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Hypergraphs Do Jump

RAHIL BABER and JOHN TALBOT†

Department of Mathematics, UCL, London, WC1E 6BT, UK
(e-mail: rahilbaber@hotmail.com, talbot@math.ucl.ac.uk)

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We say that $\alpha \in [0, 1)$ is a *jump* for an integer $r \geqslant 2$ if there exists $c(\alpha) > 0$ such that for all $\epsilon > 0$ and all $t \geqslant 1$, any r-graph with $n \geqslant n_0(\alpha, \epsilon, t)$ vertices and density at least $\alpha + \epsilon$ contains a subgraph on t vertices of density at least $\alpha + c$.

The Erdős–Stone–Simonovits theorem [4, 5] implies that for $r = 2$, every $\alpha \in [0, 1)$ is a jump. Erdős [3] showed that for all $r \geqslant 3$, every $\alpha \in [0, r!/r^r)$ is a jump. Moreover he made his famous 'jumping constant conjecture', that for all $r \geqslant 3$, every $\alpha \in [0, 1)$ is a jump. Frankl and Rödl [7] disproved this conjecture by giving a sequence of values of non-jumps for all $r \geqslant 3$.

We use Razborov's flag algebra method [9] to show that jumps exist for $r = 3$ in the interval $[2/9, 1)$. These are the first examples of jumps for any $r \geqslant 3$ in the interval $[r!/r^r, 1)$. To be precise, we show that for $r = 3$ every $\alpha \in [0.2299, 0.2316)$ is a jump.

We also give an improved upper bound for the Turán density of $K_4^- = \{123, 124, 134\}$: $\pi(K_4^-) \leqslant 0.2871$. This in turn implies that for $r = 3$ every $\alpha \in [0.2871, 8/27)$ is a jump.

1. Introduction

An *r-uniform hypergraph* (or *r-graph* for short) is a pair $F = (V(F), E(F))$, where $V(F)$ is a set of *vertices* and $E(F)$ is a family of r-subsets of $V(F)$ called *edges*. So a 2-graph is a simple graph. For ease of notation we often identify an r-graph F with its edge set. The *density* of an r-graph F is

$$d(F) = \frac{|E(F)|}{\binom{n}{r}}.$$

We say that $\alpha \in [0, 1)$ is a *jump* for an integer $r \geqslant 2$ if there exists $c(\alpha) > 0$ such that for all $\epsilon > 0$ and all $t \geqslant 1$, there exists $n_0(\alpha, \epsilon, t)$ such that any r-graph with $n \geqslant n_0(\alpha, \epsilon, t)$ vertices and at least $(\alpha + \epsilon)\binom{n}{r}$ edges contains a subgraph on t vertices with at least $(\alpha + c)\binom{t}{r}$ edges.

The Erdős–Stone–Simonovits theorem [4, 5] implies that for $r = 2$, every $\alpha \in [0, 1)$ is a jump. Erdős [3] showed that for all $r \geqslant 3$, every $\alpha \in [0, r!/r^r)$ is a jump. He went on to make his

† This author is a Royal Society University Research Fellow.

famous 'jumping constant conjecture', that for all $r \geqslant 3$, every $\alpha \in [0,1)$ is a jump. Frankl and Rödl [7] disproved this conjecture by giving a sequence of values of non-jumps for all $r \geqslant 3$. More recently a number of authors have given more examples of non-jumps for each $r \geqslant 3$ in the interval $[5r!/2r^r, 1)$ (see [6] for example). However, nothing was previously known regarding the location of jumps or non-jumps in the interval $[r!/r^r, 5r!/2r^r)$ for any $r \geqslant 3$.

We give the first examples of jumps for any $r \geqslant 3$ in the interval $[r!/r^r, 1)$.

Theorem 1.1. *If $\alpha \in [0.2299, 0.2316)$ then α is a jump for $r = 3$.*

In order to explain our proof we require some definitions and a theorem of Frankl and Rödl [7]. Let F be an r-graph with vertex set $[n] = \{1, 2, \dots, n\}$ and edge set $E(F)$. Define

$$S_n = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1, x_i \geqslant 0 \right\}.$$

For $x \in S_n$ let

$$\lambda(F, x) = \sum_{\{i_1, i_2, \dots, i_r\} \in E(F)} r! x_{i_1} x_{i_2} \cdots x_{i_r}.$$

