*

$$\int_{\partial\mathbb{D}} \mathrm{Re}\left(\frac{\dot{\Phi}_c}{\Phi_c}\right)^2 \mathrm{d}\tilde{\mu_c} \to \infty$$

when $c \to 1/4^-$.

In order to prove Proposition 8.1 we will need the three following lemmas.

Lemma 8.2. *For every* $N \in \mathbb{N}$ *we have*

$$\lim_{c \to 1/4^-} \int_{\mathcal{M}_N} |\dot{\Psi}_{N,c}|^2 \,\mathrm{d}\tilde{\mu_c} = \infty.$$

Lemma 8.3. *For every* $N \in \mathbb{N}$ *we have*

$$\lim_{c \to 1/4^-} \frac{\int_{\mathcal{B}_N} |\dot{\Phi}_c|^2 \,\mathrm{d}\tilde{\mu_c}}{\int_{\mathcal{M}_N} |\dot{\Psi}_{N,c}|^2 \,\mathrm{d}\tilde{\mu_c}} = 0.$$

Lemma 8.4. *For every* $N \in \mathbb{N}$ *we have*

$$\lim_{c \to 1/4^-} \frac{\int_{\mathcal{M}_N} |\dot{\Phi}_c^2 - \dot{\Psi}_{N,c}^2| \,\mathrm{d}\tilde{\mu_c}}{\int_{\mathcal{M}_N} |\dot{\Psi}_{N,c}|^2 \,\mathrm{d}\tilde{\mu_c}} = 0.$$

Now, using the above lemmas, we will prove Proposition 8.1.

Proof of Proposition 8.1. Let us fix $N \in \mathbb{N}$ (large enough) and $c_1 < 1/4$ (close to $1/4$).

We see from Corollary 4.5 and (6.5) that $\dot{\Psi}^2_{N,c}(s)$ is close to $|\dot{\Psi}_{N,c}(s)|^2$, where $s \in \mathcal{M}_N$ and $c \in (c_1, 1/4)$. Because we can also assume that $\Phi_c(s)$ is close to $1/2$ (see Lemma 4.2 (1)), we conclude that

$$\int_{\mathcal{M}_N} |\dot{\Psi}_{N,c}|^2 \, \mathrm{d}\tilde{\mu}_c \leqslant \frac{1}{3} \int_{\mathcal{M}_N} \operatorname{Re}\left(\frac{\dot{\Psi}_{N,c}}{\Phi_c}\right)^2 \mathrm{d}\tilde{\mu}_c.$$

Next, Lemma 8.4 allows us to replace $\dot{\Psi}_{N,c}$ by $\dot{\Phi}_c$ on the right-hand side of the above inequality (possibly after changing constant). So, we get

$$\int_{\mathcal{M}_N} |\dot{\Psi}_{N,c}|^2 \, \mathrm{d}\tilde{\mu}_c \leqslant \frac{1}{2} \int_{\mathcal{M}_N} \operatorname{Re}\left(\frac{\dot{\Phi}_c}{\Phi_c}\right)^2 \mathrm{d}\tilde{\mu}_c.$$

We know that the function Φ_c is separated from zero. Thus, Lemma 8.3 leads to

$$\int_{\mathcal{M}_N} |\dot{\Psi}_{N,c}|^2 \, \mathrm{d}\tilde{\mu}_c \leqslant \int_{\partial\mathbb{D}} \operatorname{Re}\left(\frac{\dot{\Phi}_c}{\Phi_c}\right)^2 \mathrm{d}\tilde{\mu}_c,$$

and the statement follows from Lemma 8.2. □

So, we see that the integral of $|\dot{\Psi}_{N,c}|^2$, over $\mathcal{M}_N$, has a decisive influence on $\int_{\partial\mathbb{D}} \operatorname{Re}(\dot{\Phi}_c/\Phi_c)^2 \, \mathrm{d}\tilde{\mu}_c$ (cf. scheme of the proof of Proposition 7.1).

The main difference between the proofs of Proposition 8.1 and (7.3) (or Proposition 7.1) is that we do not know if the integrals

$$\int_{\mathcal{B}_N} |\dot{\Phi}_c|^2 \, \mathrm{d}\tilde{\mu}_c, \qquad \int_{\mathcal{M}_N} |\dot{\Phi}^2_c - \dot{\Psi}^2_{N,c}| \, \mathrm{d}\tilde{\mu}_c$$

are bounded or not (cf. Lemmas 7.3 and 7.4). So we will prove that these integrals are small with respect to $\int_{\mathcal{M}_N} |\dot{\Psi}_{N,c}|^2 \, \mathrm{d}\tilde{\mu}_c$.

Now we will prove Lemmas 8.2 and 8.3, and Lemma 8.5, which is a stronger version of Lemma 8.4.

Proof of Lemma 8.2. Since $\tilde{\mu}_c$ is a probability measure, the assertion follows from Lemma 7.2. □

Proof of Lemma 8.3. Let us fix $N \in \mathbb{N}$ and $\varepsilon > 0$. For every $N_0 \geqslant 1$ and $s \in A^{N_0}_{N,n}$ we can write

$$\begin{aligned} |\dot{\Phi}_c(s)|^2 &= \left| -\sum_{k=1}^{N_0} \frac{1}{(f^k_c)'(\Phi_c(s))} - \sum_{k=N_0+1}^{N_0+n} \frac{1}{(f^k_c)'(\Phi_c(s))} + \frac{\dot{\Phi}_c(T^{N_0+n}(s))}{(f^{N_0+n}_c)'(\Phi_c(s))} \right|^2 \\ &\leqslant 3\left| \sum_{k=1}^{N_0} \frac{1}{(f^k_c)'(\Phi_c(s))} \right|^2 + 3\left| \sum_{k=N_0+1}^{N_0+n} \frac{1}{(f^k_c)'(\Phi_c(s))} \right|^2 + 3\left| \frac{\dot{\Phi}_c(T^{N_0+n}(s))}{(f^{N_0+n}_c)'(\Phi_c(s))} \right|^2 \\ &=: 3A(s) + 3B(s) + 3\Omega(s). \end{aligned} \tag{8.1}$$

Note that the sum in $B(s)$ is empty when $n = 0$.

