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Dynamic risk measures for stochastic asset processes from ruin theory

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Abstract
This article considers a dynamic version of risk measures for stochastic asset processes and gives a mathematical benchmark for required capital in a solvency regulation framework. Some dynamic risk measures, based on the expected discounted penalty function launched by Gerber and Shiu, are proposed to measure solvency risk from the company's *going-concern* point of view. This study proposes a novel mathematical justification of a risk measure for stochastic processes as a map on a functional path space of future loss processes.

Keywords
Insolvency risk; Gerber–Shiu function; Dynamic risk measure; Solvency capital requirement; Asset/liability management

JEL classification
G32; G22; C02

1. Introduction

On a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with the usual conditions, for each $u \in \mathbb{R}$, let $X^u := \left(X_t^u\right)_{t\geq 0}$ be an $(\mathcal{F}_t)$-adapted process with $\mathbb{P}\left(X_0^u = u\right) = 1$. Throughout the paper, we assume that $\mathcal{F}_0$ is generated by null sets. Thus, any $(\mathcal{F}_t)$-adapted process starts with a constant. We assume that X^u represents a surplus process of an insurance company, for example, a company's net asset; see section 5.2. Note that $X_t^u - X_t^v \equiv u - v$ for all $t \geq 0$ in this notation. To measure the *insolvency risk* of an insurance company, it would be natural to consider ruin-related risk over a fixed term. For example, a negative X_t^u means the company is insolvent at time t, the situation called *ruin* in risk theory; the time of ruin is defined as

$$\tau = \tau_u := \inf\{t > 0 | X_t^u < 0\}$$

which is an $\mathcal{F}_t$-stopping time. To evaluate how insolvent the company is, we need to consider a quantitative measure of insolvency.

Let $\mathcal{M} := \mathcal{M}(\mathbb{R})$ be a set of random variables $\Omega \to \mathbb{R}$. According to Denuit *et al.* (2005), an actuarial definition of a "risk measure" is *a functional* $\rho : \mathcal{M} \to [0, \infty]$, *representing extra cash that must be*

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added to X to make it acceptable (Definition 2.2.1 in Denuit *et al.*, 2005). More practically, it would be useful to extend ρ to an $\mathbb{R}$-valued mapping. That is, if $\rho > 0$, then the company should add the capital ρ to its position; if $\rho \leq 0$, then the company can still use the cash $-\rho$. Because insolvency is closely related to ruin, it would be natural to evaluate insolvency risks as a functional of τ.

In recent decades, insurance ruin theory has advanced, and it is possible to analyse many ruin-related quantities. For example, the following functional of τ was introduced by Gerber & Shiu (1998). For some $w : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$

$$\phi_0^X(u) = \mathbb{E}\left[e^{-\delta\tau} w(X^u_{\tau-}, \ -X^u_\tau)\mathbf{1}_{\{\tau<\infty\}}\right] \tag{1.1}$$

which is risk at ruin via the asset prior to ruin $X_{\tau-}$ and the deficit at ruin $-X_\tau$ by discounting with interest rate δ. This quantity, called the *Gerber–Shiu function*, would be a good, natural candidate for a "risk measure" for solvency evaluations because it can choose a "risk" in the vicinity of the bankruptcy of the company. Indeed, there have been several previous attempts to use ϕ_0^X as a risk measure, for example, Trufin *et al.* (2011), Eisenberg & Schmidli (2011), Schmidli (2002), Schmidli (2014). However, there remains the mathematical problem of dealing with this as a "risk measure".

There is a certain consensus regarding mathematical conditions that a risk measure should meet. Suppose that an order "$\preceq$" is equipped in $\mathcal{M}$, and let $\widetilde{\mathcal{M}}(\subset \mathcal{M})$ be a convex cone that includes all constant processes. Then, a risk measure ρ is defined as follows:

Definition 1.1. A map $\rho : \widetilde{\mathcal{M}} \to \mathbb{R}$ is called a risk measure if the following two properties hold true:

- Monotonicity (MO): $\rho(X) \leq \rho(Y)$ for any $X, Y \in \widetilde{\mathcal{M}}$ such that $X \preceq Y$.
- Translativity (TR): $\rho(X + c) = \rho(X) + c$ for any $X \in \widetilde{\mathcal{M}}$ and $c \in \mathbb{R}$.

In addition, a risk measure ρ is called coherent if ρ further satisfies the following conditions:

- Positive homogeneity (PH): $\rho(\lambda X) = \lambda\rho(X)$ for any $X \in \widetilde{\mathcal{M}}$ and $\lambda > 0$.
- Subadditivity (SA): $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for any $X, Y \in \widetilde{\mathcal{M}}$.

