Euro. Jnl of Applied Mathematics (2016), *vol.* 27, *pp.* 213–232.
doi:10.1017/S0956792515000509

Regularity of a thermoelastic problem with variable parameters

P. BARRAL[1], M. C. NAYA-RIVEIRO[2] and P. QUINTELA[1,3]

[1]*Department of Applied Mathematics, Faculty of Mathematics, Universidade de Santiago de Compostela,* 15782 *Santiago de Compostela, Spain*
email: *patricia.barral@usc.es, peregrina.quintela@usc.es*
[2]*Department of Pedagogy and Didactics, Faculty of Educational Studies*
Universidade da Coruña, 15071 *A Coruña, Spain*
email: *cristina.naya@udc.es*
[3]*ITMATI (Technological Institute for Industrial Mathematics)*
Campus Vida, 15782 *Santiago de Compostela, Spain*

(*Received 9 December 2013; revised 14 September 2015; accepted 14 September 2015; first published online 19 October 2015*)

This paper deals with a fully-coupled thermoelastic problem, in a heterogeneous medium, arising from the metallurgical industry. The aim is to prove regularity properties of the solution with respect to space and time. Regularity in space is obtained by means of regularity properties for elliptic operators. In order to prove regularity in time, a mathematical induction technique, together with an existence and uniqueness result for this type of problems, is applied.

Key words: Thermoelasticity; quasi-static problem; heterogeneous materials; regularity.
Key subject categories: 35B65; 35D30; 35Q74; 74F05.

1 Introduction

The present paper deals with the question of regularity with respect to time and space for the solution to a quasi-static-coupled thermoelastic problem arising from metallurgical industry processes, such as casting of alloys (see, for instance, Barral and Quintela [5]) or production of aluminium in electrolytic cells (see Bermúdez *et al.* [6]). The knowledge of regularity properties of the solution for the former problems is important in order to obtain their qualitative properties and to determine which methods are more suitable for their numerical solution.

Previously, in Barral *et al.* [2,3] we studied the existence, uniqueness and regularity of a mechanical problem when the behaviour law is of Maxwell-Norton type with temperature dependent coefficients. Afterwards, in Barral *et al.* [4], we studied the coupling with a thermal problem, assuming, as a first approach, that the material is linearly elastic and heterogeneous; there we considered mixed boundary conditions in both sub-models and also a Robin type boundary condition for the thermal one. This choice was suggested by some industrial applications such as the aforementioned. In that paper, the existence and uniqueness of solution were proved and here we obtain regularity properties in space and time of that solution. Specifically, assuming additional regularity on the data, we prove

H^2_{Loc} regularity in space and $W^{r,\infty}$ regularity in time, $r \in \{0\} \cup \mathbb{N}$, for displacements and temperature.

In the bibliography, there are many works that deal with regularity properties with respect to space. We refer the reader to Kačur and Ženíšek [16] and to Marzocchi *et al.* [18] for the dynamic case, when the coefficients of the mechanical behaviour law and the reference temperature are independent of the spatial variable. In Kačur and Ženíšek [16], the problem is rewritten as two equations defined by means of elliptic operators and the regularity properties in space are obtained applying regularity results for these operators. On the other hand, Marzocchi *et al.* [18] use standard Galerkin approximations and regularity properties for the elliptic transmission problem. In the case of quasi-static problems, Copetti and Elliott [7] give regularity properties with respect to space of the solution of a one-dimensional linear thermoelastic problem with unilateral contact of Signorini type using a monotonicity method. Later, Muñoz and Racke [20] and Jiang and Racke [15] studied the interior smoothing effects in the multi-dimensional case, assuming that all the coefficients are C^∞ smooth. In their works, the problem is decoupled and the energy equation is transformed into a parabolic one; then, the regularity results are obtained from the usual regularity for parabolic equations. In the present work, we apply the techniques introduced by Kačur and Ženíšek [16] to obtain regularity results in space, when the parameters depend on the spatial point.

With respect to regularity properties in time, we mention the papers of Gawinecki [9–13] and Gawinecki *et al.* [14], who present results of regularity with respect to space and time for dynamic-coupled thermoelastic problems, with homogeneous Dirichlet boundary conditions. Later, Zhelezovskiĭ [22, 23] gave results on the smoothness of solutions considering the mechanical dissipation term in the energy equation. The proofs are obtained by mathematical induction. In this paper, following their techniques, we achieve similar regularity properties in time for our problem. The main difficulty in the quasi-static case is to establish the appropriate regularity properties and compatibility conditions for the time derivatives of the solution at the initial instant.

Finally, we prove the same time regularity for the corresponding homogeneous Dirichlet problem, assuming less smoothness over the solution at the initial instant by increasing the regularity of the initial conditions. Nonetheless, these considerations cause some difficulties which will be overcomed using results of Nečas [21] and Agmon *et al.* [1].

The resulting regularity properties in time are the main contributions of this paper.

The outline of this paper is as follows. First, in Section 2, we will introduce the mathematical model and we will recall the result of existence and uniqueness of solution given in Barral *et al.* [4]. Then, in Section 3, we will prove the H^2_{Loc} regularity of displacements and temperature with respect to space. In Section 4, we will obtain the $W^{r,\infty}$ regularity properties of the solution with respect to time for $r \in \{0\} \cup \mathbb{N}$ and finally, some conclusions will be given in Section 5.

2 Mathematical model

2.1 Domain and notation

Let $\Omega \subset \mathbb{R}^3$ be an open and bounded set with smooth enough boundary $\Gamma = \partial\Omega$. We refer the motion of the body to a fixed system of rectangular Cartesian axes $Op_1p_2p_3$.

Let $g(p,t)$ be a scalar function; we represent by $g(t)$ the function $p \longrightarrow g(p,t)$ and ∇g its gradient with respect to p.

If $\mathbf{u}$, $\mathbf{v}$ are vector fields in $\mathbb{R}^3$, their scalar product is represented by $\mathbf{u} \cdot \mathbf{v}$. Furthermore, $\nabla \mathbf{u}$ and $\mathrm{Div}\, \mathbf{u}$ denote the gradient and the divergence of $\mathbf{u}$, respectively.

We denote by S_3 the space of symmetric second order tensors over $\mathbb{R}^3$ and by : its scalar product. Furthermore, if $\boldsymbol{\tau}$ is a tensor field, $|\boldsymbol{\tau}|$, $\mathrm{tr}(\boldsymbol{\tau})$ and $\mathrm{Div}\, \boldsymbol{\tau}$ denote the norm induced by this scalar product, its trace and its divergence, respectively.

We consider the notation ∂_t^r in order to denote the partial derivative with respect to t of order r, with $r \in \{0\} \cup \mathbb{N}$. As usual, for $r = 1$ we will omit the superscript r.

We represent by $[0, t_f]$ the time interval of interest. We denote by $\mathbf{u}(p,t)$ the displacement field and by $\theta(p,t)$ the temperature field at each (p,t) in $\Omega \times (0, t_f]$.

We assume that $\Gamma_{\mathbf{u},D}$, $\Gamma_{\mathbf{u},N}$, $\Gamma_{\theta,D}$, $\Gamma_{\theta,N}$ and $\Gamma_{\theta,R}$ are open subsets of Γ, such that

- $\Gamma = \overline{\Gamma}_{\mathbf{u},D} \cup \overline{\Gamma}_{\mathbf{u},N} = \overline{\Gamma}_{\theta,D} \cup \overline{\Gamma}_{\theta,N} \cup \overline{\Gamma}_{\theta,R}$,
- $\Gamma_{\mathbf{u},D} \cap \Gamma_{\mathbf{u},N} = \emptyset$, $\Gamma_{\theta,D} \cap \Gamma_{\theta,N} = \emptyset$, $\Gamma_{\theta,D} \cap \Gamma_{\theta,R} = \emptyset$, $\Gamma_{\theta,R} \cap \Gamma_{\theta,N} = \emptyset$,
- $\mathrm{meas}(\Gamma_{\mathbf{u},D}) > 0$ and $\mathrm{meas}(\Gamma_{\theta,D} \cup \Gamma_{\theta,R}) > 0$.

2.2 Problem (P)

The aim of this work is to obtain regularity properties with respect to time and space of the displacement and temperature fields, which are the solution to the following problem:
Problem (P)
Find $\mathbf{u}(p,t)$ and $\theta(p,t)$ in $\Omega \times (0, t_f]$, satisfying

$$-\mathrm{Div}\, \boldsymbol{\sigma}(\theta, \mathbf{u}) = \mathbf{b} \quad \text{in} \quad \Omega \times (0, t_f], \tag{2.1}$$

$$\rho_0 c_F \partial_t \theta = -3\theta_r \alpha K\, \mathrm{Div}\, \partial_t \mathbf{u} + \mathrm{Div}\, (k \nabla \theta) + f \quad \text{in} \quad \Omega \times (0, t_f], \tag{2.2}$$

$$\boldsymbol{\sigma}(\theta, \mathbf{u}) = \Lambda^{-1} : \boldsymbol{\varepsilon}(\mathbf{u}) - 3\alpha(\theta - \theta_r) K \mathbf{I} \quad \text{in} \quad \Omega \times (0, t_f], \tag{2.3}$$

$$\mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma_{\mathbf{u},D} \times (0, t_f], \tag{2.4}$$

$$\boldsymbol{\sigma}(\theta, \mathbf{u})\, \mathbf{n} = \mathbf{g} \quad \text{on} \quad \Gamma_{\mathbf{u},N} \times (0, t_f], \tag{2.5}$$

$$k \nabla \theta \cdot \mathbf{n} = \alpha_c(\theta^e - \theta) \quad \text{on} \quad \Gamma_{\theta,R} \times (0, t_f], \tag{2.6}$$

$$k \nabla \theta \cdot \mathbf{n} = h \quad \text{on} \quad \Gamma_{\theta,N} \times (0, t_f], \tag{2.7}$$

$$\theta = \theta_D \quad \text{on} \quad \Gamma_{\theta,D} \times (0, t_f], \tag{2.8}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \theta(0) = \theta_0 \quad \text{in} \quad \Omega. \tag{2.9}$$