First, in the same way as in the proof of Lemma 7.3, we will show that the integral of the expression $\Omega(s)$ (the 'tail') is less than

$$\frac{1}{6}\int_{\mathcal{B}_N} |\dot{\Phi}_c|^2 \, \mathrm{d}\tilde{\mu_c}$$

for N_0 large enough (see Step 1).

Next, we will estimate the first two expressions (squares of sums) on the right-hand side of (8.1). In the proof of Lemma 7.3 related sums were considered together, and each inverse of a derivative was bounded simply by 1. In our case this estimate is not enough. Of course the integral of $A(s)$ is bounded (see Step 3), but we would get an infinite upper bound of the integral of $B(s)$.

Nevertheless, the expression $B(s)$ can be estimated by $|\dot{\Psi}_{N,c}(T^{N_0})|^2$, whereas $A_{N,n}^{N_0} \subset T^{-N_0}(C_{N+n})$. Thus, roughly speaking, the integral

$$\int_{A_{N,n}^{N_0}} B(s) \, \mathrm{d}\tilde{\mu_c}$$

turns out to be small with respect to

$$\int_{C_{N+n}} |\dot{\Psi}_{N,c}|^2 \, \mathrm{d}\tilde{\mu_c},$$

because $\tilde{\mu_c}(A_{N,n}^{N_0}) \leqslant \varepsilon\tilde{\mu_c}(C_{N+n})$ (see Step 2). This is the key fact that we need to complete the proof (see Step 4).

Step 1. We have $T^{N_0+n}(A_{N,n}^{N_0}) \subset \mathcal{B}_N$, so $f_c^{N_0+n}(\Phi_c(s)) \in \mathcal{B}_N(c)$. Therefore, Lemma 4.6 and the fact that the measure $\tilde{\mu_c}$ is T-invariant give us

$$\begin{aligned}\int_{A_{N,n}^{N_0}} \left|\frac{\dot{\Phi}_c(T^{N_0+n}(s))}{(f_c^{N_0+n})'(\Phi_c(s))}\right|^2 \mathrm{d}\tilde{\mu_c}(s) &\leqslant \frac{\lambda_1(N)}{(N_0+n)^4}\int_{A_{N,n}^{N_0}} |\dot{\Phi}_c(T^{N_0+n}(s))|^2 \, \mathrm{d}\tilde{\mu_c}(s) \\ &\leqslant \frac{\lambda_1(N)}{(N_0+n)^4}\int_{\mathcal{B}_N} |\dot{\Phi}_c|^2 \, \mathrm{d}\tilde{\mu_c}.\end{aligned}$$

Hence, we obtain

$$\sum_{n=0}^{\infty}\int_{A_{N,n}^{N_0}} \left|\frac{\dot{\Phi}_c(T^{N_0+n})}{(f_c^{N_0+n})'(\Phi_c)}\right|^2 \mathrm{d}\tilde{\mu_c} \leqslant \lambda_1(N)\left(\sum_{n=N_0}^{\infty}\frac{1}{n^4}\right)\int_{\mathcal{B}_N} |\dot{\Phi}_c|^2 \, \mathrm{d}\tilde{\mu_c}.$$

Thus, for $N_0 = N_0(N)$ large enough, we have

$$\sum_{n=0}^{\infty}\int_{A_{N,n}^{N_0}} \left|\frac{\dot{\Phi}_c(T^{N_0+n})}{(f_c^{N_0+n})'(\Phi_c)}\right|^2 \mathrm{d}\tilde{\mu_c} \leqslant \frac{1}{6}\int_{\mathcal{B}_N} |\dot{\Phi}_c|^2 \, \mathrm{d}\tilde{\mu_c}. \tag{8.2}$$

https://doi.org/10.1017/S0013091516000481 Published online by Cambridge University Press

Step 2. The second sum on the right-hand side of (8.1) is non-zero for $n \geqslant 1$. Since $|(f_c^k)'(\Phi_c)| \geqslant 1$, for every $\tilde{n} \geqslant 1$ we can get

$$\begin{aligned}
\sum_{n=1}^{\infty} \int_{A_{N,n}^{N_0}} \bigg| \sum_{k=N_0+1}^{N_0+n} \frac{1}{(f_c^k)'(\Phi_c)} \bigg|^2 \,\mathrm{d}\tilde{\mu_c} &\leqslant \sum_{n=1}^{\tilde{n}} \int_{A_{N,n}^{N_0}} \bigg| \sum_{k=N_0+1}^{N_0+n} \frac{1}{(f_c^k)'(\Phi_c)} \bigg|^2 \,\mathrm{d}\tilde{\mu_c} \\
&\quad + \sum_{n=\tilde{n}+1}^{\infty} \int_{A_{N,n}^{N_0}} \bigg| \frac{1}{(f_c^{N_0})'(\Phi_c)} \sum_{k=1}^{n} \frac{1}{(f_c^k)'(\Phi_c(T^{N_0}))} \bigg|^2 \,\mathrm{d}\tilde{\mu_c} \\
&\leqslant \sum_{n=1}^{\tilde{n}} n^2 + \sum_{n=\tilde{n}+1}^{\infty} \int_{A_{N,n}^{N_0}} \bigg| \sum_{k=1}^{n} \frac{1}{(f_c^k)'(\Phi_c(T^{N_0}))} \bigg|^2 \,\mathrm{d}\tilde{\mu_c} \\
&\leqslant \tilde{n}^3 + \sum_{n=\tilde{n}+1}^{\infty} \int_{A_{N,n}^{N_0}} |\dot{\Psi}_{N,c}(T^{N_0})|^2 \,\mathrm{d}\tilde{\mu_c}.
\end{aligned}$$