Since the paper by Artzner *et al.* (1999), this concept of risk measures has been widely used in finance and insurance, and many other properties for ρ have been proposed by several authors. Although the Gerber–Shiu function ϕ_0^X can be an intuitive "risk measure" in an actuarial sense as in Denuit *et al.* (2005) for insolvency risks, the domain $\widetilde{\mathcal{M}}$ is unclear mathematically. For example, Trufin *et al.* (2011) discussed a risk measure for a "loss variable Z" as

$$\rho_\epsilon(Z) = \inf\{u \geq 0 \mid \mathbb{P}(\tau_u < \infty) \leq \epsilon\} \tag{1.2}$$

under the classical Cramér–Lundberg risk model; see also Mitric & Trufin (2015). This is intuitively a kind of risk measure based on the Gerber–Shiu function with $w(x, y) \equiv 1$ and $\delta = 0$; $\rho_\varepsilon(Z)$ is interpreted as the minimum capital such that the ultimate ruin probability is at most the specified level ε. To justify this ρ_ε as a risk measure in the sense of Definition 1.1, Z is considered as an individual claim size. However, this perspective has some drawbacks: (i) the risk for X^u depends not only on the claim size Z, but also on the claim number process and the value of the premium; (ii) the risk model must be restricted to, for example, a classical model. For example, if X^u is a Lévy process with infinite activity jumps, then the meaning of Z is now unclear.

We claim that insolvency risk should reflect a "process risk" of X^u; that is, the loss Z should depend on the entire path of X^u. To define a universal solvency risk measure, independent of a model structure, the measure should be a map on a functional space with the underlying process X^u values. On this criticism, we can find an attempt by Loisel & Trufin (2014). They consider a risk measure derived from the expected area in red for a compound Poisson surplus process and investigate these properties as a mathematical risk measure. They define a "stochastic order" for aggregate claim processes based on a stochastic order of all marginal variables of the process. Since we deal with more varied ruin-related risks under a more general class of surplus processes, we will later introduce an ordering rule for surplus processes that more flexibly depends on the risk form.

In this paper, we formulate a risk measure in the sense of Definition 1.1, using a Gerber–Shiu function without specifying the underlying process, by which ρ_ε in (1.2) is also justified, in our context, not only for the classical Lundberg model, but also for a more generalised asset model. In section 2, we define a *Gerber–Shiu risk* to evaluate insolvency risk in a finite-time interval, based on which we propose a *Gerber–Shiu risk measure* in section 3. This static risk measure (in a single period) is used to define a *dynamic risk measure (DRM)* for multiple periods in section 4. In section 5, we demonstrate the use of the DRM for asset/liability management and define the *solvency capital requirement (SCR)* based on a Gerber–Shiu risk measure for insurance companies. A simple example gives us a theoretical "benchmark" for defining the SCR in practice.

2. Gerber–Shiu Risk Processes

Considering an insurance business in practice, solvency risk is usually considered up to a certain maturity $T > 0$. For instance, suppose that an asset risk X at time $t > 0$ is measured by the value of $\varpi(X_t)$ for a penalty function ϖ. Then, considering a discount factor $e^{-\delta t}$, an insurance loss at time T (say, L_T) can be defined as follows:

$$L_T := \begin{cases} \varpi(X_T^u) & (\tau > T) \\ e^{\delta(T-\tau)}\varpi(X_\tau^u) & (\tau \le T) \end{cases} \tag{2.1}$$

Note that when ruin occurs before maturity ($\tau \le T$), the loss at time τ is inflated up to time T to measure the risk at T. Hence, the expected preset ($t = 0$) value of the risk is given by

$$\mathbb{E}\left[e^{-\delta T} L_T\right] = \mathbb{E}\left[e^{-\delta(\tau\wedge T)}\varpi\left(X^u_{\tau\wedge T}\right)\right]$$

This idea motivates us to consider a more general *finite-time Gerber–Shiu function* up to time T for measuring ruin risk at $t = 0$.