Here,

- $\boldsymbol{\sigma}(\theta, \mathbf{u})$ is the stress tensor given by the thermoelastic behaviour law (2.3). In this law, Λ^{-1} is the elasticity tensor defined as

$$\Lambda^{-1} : \boldsymbol{\tau} = \lambda \mathrm{tr}(\boldsymbol{\tau}) \mathbf{I} + 2\mu \boldsymbol{\tau}, \quad \forall \boldsymbol{\tau} \in S_3, \tag{2.10}$$

where λ, μ are the Lamé's parameters, $\mathbf{I}$ is the identity tensor, $\boldsymbol{\varepsilon}(\mathbf{u})$ denotes the linearized deformation tensor, α is the coefficient of thermal expansion, θ_r is the reference temperature and K is the bulk modulus of the material

$$K = \frac{1}{3}(3\lambda + 2\mu).$$

- $\mathbf{b}$ is the body force per unit volume at the reference configuration.
- ρ_0 is the reference density.
- c_F is the specific heat at constant deformation.
- k is the thermal conductivity of the material.
- f is the body heating per unit volume at the reference configuration.
- $\mathbf{u}_D$ is the displacement on the Dirichlet mechanical boundary $\Gamma_{\mathbf{u},D}$.
- $\mathbf{n}$ is the outward unit vector normal to the boundary of Ω.
- $\mathbf{g}$ is the density of surface forces on the Neumann mechanical boundary $\Gamma_{\mathbf{u},N}$.
- α_c is the coefficient of convective heat transfer on $\Gamma_{\theta,R}$.
- θ^e is the external convection temperature on $\Gamma_{\theta,R}$.
- h is the heat flux on the Neumann thermal boundary $\Gamma_{\theta,N}$.
- θ_D is the temperature on the Dirichlet thermal boundary $\Gamma_{\theta,D}$.
- $\mathbf{u}_0$ and θ_0 are the initial conditions, which must satisfy the following compatibility conditions:

$$\begin{cases} \boldsymbol{\sigma}(\theta_0, \mathbf{u}_0) = \Lambda^{-1} : \boldsymbol{\varepsilon}(\mathbf{u}_0) - 3\alpha(\theta_0 - \theta_r)K\mathbf{I} & \text{in} \quad \Omega, \\ -\operatorname{Div} \boldsymbol{\sigma}(\theta_0, \mathbf{u}_0) = \mathbf{b}(0) & \text{in} \quad \Omega, \\ \mathbf{u}_0 = \mathbf{u}_D(0) & \text{on} \quad \Gamma_{\mathbf{u},D}, \\ \boldsymbol{\sigma}(\theta_0, \mathbf{u}_0)\,\mathbf{n} = \mathbf{g}(0) & \text{on} \quad \Gamma_{\mathbf{u},N}, \\ \theta_0 = \theta_D(0) & \text{on} \quad \Gamma_{\theta,D}. \end{cases}$$

2.3 Existence and uniqueness of solution

Let us consider the following variational formulation of Problem (P):
Problem (VP)
Find $(\mathbf{u}(t), \theta(t)) \in \mathbf{H}^1(\Omega) \times H^1(\Omega)$ satisfying *a.e.* $t \in (0, t_f)$

$$\begin{cases} a(\mathbf{u}(t), \mathbf{v}) - m(\theta(t) - \theta_r, \mathbf{v}) = \int_{\Gamma_{\mathbf{u},N}} \mathbf{g}(t) \cdot \mathbf{v}\, d\Gamma + \int_\Omega \mathbf{b}(t) \cdot \mathbf{v}\, dp, \ \forall \mathbf{v} \in \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega), & (2.11a) \\ (\partial_t \theta(t), \phi)_2 + \kappa(\theta(t), \phi) + m(\phi, \partial_t \mathbf{u}(t)) + c(\theta(t), \phi) = \int_\Omega \frac{f(t)}{\theta_r} \phi\, dp + c(\theta^e(t), \phi) & \\ + \int_{\Gamma_{\theta,N}} \frac{h(t)}{\theta_r} \phi\, d\Gamma, \quad \forall \phi \in H^1_{0,\Gamma_{\theta,D}}(\Omega), & (2.11b) \end{cases}$$

the boundary conditions (2.4)–(2.8) and the initial conditions (2.9).
Here, the following notation is used:

- $\mathbf{H}^1(\Omega) = [H^1(\Omega)]^3$ and $\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)$ is the admissible displacement space defined as

$$\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \quad \mathbf{v}|_{\Gamma_{\mathbf{u},D}} = \mathbf{0}\}.$$

https://doi.org/10.1017/S0956792515000509 Published online by Cambridge University Press

- $H^1_{0,\Gamma_{\theta,D}}(\Omega)$ is the admissible temperature space given by

$$H^1_{0,\Gamma_{\theta,D}}(\Omega) = \{\phi \in H^1(\Omega) : \quad \phi|_{\Gamma_{\theta,D}} = 0\}.$$

- $a(\cdot,\cdot)$ is the bilinear form defined on $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$ by

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega (\Lambda^{-1} : \boldsymbol{\varepsilon}(\mathbf{u})) : \boldsymbol{\varepsilon}(\mathbf{v})\, dp. \tag{2.12}$$

- $m(\cdot,\cdot)$ is the bilinear form on $L^2(\Omega) \times \mathbf{H}^1(\Omega)$ defined as

$$m(\phi,\mathbf{v}) = \int_\Omega 3\phi\alpha K \mathbf{I} : \boldsymbol{\varepsilon}(\mathbf{v})\, dp. \tag{2.13}$$

- $(\cdot,\cdot)_2$ is the scalar product in $L^2(\Omega)$ given by

$$(\phi,\psi)_2 = \int_\Omega \frac{\rho_0 c_F}{\theta_r}\phi\psi\, dp. \tag{2.14}$$

- $\kappa(\cdot,\cdot)$ is the bilinear form defined on $H^1(\Omega) \times H^1(\Omega)$ by

$$\kappa(\phi,\psi) = \int_\Omega k\nabla\phi \cdot \nabla\left(\frac{\psi}{\theta_r}\right)\, dp.$$

- $c(\cdot,\cdot)$ is the bilinear form defined on $H^1(\Omega) \times H^1(\Omega)$ as

$$c(\phi,\psi) = \int_{\Gamma_{\theta,R}} \alpha_c \frac{\phi}{\theta_r}\psi\, d\Gamma. \tag{2.15}$$

Furthermore, from here on, we will write $\mathbf{L}^r(\Omega) = [L^r(\Omega)]^3$, $1 \leqslant r \leqslant \infty$.
Let us consider the following hypotheses:

(H1) The elasticity tensor $\Lambda^{-1} \in [\mathbf{L}^\infty(\Omega)]^4$ and there exists $a_{min} > 0$ such that

$$(\Lambda^{-1} : \boldsymbol{\tau}) : \boldsymbol{\tau} \geqslant a_{min}|\boldsymbol{\tau}|^2, \quad \forall \boldsymbol{\tau} \in S_3.$$

(H2) The reference temperature $\theta_r \in W^{1,\infty}(\Omega)$, and there exists $\theta_{r,min} > 0$ such that $\theta_r(p) \geqslant \theta_{r,min}$ in Ω.

(H3) The reference density $\rho_0 > 0$, the specific heat at constant deformation $c_F > 0$ and the coefficient of thermal expansion $\alpha > 0$.

(H4) The thermal conductivity coefficient $k \in W^{1,\infty}(\Omega)$, and there exists $k_{min} > 0$ such that $k(p) \geqslant k_{min}$ in Ω.

(H5) The body force $\mathbf{b} \in W^{2,2}(0,t_f;\mathbf{L}^2(\Omega))$.

(H6) The body heating $f \in W^{1,2}(0,t_f;L^2(\Omega))$.

(H7) $\mathbf{u}_D$ is the restriction to $\Gamma_{\mathbf{u},D} \times (0,t_f)$ of a function called $\bar{\mathbf{u}}_D$ such that $\bar{\mathbf{u}}_D \in W^{2,2}(0,t_f;\mathbf{H}^{\frac{1}{2}}(\Gamma))$.

https://doi.org/10.1017/S0956792515000509 Published online by Cambridge University Press

(H8) θ_D is the restriction to $\Gamma_{\theta,D} \times (0,t_f)$ of a function called $\bar{\theta}_D$ such that $\bar{\theta}_D \in W^{2,2}(0,t_f;H^{\frac{1}{2}}(\Gamma))$.

(H9) The surface forces $\mathbf{g} \in W^{2,2}(0,t_f;\mathbf{L}^2(\Gamma_{\mathbf{u},N}))$ and $h \in W^{1,2}(0,t_f;L^2(\Gamma_{\theta,N}))$.

(H10) The coefficient of convective heat transfer $\alpha_c \in L^\infty(\Gamma_{\theta,R})$, and there exists $\alpha_{c,min} > 0$ satisfying $\alpha_c(p) \geqslant \alpha_{c,min}$ *a.e.* on $\Gamma_{\theta,R}$.

(H11) The external convection temperature $\theta^e \in W^{1,2}(0,t_f;L^2(\Gamma_{\theta,R}))$.

(H12) The initial conditions $\mathbf{u}_0 \in \mathbf{H}^1(\Omega)$ and $\theta_0 \in H^1(\Omega)$.

(H13) The initial conditions $\mathbf{u}_0$ and θ_0 satisfy

$$a(\mathbf{u}_0,\mathbf{v}) - m(\theta_0 - \theta_r,\mathbf{v}) = \int_{\Gamma_{\mathbf{u},N}} \mathbf{g}(0)\cdot\mathbf{v}\,d\Gamma + \int_\Omega \mathbf{b}(0)\cdot\mathbf{v}\,dp,\ \forall\mathbf{v}\in\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega),$$
$$\mathbf{u}_0 = \mathbf{u}_D(0) \text{ on } \Gamma_{\mathbf{u},D}, \qquad \theta_0 = \theta_D(0) \text{ on } \Gamma_{\theta,D}.$$

In Barral *et al.* [4], the following result is proved:

Theorem 2.1 *Under assumptions* (H1)–(H13)*, there exists a unique solution* $(\mathbf{u},\theta)$ *to Problem* (VP) *such that*

$$\mathbf{u} \in L^\infty\big(0,t_f;\mathbf{H}^1(\Omega)\big),\quad \partial_t\mathbf{u} \in L^2\big(0,t_f;\mathbf{H}^1(\Omega)\big),\ \textit{and} \tag{2.16}$$
$$\theta \in L^\infty\big(0,t_f;H^1(\Omega)\big),\quad \partial_t\theta \in L^2\big(0,t_f;L^2(\Omega)\big). \tag{2.17}$$

3 Regularity of the weak solution with respect to space

In this section, we prove additional regularity properties with respect to space of the weak solution to Problem (VP). The proof is based on the methodology used by Kačur and Ženíšek [16], which consists in rewriting our coupled problem as two equations defined by means of two elliptic operators. Then, some results given by Nečas [21], Lions and Magenes [17] and Mizohata [19] can be applied in order to obtain the H^2_{Loc} regularity of the solution with respect to space.