The *Lagrangian* of F is defined to be

$$\lambda(F) = \max_{x \in S_n} \lambda(F, x).$$

Given a family of r-graphs $\mathcal{F}$, we say that an r-graph H is *$\mathcal{F}$-free* if H does not contain a subgraph isomorphic to any member of $\mathcal{F}$. For any integer $n \geqslant 1$ we define the *Turán number* of $\mathcal{F}$ to be

$$\mathrm{ex}(n, \mathcal{F}) = \max\{|E(H)| : H \text{ is } \mathcal{F}\text{-free, } |V(H)| = n\}.$$

The *Turán density* of $\mathcal{F}$ is defined to be the following limit (a simple averaging argument shows that it always exists):

$$\pi(\mathcal{F}) = \lim_{n \to \infty} \frac{\mathrm{ex}(n, \mathcal{F})}{\binom{n}{r}}.$$

We say that α is *threshold* for $\mathcal{F}$ if $\pi(\mathcal{F}) \leqslant \alpha$.

Theorem 1.2 (Frankl and Rödl [7]). *The following are equivalent:*

(i) *α is a jump for r,*

(ii) *α is threshold for a finite family $\mathcal{F}$ of r-graphs satisfying*

$$\min_{F \in \mathcal{F}} \lambda(F) > \alpha.$$

Let F_r be the r-graph consisting of a single edge. Since any $\alpha \in [0,1)$ is threshold for F_r and $\lambda(F_r) = r!/r^r$, Theorem 1.2 trivially implies Erdős's result [3] that for each $r \geqslant 3$, every $\alpha \in [0, r!/r^r)$ is a jump for r.

The original version of Erdős's jumping constant conjecture asserted that $r!/r^r$ is a jump for every $r \geqslant 3$. This fascinating problem is still open, even for $r = 3$. Erdős speculated [3] that

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$3!/3^3 = 2/9$ was threshold for the following family of 3-graphs $\mathcal{F}^* = \{F_1, F_2, F_3\}$, where

$$F_1 = \{123, 124, 134\}, \quad F_2 = \{123, 124, 125, 345\}, \quad F_3 = \{123, 124, 235, 145, 345\}.$$

It is straightforward to check that $\lambda(F_1) = 8/27$, $\lambda(F_2) = \frac{189+15\sqrt{5}}{961}$ and $\lambda(F_3) = 6/25$. Since $\min_{1\leqslant i\leqslant 3} \lambda(F_i) = \lambda(F_2) > 2/9$, if $2/9$ were threshold for $\mathcal{F}^*$ then Theorem 1.2 would imply that $2/9$ is a jump for $r = 3$.

Unfortunately Erdős's suggestion is incorrect: $2/9$ is not threshold for $\mathcal{F}^*$. There exist 7 vertex 3-graphs that are $\mathcal{F}^*$-free with Lagrangians greater than $2/9$. By taking appropriate 'blow-ups' of such 3-graphs we find that $\pi(\mathcal{F}^*) > 2/9$. (To be precise we could take blow-ups of F_4, defined below, to show that $\pi(\mathcal{F}^*) \geqslant 0.2319$.) However, Erdős's idea suggests a natural approach to proving that $2/9$ is a jump for $r = 3$. Let $\mathcal{F}'$ be a family of 3-graphs containing F_1, F_2, F_3 with the property that $\min_{F\in\mathcal{F}'} \lambda(F) > 2/9$. If we can show that $2/9$ is threshold for $\mathcal{F}'$, then (by Theorem 1.2) $2/9$ is a jump for $r = 3$.

A search of all 3-graphs with at most 7 vertices yields the following two additional 3-graphs which we can add to $\mathcal{F}'$:

$$F_4 = \{123, 135, 145, 245, 126, 246, 346, 356, 237, 147, 347, 257, 167\},$$
$$F_5 = \{123, 124, 135, 145, 236, 346, 256, 456, 247, 347, 257, 357, 167\}.$$

It is easy to check that $\lambda(F_4) \geqslant 0.2319 > \lambda(F_2)$ (to see this, set $x_1 = x_2 = x_3 = 0.164$, $x_4 = 0.154$, $x_5 = x_6 = x_7 = 0.118$) and $\lambda(F_5) \geqslant \lambda(F_2)$ (set $\mu = (18 - 3\sqrt{5})/31$, $x_1 = x_6 = x_7 = \mu/3$, $x_2 = x_3 = x_4 = x_5 = (1-\mu)/4$).

We can now ask: Is it true that $2/9$ is threshold for $\mathcal{F}' = \{F_1, F_2, F_3, F_4, F_5\}$? Unfortunately this is still false: there exist 3-graphs on 8 vertices avoiding all members of $\mathcal{F}'$ and with Lagrangians greater than $2/9$. By taking appropriate 'blow-ups' of such 3-graphs, we can show that $\pi(\mathcal{F}') > 2/9$. Moreover, by considering 8 vertex 3-graphs, numerical evidence suggests that if $2/9$ is a jump then the size of the jump is extremely small: $c(2/9) \leqslant 0.00009254$.