$$\phi_0^X(u, T) = \mathbb{E}\left[e^{-\delta(\tau\wedge T)} w\left(X^u_{(\tau\wedge T)-}, X^u_{\tau\wedge T}\right)\right] \tag{2.2}$$

for a function w on $\mathbb{R}^2$. This finite-time version was investigated by Garrido (2013) and Cojocaru *et al.* (2014) as a possible risk measure. Although there is another finite-time version by Kuznetsov & Morales (2014), version (2.2) seems natural to measure risks in a fixed time interval. Later, we consider such a risk at each time t by defining

$$\phi_t^X(u, T) = \begin{cases} \mathbb{E}\left[e^{-\delta(\tau\wedge T)} w\left(X^u_{(\tau\wedge T)-}, X^u_{\tau\wedge T}\right) \mid \mathcal{F}_t\right] & \text{on } \{\tau > t\} \\ \infty & \text{on } \{\tau \le t\} \end{cases}, \quad \text{a.s.} \tag{2.3}$$

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That is, we measure a risk by a finite-time Gerber–Shiu function when ruin still has not occurred at t: $\tau > t$, and we regard the risk as infinite if ruin has already occurred by t: $\tau \leq t$.

Note that $\phi^X(u, T) = (\phi_t(u, T))_{t \geq 0}$ is called a *Gerber–Shiu process*, which was introduced by Garrido (2010) at the International Gerber–Shiu Workshop in 2010. However, its properties and applications have been insufficiently investigated. We shall focus again on this concept and give its meaning as a mathematical risk measure with applications.

When we consider a risk by an insurance company, the risk should decrease if the initial reserve increases. Since we use functions (2.2) or (2.3) to measure a company's risk, the penalty function w should be determined so that ϕ_0^X decreases if the initial asset u increases. In this paper, we require such a condition for the penalty function w in (2.2).

Definition 2.1. Function (2.2) is called a Gerber–Shiu risk if the following condition holds true:

$$\phi_0^X(u+v, T) \leq \phi_0^X(u, T) \tag{2.4}$$

for any $u, v, T > 0$. Moreover, we say that $\phi^X = \left(\phi_t^X\right)_{t \in [0, T]}$ is a Gerber–Shiu risk process if ϕ_0^X is a Gerber–Shiu risk.

This definition means that the larger a Gerber–Shiu risk is, the riskier the corresponding company is. In practice, the company should ensure that Gerber–Shiu risk is kept low.

Example 2.1. When $w(x, y) = \mathbf{1}_{\{y<0\}}$ and $\delta = 0$, function (2.2) is given by

$$\phi_0^X(u, T) = \mathbb{P}\left(X_{\tau \wedge T}^u < 0\right) = \mathbb{P}(\tau \leq T)$$

which represents the finite-time ruin probability. This clearly satisfies

$$\phi_0^X(u, T) = \mathbb{P}(\tau_u \leq T) > \mathbb{P}(\tau_{u+v} \leq T) = \phi_0^X(u+v, T)$$

for any $v, T > 0$ and $u \in \mathbb{R}$. Therefore, this ϕ_0^X is a Gerber–Shiu risk.

Example 2.2. Let $\delta = 0$, $w(x, y) = \mathbf{1}_{\{-y<\beta\}}$ for any constant $\beta \in \mathbb{R}$:

$$\phi_0^X(u, T) = \mathbb{P}\left(-X_{\tau \wedge T}^u < \beta\right)$$

which is the distribution function of $-X_{\tau \wedge T}^u$. Suppose that X^u is a spectrally negative Lévy process starting at u. Then, $X_t^u \leq X_t^{u+v}$ a.s. for any $v, t > 0$. Therefore, $X_{\tau \wedge T}^u \leq X_{\tau \wedge T}^{u+v}$. Hence, $\phi_0^X(u, T)$ satisfies (2.4).

Example 2.3. Consider the Cramér–Lundberg model, that is

$$X_t = u + ct - \sum_{i=1}^{N_t} U_i$$

where N is a Poisson process, and the U_is are i.i.d. positive random variables with mean $\mu > 0$. When $w(x, y) = x - y$, function (2.2) is

$$\phi_0^X(u, T) = \mathbb{E}\left[e^{-\delta(\tau_u \wedge T)}\left(X_{(\tau_u \wedge T)-} - X_{\tau_u \wedge T}\right)\right]$$