Definition 3.1 *For* $\imath = (\imath_1,\imath_2,\imath_3)$*, a 3-tuple of non-negative integers,* $\mathscr{C}^r(\overline{\Omega})$ *denotes the vectorial space consisting of all functions* $\phi : \overline{\Omega} \mapsto \mathbb{R}$ *which, together with all their partial derivatives of orders* $|\imath| \leqslant r$*, are continuous on* $\overline{\Omega}$*.*

Definition 3.2 *If* $0 < \delta \leqslant 1$*, we define* $\mathscr{C}^{r,\delta}(\overline{\Omega})$ *as the subspace of* $\mathscr{C}^r(\overline{\Omega})$ *consisting of those functions* ϕ *for which, for* $0 \leqslant |\imath| \leqslant r$*, the partial derivative of order* $|\imath|$ *satisfies in* Ω *a Hölder condition of exponent* δ*.*

In order to prove the space regularity, we increase the regularity properties for some of the hypotheses imposed in the previous section. In particular,

$(\mathrm{H1})_s$ The elasticity tensor satisfies (H1) and $\Lambda^{-1} \in [\mathscr{C}^{0,1}(\overline{\Omega})]^{3^4} \cap [\mathscr{C}^1(\overline{\Omega})]^{3^4}$.

$(\mathrm{H2})_s$ The reference temperature satisfies (H2) and $\theta_r \in \mathscr{C}^{0,1}(\overline{\Omega}) \cap \mathscr{C}^1(\overline{\Omega})$.

$(\mathrm{H4})_s$ The thermal conductivity coefficient satisfies (H4) and $k \in \mathscr{C}^{0,1}(\overline{\Omega}) \cap \mathscr{C}^1(\overline{\Omega})$.

$(\mathrm{H7})_s$ $\mathbf{u}_D$ satisfies (H7) with $\bar{\mathbf{u}}_D \in W^{2,2}(0,t_f;\mathbf{H}^{\frac{3}{2}}(\Gamma))$.

$(\mathrm{H8})_s$ θ_D satisfies (H8) with $\bar{\theta}_D \in W^{2,2}(0,t_f;H^{\frac{3}{2}}(\Gamma))$.

Remark 3.3 *Taking into account hypotheses* $(\mathrm{H1})_s$, $(\mathrm{H2})_s$ *and* $(\mathrm{H4})_s$, *the Lamé's parameters, the reference temperature and the thermal conductivity satisfy a Hölder condition of exponent 1, with Lipschitz constants* $a_{\lambda,h}$, $a_{\mu,h}$, $\theta_{r,h}$ *and* k_h, *respectively.*

Theorem 3.4 *Under assumptions* $(\mathrm{H1})_s$, $(\mathrm{H2})_s$, (H3), $(\mathrm{H4})_s$, (H5), (H6), $(\mathrm{H7})_s$, $(\mathrm{H8})_s$ *and* (H9)–(H13), *the solution to Problem (VP) satisfies*

$$\mathbf{u} \in L^\infty(0,t_f;\mathbf{H}^1(\Omega)\cap \mathbf{H}^2_{Loc}(\Omega)) \text{ and } \theta \in L^\infty(0,t_f;H^1(\Omega)\cap H^2_{Loc}(\Omega)). \tag{3.1}$$

Proof First, we introduce the change of variable by translation

$$\tilde{\mathbf{u}} = \mathbf{u} - \underline{\mathbf{u}}, \quad \tilde{\mathbf{u}}_0 = \mathbf{u}_0 - \underline{\mathbf{u}}(0), \quad \tilde{\theta} = \theta - \underline{\theta}, \quad \tilde{\theta}_0 = \theta_0 - \underline{\theta}(0), \tag{3.2}$$

in order to obtain a homogeneous Dirichlet problem. We notice that the existence of $\underline{\mathbf{u}}$ and $\underline{\theta}$ is guaranteed by assumptions $(\mathrm{H7})_s$, $(\mathrm{H8})_s$; furthermore, they satisfy (see Duvaut and Lions [8]),

$$\underline{\mathbf{u}} \in W^{2,2}(0,t_f;\mathbf{H}^2(\Omega)), \quad \underline{\mathbf{u}} = \mathbf{u}_D \text{ on } \Gamma_{\mathbf{u},D}\times(0,t_f], \tag{3.3}$$

$$\underline{\theta} \in W^{2,2}(0,t_f;H^2(\Omega)), \quad \underline{\theta} = \theta_D \text{ on } \Gamma_{\theta,D}\times(0,t_f]. \tag{3.4}$$

With respect to these new unknowns, we introduce the following problem:
Problem $(\widetilde{VP})$
Find $(\tilde{\mathbf{u}}(t),\tilde{\theta}(t)) \in \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)\times H^1_{0,\Gamma_{\theta,D}}(\Omega)$ satisfying *a.e.* $t\in(0,t_f)$

$$\begin{cases} a(\tilde{\mathbf{u}}(t),\mathbf{v}) - m(\tilde{\theta}(t),\mathbf{v}) = \displaystyle\int_{\Gamma_{\mathbf{u},N}} \mathbf{g}(t)\cdot\mathbf{v}\,d\Gamma + \int_\Omega \mathbf{b}(t)\cdot\mathbf{v}\,dp - a(\underline{\mathbf{u}}(t),\mathbf{v}) \\ +m(\underline{\theta}(t)-\theta_r,\mathbf{v}), \ \forall \mathbf{v}\in\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega), \qquad\qquad (3.5a) \\ (\partial_t\tilde{\theta}(t),\phi)_2 + \kappa(\tilde{\theta}(t),\phi) + m(\phi,\partial_t\tilde{\mathbf{u}}(t)) + c(\tilde{\theta}(t),\phi) = \displaystyle\int_\Omega \frac{f(t)}{\theta_r}\phi\,dp + c(\theta^e(t),\phi) \\ + \displaystyle\int_{\Gamma_{\theta,N}} \frac{h(t)}{\theta_r}\phi\,d\Gamma - (\partial_t\underline{\theta}(t),\phi)_2 - \kappa(\underline{\theta}(t),\phi) - m(\phi,\partial_t\underline{\mathbf{u}}(t)) - c(\underline{\theta}(t),\phi), \\ \qquad \forall\phi\in H^1_{0,\Gamma_{\theta,D}}(\Omega), \qquad\qquad (3.5b) \end{cases}$$

with the initial conditions

$$\tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}_0, \quad \tilde{\theta}(0) = \tilde{\theta}_0. \tag{3.6}$$

https://doi.org/10.1017/S0956792515000509 Published online by Cambridge University Press

Since Problems (VP) and $(\widetilde{VP})$ are equivalent, from Theorem 2.1, we get the existence and uniqueness of $\tilde{\mathbf{u}}$ and $\tilde{\theta}$ solution to Problem $(\widetilde{VP})$ such that

$$\begin{aligned} &\tilde{\mathbf{u}} \in L^\infty(0,t_f;\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)), \quad \partial_t\tilde{\mathbf{u}} \in L^2(0,t_f;\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)), \text{ and} \\ &\tilde{\theta} \in L^\infty(0,t_f;H^1_{0,\Gamma_{\theta,D}}(\Omega)), \quad \partial_t\tilde{\theta} \in L^2(0,t_f;L^2(\Omega)). \end{aligned} \tag{3.7}$$

Taking into account equations (3.5a), (3.5b) of Problem $(\widetilde{VP})$ and applying a Green's formula, we can deduce that its solution $(\tilde{\mathbf{u}}(t),\tilde{\theta}(t)) \in \mathbf{H}^1(\Omega)\times H^1(\Omega)$ is a weak solution in the sense of distributions of the equations

$$A_{\mathbf{u}}\tilde{\mathbf{u}}(t) = \mathbf{b}_{\mathbf{u}}(t), \quad A_\theta\tilde{\theta}(t) = f_\theta(t), \tag{3.8}$$

where

$$\begin{cases} A_{\mathbf{u}}\tilde{\mathbf{u}}(t) = -\mathrm{Div}\Big(\Lambda^{-1} : \boldsymbol{\varepsilon}(\tilde{\mathbf{u}}(t))\Big), \quad A_\theta\tilde{\theta}(t) = \mathrm{Div}\Big(\dfrac{k}{\theta_r}\nabla\tilde{\theta}(t)\Big), \\ \mathbf{b}_{\mathbf{u}}(t) = -\mathrm{Div}\Big(3\alpha\tilde{\theta}(t)K\mathbf{I}\Big) + \mathbf{b}(t) + \mathrm{Div}\Big(\Lambda^{-1} : \boldsymbol{\varepsilon}(\underline{\mathbf{u}}(t))\Big) - \mathrm{Div}\big(3\alpha(\underline{\theta}(t)-\theta_r)K\mathbf{I}\big), \\ f_\theta(t) = -\dfrac{\rho_0 c_F}{\theta_r}\partial_t\tilde{\theta}(t) - 3\alpha K\mathbf{I} : \boldsymbol{\varepsilon}(\partial_t\tilde{\mathbf{u}}(t)) - k\nabla\tilde{\theta}(t)\cdot\dfrac{\nabla\theta_r}{\theta_r^2} + \dfrac{f(t)}{\theta_r} - \dfrac{\rho_0 c_F\partial_t\underline{\theta}(t)}{\theta_r} \\ \qquad +\mathrm{Div}\left(\dfrac{k}{\theta_r}\nabla\underline{\theta}(t)\right) - k\nabla\underline{\theta}(t)\cdot\dfrac{\nabla\theta_r}{\theta_r^2} - 3\alpha K\mathbf{I} : \boldsymbol{\varepsilon}(\partial_t\underline{\mathbf{u}}(t)). \end{cases}$$

Theorem 1.1 of Chapter 4 given in Nečas [21], guarantees the H^2_{Loc} regularity of the solution to equations (3.8), due to the following properties:

- The operator $A_{\mathbf{u}}$ is $\mathbf{H}^1_0(\Omega)$-elliptic with coefficients of $\mathscr{C}^{0,1}(\overline{\Omega})$ thanks to hypothesis $(\mathrm{H1})_s$.
- The operator A_θ is also $H^1_0(\Omega)$-elliptic with coefficients of $\mathscr{C}^{0,1}(\overline{\Omega})$ taking into account Remark 3.3 and assumptions $(\mathrm{H2})_s$ and $(\mathrm{H4})_s$.
- $\mathbf{b}_{\mathbf{u}}(t) \in \mathbf{L}^2(\Omega)$ *a.e.* $t \in (0,t_f)$ from hypotheses $(\mathrm{H1})_s$, $(\mathrm{H2})_s$, (H3), $(\mathrm{H4})_s$, (H5), (H6), $(\mathrm{H7})_s$, $(\mathrm{H8})_s$ and (H9)–(H11).
- $f_\theta(t) \in L^2(\Omega)$ *a.e.* $t \in (0,t_f)$ thanks to assumptions $(\mathrm{H1})_s$, $(\mathrm{H2})_s$, (H3), $(\mathrm{H4})_s$, (H5), (H6), $(\mathrm{H7})_s$, $(\mathrm{H8})_s$ and (H9)–(H11).