However, although $2/9$ is not threshold for $\mathcal{F}'$, we can show the following upper bound on the Turán density of $\mathcal{F}'$.

Lemma 1.3. *The Turán density of $\mathcal{F}'$ satisfies $\pi(\mathcal{F}') \leqslant 0.2299$.*

Since $0.2299 < \min_{F\in\mathcal{F}'} \lambda(F) = \lambda(F_2) = 0.2316$, Theorem 1.1 is an immediate corollary of Lemma 1.3 and Theorem 1.2.

It remains to prove Lemma 1.3. For this we make use of recent work of Razborov [9] on flag algebras that introduces a new technique that drastically improves our ability to compute (and approximate) Turán densities. We outline the necessary background in the next section but emphasize that the reader should consult Razborov [8] and [9] for a full description of his work.

2. Computing Turán densities via flag algebras

2.1. Razborov's method

Let $\mathcal{F}$ be a family of r-graphs whose Turán density we wish to compute (or at least approximate). Razborov [9], describes a method for attacking this problem that can be thought of as a general application of Cauchy–Schwarz using the information given by small $\mathcal{F}$-free r-graphs.

Let $\mathcal{H}$ be the family of all $\mathcal{F}$-free r-graphs of order l, up to isomorphism. If l is sufficiently small we can explicitly determine $\mathcal{H}$ (by computer search if necessary).

For $H \in \mathcal{H}$ and a large $\mathcal{F}$-free r-graph G, we define $p(H;G)$ to be the probability that a random l-set from $V(G)$ induces a subgraph isomorphic to H. Trivially, the density of G is equal to the probability that a random r-set from $V(G)$ forms an edge in G. Thus, averaging over l-sets in $V(G)$, we can express the density of G as

$$d(G) = \sum_{H \in \mathcal{H}} d(H)p(H;G), \tag{2.1}$$

and hence $d(G) \leqslant \max_{H\in\mathcal{H}} d(H)$.

This 'averaging' bound on $d(G)$ is in general rather poor: clearly it could only be sharp if all subgraphs of G of order l are as dense as possible. It also fails to consider how different subgraphs of G can overlap. Razborov's flag algebras method allows us to make use of the information given by examining overlapping subgraphs of G to give far stronger bounds.

A *flag*, $F = (G_F, \theta)$, is an r-graph G_F together with an injective map θ: $[s] \to V(G_F)$. If θ is bijective (and so $|V(G_F)| = s$) we call the flag a *type*. For ease of notation, given a flag $F = (G_F, \theta)$ we define its order $|F|$ to be $|V(G_F)|$.

Given a type σ we call a flag $F = (G_F, \theta)$ a *σ-flag* if the induced labelled subgraph of G_F given by θ is σ. A flag $F = (G_F, \theta)$ is *admissible* if G_F is $\mathcal{F}$-free.

Fix a type σ and an integer $m \leqslant (l + |\sigma|)/2$. (The bound on m ensures that an l-vertex r-graph can contain two m-vertex subgraphs overlapping in $|\sigma|$ vertices.) Let $\mathcal{F}_m^\sigma$ be the set of all admissible σ-flags of order m, up to isomorphism. Let Θ be the set of all injective functions from $[|\sigma|]$ to $V(G)$. Given $F \in \mathcal{F}_m^\sigma$ and $\theta \in \Theta$, we define $p(F, \theta; G)$ to be the probability that an m-set V', chosen uniformly at random from $V(G)$ subject to $\mathrm{im}(\theta) \subseteq V'$, induces a σ-flag $(G[V'], \theta)$ that is isomorphic to F.

If $F_a, F_b \in \mathcal{F}_m^\sigma$ and $\theta \in \Theta$ then $p(F_a, \theta; G)p(F_b, \theta; G)$ is the probability that two m-sets $V_a, V_b \subseteq V(G)$, chosen independently at random subject to $\mathrm{im}(\theta) \subseteq V_a \cap V_b$, induce σ-flags $(G[V_a], \theta)$, $(G[V_b], \theta)$ that are isomorphic to F_a, F_b respectively. We define a related probability, $p(F_a, F_b, \theta; G)$, to be the probability that if we choose a random m-set $V_a \subseteq V(G)$, subject to $\mathrm{im}(\theta) \subseteq V_a$ and then choose a random m-set $V_b \subseteq V(G)$ such that $V_a \cap V_b = \mathrm{im}(\theta)$, then $(G[V_a], \theta)$, $(G[V_b], \theta)$ are isomorphic to F_a, F_b respectively. Note that the difference between $p(F_a, \theta; G)p(F_b, \theta; G)$ and $p(F_a, F_b, \theta; G)$ is due to the effect of sampling *with* or *without* replacement. When G is large this difference will be negligible, as the following lemma tells us. (This is a very special case of Lemma 2.3 in [8].)