By Lemma 5.6 we have

$$\int_{A_{N,n}^{N_0}} |\dot{\Psi}_{N,c}(T^{N_0})|^2 \,\mathrm{d}\tilde{\mu_c} \leqslant K \frac{\tilde{\mu_c}(A_{N,n}^{N_0})}{\tilde{\mu_c}(C_{N+n})} \int_{C_{N+n}} |\dot{\Psi}_{N,c}|^2 \,\mathrm{d}\tilde{\mu_c},$$

where $K > 0$ is a universal constant. Next, we see from Corollary 5.3 that

$$\frac{\tilde{\mu_c}(A_{N,n}^{N_0})}{\tilde{\mu_c}(C_{N+n})} \leqslant \varepsilon \quad \text{for } n > \tilde{n}$$

(where $\tilde{n} = \tilde{n}(N_0)$ is large enough) and $c < 1/4$ is close to $1/4$. Hence, the above estimates lead to

$$\sum_{n=1}^{\infty} \int_{A_{N,n}^{N_0}} \bigg| \sum_{k=N_0+1}^{N_0+n} \frac{1}{(f_c^k)'(\Phi_c)} \bigg|^2 \,\mathrm{d}\tilde{\mu_c} \leqslant \tilde{n}^3 + \varepsilon K \int_{\mathcal{M}_{N+\tilde{n}}} |\dot{\Psi}_{N,c}|^2 \,\mathrm{d}\tilde{\mu_c}. \tag{8.3}$$

Step 3. The first expression on the right-hand side of (8.1) can be easily estimated as

$$\int_{\mathcal{B}_N} \bigg| \sum_{k=1}^{N_0} \frac{1}{(f_c^k)'(\Phi_c)} \bigg|^2 \,\mathrm{d}\tilde{\mu_c} \leqslant \int_{\mathcal{B}_N} |N_0|^2 \,\mathrm{d}\tilde{\mu_c} \leqslant N_0^2. \tag{8.4}$$

Step 4. Combining (8.1) with (8.2)–(8.4), we see that for every $N \in \mathbb{N}$ and $\varepsilon > 0$ there exist $N_0 = N_0(N)$, $\tilde{n} = \tilde{n}(N_0)$ such that

$$\int_{\mathcal{B}_N} |\dot{\Phi}_c|^2 \,\mathrm{d}\tilde{\mu_c} \leqslant 3\bigg(N_0^2 + \tilde{n}^3 + \varepsilon K \int_{\mathcal{M}_{N+\tilde{n}}} |\dot{\Psi}_{N,c}|^2 \,\mathrm{d}\tilde{\mu_c} + \frac{1}{6} \int_{\mathcal{B}_N} |\dot{\Phi}_c|^2 \,\mathrm{d}\tilde{\mu_c} \bigg), \tag{8.5}$$

where c is close to $1/4$. Because

$$\int_{\mathcal{M}_{N+\tilde{n}}} |\dot{\Psi}_{N,c}|^2 \,\mathrm{d}\tilde{\mu_c} \geqslant \int_{\mathcal{M}_{N+\tilde{n}}} |\dot{\Psi}_{N+\tilde{n},c}|^2 \,\mathrm{d}\tilde{\mu_c} \to \infty,$$

when $c \to 1/4^-$ (see Lemma 8.2) we obtain

$$\frac{1}{2}\int_{\mathcal{B}_N} |\dot{\Phi}_c|^2 \,\mathrm{d}\tilde{\mu_c} \leqslant 4\varepsilon K \int_{\mathcal{M}_{N+\tilde{n}}} |\dot{\Psi}_{N,c}|^2 \,\mathrm{d}\tilde{\mu_c} \leqslant 4\varepsilon K \int_{\mathcal{M}_N} |\dot{\Psi}_{N,c}|^2 \,\mathrm{d}\tilde{\mu_c},$$

where the parameter $c < 1/4$ is close to $1/4$. Since K is a universal constant, the assertion follows. □

If $z \in C_{N+n}(c)$, then we define the function $f^*_{N,c}$ by setting

$$f^*_{N,c}(z) := f^n_c(z).$$

We need to prove Lemma 8.4, but it is an immediate consequence of the following fact, which will be also used later on.

Lemma 8.5. *For every $N \in \mathbb{N}$ we have*

$$\lim_{c\to 1/4^-} \frac{\int_{\mathcal{M}_N} |(f^*_{N,c})'(\Phi_c)|\,|\dot{\Phi}^2_c - \dot{\Psi}^2_{N,c}| \,\mathrm{d}\tilde{\mu_c}}{\int_{\mathcal{M}_N} |\dot{\Psi}_{N,c}|^2 \,\mathrm{d}\tilde{\mu_c}} = 0.$$

Proof. Let $N \in \mathbb{N}$. If $s \in C_{N+n}$, then

$$\begin{aligned}(\dot{\Phi}_c(s))^2 - (\dot{\Psi}_{N,c}(s))^2 &= \left(\dot{\Psi}_{N,c}(s) + \frac{\dot{\Phi}_c(T^n(s))}{(f^n_c)'(\Phi_c(s))}\right)^2 - (\dot{\Psi}_{N,c}(s))^2 \\ &= 2\dot{\Psi}_{N,c}(s)\frac{\dot{\Phi}_c(T^n(s))}{(f^n_c)'(\Phi_c(s))} + \left(\frac{\dot{\Phi}_c(T^n(s))}{(f^n_c)'(\Phi_c(s))}\right)^2.\end{aligned}$$

Thus, we see that

$$\begin{aligned}&\int_{\mathcal{M}_N} |(f^*_{N,c})'(\Phi_c)|\,|\dot{\Phi}^2_c - \dot{\Psi}^2_{N,c}| \,\mathrm{d}\tilde{\mu_c} \\ &\qquad \leqslant \sum_{n=1}^{\infty} \int_{C_{N+n}} 2|\dot{\Psi}_{N,c}\dot{\Phi}_c(T^n)| \,\mathrm{d}\tilde{\mu_c} + \sum_{n=1}^{\infty} \int_{C_{N+n}} \frac{|\dot{\Phi}_c(T^n(s))|^2}{|(f^n_c)'(\Phi_c(s))|} \,\mathrm{d}\tilde{\mu_c}. \quad (8.6)\end{aligned}$$

First, we will estimate the rightmost expression. Note that this part of the proof is similar to the proof of Lemma 7.4 (and estimates of 'tails').