Therefore, $(\tilde{\mathbf{u}}(t),\tilde{\theta}(t))$ is the weak solution in the sense of distributions to problem (3.8), satisfying *a.e.* $t \in (0,t_f)$

$$(\tilde{\mathbf{u}}(t),\tilde{\theta}(t)) \in \mathbf{H}^2_{Loc}(\Omega)\times H^2_{Loc}(\Omega).$$

Finally, from equalities (3.2)–(3.4), we can conclude the regularity properties (3.1). □

4 Regularity with respect to time

The aim of this section is to prove the $W^{r,\infty}$ regularity of the solution to Problem (VP) (see equations (2.11a), (2.11b)) with respect to time for $r \in \{0\}\cup\mathbb{N}$. The main difficulty is to establish the assumptions of regularity at the initial instant. In the first subsection, we study the regularity in time of the solution when the smooth properties of the data and the solution at the initial instant are increased. In the following subsection, we analyse the regularity with respect to time of the solution to the associated homogeneous Dirichlet problem; in this case, we propose to improve the smooth properties in space of the initial data of the problem instead of increasing the regularity properties of the solution at the initial instant.

4.1 Regularity of the weak solution with respect to time

Let us generalize assumptions (H1), (H5)–(H9) and (H11)–(H13) as follows:

$(\mathrm{H1})_{\mathrm{t}}$ The elasticity tensor satisfies (H1) and $\Lambda^{-1} \in [\mathbf{W}^{1,\infty}(\Omega)]^4$.

$(\mathrm{H5})_{\mathrm{t}}$ The body force $\mathbf{b} \in W^{r+2,2}(0, t_f; \mathbf{L}^2(\Omega))$.

$(\mathrm{H6})_{\mathrm{t}}$ The body heating $f \in W^{r+1,2}(0, t_f; L^2(\Omega))$.

$(\mathrm{H7})_{\mathrm{t}}$ $\mathbf{u}_D$ satisfies (H7) with $\bar{\mathbf{u}}_D \in W^{r+2,2}(0, t_f; \mathbf{H}^{\frac{1}{2}}(\Gamma))$.

$(\mathrm{H8})_{\mathrm{t}}$ θ_D satisfies (H8) with $\bar{\theta}_D \in W^{r+2,2}(0, t_f; H^{\frac{1}{2}}(\Gamma))$.

$(\mathrm{H9})_{\mathrm{t}}$ The surface forces $\mathbf{g} \in W^{r+2,2}(0, t_f; \mathbf{L}^2(\Gamma_{\mathbf{u},N}))$ and $h \in W^{r+1,2}(0, t_f; L^2(\Gamma_{\theta,N}))$.

$(\mathrm{H11})_{\mathrm{t}}$ The external convection temperature $\theta^e \in W^{r+1,2}(0, t_f; L^2(\Gamma_{\theta,R}))$.

$(\mathrm{H12})_{\mathrm{t}}$ The displacements and temperature satisfy at time $t = 0$

$$\partial_t^l \mathbf{u}(0) \in \mathbf{H}^1(\Omega) \text{ and } \partial_t^l \theta(0) \in H^1(\Omega), \quad 0 \leqslant l \leqslant r.$$

$(\mathrm{H13})_{\mathrm{t}}$ The displacements and temperature satisfy at time $t = 0$ for all $0 \leqslant l \leqslant r$

$$a(\partial_t^l \mathbf{u}(0), \mathbf{v}) - m(\partial_t^l \theta(0) - \partial_t^l \theta_r, \mathbf{v}) = \int_{\Gamma_{\mathbf{u},N}} \partial_t^l \mathbf{g}(0) \cdot \mathbf{v}\, d\Gamma + \int_\Omega \partial_t^l \mathbf{b}(0) \cdot \mathbf{v}\, dp, \ \forall \mathbf{v} \in \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega),$$
$$\partial_t^l \mathbf{u}(0) = \partial_t^l \mathbf{u}_D(0) \text{ on } \Gamma_{\mathbf{u},D}, \qquad \partial_t^l \theta(0) = \partial_t^l \theta_D(0) \text{ on } \Gamma_{\theta,D}.$$

Furthermore, for $0 < l \leqslant r$,

$$(\partial_t^l \theta(0), \phi)_2 + \kappa(\partial_t^{l-1} \theta(0), \phi) + m(\phi, \partial_t^l \mathbf{u}(0)) + c(\partial_t^{l-1} \theta(0), \phi)$$
$$= \int_\Omega \frac{\partial_t^{l-1} f(0)}{\theta_r} \phi\, dp + c(\partial_t^{l-1} \theta^e(0), \phi) + \int_{\Gamma_{\theta,N}} \frac{\partial_t^{l-1} h(0)}{\theta_r} \phi\, d\Gamma, \quad \forall \phi \in H^1_{0,\Gamma_{\theta,D}}(\Omega).$$

Remark 4.1 *We notice that in hypothesis* $(\mathrm{H13})_{\mathrm{t}}$ *the term* $\partial_t^l \theta_r$ *of the first member of the first equality is only not null when* $l = 0$.

Theorem 4.2 *Let* $r \in \{0\} \cup \mathbb{N}$ *be a fixed parameter. Under assumptions* $(\mathrm{H1})_{\mathrm{t}}$, (H2)–(H4), $(\mathrm{H5})_{\mathrm{t}}$–$(\mathrm{H9})_{\mathrm{t}}$, (H10) *and* $(\mathrm{H11})_{\mathrm{t}}$–$(\mathrm{H13})_{\mathrm{t}}$, *the solution to Problem* (VP) *satisfies*

$$\mathbf{u} \in W^{r,\infty}(0, t_f; \mathbf{H}^1(\Omega)), \quad \partial_t^{r+1} \mathbf{u} \in L^2(0, t_f; \mathbf{H}^1(\Omega)) \text{ and} \tag{4.1}$$
$$\theta \in W^{r,\infty}(0, t_f; H^1(\Omega)), \quad \partial_t^{r+1} \theta \in L^2(0, t_f; L^2(\Omega)). \tag{4.2}$$

https://doi.org/10.1017/S0956792515000509 Published online by Cambridge University Press

Proof We prove this result using the methodology of mathematical induction. For this purpose, we show the induction from $r = 0$ to $r = 1$ and the induction from r to $r + 1$ runs in the same way. The proof is divided into two steps following the scheme:

- *Step* $r = 0$. This is directly deduced from Theorem 2.1.
- *Step* $r = 1$. In order to obtain the regularity of the first derivative with respect to time, we define an auxiliary problem, where the second members are the derivatives in time of the Problem (VP). We will prove that this problem satisfies the assumptions of Theorem 2.1, and we will show that its unique solution is the derivative in time of the solution to Problem (VP).

Auxiliary problem.

If we formally differentiate the second member of equations (2.11*a*) and (2.11*b*) of Problem (VP) with respect to time, we can define the following problem:

Problem $(\widehat{VP})_t$

Find $(\widehat{\mathbf{u}}(t), \widehat{\theta}(t)) \in \mathbf{H}^1(\Omega) \times H^1(\Omega)$ satisfying *a.e.* $t \in (0, t_f)$

$$\begin{cases} a(\widehat{\mathbf{u}}(t), \mathbf{v}) - m(\widehat{\theta}(t) - \theta_r, \mathbf{v}) = \displaystyle\int_{\Gamma_{\mathbf{u},N}} \left[\partial_t \mathbf{g}(t) + \left(3\alpha\theta_r K \mathbf{I}\right)\mathbf{n}\right] \cdot \mathbf{v}\, d\Gamma \\ + \displaystyle\int_\Omega \left[\partial_t \mathbf{b}(t) - 3\alpha\nabla\left(\theta_r K\right)\right] \cdot \mathbf{v}\, dp, \quad \forall \mathbf{v} \in \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega), & (4.3a) \\ \widehat{\mathbf{u}}(t) = \partial_t \mathbf{u}_D(t) \text{ on } \Gamma_{\mathbf{u},D}, & (4.3b) \\ (\partial_t \widehat{\theta}(t), \phi)_2 + \kappa(\widehat{\theta}(t), \phi) + m(\phi, \partial_t \widehat{\mathbf{u}}(t)) + c(\widehat{\theta}(t), \phi) = \displaystyle\int_\Omega \frac{\partial_t f(t)}{\theta_r} \phi\, dp \\ + c(\partial_t \theta^e(t), \phi) + \displaystyle\int_{\Gamma_{\theta,N}} \frac{\partial_t h(t)}{\theta_r} \phi\, d\Gamma, \quad \forall \phi \in H^1_{0,\Gamma_{\theta,D}}(\Omega), & (4.3c) \\ \widehat{\theta}(t) = \partial_t \theta_D(t) \text{ on } \Gamma_{\theta,D}, & (4.3d) \end{cases}$$

and the initial conditions $\widehat{\mathbf{u}}(0) = \widehat{\mathbf{u}}_0$, $\widehat{\theta}(0) = \widehat{\theta}_0$ in Ω, where these initial conditions are defined as $\widehat{\mathbf{u}}_0 = \partial_t \mathbf{u}(0)$ and $\widehat{\theta}_0 = \partial_t \theta(0)$, which satisfy assumption (H12) of Theorem 2.1 thanks to hypothesis $(\mathrm{H12})_\mathrm{t}$.

Existence and uniqueness of solution for Problem $(\widehat{VP})_t$.