Lemma 2.1 (Razborov [8]). *For any $F_a, F_b \in \mathcal{F}_m^\sigma$, and $\theta \in \Theta$,*

$$p(F_a, \theta; G)p(F_b, \theta; G) = p(F_a, F_b, \theta; G) + o(1),$$

where the $o(1)$ term tends to 0 as $|V(G)|$ tends to infinity.

Proof. Choose random m-sets $V_a, V_b \subseteq V(G)$, independently, subject to $\mathrm{im}(\theta) \subseteq V_a \cap V_b$. Let E be the event that $V_a \cap V_b = \mathrm{im}(\theta)$. Then

$$p(F_a, F_b, \theta; G)\mathbf{P}[E] \leqslant p(F_a, \theta; G)p(F_b, \theta; G) \leqslant p(F_a, F_b, \theta; G)\mathbf{P}[E] + \mathbf{P}[\bar{E}].$$

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If $|V(G)| = n$ then

$$\mathbf{P}[E] = \frac{\binom{n-|\sigma|}{m-|\sigma|}\binom{n-m}{m-|\sigma|}}{\binom{n-|\sigma|}{m-|\sigma|}^2} = 1 - o(1). \qquad \square$$

Averaging over a uniformly random choice of $\theta \in \Theta$, we have

$$\mathbf{E}_{\theta\in\Theta}\left[p(F_a,\theta;G)p(F_b,\theta;G)\right] = \mathbf{E}_{\theta\in\Theta}\left[p(F_a,F_b,\theta;G)\right] + o(1). \tag{2.2}$$

Note that this expectation can be computed by averaging over l-vertex subgraphs of G. For an l-vertex subgraph $H \in \mathcal{H}$, let Θ_H be the set of all injective maps $\theta : [|\sigma|] \to V(H)$. Recall that, for $H \in \mathcal{H}$, $p(H;G)$ is the probability that a random l-set from $V(G)$ induces a subgraph isomorphic to H. Thus,

$$\mathbf{E}_{\theta\in\Theta}\left[p(F_a,F_b,\theta;G)\right] = \sum_{H\in\mathcal{H}} \mathbf{E}_{\theta\in\Theta_H}\left[p(F_a,F_b,\theta;H)\right]p(H;G). \tag{2.3}$$

Consider a positive semidefinite matrix $Q = (q_{ab})$ of dimension $|\mathcal{F}_m^\sigma|$. For $\theta \in \Theta$ define $\mathbf{p}_\theta = (p(F,\theta;G) : F \in \mathcal{F}_m^\sigma)$. Using (2.2), (2.3) and linearity of expectation, we have

$$\mathbf{E}_{\theta\in\Theta}[\mathbf{p}_\theta^T Q\mathbf{p}_\theta] = \sum_{F_a,F_b\in\mathcal{F}_m^\sigma}\sum_{H\in\mathcal{H}} q_{ab}\mathbf{E}_{\theta\in\Theta_H}[p(F_a,F_b,\theta;H)]p(H;G) + o(1). \tag{2.4}$$

For $H \in \mathcal{H}$ define the coefficient of $p(H;G)$ in (2.4) by

$$c_H(\sigma,m,Q) = \sum_{F_a,F_b\in\mathcal{F}_m^\sigma} q_{ab}\mathbf{E}_{\theta\in\Theta_H}[p(F_a,F_b,\theta;H)]. \tag{2.5}$$

Suppose we have t choices of (σ_i,m_i,Q_i), where each σ_i is a type, each $m_i \leqslant (l+|\sigma_i|)/2$ is an integer, and each Q_i is a positive semidefinite matrix of dimension $|\mathcal{F}_{m_i}^{\sigma_i}|$. For $H \in \mathcal{H}$ define

$$c_H = \sum_{i=1}^{t} c_H(\sigma_i,m_i,Q_i).$$

Note that c_H is independent of G.