Step 1. The measure $\tilde{\mu_c}$ is T-invariant, so the fact that $T^n(C_{N+n}) \subset \mathcal{B}_N$ and Lemma 4.6 lead to

$$\sum_{n=1}^{\infty} \int_{C_{N+n}} \frac{|\dot{\Phi}_c(T^n)|^2}{|(f^n_c)'(\Phi_c)|} \,\mathrm{d}\tilde{\mu_c} \leqslant \sum_{n=1}^{\infty} \frac{\lambda_1(N)}{n^2} \int_{\mathcal{B}_N} |\dot{\Phi}_c|^2 \,\mathrm{d}\tilde{\mu_c}.$$

Since the series $\sum_{n=1}^{\infty} 1/n^2$ converges, using Lemma 8.3 we observe that the latter expression is less than or equal to

$$K_1(N) \int_{\mathcal{B}_N} |\dot{\Phi}_c|^2 \,\mathrm{d}\tilde{\mu_c} \leqslant \delta_N(c) K_1(N) \int_{\mathcal{M}_N} |\dot{\Psi}_{N,c}|^2 \,\mathrm{d}\tilde{\mu_c},$$

where $\delta_N(c)$ is a positive function such that $\delta_N(c) \to 0$ when $c \to 1/4^-$.

Step 2. If $s \in C_{N+n}$, then write $T^*(s) := T^n(s)$. By the Schwarz inequality, we obtain

$$\sum_{n=1}^{\infty} \int_{C_{N+n}} 2|\dot{\Psi}_{N,c} \cdot \dot{\Phi}_c(T^n)| \, \mathrm{d}\tilde{\mu_c} = 2 \int_{\mathcal{M}_N} |\dot{\Psi}_{N,c} \cdot \dot{\Phi}_c(T^*)| \, \mathrm{d}\tilde{\mu_c}$$
$$\leqslant 2\biggl(\int_{\mathcal{M}_N} |\dot{\Psi}_{N,c}|^2 \, \mathrm{d}\tilde{\mu_c} \biggr)^{1/2} \biggl(\int_{\mathcal{M}_N} |\dot{\Phi}_c(T^*)|^2 \, \mathrm{d}\tilde{\mu_c} \biggr)^{1/2}.$$

Now, we need to estimate the integral of $|\dot{\Phi}_c(T^*)|^2$. Using Lemma 5.6 and then Lemma 8.3 we get

$$\begin{aligned}
\int_{\mathcal{M}_N} |\dot{\Phi}_c(T^*)|^2 \, \mathrm{d}\tilde{\mu_c} &= \sum_{n=1}^{\infty} \int_{C_{N+n}} |\dot{\Phi}_c(T^n)|^2 \, \mathrm{d}\tilde{\mu_c} \\
&\leqslant K \sum_{n=1}^{\infty} \frac{\tilde{\mu_c}(C_{N+n})}{\tilde{\mu_c}(C_N)} \int_{C_N} |\dot{\Phi}_c|^2 \, \mathrm{d}\tilde{\mu_c} \\
&\leqslant K \frac{\tilde{\mu_c}(\mathcal{M}_N)}{\tilde{\mu_c}(C_N)} \int_{\mathcal{B}_N} |\dot{\Phi}_c|^2 \, \mathrm{d}\tilde{\mu_c} \\
&\leqslant K_2(N) \int_{\mathcal{B}_N} |\dot{\Phi}_c|^2 \, \mathrm{d}\tilde{\mu_c} \\
&\leqslant \delta_N(c) K_2(N) \int_{\mathcal{M}_N} |\dot{\Psi}_{N,c}|^2 \, \mathrm{d}\tilde{\mu_c},
\end{aligned}$$

where $K_2(N)$ is a constant that depends on N and $\delta_N(c) \to 0$ if $c \to 1/4^-$. Thus, we conclude that

$$\sum_{n=1}^{\infty} \int_{C_{N+n}} 2|\dot{\Psi}_{N,c} \cdot \dot{\Phi}_c(T^n)| \, \mathrm{d}\tilde{\mu_c} \leqslant 2(\delta_N(c) K_2(N))^{1/2} \int_{\mathcal{M}_N} |\dot{\Psi}_{N,c}|^2 \, \mathrm{d}\tilde{\mu_c}.$$

Step 3. Since we can assume that $\delta_N(c) \leqslant (\delta_N(c))^{1/2}$, (8.6) combined with the above and the estimate from Step 1 gives us

$$\int_{\mathcal{M}_N} |(f_{N,c}^*)'(\Phi_c) \cdot (\dot{\Phi}_c^2 - \dot{\Psi}_{N,c}^2)| \, \mathrm{d}\tilde{\mu_c} \leqslant K_3(N)(\delta_N(c))^{1/2} \int_{\mathcal{M}_N} |\dot{\Psi}_{N,c}|^2 \, \mathrm{d}\tilde{\mu_c}. \tag{8.7}$$

Thus, the statement follows. □

Lemmas 8.3 and 8.4 lead to the following corollary.

Corollary 8.6. *For every $N \in \mathbb{N}$ we have*

$$\lim_{c \to 1/4^-} \frac{\int_{\partial \mathbb{D}} |\dot{\Phi}_c|^2 \, \mathrm{d}\tilde{\mu_c}}{\int_{\mathcal{M}_N} |\dot{\Psi}_{N,c}|^2 \, \mathrm{d}\tilde{\mu_c}} = 1.$$

9. The integral of $\mathrm{Re}(\ddot{\Phi}_c/\Phi_c)$ is negative

Proposition 9.1. *There exists* $c_1 < 1/4$ *such that*

$$\int_{\partial\mathbb{D}} \mathrm{Re}\left(\frac{\ddot{\Phi}_c}{\Phi_c}\right) \mathrm{d}\tilde{\mu}_c < 0,$$

provided that $c \in (c_1, 1/4)$. *Moreover,*

$$\int_{\partial\mathbb{D}} \mathrm{Re}\left(\frac{\ddot{\Phi}_c}{\Phi_c}\right) \mathrm{d}\tilde{\mu}_c \to -\infty \quad \textit{when } c \to 1/4^-.$$

In order to prove Proposition 9.1 we will need the following facts.