Next, we prove that the data of Problem $(\widehat{VP})_t$ satisfy the assumptions of Theorem 2.1. Indeed,

- Taking into account hypotheses $(\mathrm{H1})_\mathrm{t}$, (H2), (H3) and $(\mathrm{H5})_\mathrm{t}$, we deduce that the body force associated to Problem $(\widehat{VP})_t$ satisfies

$$\partial_t \mathbf{b}(t) - 3\alpha\nabla\left(\theta_r K\right) \in W^{2,2}(0, t_f; \mathbf{L}^2(\Omega)).$$

- Thanks to assumption $(\mathrm{H6})_\mathrm{t}$, the body heating for Problem $(\widehat{VP})_t$ satisfies

$$\partial_t f \in W^{1,2}(0, t_f; L^2(\Omega)).$$

- Considering hypotheses $(\mathrm{H1})_\mathrm{t}$, (H2), (H3) and $(\mathrm{H9})_\mathrm{t}$, the density of surface forces associated to Problem $(\widehat{VP})_t$ satisfies hypothesis (H9):

$$\partial_t \mathbf{g}(t) + \big(3\alpha\theta_r K\mathbf{I}\big)\mathbf{n} \in W^{2,2}(0,t_f;\mathbf{L}^2(\Gamma_{\mathbf{u},N})), \text{ and } \partial_t h(t) \in W^{1,2}(0,t_f;L^2(\Gamma_{\theta,N})).$$

- Under hypothesis $(\mathrm{H11})_\mathrm{t}$, the external convection temperature for Problem $(\widehat{VP})_t$ satisfies

$$\partial_t \theta^e(t) \in W^{1,2}(0,t_f;L^2(\Gamma_{\theta,R})).$$

- Finally, thanks to hypotheses $(\mathrm{H12})_\mathrm{t}$ and $(\mathrm{H13})_\mathrm{t}$ for $r=1$, the initial conditions $\widehat{\mathbf{u}}_0$ and $\widehat{\theta}_0$ satisfy hypotheses (H12) and (H13) of Theorem 2.1

$$a(\widehat{\mathbf{u}}_0,\mathbf{v}) - m(\widehat{\theta}_0 - \theta_r,\mathbf{v}) = \int_{\Gamma_{\mathbf{u},N}} \Big[\partial_t \mathbf{g}(0) + \big(3\alpha\theta_r K\mathbf{I}\big)\mathbf{n}\Big]\cdot\mathbf{v}\,d\Gamma$$
$$+ \int_\Omega \big(\partial_t \mathbf{b}(0) - \mathrm{Div}\big(3\theta_r\alpha K\mathbf{I}\big)\big)\cdot\mathbf{v}\,dp, \text{ for all } \mathbf{v}\in\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega),$$
$$\widehat{\mathbf{u}}_0 = \partial_t \mathbf{u}_D(0) \text{ on } \Gamma_{\mathbf{u},D}, \quad \widehat{\theta}_0 = \partial_t\theta_D(0) \text{ on } \Gamma_{\theta,D}.$$

Therefore, we can deduce the existence of a unique solution $(\widehat{\mathbf{u}},\widehat{\theta})$ to Problem $(\widehat{VP})_t$ such that

$$\widehat{\mathbf{u}} \in W^{0,\infty}(0,t_f;\mathbf{H}^1(\Omega)), \quad \partial_t\widehat{\mathbf{u}} \in L^2(0,t_f;\mathbf{H}^1(\Omega)) \text{ and} \tag{4.4}$$
$$\widehat{\theta} \in W^{0,\infty}(0,t_f;H^1(\Omega)), \quad \partial_t\widehat{\theta} \in L^2(0,t_f;L^2(\Omega)). \tag{4.5}$$

The solution to Problem $(\widehat{VP})_t$ *is the derivative in time of the solution to Problem* (VP).

Let us introduce the helpful functions

$$\mathbf{w}(t) = \mathbf{u}_0 + \int_0^t \widehat{\mathbf{u}}(s)ds \quad \text{and} \quad \Theta(t) = \theta_0 + \int_0^t \widehat{\theta}(s)ds. \tag{4.6}$$

From the regularity properties (4.4) and (4.5), we deduce that

$$\mathbf{w} \in W^{1,\infty}(0,t_f;\mathbf{H}^1(\Omega)), \quad \partial_t^2\mathbf{w} \in L^2(0,t_f;\mathbf{H}^1(\Omega)) \text{ with } \mathbf{w}(0) = \mathbf{u}_0,$$

and

$$\Theta \in W^{1,\infty}(0,t_f;H^1(\Omega)), \quad \partial_t^2\Theta \in L^2(0,t_f;L^2(\Omega)) \text{ with } \Theta(0) = \theta_0.$$

Therefore, if we integrate the equations of Problem $(\widehat{VP})_t$ over $(0,t)$ and we apply a Green's formula to term $m(\theta_r,\mathbf{v})$ of equation (4.3a), we arrive at

$$\int_0^t a(\widehat{\mathbf{u}}(s),\mathbf{v})ds - \int_0^t m(\widehat{\theta}(s),\mathbf{v})ds = \int_\Omega \big(\mathbf{b}(t)-\mathbf{b}(0)\big)\cdot\mathbf{v}\,dp + \int_{\Gamma_{\mathbf{u},N}} (\mathbf{g}(t)-\mathbf{g}(0))\cdot\mathbf{v}\,d\Gamma, \quad \forall \mathbf{v}\in \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega), \tag{4.7}$$

$$(\widehat{\theta}(t)-\widehat{\theta}(0),\phi)_2 + \int_0^t \kappa(\widehat{\theta}(s),\phi)ds + m(\phi,\widehat{\mathbf{u}}(t)-\widehat{\mathbf{u}}(0)) + \int_0^t c(\widehat{\theta}(s),\phi)ds = \int_\Omega \frac{(f(t)-f(0))}{\theta_r}\phi\,dp + c(\theta^e(t)-\theta^e(0),\phi) + \int_{\Gamma_{\theta,N}} \frac{(h(t)-h(0))}{\theta_r}\phi d\Gamma, \quad \forall\phi\in H^1_{0,\Gamma_{\theta,D}}(\Omega)). \tag{4.8}$$

On the other hand, considering hypothesis $(\mathrm{H13})_\mathrm{t}$ for $l=0$ in displacements and for $l=1$ in temperature, we get

$$\int_\Omega \mathbf{b}(0)\cdot\mathbf{v}\,dp + \int_{\Gamma_{\mathbf{u},N}} \mathbf{g}(0)\cdot\mathbf{v}\,d\Gamma = a(\mathbf{u}_0,\mathbf{v}) - m(\theta_0-\theta_r,\mathbf{v}), \ \forall\mathbf{v}\in\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega),$$

$$(\widehat{\theta}_0,\phi)_2 = -\kappa(\theta_0,\phi) - m(\phi,\widehat{\mathbf{u}}_0) - c(\theta_0,\phi) + \int_\Omega \frac{f(0)}{\theta_r}\phi dp + c(\theta^e(0),\phi) + \int_{\Gamma_{\theta,N}} \frac{h(0)}{\theta_r}\phi\,d\Gamma, \quad \forall\phi\in H^1_{0,\Gamma_{\theta,D}}(\Omega).$$

Thus, if we replace the previous equalities in expressions (4.7), (4.8), we obtain

$$a(\mathbf{u}_0 + \int_0^t \widehat{\mathbf{u}}(s)ds,\mathbf{v}) - m(\theta_0 + \int_0^t \widehat{\theta}(s)ds - \theta_r,\mathbf{v}) = \int_\Omega \mathbf{b}(t)\cdot\mathbf{v}\,dp + \int_{\Gamma_{\mathbf{u},N}} \mathbf{g}(t)\cdot\mathbf{v}\,d\Gamma, \quad \forall\mathbf{v}\in\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega), \tag{4.9}$$

$$(\widehat{\theta}(t),\phi)_2 + \kappa(\theta_0 + \int_0^t \widehat{\theta}(s)ds,\phi) + m(\phi,\widehat{\mathbf{u}}(t)) + c\left(\theta_0 + \int_0^t \widehat{\theta}(s)ds,\phi\right) = \int_\Omega \frac{f(t)}{\theta_r}\phi\,dp + c(\theta^e(t),\phi) + \int_{\Gamma_{\theta,N}} \frac{h(t)}{\theta_r}\phi d\Gamma, \quad \forall\phi\in H^1_{0,\Gamma_{\theta,D}}(\Omega). \tag{4.10}$$

Therefore, $\mathbf{w}$ and Θ solve Problem (VP) (see equations (2.11a), (2.11b)). Since this problem has a unique solution, we can conclude that $\mathbf{w}(t)=\mathbf{u}(t)$ and $\Theta(t)=\theta(t)$. Furthermore, from (4.6) we deduce that $\widehat{\mathbf{u}}(t)=\partial_t\mathbf{u}(t)$ and $\widehat{\theta}(t)=\partial_t\theta(t)$.

Finally, from the regularity properties (4.4) and (4.5), we obtain (4.1), (4.2) for $r=1$.

□

To conclude this subsection, we summarize the regularity properties in time and space for the solution to Problem (VP) from Theorems 3.4 and 4.2. To do so, we replace $(\mathrm{H7})_\mathrm{t}$ and $(\mathrm{H8})_\mathrm{t}$ by the following hypotheses with $r\in\{0\}\cup\mathbb{N}$:

$(\mathrm{H7})_{\mathrm{st}}$ $\mathbf{u}_D$ satisfies (H7) with $\bar{\mathbf{u}}_D \in W^{r+2,2}(0, t_f; \mathbf{H}^{\frac{3}{2}}(\Gamma))$.
$(\mathrm{H8})_{\mathrm{st}}$ θ_D satisfies (H8) with $\bar{\theta}_D \in W^{r+2,2}(0, t_f; H^{\frac{3}{2}}(\Gamma))$.

Theorem 4.3 *Let* $r \in \{0\} \cup \mathbb{N}$ *be a fixed parameter. Under assumptions* $(\mathrm{H1})_s$, $(\mathrm{H2})_s$, (H3), $(\mathrm{H4})_s$, $(\mathrm{H5})_{\mathrm{t}}$, $(\mathrm{H6})_{\mathrm{t}}$, $(\mathrm{H7})_{\mathrm{st}}$, $(\mathrm{H8})_{\mathrm{st}}$, $(\mathrm{H9})_{\mathrm{t}}$, (H10) *and* $(\mathrm{H11})_{\mathrm{t}}$–$(\mathrm{H13})_{\mathrm{t}}$, *the solution* $(\mathbf{u}, \theta)$ *to Problem* (VP) *satisfies*

$$\mathbf{u} \in W^{r,\infty}(0, t_f; \mathbf{H}^1(\Omega) \cap \mathbf{H}^2_{Loc}(\Omega)), \quad \partial_t^{r+1}\mathbf{u} \in L^2(0, t_f; \mathbf{H}^1(\Omega)) \text{ and}$$
$$\theta \in W^{r,\infty}(0, t_f; H^1(\Omega) \cap H^2_{Loc}(\Omega)), \quad \partial_t^{r+1}\theta \in L^2(0, t_f; L^2(\Omega)).$$

Proof The proof is deduced directly from Theorems 3.4 and 4.2. □

4.2 Regularity of the Dirichlet problem with respect to time

In this subsection, we consider a particular case of Problem (P) with homogeneous Dirichlet boundary conditions in displacements and temperature. We are going to prove that if we replace hypotheses $(\mathrm{H12})_t$ and $(\mathrm{H13})_t$ on the initial data by others, we can also obtain the $W^{r,\infty}$ regularity in time for the Dirichlet case. For this purpose, we introduce the following results.

Definition 4.4 *Let* γ *be a non-negative scalar function defined in* Ω. *We define the operator* $\widehat{\Lambda^{-1}}$ *as the perturbation of the tensor* Λ^{-1} *given by*

$$\widehat{\Lambda^{-1}} : \boldsymbol{\tau} = \Lambda^{-1} : \boldsymbol{\tau} + \gamma \boldsymbol{\tau}_s, \quad \boldsymbol{\tau} \in S_3, \tag{4.11}$$

where $\boldsymbol{\tau}_s$ *denotes the spherical part of* $\boldsymbol{\tau}$.