Since each Q_i is positive semidefinite, (2.4) implies that

$$\sum_{H\in\mathcal{H}} c_H p(H;G) + o(1) \geqslant 0.$$

Thus, using (2.1), we have

$$d(G) \leqslant \sum_{H\in\mathcal{H}} (d(H) + c_H)p(H;G) + o(1).$$

Hence the Turán density satisfies

$$\pi(\mathcal{F}) \leqslant \max_{H\in\mathcal{H}}(d(H) + c_H). \tag{2.6}$$

Since the c_H may be negative, for an appropriate choice of the (σ_i,m_i,Q_i), this bound may be significantly better than the trivial averaging bound given by (2.1).

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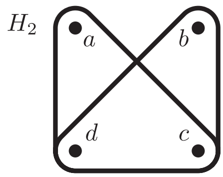

Figure 1. The 3-graph H_2, with vertices labelled a, b, c, d. Its two edges are acd and bcd.

Note that we now have a semidefinite programming problem: given any particular choice of the (σ_i, m_i), find positive semidefinite matrices Q_i so as to minimize the bound for $\pi(\mathcal{F})$ given by (2.6).

2.2. An example

We now illustrate Razborov's method with a simple example. Let $K_4^- = \{123, 124, 134\}$. We will reprove de Caen's [2] bound: $\pi(K_4^-) \leqslant 1/3$.

Let $l = 4$, so $\mathcal{H}$ consists of all K_4^--free 3-graphs of order 4, up to isomorphism. There are three such 3-graphs, which we will refer to as H_0, H_1, and H_2; they have 0, 1, and 2 edges, respectively (this is enough information to identify them uniquely). We will use a single type: $\sigma = (G_\sigma, \theta)$ where $V(G_\sigma) = [2]$, $E(G_\sigma) = \emptyset$ and $\theta(x) = x$. Taking $m = 3$, there are only two admissible σ-flags of order 3 up to isomorphism, F_0 and F_1, containing 0 and 1 edge, respectively.

In order to calculate the coefficients c_H we need to compute $\mathbf{E}_{\theta\in\Theta_H}[p(F_a, F_b, \theta; H)]$, for each $H \in \{H_0, H_1, H_2\}$ and each pair $F_a, F_b \in \{F_0, F_1\}$. Their values are given in the following table:

	H_0	H_1	H_2
F_0, F_0	1	1/2	1/6
F_0, F_1	0	1/4	1/3
F_1, F_1	0	0	1/6

As an example of how these values are computed, consider $\mathbf{E}_{\theta\in\Theta_{H_2}}[p(F_0, F_1, \theta; H_2)]$. This is the probability that a random choice of $\theta \in \Theta_{H_2}$ and 3-sets $V_0, V_1 \subset V(H_2)$, such that $V_0 \cap V_1 = \mathrm{im}(\theta)$, induce σ-flags $(H_2[V_0], \theta), (H_2[V_1], \theta)$ that are isomorphic to F_0, F_1, respectively. A random of choice of $\theta \in \Theta_{H_2}$ is equivalent to picking a random ordered pair of vertices (u, v) from H_2, and setting $\theta(1) = u$ and $\theta(2) = v$. To form the random 3-sets V_0, V_1, we pick the remaining two vertices of $V(H_2) \setminus \{u, v\}$ randomly in the order x, y and set $V_0 = \{u, v, x\}, V_1 = \{u, v, y\}$. The σ-flags $(H_2[V_0], \theta), (H_2[V_1], \theta)$ are isomorphic to F_0, F_1 if and only if $V_0 \notin E(H_2)$ and $V_1 \in E(H_2)$ respectively. Consequently $\mathbf{E}_{\theta\in\Theta_{H_2}}[p(F_0, F_1, \theta; H_2)]$ is the probability that a random permutation (u, v, x, y) of $V(H_2)$ satisfies $\{u, v, x\} \notin E(H_2)$ and $\{u, v, y\} \in E(H_2)$. Of the 24 permutations of $V(H_2) = \{a, b, c, d\}$ (see Figure 1), the following 8 have this property:

$$(a,c,b,d),\quad (a,d,b,c),\quad (b,c,a,d),\quad (b,d,a,c),$$
$$(c,a,b,d),\quad (d,a,b,c),\quad (c,b,a,d),\quad (d,b,a,c).$$

Hence $\mathbf{E}_{\theta\in\Theta_{H_2}}[p(F_0, F_1, \theta; H_2)] = 8/24 = 1/3$.