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which is a Gerber–Shiu risk. Indeed, if the initial level u increases to $u+v$ for $v>0$, then it follows that $\tau_u < \tau_{u+v}$ a.s., and that

$$X_{(\tau_u \wedge T)-} - X_{\tau_u \wedge T} = U_\sigma \mathbf{1}_{\{\tau_u \leq T\}}$$

because $\Delta X_T = 0$ a.s., where U_σ is the claim size causing ruin. Therefore

$$\begin{aligned}\phi_0^X(u, T) &= \mathbb{E}\left[\mathbb{E}\left[e^{-\delta\tau_u} U_\sigma \mathbf{1}_{\{\tau_u \leq T\}} \middle| \tau_u\right]\right] \\ &= \mu\mathbb{E}\left[e^{-\delta\tau_u} \mathbf{1}_{\{\tau_u \leq T\}}\right] \\ &> \mu\mathbb{E}\left[e^{-\delta\tau_{u+v}} \mathbf{1}_{\{\tau_{u+v} \leq T\}}\right] \\ &= \phi_0^X(u+v, T)\end{aligned}$$

which is (2.4). This Gerber–Shiu risk measures a risk by the expected present value of the claim size that causes ruin if ruin occurs, and indicates "no risk" otherwise.

3. Gerber–Shiu Risk Measures

Hereafter, we use the following notation:

- $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.
- $\mathbb{D} := \mathbb{D}[0, \infty)$: a space of càdlàg functions with a suitable metric $\|\cdot\|$. Space $\mathbb{D}$ becomes a measurable space with the σ-field generated by open balls.
- Given a measurable space (E, E), we denote, using $\mathcal{M}_t(E)$, a family of $\mathcal{F}_t/E$-measurable maps from Ω to E for each $t \geq 0$. Note that $\mathcal{M}_s(E) \subset \mathcal{M}_t(E)$ for any $s \leq t$.
- For stochastic processes $X = (X_t)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$, we denote $X + Y = (X_t + Y_t)_{t \geq 0}$ and $X \cdot Y = (X_t Y_t)_{t \geq 0}$, an additive operation and a multiplicative operation, respectively, in $\mathcal{M}_t(\mathbb{D})$ for each $t \in [0, T]$.
- For a constant $c \in \mathbb{R}$, we use the same notation for the constant process $c = (c)_{t \geq 0}$.
- For an asset process X, we define a *(dual) loss process* by

  $$\widetilde{X}^u := \left(-X_t^u\right)_{t \in [0, T]}$$

  Note that $\widetilde{X}_0^u = -X_0^u = -u$ a.s.
- For random variables U and V, we define the *stochastic order* $U \leq_{st} V$ as

  $$F_V(x) \leq F_U(x)$$

  for all $x \in \mathbb{R}$, where F_U is the distribution function of U.

Definition 3.1. For a given $\varepsilon > 0$ and Gerber–Shiu risk ϕ_0^X, map $GS_T^\epsilon : \mathcal{M}_T(\mathbb{D}) \to \overline{\mathbb{R}}$, is defined by

$$GS_T^\epsilon\left(\widetilde{X}^u\right) := \inf\left\{z \in \mathbb{R} \middle| \phi_0^X(u+z, T) < \epsilon\right\}$$

GS_T^ϵ is the minimum extra capital to be added to the initial asset to maintain a Gerber–Shiu risk less than $\varepsilon > 0$, which is given because of a strategy of the insurance company. This is a risk measure in the actuarial sense; ε is a level that makes the company's position acceptable. This is motivated from Value-at-Risk (VaR); the concept is not quite new but is a natural extension of the VaR risk measure of ruin theory in Trufin *et al.* (2011) or Loisel & Trufin (2014).

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Lemma 3.1. The map GS_T^ϵ is translation invariant:

$$GS_T^\epsilon\left(\widetilde{X}^u + c\right) = GS_T^\epsilon\left(\widetilde{X}^u\right) + c$$

for any c, $u \in \mathbb{R}$.

Proof: Since $\widetilde{X}^u + c = \widetilde{X}^{u-c}$ by the definition of a loss process, we see that

$$\begin{aligned} GS_T^\epsilon\left(\widetilde{X}^u + c\right) &= \inf\left\{z \in \mathbb{R} \middle| \phi_0^X(u + (z - c),\ T) < \epsilon\right\} \\ &= \inf\left\{v + c \in \mathbb{R} \middle| \phi_0^X(u + v,\ T) < \epsilon\right\} \\ &= c + \inf\left\{v \in \mathbb{R} \middle| \phi_0^X(u + v,\ T) < \epsilon\right\} \\ &= GS_T^\epsilon(\widetilde{X}^u) + c \end{aligned}$$

□

Theorem 3.1. Suppose that an order $\preceq$ in $\mathcal{M}_T(\mathbb{D})$ is defined such that, for any $u \in \mathbb{R}$ and $T > 0$

$$\widetilde{X}^u \preceq \widetilde{Y}^u \quad \Rightarrow \quad \phi_0^X(u,\ T) \le \phi_0^Y(u,\ T) \tag{3.1}$$

Then, map GS_T^ϵ satisfies the monotonicity property: for any $u \in \mathbb{R}$

$$\widetilde{X}^u \preceq \widetilde{Y}^u \quad \Rightarrow \quad GS_T^\epsilon\left(\widetilde{X}^u\right) \le GS_T^\epsilon\left(\widetilde{Y}^u\right)$$

That is, GS_T^ϵ is a risk measure under condition (3.1). Hence, under conditions (2.4) and (3.1), GS_T^ϵ is a risk measure in the sense of Definition 1.1.