Lemma 1 *Let us consider* $\mathrm{m} \in \mathbb{N}$ *a fixed parameter. We suppose that*

- *the elasticity tensor satisfies*

$$\Lambda^{-1} \in \left[\mathscr{C}^{0,1}(\overline{\Omega})\right]^{3^4} \text{ and, if } \mathrm{m} \geqslant 2, \ \Lambda^{-1} \in \left[\mathscr{C}^{2\mathrm{m}-1}(\overline{\Omega})\right]^{3^4}, \tag{4.12}$$

and there exists $a_{min} > 0$ *such that*

$$(\Lambda^{-1} : \boldsymbol{\tau}) : \boldsymbol{\tau} \geqslant a_{min}|\boldsymbol{\tau}|^2, \quad \forall \boldsymbol{\tau} \in S_3;$$

- *the body force* $\overline{\mathbf{b}} \in \mathbf{H}^{2(\mathrm{m}-1)}(\Omega)$;
- γ *is a non-negative scalar function such that*

$$\gamma \in \mathscr{C}^{0,1}(\overline{\Omega}) \text{ and, if } \mathrm{m} \geqslant 2, \quad \gamma \in \mathscr{C}^{2\mathrm{m}-1}(\overline{\Omega}). \tag{4.13}$$

Then, there exists a unique weak solution $\bar{\mathbf{u}} \in \mathbf{H}^1_0(\Omega) \cap \mathbf{H}^{2\mathrm{m}}(\Omega)$ *of the following equation:*

$$-\mathrm{Div}\left(\widehat{\Lambda^{-1}} : \boldsymbol{\varepsilon}(\bar{\mathbf{u}})\right) = \overline{\mathbf{b}} \ \text{ in } \ \Omega.$$

Proof Notice that, since $\varLambda^{-1}$ satisfies (4.12) and γ satisfies (4.13) then

$$\widehat{\varLambda^{-1}} \in \begin{cases} \left[\mathscr{C}^{0,1}(\overline{\Omega})\right]^{3^4} & \text{if } \mathrm{m}=1, \\ \left[\mathscr{C}^{0,1}(\overline{\Omega})\right]^{3^4} \cap \left[\mathscr{C}^{2\mathrm{m}-1}(\overline{\Omega})\right]^{3^4} & \text{if } \mathrm{m} \geqslant 2. \end{cases}$$

Indeed, from definition (4.11) the Lamé's parameters of $\widehat{\varLambda^{-1}}$ are μ and $\lambda + \frac{\gamma}{3}$, which belong to $\mathscr{C}^{0,1}(\overline{\Omega})$ if $\mathrm{m}=1$ and $\mathscr{C}^{0,1}(\overline{\Omega}) \cap \mathscr{C}^{2\mathrm{m}-1}(\overline{\Omega})$ if $\mathrm{m} \geqslant 2$.

Thus, the result is true for $\mathrm{m}=1$ thanks to Theorems 3.7.2, 2.4.10 and Lemma 3.2 of Chapter 5 of Nečas [21]. For $\mathrm{m} \geqslant 2$, the $\mathbf{H}^{2\mathrm{m}}$ regularity is obtained thanks to Theorem 10.5 of Agmon *et al.* [1]. □

From here on, let us denote by $r \in \mathbb{N}$ a fixed parameter. We define the following static problem:

Problem $(\bar{P}^r)$

Find $\bar{\mathbf{u}}^r$ in Ω, satisfying

$$-\mathrm{Div}\left(\widehat{\varLambda^{-1}} : \boldsymbol{\varepsilon}(\bar{\mathbf{u}}^r)\right) = \bar{\mathbf{b}}^r \quad \text{in} \quad \Omega, \tag{4.14}$$

$$\bar{\mathbf{u}}^r = \mathbf{0} \quad \text{on} \quad \Gamma, \tag{4.15}$$

where $\widehat{\varLambda^{-1}}$ is the perturbed operator defined in expression (4.11), with $\gamma = \dfrac{27\theta_r \alpha^2 K^2}{\rho_0 c_F}$ and

$$\bar{\mathbf{b}}^r = \partial_t^r \mathbf{b}(0) - \mathrm{Div}\left(\frac{3\alpha K}{\rho_0 c_F} \mathrm{Div}\,(k\nabla\bar{\theta}^{r-1})\mathbf{I}\right) - \mathrm{Div}\left(\frac{3\alpha K \,\partial_t^{r-1} f(0)}{\rho_0 c_F}\mathbf{I}\right). \tag{4.16}$$

Here, $\bar{\theta}^0 = \theta_0$ and for $r \geqslant 1$

$$\bar{\theta}^r = -\frac{3\theta_r \alpha K}{\rho_0 c_F} \mathrm{Div}\bar{\mathbf{u}}^r + \frac{\mathrm{Div}\,(k\nabla\bar{\theta}^{r-1})}{\rho_0 c_F} + \frac{\partial_t^{r-1} f(0)}{\rho_0 c_F} \quad \text{in} \quad \Omega. \tag{4.17}$$

Corollary 1 *Let* $1 \leqslant l \leqslant r$. *Under hypothesis* (H3) *and the following assumptions:*

($h1$) θ_r, λ *and* μ *are strictly positive functions in* $\mathscr{C}^{0,1}(\overline{\Omega}) \cap \mathscr{C}^{2r-1}(\overline{\Omega})$,

($h2$) $k \in H^{2r}(\Omega)$,

($h3$) $\theta_0 \in H^{2r+1}(\Omega)$,

($h4$) $\partial_t^l \mathbf{b}(0) \in \mathbf{H}^{2r-2l}(\Omega)$,

($h5$) $\partial_t^{l-1} f(0) \in H^{2r-2l+1}(\Omega)$,

there exists a unique weak solution $\bar{\mathbf{u}}^l \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^{2r-2l+2}(\Omega)$ *to each Problem* $(\bar{P}^l)$.

Proof This result is proved using the methodology of mathematical induction on the parameter l for a fixed $r \in \mathbb{N}$.

- If $r=1$ then $l=1$ and the result is deduced from the previous lemma taking $\mathrm{m}=1$. Indeed, thanks to assumptions (H3) and ($h1$)–($h5$) for $r=1$, the body force associated with Problem ($\bar{P}^1$), given by (4.16), belongs to $\mathbf{L}^2(\Omega)$. Furthermore, under hypotheses (H3) and ($h1$), we get

$$\gamma \in \mathscr{C}^1(\overline{\Omega}) \cap \mathscr{C}^{0,1}(\overline{\Omega}) \text{ and } \gamma(p) \geqslant \frac{27\theta_{r,min}\alpha^2 K^2(p)}{\rho_0 c_F} \geqslant 0, \quad \text{for all } p \in \Omega.$$

- If $r \geqslant 2$, we show the induction from $l = 1$ to $l = 2$ and the induction from l to $l+1$ runs in the same way. Notice that, at each step l, we can apply Lemma 1 for $\mathrm{m} = r - l + 1$.
  - *Step* $l = 1$. It is obtained directly from Lemma 1 for $\mathrm{m} = r$.
  - *Step* $l = 2$. In this case, the body force of Problem ($\bar{P}^2$) depends on $\bar{\theta}^1$ (see equation (4.17)) and, therefore, on $\bar{\mathbf{u}}^1$, solution to Problem ($\bar{P}^1$). Since from the previous step $\bar{\mathbf{u}}^1 \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^{2r}(\Omega)$ and $\bar{\theta}^1$ belongs to $H^{2r-1}(\Omega)$, we can deduce that $\bar{\mathbf{b}}^2 \in \mathbf{H}^{2r-4}(\Omega)$ (see equation (4.16)). Thus, the existence and uniqueness of $\bar{\mathbf{u}}^2 \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^{2r-2}(\Omega)$, solution to Problem ($\bar{P}^2$), is deduced from Lemma 1 with $\mathrm{m} = r - 1$.

□

Remark 4.5 *We notice that the result is also valid when θ_r, λ and μ are strictly positive functions in $W^{2r-1,\infty}(\Omega) \cap \mathscr{C}^{0,1}(\overline{\Omega})$.*

As we have stated, throughout this subsection we consider the following Dirichlet problem:
<u>Problem</u> (P_D)
Find $\mathbf{u}$ and θ in $\Omega \times (0, t_f]$, satisfying (2.1)–(2.3), the initial conditions (2.9) and

$$\mathbf{u} = \mathbf{0}, \quad \theta = 0 \ \text{ on } \ \Gamma \times (0, t_f].$$

Following the reasoning used in Barral *et al.* [4], we propose the following weak variational formulation:
<u>Problem</u> (VP_D)
Find $(\mathbf{u}(t), \theta(t)) \in \mathbf{H}_0^1(\Omega) \times H_0^1(\Omega)$, satisfying *a.e.* $t \in (0, t_f)$

$$\begin{cases} a(\mathbf{u}(t), \mathbf{v}) - m(\theta(t) - \theta_r, \mathbf{v}) = \displaystyle\int_\Omega \mathbf{b}(t) \cdot \mathbf{v}\, dp, \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), & (4.18a) \\ (\partial_t \theta(t), \phi)_2 + \kappa(\theta(t), \phi) + m(\phi, \partial_t \mathbf{u}(t)) = \displaystyle\int_\Omega \frac{f(t)}{\theta_r} \phi\, dp, \quad \forall \phi \in H_0^1(\Omega), & (4.18b) \end{cases}$$

and the initial conditions (2.9).
Under hypotheses (H1)–(H6), (H12), (H13), Theorem 2.1 implies the existence and uniqueness of solution $(\mathbf{u}, \theta)$ to Problem (VP_D), satisfying (2.16), (2.17), or equivalently (4.1), (4.2) for $r = 0$.
In the following, we modify some hypotheses to successfully treat this second result of regularity in time:

$(\mathrm{H1})_{\mathrm{t2}}$ The elasticity tensor satisfies (H1) and $\Lambda^{-1} \in \left[\mathscr{C}^{0,1}(\overline{\Omega})\right]^{3^4} \cap \left[\mathscr{C}^{2r-1}(\overline{\Omega})\right]^{3^4}$.

$(\mathrm{H2})_{\mathrm{t2}}$ The reference temperature satisfies (H2) and $\theta_r \in \mathscr{C}^{0,1}(\overline{\Omega}) \cap \mathscr{C}^{2r-1}(\overline{\Omega})$.

https://doi.org/10.1017/S0956792515000509 Published online by Cambridge University Press

$(\mathrm{H4})_{t2}$ The thermal conductivity coefficient satisfies (H4) and $k \in W^{2r,\infty}(\Omega)$.