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We now need to find a positive semidefinite matrix

$$Q = \begin{pmatrix} q_{00} & q_{01} \\ q_{01} & q_{11} \end{pmatrix},$$

to minimize the bound given by (2.6). Note that

$$\begin{aligned} c_{H_0} &= q_{00}, \\ c_{H_1} &= \frac{1}{2}q_{00} + \frac{1}{2}q_{01}, \\ c_{H_2} &= \frac{1}{6}q_{00} + \frac{2}{3}q_{01} + \frac{1}{6}q_{11}. \end{aligned}$$

The bound on $\pi(K_4^-)$ given by (2.6) is now

$$\pi(K_4^-) \leqslant \max\left\{q_{00},\quad \frac{q_{00}}{2} + \frac{q_{01}}{2} + \frac{1}{4},\quad \frac{q_{00}}{6} + \frac{2q_{01}}{3} + \frac{q_{11}}{6} + \frac{1}{2}\right\}.$$

This can be expressed as a semidefinite programming problem, the solution to which is

$$Q = \frac{1}{3}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}.$$

Consequently $\pi(K_4^-) \leqslant \max\{1/3, 1/12, 1/3\} = 1/3$.

2.3. Proof of Lemma 1.3

To prove $\pi(\mathcal{F}') \leqslant 0.2299$, we use Razborov's flag algebras method, as outlined above. We set $l = 7$, so $\mathcal{H}$ consists of all 7 vertex 3-graphs that do not contain any $F \in \mathcal{F}'$, up to isomorphism. There are 4042 such 3-graphs, which are explicitly determined by the C++ program DensityBounder.[1] To calculate the coefficients c_H we take six choices of (σ_i, m_i, Q_i). The types are $\sigma_i = ((V_i, E_i), \theta_i)$, where

$$\begin{aligned} V_1 &= [1], \quad E_1 = \emptyset, \\ V_2 &= [3], \quad E_2 = \emptyset, \\ V_3 &= [3], \quad E_3 = \{123\}, \\ V_4 &= [5], \quad E_4 = \{123, 124, 135\}, \\ V_5 &= [5], \quad E_5 = \{123, 124, 345\}, \\ V_6 &= [5], \quad E_6 = \{123, 124, 135, 245\}, \end{aligned}$$

and $\theta_i : [|V_i|] \to V_i$, maps $x \mapsto x$. Ideally we would use all types of size at most $l - 2 = 5$. However, this yields a computationally intractable semidefinite program. Our actual choice was made by experiment, in each case taking the value of $m_i = \lfloor (7 + |\sigma_i|)/2 \rfloor$. DensityBounder determines the positive semidefinite matrices Q_i by creating a semidefinite programming problem. Several implementations of semidefinite program solvers exist. We chose the CSDP library [1] to solve the problem. The CSDP library uses floating-point arithmetic, which may introduce

[1] This, along with `HypergraphsDoJump.soln`, can be downloaded from: `http://www.ucl.ac.uk/~ucahjmt/SolnFiles.zip`.

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rounding errors. DensityBounder takes the output of the CSDP program and uses it to construct the Q_i (removing any rounding errors). Our results can, however, be verified without needing to solve a semidefinite program: DensityBounder can load pre-computed matrices Q_i from the file `HypergraphsDoJump.soln`.

For each $H \in \mathcal{H}$, $d(H)$ and c_H are calculated by DensityBounder, and using (2.6) it computes that 0.2299 is an upper bound for $\pi(\mathcal{F}')$. Note that although floating-point operations are used by the semidefinite program solver, our final computer proof consists of positive semidefinite matrices with rational coefficients and our proof can be verified using only integer operations, and thus there is no issue of numerical accuracy.

2.4. Other results

The program DensityBounder can be used to calculate upper bounds on the Turán density of other families of 3-graphs. In particular, we have used it to reproduce Razborov's bound: $\pi(K_4^{(3)}) \leqslant 0.561666$ [9].

The conjectured value of $\pi(K_4^-)$ is $2/7 = 0.2857$. Razborov [9] showed that $\pi(K_4^-) \leqslant 0.2978$. Using DensityBounder we obtain a new upper bound of 0.2871 by taking $l = 7$ and considering the following four types $\sigma_i = ((V_i, E_i), \theta_i)$ with the given values of m_i (in each case θ_i is the identity map):

$$\begin{aligned} V_1 &= [3], & E_1 &= \emptyset, & m_1 &= 5,\\ V_2 &= [3], & E_2 &= \{123\}, & m_2 &= 5,\\ V_3 &= [4], & E_3 &= \{123\}, & m_3 &= 5,\\ V_4 &= [5], & E_4 &= \{123, 124, 125\}, & m_4 &= 6. \end{aligned}$$

As before, the positive semidefinite matrices Q_i are determined by solving a semidefinite programming problem.