Lemma 9.2. *For every* $N \in \mathbb{N}$ *we have*

(1)

$$\lim_{c\to 1/4^-} \frac{\int_{\partial\mathbb{D}} |\dot{\Phi}_c|^2 \,\mathrm{d}\tilde{\mu}_c}{\int_{\mathcal{M}_N} |\ddot{A}_{N,c}| \,\mathrm{d}\tilde{\mu}_c} = 0$$

and, in particular, $\int_{\mathcal{M}_N} |\ddot{A}_{N,c}| \,\mathrm{d}\tilde{\mu}_c \to \infty$ *when* $c \to 1/4^-$;

(2)

$$\lim_{c\to 1/4^-} \frac{\int_{\mathcal{M}_N} |\ddot{\Theta}_{N,c} - \ddot{A}_{N,c}| \,\mathrm{d}\tilde{\mu}_c}{\int_{\mathcal{M}_N} |\ddot{A}_{N,c}| \,\mathrm{d}\tilde{\mu}_c} = 0.$$

Lemma 9.2 (2) leads to the following corollary.

Corollary 9.3. *For every* $N \in \mathbb{N}$ *we have*

$$\lim_{c\to 1/4^-} \frac{\int_{\mathcal{M}_N} |\ddot{\Theta}_{N,c}| \,\mathrm{d}\tilde{\mu}_c}{\int_{\mathcal{M}_N} |\ddot{A}_{N,c}| \,\mathrm{d}\tilde{\mu}_c} = 1.$$

Lemma 9.4. *For every* $N \in \mathbb{N}$ *we have*

$$\lim_{c\to 1/4^-} \frac{\int_{\mathcal{B}_N} |\ddot{\Phi}_c| \,\mathrm{d}\tilde{\mu}_c}{\int_{\mathcal{M}_N} |\ddot{A}_{N,c}| \,\mathrm{d}\tilde{\mu}_c} = 0.$$

Lemma 9.5. *For every* $N \in \mathbb{N}$ *we have*

$$\lim_{c\to 1/4^-} \frac{\int_{\mathcal{M}_N} |\ddot{\Phi}_c - \ddot{\Theta}_{N,c}| \,\mathrm{d}\tilde{\mu}_c}{\int_{\mathcal{M}_N} |\ddot{A}_{N,c}| \,\mathrm{d}\tilde{\mu}_c} = 0.$$

Now we will prove Proposition 9.1.

Proof of Proposition 9.1. Let us fix $N \in \mathbb{N}$ (large enough) and $c_1 < 1/4$ (close to 1/4).

https://doi.org/10.1017/S0013091516000481 Published online by Cambridge University Press

We see from Corollary 4.5 and definition (6.7) that $-\ddot{A}_{N,c}(s)$ is close to $|\ddot{A}_{N,c}(s)|$, where $s \in \mathcal{M}_N$ and $c \in (c_1, 1/4)$. Because we can also assume that $\Phi_c(s)$ is close to $1/2$ (see Lemma 4.2 (1)), we conclude that

$$\int_{\mathcal{M}_N} |\ddot{A}_{N,c}| \, \mathrm{d}\tilde{\mu_c} \leqslant -\frac{5}{8} \int_{\mathcal{M}_N} \mathrm{Re}\left(\frac{\ddot{A}_{N,c}}{\Phi_c}\right) \mathrm{d}\tilde{\mu_c}.$$

Using Lemma 9.2 (2) we get

$$\int_{\mathcal{M}_N} |\ddot{A}_{N,c}| \, \mathrm{d}\tilde{\mu_c} \leqslant -\frac{6}{8} \int_{\mathcal{M}_N} \mathrm{Re}\left(\frac{\ddot{\Theta}_{N,c}}{\Phi_c}\right) \mathrm{d}\tilde{\mu_c}.$$

Combining this with Lemma 9.5 we see that

$$\int_{\mathcal{M}_N} |\ddot{A}_{N,c}| \, \mathrm{d}\tilde{\mu_c} \leqslant -\frac{7}{8} \int_{\mathcal{M}_N} \mathrm{Re}\left(\frac{\ddot{\Phi}_c}{\Phi_c}\right) \mathrm{d}\tilde{\mu_c}.$$

Next, Lemma 9.4 leads to

$$\int_{\mathcal{M}_N} |\ddot{A}_{N,c}| \, \mathrm{d}\tilde{\mu_c} \leqslant - \int_{\partial\mathbb{D}} \mathrm{Re}\left(\frac{\ddot{\Phi}_c}{\Phi_c}\right) \mathrm{d}\tilde{\mu_c}.$$

We see from Lemma 9.2 (1) that $\int_{\mathcal{M}_N} |\ddot{A}_{N,c}| \, \mathrm{d}\tilde{\mu_c} \to \infty$ when $c \to 1/4^-$. Thus, the statement follows. $\square$

Let us note that Lemmas 9.4 and 9.5 play similar roles to Lemmas 8.3 and 8.4, respectively. So, we will prove that the integrals

$$\int_{\mathcal{B}_N} |\ddot{\Phi}_c| \, \mathrm{d}\tilde{\mu_c}, \qquad \int_{\mathcal{M}_N} |\ddot{\Phi}_c - \dot{\Theta}_{N,c}| \, \mathrm{d}\tilde{\mu_c}$$

are small with respect to the integral $\int_{\mathcal{M}_N} |\ddot{A}_{N,c}| \, \mathrm{d}\tilde{\mu_c}$, which has a decisive influence on $\int_{\partial\mathbb{D}} \mathrm{Re}(\ddot{\Phi}_c/\Phi_c) \, \mathrm{d}\tilde{\mu_c}$.