$(\mathrm{H5})_{t2}$ The body force $\mathbf{b} \in W^{r+2,2}(0,t_f;\mathbf{L}^2(\Omega))$, and $\partial_t^l \mathbf{b}(0) \in \mathbf{H}^{2r-2l}(\Omega)$, $1 \leqslant l \leqslant r$.

$(\mathrm{H6})_{t2}$ The body heating $f \in W^{r+1,2}(0,t_f;L^2(\Omega))$, and $\partial_t^l f(0) \in H_0^{2r-(2l+1)}(\Omega)$, $0 \leqslant l \leqslant r-1$.

$(\mathrm{H12})_{t2}$ The initial conditions $\mathbf{u}_0 \in \mathbf{H}_0^1(\Omega)$ and $\theta_0 \in H_0^{2r+1}(\Omega)$.

$(\mathrm{H13})_{t2}$ The initial conditions $\mathbf{u}_0$ and θ_0 satisfy

$$a(\mathbf{u}_0,\mathbf{v}) - m(\theta_0-\theta_r,\mathbf{v}) = \int_\Omega \mathbf{b}(0)\cdot\mathbf{v}\,dp, \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega).$$

$(\mathrm{H14})_{t2}$. For $1 \leqslant l \leqslant r$, the solution $\bar{\mathbf{u}}^l$ to each Problem $(\bar{P}^l)$ satisfies

$$\operatorname{Div} \bar{\mathbf{u}}^l \in \mathbf{H}_0^{2r-2l+1}(\Omega).$$

Remark 4.6 *We notice that in hypotheses* $(\mathrm{H1})_{t2}$ *and* $(\mathrm{H2})_{t2}$, *it would be enough to consider* $\Lambda^{-1} \in [\mathscr{C}^{0,1}(\overline{\Omega})]^{3^4} \cap [\mathbf{W}^{2r-1,\infty}(\Omega)]^4$ *and* $\theta_r \in \mathscr{C}^{0,1}(\overline{\Omega}) \cap W^{2r-1,\infty}(\Omega)$.

Theorem 4.7 *Let* $r \in \mathbb{N}$ *be a fixed parameter. Under assumptions* $(\mathrm{H1})_{t2}$, $(\mathrm{H2})_{t2}$, (H3), $(\mathrm{H4})_{t2}$–$(\mathrm{H6})_{t2}$ *and* $(\mathrm{H12})_{t2}$–$(\mathrm{H14})_{t2}$, *the solution to Problem* (VP_D) *satisfies*

$$\mathbf{u} \in W^{r,\infty}(0,t_f;\mathbf{H}_0^1(\Omega)), \quad \partial_t^{r+1}\mathbf{u} \in L^2(0,t_f;\mathbf{H}_0^1(\Omega)) \text{ and}$$
$$\theta \in W^{r,\infty}(0,t_f;H_0^1(\Omega)), \quad \partial_t^{r+1}\theta \in L^2(0,t_f;L^2(\Omega)).$$

Proof The proof follows the scheme of Theorem 4.2. Therefore, we give the proof for $r = 1$.

Auxiliary problem.

Using formal derivation with respect to time in Problem (VP_D), we define the following problem:

<u>**Problem**</u> $(\widehat{VP}_D)_t$

Find $(\widehat{\mathbf{u}}(t),\widehat{\theta}(t)) \in \mathbf{H}_0^1(\Omega) \times H_0^1(\Omega)$, satisfying *a.e.* $t \in (0,t_f)$:

$$\begin{cases} a(\widehat{\mathbf{u}}(t),\mathbf{v}) - m(\widehat{\theta}(t)-\theta_r,\mathbf{v}) = \displaystyle\int_\Omega \Big[\partial_t \mathbf{b}(t) - 3\alpha\nabla\big(\theta_r K\big)\Big]\cdot\mathbf{v}\,dp, \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), & (4.19a)\\ (\partial_t\widehat{\theta}(t),\phi)_2 + \kappa(\widehat{\theta}(t),\phi) + m(\phi,\partial_t\widehat{\mathbf{u}}(t)) = \displaystyle\int_\Omega \frac{\partial_t f(t)}{\theta_r}\phi\,dp, \quad \forall \phi \in H_0^1(\Omega), & (4.19b)\end{cases}$$

and the initial conditions $\widehat{\mathbf{u}}(0) = \widehat{\mathbf{u}}_0$, $\widehat{\theta}(0) = \widehat{\theta}_0$ in Ω, where

$$\widehat{\mathbf{u}}_0 = \bar{\mathbf{u}}^1, \quad \widehat{\theta}_0 = \bar{\theta}^1. \tag{4.20}$$

Here, $\bar{\mathbf{u}}^1$ is the weak solution to Problem $(\bar{P}^1)$ and $\bar{\theta}^1$ is defined from equality (4.17). Notice that $\widehat{\mathbf{u}}_0$ and $\widehat{\theta}_0$ are defined to coincide formally with the derivatives with

https://doi.org/10.1017/S0956792515000509 Published online by Cambridge University Press

respect to time of $\mathbf{u}$ and θ at $t=0$. In effect, if we evaluate energy equation (2.2) at time $t=0$, we obtain

$$\partial_t\theta(0) = -\frac{3\theta_r\alpha K \operatorname{Div}\partial_t\mathbf{u}(0)}{\rho_0 c_F} + \frac{\operatorname{Div}(k\nabla\theta_0)}{\rho_0 c_F} + \frac{f(0)}{\rho_0 c_F} \quad \text{in} \quad \Omega.$$

In addition, if we formally differentiate motion equation (2.1) with respect to time, we consider $t=0$ and we replace the previous expression, we obtain equation (4.14) for $\bar{\mathbf{u}}^1$ playing the role of $\partial_t\mathbf{u}(0)$ with $\bar{\mathbf{b}}^1$ given by equality (4.16).

Lemma 2 *Under assumptions of Theorem* 4.7 *for* $r=1$, *the initial conditions* $\widehat{\mathbf{u}}_0$ *and* $\widehat{\theta}_0$, *given in* (4.20), *are well defined.*

Proof Under hypotheses $(\mathrm{H1})_{t2}$, $(\mathrm{H2})_{t2}$, (H3), $(\mathrm{H4})_{t2}$–$(\mathrm{H6})_{t2}$ and $(\mathrm{H12})_{t2}$ for $r=1$, the assumptions of Corollary 1 are true for $r=1$ and we can conclude that $\widehat{\mathbf{u}}_0 \in \mathbf{H}_0^1(\Omega)\cap\mathbf{H}^2(\Omega)$ is the unique solution to equation (4.14) for $r=1$. Therefore, $\widehat{\theta}_0 = \bar{\theta}^1$ can be defined from equality (4.17) and $\widehat{\theta}_0 \in H^1(\Omega)$. □

Existence and uniqueness of solution to Problem $(\widehat{VP_D})_t$.

In the following, we prove the existence and uniqueness of solution to Problem $(\widehat{VP_D})_t$ from Theorem 2.1; the main difficulty is to verify that the initial condition of this problem, $(\widehat{\mathbf{u}}_0, \widehat{\theta}_0)$, satisfies the required hypotheses. Indeed,

- Taking into account hypotheses $(\mathrm{H1})_{t2}$, $(\mathrm{H2})_{t2}$ and $(\mathrm{H5})_{t2}$, we easily deduce that the body force associated to Problem $(\widehat{VP_D})_t$ satisfies $\partial_t\mathbf{b} - 3\alpha\nabla\big(\theta_r K\big) \in W^{2,2}(0,t_f;\mathbf{L}^2(\Omega))$.
- In the same way, considering assumption $(\mathrm{H6})_{t2}$, we obtain $\partial_t f \in W^{1,2}(0,t_f;L^2(\Omega))$.
- From Lemma 2, $\widehat{\mathbf{u}}_0 \in \mathbf{H}_0^1(\Omega)\cap\mathbf{H}^2(\Omega)$ and $\widehat{\theta}_0 \in H^1(\Omega)$.
- Finally, it is necessary to check hypothesis (H13). From definition of $\widehat{\mathbf{u}}_0$, we can deduce that $(\widehat{\mathbf{u}}_0, \widehat{\theta}_0)$ satisfies the weak equality

$$a(\widehat{\mathbf{u}}_0,\mathbf{v}) - m(\widehat{\theta}_0 - \theta_r,\mathbf{v}) = \int_\Omega \big(\partial_t\mathbf{b}(0) - \operatorname{Div}\big(3\theta_r\alpha K\mathbf{I}\big)\big)\cdot\mathbf{v}\,dp, \tag{4.21}$$

for all $\mathbf{v}\in\mathbf{H}_0^1(\Omega)$. Indeed, since $\widehat{\mathbf{u}}_0$ is the weak solution to Problem $(\bar{P}^1)$, given by equation (4.14), considering the definition of $\bar{\mathbf{b}}^1$ (see (4.16)) and taking into account Definition 4.4 for $\gamma = \dfrac{27\theta_r\alpha^2K^2}{\rho_0 c_F}$,

$$-\int_\Omega \operatorname{Div}\left(\Lambda^{-1}:\boldsymbol{\varepsilon}(\widehat{\mathbf{u}}_0) - 3\alpha K\left[-\frac{3\theta_r\alpha K}{\rho_0 c_F}\operatorname{Div}(\widehat{\mathbf{u}}_0)\mathbf{I} + \frac{\operatorname{Div}(k\nabla\theta_0)\mathbf{I}}{\rho_0 c_F} + \frac{f(0)}{\rho_0 c_F}\mathbf{I}\right]\right)\cdot\mathbf{v}\,dp$$
$$= \int_\Omega \partial_t\mathbf{b}(0)\cdot\mathbf{v}\,dp, \text{ for all } \mathbf{v}\in\mathbf{H}_0^1(\Omega). \tag{4.22}$$

https://doi.org/10.1017/S0956792515000509 Published online by Cambridge University Press

Furthermore, considering expression (4.17) and taking into account that $\widehat{\theta}_0 = \bar{\theta}^1$, equation (4.22) can be rewritten as

$$-\int_\Omega \mathrm{Div}\left(\Lambda^{-1} : \boldsymbol{\varepsilon}(\widehat{\mathbf{u}}_0) - 3\alpha K(\widehat{\theta}_0 - \theta_r)\mathbf{I}\right) \cdot \mathbf{v}\, dp$$
$$= \int_\Omega \left[\partial_t \mathbf{b}(0) - \mathrm{Div}\left(3\alpha\theta_r K\mathbf{I}\right)\right] \cdot \mathbf{v}\, dp, \text{ for all } \mathbf{v} \in \mathbf{H}^1_0(\Omega),$$

which proves (4.21).