Theorem 2.2. *Let K_4^- be the 3-graph on four vertices with three edges. The Turán density of K_4^- satisfies*

$$0.2857\cdots = \frac{2}{7} \leqslant \pi(K_4^-) \leqslant 0.2871.$$

As with our main result, our computations can be verified without any floating-point operations, so there is no issue of numerical accuracy in these results. Theorem 2.2 yields a second new interval of jumps for $r = 3$.

Corollary 2.3. *If $\alpha \in [0.2871, 8/27)$ then α is a jump for $r = 3$.*

Proof. Since $\lambda(K_4^-) = 8/27$, this follows directly from Theorem 2.2 and Theorem 1.2. □

2.5. Solving the semidefinite program

Razborov's method, as outlined above, reduces the problem of computing an upper bound on a Turán density to solving a semidefinite programming problem. In practice this may be computationally difficult. Razborov [9] describes a number of ways in which this problem can be

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simplified to make the computation more tractable. Below, we outline one of these ideas, which we have used in our work.

For a type σ and the collection of all admissible σ-flags of order m, $\mathcal{F}_m^\sigma$, define $\mathbb{R}\mathcal{F}_m^\sigma$ to be the real vector space of formal linear combinations of σ-flags of order m. Let $\mathcal{H}$ be the collection of all admissible r-graphs of order l.

Let us introduce Razborov's $[\![\cdot]\!]_\sigma$ notation (which will make our expressions easier to read). Define $[\![\cdot]\!]_\sigma : \mathbb{R}\mathcal{F}_m^\sigma \times \mathbb{R}\mathcal{F}_m^\sigma \to \mathbb{R}^{|\mathcal{H}|}$, by

$$[\![F_a F_b]\!]_\sigma = (\mathbf{E}_{\theta \in \Theta_H}[p(F_a, F_b, \theta; H)] : H \in \mathcal{H}),$$

for $F_a, F_b \in \mathcal{F}_m^\sigma$ and extend it to be bilinear.

For a positive semidefinite matrix Q and $\mathbf{p} = (F : F \in \mathcal{F}_m^\sigma)$, the vector of all admissible σ-flags (in an arbitrary but fixed order), we have

$$[\![\mathbf{p}^T Q \mathbf{p}]\!]_\sigma = (c_H(\sigma, m, Q) : H \in \mathcal{H}),$$

where the c_H are as defined in (2.5).

Razborov [9] describes a natural change of basis for $\mathbb{R}\mathcal{F}_m^\sigma$. The important property (in terms of reducing the computational complexity of the associated semidefinite program) is that the new basis is of the form $\mathcal{B} = \mathcal{B}^+ \dot\cup \mathcal{B}^-$, and for all $B^+ \in \mathcal{B}^+$ and $B^- \in \mathcal{B}^-$ we have $[\![B^+B^-]\!]_\sigma = \mathbf{0}$. Thus, in our new basis the corresponding semidefinite program has a solution Q', which is a block diagonal matrix with two blocks, of sizes $|\mathcal{B}^+|$ and $|\mathcal{B}^-|$, respectively. Since the best algorithms for solving semidefinite programs scale like the square of the size of block matrices, this change of basis can potentially simplify our computation significantly.

For a type $\sigma = (G_\sigma, \theta_\sigma)$ we construct the basis $\mathcal{B}$ as follows. First construct Γ_σ, the automorphism group of σ, whose elements are bijective maps $\alpha : [|\sigma|] \to [|\sigma|]$ such that $(G_\sigma, \theta_\sigma \alpha)$ is isomorphic to σ. The elements of Γ_σ act on σ-flags in an obvious way: for $\alpha \in \Gamma_\sigma$ and σ-flag $F = (G_F, \theta_F)$ we define $F\alpha$ to be the σ-flag $(G_F, \theta_F \alpha)$. Define subspaces

$$\mathbb{R}\mathcal{F}_m^{\sigma+} = \{L \in \mathbb{R}\mathcal{F}_m^\sigma : L\alpha = L \ \forall \alpha \in \Gamma_\sigma\}$$

and

$$\mathbb{R}\mathcal{F}_m^{\sigma-} = \left\{L \in \mathbb{R}\mathcal{F}_m^\sigma : \sum_{\alpha \in \Gamma_\sigma} L\alpha = \mathbf{0}\right\}.$$

Below we describe how to find bases $\mathcal{B}^+, \mathcal{B}^-$ for these subspaces. By the construction of these bases it will be clear that $\mathbb{R}\mathcal{F}_m^\sigma = \mathbb{R}\mathcal{F}_m^{\sigma+} \oplus \mathbb{R}\mathcal{F}_m^{\sigma-}$. Finally we will verify that for all $B^+ \in \mathcal{B}^+$ and $B^- \in \mathcal{B}^-$ we have $[\![B^+B^-]\!]_\sigma = \mathbf{0}$.