Moreover, the schemes of the proofs of Lemmas 9.4 and 8.3 are the same, whereas the proof of Lemma 9.5 is similar to the first step of the proof of Lemma 8.5 (which is a stronger version of Lemma 8.4).

On the other hand, the formula for $\ddot{\Phi}_c$ is more complicated than the formula for $\dot{\Phi}_c$ (cf. formulae (6.4) and (6.6)). Thus, in particular, we will have to deal with 'tails' of the functions $\dot{\Phi}_c^2(T^{k-1})$ (see Lemma 9.2 (2)).

Proof of Lemma 9.2. Fix $N \in \mathbb{N}$ and $\varepsilon > 0$.

Step 1. We can assume that cylinders with sufficiently large indexes are as close to the point $1/2$ as we want (for $c < 1/4$ close to $1/4$). Thus, the derivative f_c' is close to 1 on these cylinders, and then we can find $\tilde{n} \in \mathbb{N}$ and c_1 such that

$$|\dot{\Psi}_{N,c}^2(s)| \leqslant \varepsilon \left| \sum_{k=1}^{n} \frac{\dot{\Psi}_{N,c}^2(T^{k-1}(s))}{(f_c^k)'(\Phi_c(s))} \right| = \varepsilon |\ddot{A}_{N,c}(s)|,$$

where $s \in C_{N+n}$, $n > \tilde{n}$, and $c \in (c_1, 1/4)$. So, combining this with Corollary 8.6 (possibly changing c_1), we get

$$\begin{aligned}\int_{\partial\mathbb{D}} |\dot{\Phi}_c|^2 \,\mathrm{d}\tilde{\mu_c} &\leqslant (1+\varepsilon) \int_{\mathcal{M}_{N+\tilde{n}}} |\dot{\Psi}_{N,c}|^2 \,\mathrm{d}\tilde{\mu_c} \\ &\leqslant \varepsilon(1+\varepsilon) \int_{\mathcal{M}_{N+\tilde{n}}} |\ddot{A}_{N,c}| \,\mathrm{d}\tilde{\mu_c} \\ &\leqslant \varepsilon(1+\varepsilon) \int_{\mathcal{M}_{N}} |\ddot{A}_{N,c}| \,\mathrm{d}\tilde{\mu_c}. \end{aligned} \tag{9.1}$$

Hence, the first statement follows.

Step 2. We have

$$\begin{aligned} f_c'(f_c^{k-1}(\Phi_c)) \cdot (f_c^n)'(\Phi_c) &= f_c'(f_c^{k-1}(\Phi_c)) \cdot (f_c^{n-k})'(f_c^k(\Phi_c)) \cdot (f_c^k)'(\Phi_c) \\ &= (f_c^{n-k+1})'(f_c^{k-1}(\Phi_c)) \cdot (f_c^k)'(\Phi_c). \end{aligned} \tag{9.2}$$

Let

$$\ddot{\Omega}_{N,c}(s) := \ddot{\Theta}_{N,c}(s) - \ddot{A}_{N,c}(s) = -\sum_{k=1}^{n} \frac{\dot{\Phi}_c^2(T^{k-1}(s)) - \dot{\Psi}_{N,c}^2(T^{k-1}(s))}{(f_c^k)'(\Phi_c(s))}. \tag{9.3}$$

Then, using (9.2), we obtain

$$\begin{aligned}\int_{C_{N+n}} |\ddot{\Omega}_{N,c}| \,\mathrm{d}\tilde{\mu_c} &\leqslant \int_{C_{N+n}} \sum_{k=1}^{n} \frac{|(f_c^{n-k+1})'(f_c^{k-1}(\Phi_c))|}{|(f_c^{n-k+1})'(f_c^{k-1}(\Phi_c))|} \cdot \frac{|\dot{\Phi}_c^2(T^{k-1}) - \dot{\Psi}_{N,c}^2(T^{k-1})|}{|(f_c^k)'(\Phi_c)|} \,\mathrm{d}\tilde{\mu_c} \\ &\leqslant \int_{C_{N+n}} \sum_{k=1}^{n} \frac{|(f_c^{n-k+1})'(\Phi_c(T^{k-1}))| \cdot |\dot{\Phi}_c^2(T^{k-1}) - \dot{\Psi}_{N,c}^2(T^{k-1})|}{|(f_c^n)'(\Phi_c)|} \,\mathrm{d}\tilde{\mu_c}. \end{aligned}$$

Next, the fact that $T^{k-1}(C_{N+n}) = C_{N+n-k+1}$ and Lemma 4.6 lead to the following estimate of the latter expression

$$\begin{aligned} &\frac{\lambda(N)}{n^2} \sum_{k=1}^{n} \int_{C_{N+n-k+1}} |(f_c^{n-k+1})'(\Phi_c)| \cdot |\dot{\Phi}_c^2 - \dot{\Psi}_{N,c}^2| \,\mathrm{d}\tilde{\mu_c} \\ &\qquad\qquad \leqslant \frac{\lambda(N)}{n^2} \int_{\mathcal{M}_N} |(f_{N,c}^*)'(\Phi_c)| \cdot |\dot{\Phi}_c^2 - \dot{\Psi}_{N,c}^2| \,\mathrm{d}\tilde{\mu_c}, \end{aligned}$$

where $f_{N,c}^*(z) = f_c^m(z)$ if $z \in C_{N+m}(c)$.