Proof: Suppose that $\widetilde{X}^u \preceq \widetilde{Y}^u$ and $x_\epsilon := GS_T^\epsilon\left(\widetilde{X}^u\right)$. From the definition of GS_T^ϵ and property (3.1), it follows for any $\delta > 0$ that

$$\epsilon \le \phi_0^X(x + x_\epsilon + \delta,\ T) \le \phi_0^Y(x + x_\epsilon + \delta,\ T)$$

Since the Gerber–Shiu risk $\phi_0^Y(\cdot,\ T)$ satisfies condition (2.4), we have that

$$x_\epsilon + \delta \le \inf\left\{z \in \mathbb{R} \middle| \phi_0^Y(u + z, T) < \epsilon\right\} = GS_T^\epsilon\left(\widetilde{Y}^u\right)$$

Then, letting $\delta \to 0$, we have confirmed the monotonicity of GS_T^ϵ. With Lemma 3.1, GS_T^ϵ is a risk measure in the sense of Definition 1.1.

□

Remark 3.1. The order $\widetilde{X}^u \preceq \widetilde{Y}^u$ means that process Y is riskier than process X. Since we consider that a process X is riskier if ϕ_0^X is larger (see (2.4)), ordering as in (3.1) is natural. However, how to define ordering (3.1) is not unique, because there could be many senses in which "process X is risky", which should depend on what we consider risk to be. For example, if we consider that "risky" means "earlier ruin", we use ruin probability as a measure of risk. In that case, the risk ordering should be based on a time-of-ruin order; see Example 3.1. In this way, ordering (3.1) should be linked to penalty function w in ϕ_0.

Theorem 3.2. Suppose a Gerber–Shiu risk ϕ_0^X satisfies the following condition:

$$\phi_0^{\lambda X}(\lambda u,\ T) = \phi_0^X(u,\ T) \tag{3.2}$$

for any λ, $T > 0$ and $u \in \mathbb{R}$

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Then, map GS_T^ϵ satisfies the positive homogeneity

$$GS_T^\epsilon\left(\lambda \cdot \widetilde{X}^u\right) = \lambda \cdot GS_T^\epsilon\left(\widetilde{X}\right)^u$$

for any $\lambda > 0$ and $u \in \mathbb{R}$

Proof: By condition (3.2), it follows that

$$\begin{aligned} GS_T^\epsilon\left(\lambda \cdot \widetilde{X}^u\right) &= \inf\{z \in \mathbb{R} | \phi_0^{\lambda X}(\lambda u + z, T) < \epsilon\} \\ &= \inf\{z \in \mathbb{R} | \phi_0^X(u + z/\lambda, T) < \epsilon\} \\ &= \inf\{\lambda v \in \mathbb{R} | \phi_0^X(u + v, T) < \epsilon\} \\ &= \lambda GS_T^\epsilon\left(\widetilde{X}^u\right) \end{aligned}$$

□

Example 3.1. (Finite-time ruin probability). This is a continuation of Example 2.1: $w(x, y) = \mathbf{1}_{\{y<0\}}$ and $\delta = 0$. Let τ_u^X be the time of ruin of asset X^u. We define order $\widetilde{X}^u \preceq \widetilde{Y}^u$ if and only if

$$\tau_u^Y \leq_{st} \tau_u^X$$

which means that a portfolio with an earlier time of ruin is riskier than one with a later time of ruin. Then, $\tau_u^Y \leq_{st} \tau_u^X$ indicates that

$$\phi_0^X(u, T) = \mathbb{P}\left(\tau_u^X \leq T\right) \leq \mathbb{P}\left(\tau_u^Y \leq T\right) = \phi_0^Y(u, T)$$

which is condition (3.1). Moreover, it follows for any $\lambda > 0$ that

$$\phi_0^{\lambda X}(\lambda u, T) = \mathbb{P}\left(\inf_{t \in [0,T]} \lambda X_t^u < 0\right) = \mathbb{P}\left(\inf_{t \in [0,T]} X_t^u < 0\right) = \phi_0^X(u, T)$$

which is condition (3.2). As a result, GS_T^ϵ with $w(x, y) = \mathbf{1}_{\{y<0\}}$ and $\delta = 0$ satisfies the monotonicity, cash invariant, and positive homogeneity properties.