We start with the canonical basis for $\mathbb{R}\mathcal{F}_m^\sigma$, given by $\mathcal{F}_m^\sigma = \{F_1, F_2, \dots, F_t\}$. For each $F_i \in \mathcal{F}_m^\sigma$ define the orbit of F_i under Γ_σ by

$$F_i\Gamma_\sigma = \{F\alpha : \alpha \in \Gamma_\sigma\}.$$

Any two orbits are either equal or disjoint. Suppose there are u distinct orbits: $O_1, \dots, O_u$. For $i \in [u]$ let $B_i^+ = \sum_{F \in O_i} F$. Then $\mathcal{B}^+ = \{B_1^+, \dots, B_u^+\}$ is easily seen to be a basis for $\mathbb{R}\mathcal{F}_m^{\sigma+}$. Moreover, if $O_i = \{F_{i_1}, \dots, F_{i_q}\}$ then $F_{i_1} - F_{i_z} \in \mathbb{R}\mathcal{F}_m^{\sigma-}$ for $2 \leqslant z \leqslant q$, and the union of all such vectors forms a basis $\mathcal{B}^-$ for $\mathbb{R}\mathcal{F}_m^{\sigma-}$.

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We now need to check that if $B^+ \in \mathcal{B}^+$ and $B^- \in \mathcal{B}^-$ then $[\![B^+B^-]\!]_\sigma = \mathbf{0}$. If $B^- \in \mathcal{B}^-$ then by construction $B^- = F_b\alpha - F_b$ for some $F_b \in \mathcal{F}_m^\sigma$ and $\alpha \in \Gamma_\sigma$. Moreover, $B^+\alpha = B^+$. Hence, by linearity,

$$[\![B^+B^-]\!]_\sigma = [\![B^+(F_b\alpha - F_b)]\!]_\sigma = [\![(B^+\alpha)(F_b\alpha) - B^+F_b]\!]_\sigma.$$

We observe that for any $F_a \in \mathcal{F}_m^\sigma$,

$$\begin{aligned}[\![(F_a\alpha)(F_b\alpha)]\!]_\sigma &= (\mathbf{E}_{\theta\in\Theta_H}[p(F_a, F_b, \theta\alpha^{-1}; H)] : H \in \mathcal{H}) \\ &= (\mathbf{E}_{\theta\in\Theta_H\alpha^{-1}}[p(F_a, F_b, \theta; H)] : H \in \mathcal{H}),\end{aligned}$$

where $\Theta_H\alpha^{-1} = \{\theta\alpha^{-1} : \theta \in \Theta_H\}$. Since $\Theta_H\alpha^{-1} = \Theta_H$, we must have $[\![(F_a\alpha)(F_b\alpha)]\!]_\sigma = [\![F_aF_b]\!]_\sigma$. Thus, since $B^+ = F_{a_1} + F_{a_2} + \cdots + F_{a_s}$, we have $[\![(B^+\alpha)(F_b\alpha) - B^+F_b]\!]_\sigma = \mathbf{0}$, and hence $[\![B^+B^-]\!]_\sigma = \mathbf{0}$.

3. Open problems

We have shown that $[0.2299, 0.2316)$ is an interval of jumps for $r = 3$. If we were able to compute $\pi(\mathcal{F}')$ precisely we could quite possibly extend this interval below 0.2299. However, as noted in the Introduction, we know that $\pi(\mathcal{F}') > 2/9$, so our approach could never resolve the most important open question in this area: Is $2/9$ a jump?

Indeed the question of whether $2/9$ is a jump for $r = 3$ seems remarkably difficult to resolve. If $2/9$ is a jump then the size of this jump is very small, and so to give a proof along the same lines as the proof of Theorem 1.1 would appear to require a very precise approximation of the Turán density of some unknown family of 3-graphs. On the other hand, the only current technique for showing a value is *not* a jump is to follow the method of Frankl and Rödl [7], but this trivially fails for $2/9$ (or indeed $r!/r^r$ for any $r \geqslant 3$).

Another obvious open problem is to compute $\pi(K_4^-)$ exactly. It is likely that improvements over our bound of 0.2871 could be made by applying Razborov's method with larger flags or by considering different types of order 5. Similarly improved bounds for the central problem in this area, determining $\pi(K_4^{(3)})$, could quite probably be found by the use of larger flags.

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