The above estimates, Lemma 8.5 and then Corollary 8.6 give us

$$\begin{aligned}\int_{\mathcal{M}_N} |\ddot{\Omega}_{N,c}| \,\mathrm{d}\tilde{\mu_c} &= \sum_{n=1}^{\infty} \int_{C_{N+n}} |\ddot{\Omega}_{N,c}| \,\mathrm{d}\tilde{\mu_c} \\ &\leqslant \sum_{n=1}^{\infty} \frac{\lambda(N)}{n^2} \int_{\mathcal{M}_N} |(f_{N,c}^*)'(\Phi_c)| \cdot |\dot{\Phi}_c^2 - \dot{\Psi}_{N,c}^2| \,\mathrm{d}\tilde{\mu_c} \end{aligned}$$

https://doi.org/10.1017/S0013091516000481 Published online by Cambridge University Press

$$\leqslant \delta_N(c) \sum_{n=1}^{\infty} \frac{\lambda(N)}{n^2} \int_{\mathcal{M}_N} |\dot{\Psi}_{N,c}|^2 \,\mathrm{d}\tilde{\mu_c}$$
$$= \delta_N(c) K(N) \int_{\partial\mathbb{D}} |\dot{\Phi}_c|^2 \,\mathrm{d}\tilde{\mu_c},$$

where $\delta_N(c) \to 0$ when $c \to 1/4^-$. Thus, we conclude from (9.1) that

$$\int_{\mathcal{M}_N} |\ddot{\Omega}_{N,c}| \,\mathrm{d}\tilde{\mu_c} \leqslant \delta_N(c) K(N) \int_{\mathcal{M}_N} |\ddot{A}_{N,c}| \,\mathrm{d}\tilde{\mu_c}.$$

So, the second statement follows from definition (9.3). □

Proof of Lemma 9.4. Let us fix $\varepsilon > 0$ and $N \in \mathbb{N}$. For every $N_0 \geqslant 1$ and $s \in A_{N,n}^{N_0}$ we have

$$\ddot{\Phi}_c(s) = -\sum_{k=1}^{N_0} \frac{2\dot{\Phi}_c^2(T^{k-1}(s))}{(f_c^k)'(\Phi_c(s))} - \sum_{k=N_0+1}^{N_0+n} \frac{2\dot{\Phi}_c^2(T^{k-1}(s))}{(f_c^k)'(\Phi_c(s))} + \frac{\ddot{\Phi}_c(T^{N_0+n}(s))}{(f_c^{N_0+n})'(\Phi_c(s))}. \tag{9.4}$$

Note that if $n = 0$, then the second sum is empty.

Step 1. We have $T^{N_0+n}(A_{N,n}^{N_0}) \subset \mathcal{B}_N$ and $f_c^{N_0+n}(\Phi_c(s)) \in \mathcal{B}_N(c)$. Thus, Lemma 4.6 and the fact that $\tilde{\mu_c}$ is T-invariant lead to

$$\int_{A_{N,n}^{N_0}} \left| \frac{\ddot{\Phi}_c(T^{N_0+n}(s))}{(f_c^{N_0+n})'(\Phi_c(s))} \right| \mathrm{d}\tilde{\mu_c}(s) \leqslant \frac{\lambda_1(N)}{(N_0+n)^2} \int_{\mathcal{B}_N} |\ddot{\Phi}_c(s)| \,\mathrm{d}\tilde{\mu_c}(s).$$

So, for $N_0 = N_0(N)$ large enough, we obtain

$$\sum_{n=0}^{\infty} \int_{A_{N,n}^{N_0}} \left| \frac{\ddot{\Phi}_c(T^{N_0+n})}{(f_c^{N_0+n})'(\Phi_c)} \right| \mathrm{d}\tilde{\mu_c} \leqslant \left(\sum_{n=0}^{\infty} \frac{\lambda_1(N)}{(N_0+n)^2} \right) \int_{\mathcal{B}_N} |\ddot{\Phi}_c| \,\mathrm{d}\tilde{\mu_c} \leqslant \frac{1}{2} \int_{\mathcal{B}_N} |\ddot{\Phi}_c| \,\mathrm{d}\tilde{\mu_c}. \tag{9.5}$$

Step 2. First, note that

$$\sum_{n=1}^{\tilde{n}} \int_{A_{N,n}^{N_0}} \left| \sum_{k=N_0+1}^{N_0+n} \frac{2\dot{\Phi}_c^2(T^{k-1})}{(f_c^k)'(\Phi_c)} \right| \mathrm{d}\tilde{\mu_c} \leqslant \sum_{n=1}^{\tilde{n}} \int_{\partial\mathbb{D}} 2|\dot{\Phi}_c|^2 \,\mathrm{d}\tilde{\mu_c} = 2\tilde{n} \int_{\partial\mathbb{D}} |\dot{\Phi}_c|^2 \,\mathrm{d}\tilde{\mu_c}. \tag{9.6}$$

Next, the fact that $|(f_c^{N_0})(\Phi_c)| > 1$, Lemma 5.6 and Corollary 5.3 lead to

$$\begin{aligned} \sum_{n=\tilde{n}+1}^{\infty} \int_{A_{N,n}^{N_0}} & \left| \frac{1}{(f_c^{N_0})'(\Phi_c)} \sum_{k=1}^{n} \frac{2\dot{\Phi}_c^2(T^{N_0+k-1})}{(f_c^k)'(\Phi_c(T^{N_0}))} \right| \mathrm{d}\tilde{\mu_c} \\ & \leqslant K \frac{\tilde{\mu_c}(A_{N,n}^{N_0})}{\tilde{\mu_c}(C_{N+n})} \sum_{n=\tilde{n}+1}^{\infty} \int_{C_{N+n}} \left| \sum_{k=1}^{n} \frac{2\dot{\Phi}_c^2(T^{k-1})}{(f_c^k)'(\Phi_c)} \right| \mathrm{d}\tilde{\mu_c} \\ & \leqslant K\varepsilon \int_{\mathcal{M}_{N+\tilde{n}}} |\ddot{\Theta}_{N,c}| \,\mathrm{d}\tilde{\mu_c}, \end{aligned} \tag{9.7}$$

where K is a universal constant.

https://doi.org/10.1017/S0013091516000481 Published online by Cambridge University Press

Step 3. We have

$$\int_{\mathcal{B}_N} \left| \sum_{k=1}^{N_0} \frac{2\dot{\Phi}_c^2(T^{k-1})}{(f_c^k)'(\Phi_c)} \right| \mathrm{d}\tilde{\mu_c} \leqslant 2N_0 \int_{\partial\mathbb{D}} |\dot{\Phi}_c|^2 \, \mathrm{d}\tilde{\mu_c}. \tag{9.8}$$