Example 3.2. (Ultimate ruin probability). Consider the case in which $T = \infty$ in the above example; that is, ϕ_0^X is an ultimate ruin probability. Then

$$\begin{aligned} GS_T^\epsilon\left(\widetilde{X}^u\right) &= \inf\{z \in \mathbb{R} | \mathbb{P}(\tau_{u+z} < \infty) \leq \epsilon\} \\ &= \inf\left\{z \in \mathbb{R} \middle| \mathbb{P}\left(\inf_{t \geq 0}(X_t^u + z) < 0\right) \leq \epsilon\right\} \\ &= \inf\left\{z \in \mathbb{R} \middle| \mathbb{P}\left(\sup_{t \geq 0} \widetilde{X}_t^u \leq z\right) \geq 1 - \epsilon\right\} \\ &= VaR_{1-\epsilon}\left(\widetilde{X}_*^u\right) \end{aligned} \tag{3.3}$$

where $\widetilde{X}_*^u := sup_{t \geq 0} \widetilde{X}_t^u$. This is a VaR risk measure for the supremum risk $\widetilde{X}_*^u$. As described in section 1, this measure was investigated by Trufin et al. (2011), who regarded it as a risk measure for individual claim sizes. However, we can justify it mathematically as a risk measure for loss processes belonging to $\mathcal{M}_T(\mathbb{D})$.

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Example 3.3. (Deficit distribution at ruin). This is a continuation of Example 2.2: $\delta = 0$, $w(x, y) = \mathbf{1}_{\{y<\beta\}}$ for any constant $\beta \in \mathbb{R}$. Taking $\epsilon = 1 - \alpha > 0$, we have that

$$GS_T^\epsilon\left(\widetilde{X}^u\right) = \inf\left\{z \in \mathbb{R} \middle| \mathbb{P}\left(-X_{\tau\wedge T}^{u+z} \le \beta\right) \ge \alpha\right\}$$

which is the extra initial capital to keep the probability that the deficit at ruin $(-X_{\tau_u})$ or loss at maturity $(-X_T)$ is less than a given β is larger than a certain level α. In this case, we define order $\widetilde{X}^u \preceq \widetilde{Y}^u$ if and only if

$$-X^u_{\tau^X\wedge T} \le_{st} -Y^u_{\tau^Y\wedge T} \tag{3.4}$$

where τ^X is the ruin time for asset X. That is, we regard the portfolio with stochastically larger loss as riskier than the one with smaller loss. Then, condition (3.1) holds true by the definition of GS_T^ϵ. Hence, GS_T^ϵ is a risk measure in the sense of Definition 1.1. However, positive homogeneity does not necessarily hold.

Remark 3.2. As described in (3.3), measure GS_T^ϵ is a VaR-type risk measure. Since risk measure $VaR_{1-\epsilon} : \mathcal{M}_\infty(\mathbb{R}) \to \mathbb{R}$ is not necessarily subadditive, we cannot expect the subadditivity of GS_T^ϵ either, even if we regard it as a map on $\mathcal{M}_T(\mathbb{D})$.

4. DRMs

4.1. Gerber–Shiu DRMs

In the previous section, we introduce a "static" Gerber–Shiu risk measure. However, in insurance businesses with a maturity T, risks should be measured not only at the beginning of the period, but also dynamically up to maturity. Indeed, market-consistent evaluation of technical provisions (TPs) (as the best estimate plus a risk margin) and the SCR are required in *Solvency II*, or the Swiss solvency test. There has been research using this approach for an actuarial context by Hardy & Wirch (2004), in which the static conditional tail expectation (CTE) measure is modified to a dynamic version as an iterated CTE. Since the purpose of the SCR is to prevent the ruin of the insurance company, it would be natural to use ruin theory (see also Gerber & Loisel, 2012). This motivates us to create a DRM for solvency based on the finite-time Gerber–Shiu risk.

In this section, we construct a risk measure for a loss process $\widetilde{X}^u$ in a certain period, dynamically in time, based on the Gerber–Shiu risk measure GS_T^ϵ, and define an SCR at an arbitrary given time. Initially, we give a definition of *DRM* after the manner of Kriele & Wolf (2014).

