Eliminating dt and ds from (13), (15) and (16) one finds

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = h^{-2}\{c^2 - (1-mu)^2\}. \tag{18}$$

The equation possesses the first integral

$$u = mh^{-2}(1+m^2h^{-2})^{-1} + K \sin(1+m^2h^{-2})^{\frac{1}{2}}(\phi - \phi_0), \tag{19}$$

where $$K = (1+m^2h^{-2})^{-1}[(1+m^2h^{-2})(c^2-1)h^{-2} + m^2h^{-4}]^{\frac{1}{2}}$$

and ϕ_0 is a constant of integration. Equation (19) represents a quasi-elliptical orbit with apsidal angle $(1+m^2h^{-2})^{-\frac{1}{2}}\pi$, so that the fractional advance of perihelion per revolution is approximately

$$-\tfrac{1}{2}m^2h^{-2}. \tag{20}$$

This is one-sixth of the amount derived from the Schwarzschild solution, and has the opposite sign. It is definitely in contradiction to observation, with which the Schwarzschild result agrees within observational error.

The only hope of salvaging Littlewood's theory would seem to be the discovery of a cosmological solution representing a spherically symmetric expanding distribution of matter with density uniform at large distances, described by some $\theta(r, t)$ which gave the correct perihelion motion.

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ON RIEMANN INTEGRABILITY

By B. D. GEE
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It is well known that a necessary and sufficient condition for a bounded function $f(x)$ to have a Riemann integral over a closed interval I is that it is continuous at every point of I, except possibly on a set E of points of I of measure zero. We give here an elementary proof of the sufficient condition which is simpler than any we have found in the literature.

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Given $\omega > 0$, we can find an enumerable set of open intervals covering E such that the total measure of these intervals is less than ω. And since $f(x)$ is continuous at every point of $I-E$, given $\eta > 0$, we can find for each x of $I-E$ an open interval J_x containing x such that the oscillation of $f(x)$ in $\bar{J}_x I$ is less than η.

By the Heine–Borel theorem we can choose from this covering of I a finite set, say S, of intervals which covers I. We proceed by taking the end-points of the intervals of S, when these points lie in I, together with the end-points of I as points of subdivision of I and find that with this subdivision of I the difference between the upper and lower Riemann sums of $f(x)$ is less than $2M.\omega+\eta.mI$, where M is the upper bound of $|f(x)|$ on I. So $f(x)$ has a Riemann integral over I.

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SOME FORMULAE FOR THE ASSOCIATED LEGENDRE FUNCTIONS OF THE FIRST KIND

By F. M. Ragab

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In this paper an integral involving an associated Legendre function and an E-function is evaluated by means of a known formula for a generalized hypergeometric function, and from this result some integrals involving Bessel functions are deduced. For the definitions and properties of the E-functions see (1), §§3·12 and 5·2.

The formula to be proved is

$$\int_0^\pi (\sin\theta)^{l-1}\, T_n^{-m}(\cos\theta)\, E(p;\ \alpha_r{:}q;\ \rho_s{:}x/\sin^2\theta)\, d\theta$$
$$= \frac{\pi}{2^m\Gamma(1+\frac{1}{2}m+\frac{1}{2}n)\,\Gamma(\frac{1}{2}+\frac{1}{2}m-\frac{1}{2}n)}\, E\begin{pmatrix}\alpha_1,\ldots,\alpha_p,\ \frac{1}{2}l+\frac{1}{2}m,\ \frac{1}{2}l-\frac{1}{2}m{:}x\\ \rho_1,\ldots,\rho_q,\ \frac{1}{2}+\frac{1}{2}l+\frac{1}{2}n,\ \frac{1}{2}l-\frac{1}{2}n\end{pmatrix}, \quad (1)$$

provided that $\Re(l+m) > 0$ and $|\operatorname{amp} x| < \pi$,

where
$$T_n^{-m}(\lambda) = \frac{(1-\lambda^2)^{\frac{1}{2}m}}{2^m\Gamma(m+1)}\, {}_2F_1\begin{pmatrix}m-n,\ m+n+1;\ \frac{1}{2}-\frac{1}{2}\lambda\\ m+1;\end{pmatrix}$$

and
$$E(p;\alpha_r{:}q;\rho_s{:}x) = \frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_p)}{\Gamma(\rho_1)\ldots\Gamma(\rho_q)}\, {}_pF_q(p;\alpha_r{:}q;\rho_s{:}-1/x),$$

for $p \leqslant q$.

To prove (1), consider the special case with $p = q = 0$; then the integral becomes

$$\int_0^\pi \exp(-\sin^2\theta/x)\,(\sin\theta)^{l-1}\, T_n^{-m}(\cos\theta)\, d\theta. \quad (2)$$