Step 4. Combining (9.5)–(9.8) with (9.4) we obtain

$$\begin{aligned}\int_{\mathcal{B}_N} |\ddot{\Phi}_c| \, \mathrm{d}\tilde{\mu_c} \leqslant 2N_0 \int_{\partial\mathbb{D}} |\dot{\Phi}_c|^2 \, \mathrm{d}\tilde{\mu_c} + 2\tilde{n} \int_{\partial\mathbb{D}} |\dot{\Phi}_c|^2 \, \mathrm{d}\tilde{\mu_c} \\ + \varepsilon K \int_{\mathcal{M}_{N+\tilde{n}}} |\ddot{\Theta}_{N,c}| \, \mathrm{d}\tilde{\mu_c} + \frac{1}{2} \int_{\mathcal{B}_N} |\ddot{\Phi}_c| \, \mathrm{d}\tilde{\mu_c}.\end{aligned}$$

Thus, Lemma 9.2 leads to

$$\begin{aligned}\frac{1}{2} \int_{\mathcal{B}_N} |\ddot{\Phi}_c| \, \mathrm{d}\tilde{\mu_c} &\leqslant \varepsilon \int_{\mathcal{M}_N} |\ddot{\Theta}_{N,c}| \, \mathrm{d}\tilde{\mu_c} + \varepsilon K \int_{\mathcal{M}_{N+\tilde{n}}} |\ddot{\Theta}_{N,c}| \, \mathrm{d}\tilde{\mu_c} \\ &\leqslant \varepsilon(K+1) \int_{\mathcal{M}_N} |\ddot{\Theta}_{N,c}| \, \mathrm{d}\tilde{\mu_c} \\ &\leqslant 2\varepsilon(K+1) \int_{\mathcal{M}_N} |\ddot{A}_{N,c}| \, \mathrm{d}\tilde{\mu_c},\end{aligned}$$

and the statement follows. □

Proof of Lemma 9.5. Since $T^n(C_{N+n}) \subset \mathcal{B}_N$, we see from Lemma 4.6 and then from Lemma 9.4 that

$$\begin{aligned}\int_{\mathcal{M}_N} |\ddot{\Phi}_c - \ddot{\Theta}_{N,c}| \, \mathrm{d}\tilde{\mu_c} &= \sum_{n=1}^{\infty} \int_{C_{N+n}} \left| \frac{\ddot{\Phi}_c(T^n(s))}{(f_c^n)'(\Phi_c(s))} \right| \mathrm{d}\tilde{\mu_c} \\ &\leqslant \sum_{n=1}^{\infty} \frac{\lambda_1(N)}{n^2} \int_{\mathcal{B}_N} |\ddot{\Phi}_c| \, \mathrm{d}\tilde{\mu_c} \\ &\leqslant K(N) \int_{\mathcal{B}_N} |\ddot{\Phi}_c| \, \mathrm{d}\tilde{\mu_c} \\ &\leqslant \delta_N(c) K(N) \int_{\mathcal{M}_N} |\ddot{A}_{N,c}| \, \mathrm{d}\tilde{\mu_c},\end{aligned}$$

where $K(N)$ is a constant that depends on N and $\delta_N(c) \to 0$ when $c \to 1/4^-$. □

Thus, Theorem 1.1 follows from Propositions 8.1 and 9.1 combined with formula (6.2), and Lemmas 3.2 and 3.3.

Acknowledgements. The author is partly supported by Polish NCN Grants NN201 607 640 and UMO-2014/13/B/ST1/01033.

References

1. O. Bodart and M. Zinsmeister, Quelques résultats sur la dimension de Hausdorff des ensembles de Julia des polynômes quadratiques, *Fund. Math.* **151**(2) (1996), 121–137.
2. A. Douady, P. Sentenac and M. Zinsmeister, Implosion parabolique et dimension de Hausdorff, *C. R. Acad. Sci. Paris Sér. I* **325** (1997), 765–772.
3. G. Havard and M. Zinsmeister, Le chou-fleur a une dimension de Hausdorff inférieure à 1,295, Preprint (available at www.univ-orleans.fr/mapmo/publications/mapmo2000/mapmo0001.pdf; 2000).
4. G. Havard and M. Zinsmeister, Thermodynamic formalism and variations of the Hausdorff dimension of quadratic Julia sets, *Commun. Math. Phys.* **210** (2000), 225–247.
5. J. H. Hubbard, *Teichmüller theory and applications to geometry, topology and dynamics*, Teichmüller Theory, Volume 1 (Matrix Editions, Ithaca, NY, 2006).
6. L. Jaksztas, On the derivative of the Hausdorff dimension of the quadratic Julia sets, *Trans. Am. Math. Soc.* **363** (2011), 5251–5291.
7. L. Jaksztas, On the left side derivative of the Hausdorff dimension of the Julia sets for $z^2 + c$ at $c = -3/4$, *Israel J. Math.* **199**(1) (2014), 1–43.
8. C. McMullen, Hausdorff dimension and conformal dynamics III: computation of dimension, *Am. J. Math.* **120** (1998), 471–515.
9. C. McMullen, Hausdorff dimension and conformal dynamics II: geometrically finite rational maps, *Comment. Math. Helv.* **75** (2000), 535–593.
10. F. Przytycki and M. Urbanski, *Conformal fractals: ergodic theory methods*, London Mathematical Society Lecture Note Series, Volume 371 (Cambridge University Press, 2010).
11. T. Ransford, *Potential theory in the complex plane*, London Mathematical Society Student Texts, Volume 28 (Cambridge University Press, 1995).
12. D. Ruelle, Repellers for real analytic maps, *Ergod. Theory Dynam. Syst.* **2** (1982), 99–107.
13. M. Zinsmeister, *Thermodynamic formalism and holomorphic dynamical systems*, SMF/AMS Texts and Monographs, Volume 2 (Société Mathématique de France/American Mathematical Society, 1996).