Subsequently, let $\widetilde{\mathcal{M}}_t(\mathbb{D})$ be a subset of $\mathcal{M}_t(\mathbb{D})$ for each $t \ge 0$, such that

$$\widetilde{\mathcal{M}}_s(\mathbb{D}) \subset \widetilde{\mathcal{M}}_t(\mathbb{D}), \qquad s \le t$$

Definition 4.1. A DRM on $\widetilde{\mathcal{M}}(\mathbb{D})$ is a family of $\rho = (\rho_t)_{t\in[0,\,T]}$, each of which is a map

$$\rho_t : \widetilde{M}_T(\mathbb{D}) \to \mathcal{M}_t\left(\overline{\mathbb{R}}\right)$$

such that the following two properties hold true:

MO: $\rho_t(X) \le \rho_t(Y)$ a.s. for any $X, Y \in \widetilde{\mathcal{M}}_T(\mathbb{D})$ such that $X \preceq Y$;

TR: $\rho_t(X + C) = \rho_t(X) + C_t$ a.s. for any $C \in \widetilde{\mathcal{M}}_t(\mathbb{D})$.

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In the above, MO is monotonicity with respect to the pre-defined order in $\mathcal{M}_T(\mathbb{D})$, and TR is a dynamic translativity property (in which C_t is simply an $\mathcal{F}_t$-measurable random variable).

The *coherency* of ρ is also defined as follows. A DRM $\rho = (\rho_t)_{t \in [0,\, T]}$ is coherent if the following two properties hold true:

PH: $\rho_t(K \cdot X) = K_t \rho_t(X)$ a.s. for any $K \in \widetilde{\mathcal{M}}_t(\mathbb{D})$ with $K > 0$ a.s.;

SA: $\rho_t(X_1 + X_2) \leq \rho_t(X_1) + \rho_t(X_2)$ a.s. for any $X_1, X_2 \in \widetilde{\mathcal{M}}_T(\mathbb{D})$.

PH is positive homogeneity, in which K_t is an $\mathcal{F}_t$-measurable random variable. SA is subadditivity for process risks.

Now, we define a concrete risk measure due to the Gerber–Shiu function.

Definition 4.2. For each $t \in [0, T]$, we define a map $GS^{\epsilon}_{t,T} : \mathcal{M}_T(\mathbb{D}) \to \mathcal{M}_t(\overline{\mathbb{R}})$ as

$$GS^{\epsilon}_{t,T}\left(\widetilde{X}^u\right) = \inf\left\{z \in \mathbb{R} \middle| \phi^X_t(u+z,\, T) < \epsilon\right\} \quad \text{a.s.}$$

As a convention, we set $\inf\{\varnothing\} = \infty$.

Remark 4.1 As $\tau \leq T$, $GS^{\epsilon}_{t,T}\left(\widetilde{X}^u\right) = \infty$, since $\left\{z \in \mathbb{R} \middle| \phi^X_t(u+z,\, T) < \epsilon\right\} = \varnothing$. As $\tau < T$, $GS^{\epsilon}_{t,T}\left(\widetilde{X}^u\right)$ is $\mathcal{F}_t$-measurable since so is $\phi^X_t(u+z,\, T)$. Hence, $GS^{\epsilon}_{t,T} : \mathcal{M}_T(\mathbb{D}) \to \mathcal{M}_t(\overline{\mathbb{R}})$ is well defined.

That is, $GS^{\epsilon}_{t,T}$ is the minimum extra capital to be added to surplus u (at time $t = 0$) in order to keep the Gerber–Shiu risk in $[t, T]$ less than a level $\epsilon > 0$. The family $GS^{\epsilon}_{\cdot,T} = (GS^{\epsilon}_{t,T})_{t \in [0,T]}$ can be a DRM in the sense of Definition 4.1. Indeed, the following results are immediate from Lemma 3.1 and Theorems 3.1 and 3.2.

Subsequently, let $\mathcal{M}^*$ be a family of Markov processes and let $\mathcal{M}^*_T(\mathbb{D}) = \mathcal{M}_T(\mathbb{D}) \cap \mathcal{M}^*$.

Theorem 4.1. Suppose that an order $\preceq$ is equipped in $\mathcal{M}^*_T(\mathbb{D})$, and that ϕ^X is a Gerber–Shiu risk process for any $X \in \mathcal{M}^*_T(\mathbb{D})$. Then, a restricted map

$$GS^{\epsilon}_{t,T} : \mathcal{M}^*_T(\mathbb{D}) \to \mathcal{M}_t(\mathbb{R})$$

satisfies TR. In addition, if ϕ^X_0 satisfies (3.1) then MO also holds true. That is, $GS^{\epsilon}_{\cdot,T}$ is a DRM on $\mathcal{M}^*_T(\mathbb{D})$ in the sense of Definition 4.1.