Then, by definition of Problem $(\bar{P}^1)$, $\widehat{\mathbf{u}}_0 = \mathbf{0}$ on Γ and thanks to $(\mathrm{H2})_{t2}$, (H3), $(\mathrm{H6})_{t2}$, $(\mathrm{H12})_{t2}$ and $(\mathrm{H14})_{t2}$, from equality (4.17) we deduce that $\widehat{\theta}_0 = 0$ on Γ.

Summing up, from Theorem 2.1, we can conclude the existence of a unique weak solution $(\widehat{\mathbf{u}}, \widehat{\theta})$ to Problem $(\widehat{VP_D})_t$ satisfying (2.16), (2.17).

The solution to Problem $(\widehat{VP_D})_t$ is the derivative in time of the solution to Problem (VP_D). Following the proof of Theorem 4.2, we introduce the helpful functions $\mathbf{w}(t)$ and $\Theta(t)$ defined in (4.6).

Let us integrate the equations of Problem $(\widehat{VP_D})_t$ over $(0, t)$, and apply a Green's formula to the term $m(\theta_r, \mathbf{v})$

$$\int_0^t a(\widehat{\mathbf{u}}(s), \mathbf{v})ds - \int_0^t m(\widehat{\theta}(s), \mathbf{v})ds = \int_\Omega \left(\mathbf{b}(t) - \mathbf{b}(0)\right) \cdot \mathbf{v}\, dp, \ \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega), \tag{4.23}$$

$$(\widehat{\theta}(t) - \widehat{\theta}(0), \phi)_2 + \int_0^t \kappa(\widehat{\theta}(s), \phi)ds + m(\phi, \widehat{\mathbf{u}}(t) - \widehat{\mathbf{u}}(0))$$
$$= \int_\Omega \frac{\left(f(t) - f(0)\right)}{\theta_r} \phi\, dp, \ \forall \phi \in H^1_0(\Omega). \tag{4.24}$$

On the other hand, considering $(\mathrm{H13})_{t2}$, we have

$$\int_\Omega \mathbf{b}(0) \cdot \mathbf{v}\, dp = a(\mathbf{u}_0, \mathbf{v}) - m(\theta_0 - \theta_r, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega).$$

In addition, due to expression of $\widehat{\theta}_0 = \bar{\theta}^1$ (see equation (4.17)), we can deduce

$$(\widehat{\theta}_0, \phi)_2 = -m(\phi, \widehat{\mathbf{u}}_0) - \kappa(\theta_0, \phi) + \int_\Omega \frac{f(0)}{\theta_r} \phi\, dp, \quad \forall \phi \in H^1_0(\Omega).$$

Thus, if we replace the previous equalities in expressions (4.23) and (4.24), we obtain

$$a(\mathbf{u}_0 + \int_0^t \widehat{\mathbf{u}}(s)ds, \mathbf{v}) - m(\theta_0 + \int_0^t \widehat{\theta}(s)ds - \theta_r, \mathbf{v}) = \int_\Omega \mathbf{b}(t) \cdot \mathbf{v}\, dp, \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega),$$
$$(\widehat{\theta}(t), \phi)_2 + \kappa(\theta_0 + \int_0^t \widehat{\theta}(s)ds, \phi) + m(\phi, \widehat{\mathbf{u}}(t)) = \int_\Omega \frac{f(t)}{\theta_r} \phi\, dp, \quad \forall \phi \in H^1_0(\Omega).$$

Therefore, $\mathbf{w}$ and Θ are solution to Problem (VP_D). An argument similar to that of Theorem 4.2 completes the proof.

□

https://doi.org/10.1017/S0956792515000509 Published online by Cambridge University Press

In order to conclude this part, we summarize the regularity properties in space and time for the solution to Problem (VP_D) given in Theorems 3.4 and 4.7. For that purpose, we replace $(H4)_{t2}$ by the following hypothesis:

$(H4)_{st2}$ The thermal conductivity coefficient satisfies (H4) and $k \in \mathscr{C}^{0,1}(\overline{\Omega}) \cap \mathscr{C}^{2r}(\overline{\Omega})$.

Theorem 4.8 *Let $r \in \mathbb{N}$ be a fixed parameter. Under assumptions* $(H1)_{t2}$, $(H2)_{t2}$, (H3), $(H4)_{st2}$, $(H5)_{t2}$, $(H6)_{t2}$, *and* $(H12)_{t2}$-$(H14)_{t2}$, *the solution* $(\mathbf{u}, \theta)$ *to Problem* (VP_D) *satisfies the following regularity properties:*

$$\mathbf{u} \in W^{r,\infty}(0, t_f; \mathbf{H}_0^1(\Omega) \cap \mathbf{H}_{Loc}^2(\Omega)), \quad \partial_t^{r+1}\mathbf{u} \in L^2(0, t_f; \mathbf{H}_0^1(\Omega)) \text{ and}$$
$$\theta \in W^{r,\infty}(0, t_f; H_0^1(\Omega) \cap H_{Loc}^2(\Omega)), \quad \partial_t^{r+1}\theta \in L^2(0, t_f; L^2(\Omega)).$$

Proof The proof is deduced directly from Theorems 3.4 and 4.7. □

5 Conclusions

In this paper, we have obtained regularity properties of the solution to a quasi-static fully-coupled linear thermoelastic problem for heterogeneous materials with mixed displacement-traction boundary conditions for the mechanical sub-model and mixed Dirichlet–Neumann–Robin for the thermal one.

Specifically, we have proved H_{Loc}^2 regularity in space for displacements and temperature assuming additional regularity on the data, and we have achieved $W^{r,\infty}$ regularity in time, $r \in \{0\} \cup \mathbb{N}$, assuming also more regularity of the solution at the initial instant. Furthermore, for the corresponding homogeneous Dirichlet problem, we have obtained the same regularity in time by increasing the smooth properties in space of the initial data without considering smoother properties of the solution at the initial instant.

Acknowledgements

This work was partially funded by CICYT-FEDER (MTM2008-05682/MTM and MTM2011-23976) and Xunta de Galicia-FEDER (INCITE09PXIB207093PR and GRC2013-014).

References

[1] Agmon, S., Douglis, A. & Nirenberg, L. (1964) Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II. *Commun. Pure Appl. Math.* **17**(1), 35–92.

[2] Barral, P., Naya-Riveiro, M. C. & Quintela, P. (2007) Mathematical analysis of a viscoelastic problem with temperature-dependent coefficients. I. Existence and uniqueness. *Math. Meth. Appl. Sci.* **30**(13), 1545–1568.

[3] Barral, P., Naya-Riveiro, M. C. & Quintela, P. (2007) Mathematical analysis of a viscoelastic problem with temperature-dependent coefficients. II. Regularity. *Math. Meth. Appl. Sci.* **30**(13), 1569–1592.

[4] Barral, P., Naya-Riveiro, M. C. & Quintela, P. (2015) Existence and uniqueness of a thermoelastic problem with variable parameters. *Eur. J. Appl. Math.* **26**(04), 497–520.

[5] Barral, P. & Quintela, P. (1999) A numerical method for simulation of thermal stresses during casting of aluminium slabs. *Comput. Meth. Appl. Mech. Eng.* **178**(1–2), 69–88.

[6] Bermúdez, A., Muñiz, M. C. & Quintela, P. (1993) Numerical solution of a three-dimensional thermoelectric problem taking place in an aluminum electrolytic cell. *Comput. Meth. Appl. Mech. Eng.* **106**(1–2), 129–142.

[7] Copetti, M. I. M. & Elliott, C. M. (1993) A one-dimensional quasi-static contact problem in linear thermoelasticity. *Eur. J. Appl. Math.* **4**(2), 151–174.

[8] Duvaut, G. & Lions, J. L. (1972) Inéquations en thermoélasticité et magnétohydrodynamique. *Arch. Ration. Mech. Anal.* **46**(4), 241–279.

[9] Gawinecki, J. (1981) Uniqueness and regularity of the solution of the first boundary-initial value problem for thermal stresses equations of classical and generalized thermomechanics. *Bull. Acad. Polon. Sci. Sér. Sci. Tech.* **29**(11–12), 231–238.

[10] Gawinecki, J. (1983) Existence, uniqueness and regularity of the first boundary-initial value problem for thermal stresses equations of classical and generalized thermomechanics. *J. Tech. Phys.* **24**(4), 467–479.

[11] Gawinecki, J. (1986) Existence, uniqueness and regularity of the solution of the first boundary-initial value problem for the equations of linear thermomicroelasticity. *Bull. Polish Acad. Sci. Tech. Sci.* **34**(7–8), 447–460.

[12] Gawinecki, J. (1987) Existence, uniqueness and regularity of the first boundary-initial value problem for hyperbolic equations system of the thermal stresses theory for temperature-rate-dependent solids. *Bull. Polish Acad. Sci. Tech. Sci.* **35**(7–8), 411–419.

[13] Gawinecki, J. (1987) The faedo-galerkin method in thermal stresses theory. *Comment. Math. Prace Mat.* **27**(1), 83–107.

[14] Gawinecki, J., Kowalski, T. & Litewska, K. (1982) Existence and uniqueness of the solution of the mixed boundary-initial value problem in linear thermoelasticity. *Bull. Acad. Polon. Sci. Sér. Sci. Tech.* **30**(11–12), 173–178.

[15] Jiang, S. & Racke, R. (2000) *Evolution Equations in Thermoelasticity*, Monographs and Surveys in Pure and Applied Mathematics, Chapman & Hall/CRC, Boca Raton, FL.

[16] Kačur, J. & Ženíšek, A. (1986) Analysis of approximate solutions of coupled dynamical thermoelasticity and related problems. *Apl. Mat.* **31**(3), 190–223.

[17] Lions, J. & Magenes, E. (1968) *Problèmes aux limites non-homogènes et applications*, Dunod, Paris.

[18] Marzocchi, A., Muñoz Rivera, J. E. & Naso, M. G. (2003) Transmission problem in thermoelasticity with symmetry. *IMA J. Appl. Math.* **68**(1), 23–46.

[19] Mizohata, S. (1973) *The Theory of Partial Differential Equations*, Cambridge University Press, New York.

[20] Muñoz Rivera, J. E. & Racke, R. (1998) Multidimensional contact problems in thermoelasticity. *SIAM J. Appl. Math.* **58**(4), 1307–1337.

[21] Nečas, J. (1967) *Les méthodes directes en théorie des équations elliptiques*, Masson et Cie, Éditeurs, Paris.

[22] Zhelezovskiĭ, S. E. (2010) On the high smoothness of the solution of an abstract hyperbolic-parabolic system of equations of thermoelasticity system type. *Differ. Equ.* **46**(6), 840–852.

[23] Zhelezovskiĭ, S. E. (2010) On the smoothness of the solution of an abstract coupled problem of thermoelasticity type. *Comput. Math. Math. Phys.* **50**(7), 1178–1194.