Proof: Note that it follows for any $\widetilde{X} \in \mathcal{M}^*_T(\mathbb{D})$ that

$$GS^{\epsilon}_{t,T}\left(\widetilde{X}^u\right) = \begin{cases} \inf\left\{z \in \mathbb{R} \middle| \phi^X_0(X_t + z,\, T - t) < \epsilon\right\} & \text{on } \{\tau > t\} \\ \infty & \text{on } \{\tau \leq t\} \end{cases}$$

by the Markov property and that it is a member of $\mathcal{M}_t(\mathbb{R})$. Then, the proof of MO is the same as in the proof of Theorem 3.1.

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As for the proof of TR, note that for any $C \in \mathcal{M}_t^*(\mathbb{R})$

$$GS_{t,T}^{\epsilon}\left(\widetilde{X}^u + C\right) = \inf\{z \in \mathbb{R} \mid \phi_0^X(X_t - C_t + z,\ T - t) < \epsilon\} \quad \text{a.s.}$$

Since $C \in \mathcal{M}_t^*(\mathbb{D})$, C_t in the above can be treated as a constant with probability one, and TR holds as follows:

$$\begin{aligned} GS_{t,T}^{\epsilon}\left(\widetilde{X}^u + C\right) &= \inf\{z \in \mathbb{R} | \varphi_0^X(X_t + (z - C_t),\ T - t) < \epsilon\} \\ &= \inf\{v + C_t \in \mathbb{R} | \phi_0^X(X_t + v,\ T - t) < \epsilon\} \\ &= GS_{t,\,T}^{\epsilon}\left(\widetilde{X}^u\right) + C_t \end{aligned}$$

□

Theorem 4.2 Suppose $\phi_0^X(u, T)$ satisfies (2.4) and (3.2) for any $X \in \mathcal{M}_T^*(\mathbb{D})$. Then, $GS_{\cdot,T}^{\epsilon}$ satisfies PH.

Proof: Under (3.2), it follows for any $K \in \mathcal{M}_t^*(\mathbb{D})$ that

$$\begin{aligned} GS_{t,T}^{\epsilon}(K \cdot \widetilde{X}) &= \inf\{z \in \mathbb{R} | \phi_0^{K \cdot X}(K_t X_t + z,\ T - t) < \epsilon\} \\ &= \inf\{z \in \mathbb{R} | \phi_0^X(X_t + z / K_t,\ T - t) < \epsilon\} \\ &= \inf\{K_t v \in \mathbb{R} | \phi_0^X(X_t + v,\ T - t) < \epsilon\} \\ &= K_t GS_{t,T}^{\epsilon}(\widetilde{X}) \end{aligned}$$

which is PH. □

From Example 2.1, $GS_{t,T}^{\epsilon}$ with $w(x, y) = \mathbf{1}_{\{y<0\}}$ and $\delta = 0$ is a DRM with positive homogeneity. Moreover, $GS_{t,T}^{\epsilon}$ with $\delta = 0$ and $w(x, y) = \mathbf{1}_{\{y<\beta\}}$ for some $\beta \in \mathbb{R}$ is also a DRM on $\mathcal{M}_T^*(\mathbb{D})$.

4.2. A possible Gerber–Shiu coherent risk measure

As in Remark 3.2, we cannot expect subadditivity for ρ. In this section, we introduce a simple DRM that can be coherent in some sense.

By the same argument as in (2.2), the discounted present value of insurance loss at time t is given by

$$q_t := \mathbb{E}[e^{-\delta(T-t)} L_T | \mathcal{F}_t], \quad t \in [0, T]$$

where

$$L_T := \begin{cases} w(X_{T-}, X_T) & (\tau > T) \\ e^{\delta(T-\tau)} w(X_{\tau-}, X_\tau) & (\tau \le T) \end{cases}$$

That is,

$$q_t = e^{\delta t} \cdot \mathbb{E}\left[e^{-\delta(\tau \wedge T)} w(X_{(\tau\wedge T)-}, X_{\tau \wedge T}) \middle| \mathcal{F}_t\right]$$

We call the random variable

$$Z_{\delta,w,T}^X := e^{-\delta(\tau\wedge T)} w(X_{(\tau\wedge T)-}, X_{\tau\wedge T}) \in \mathcal{M}_T(\mathbb{R})$$

a *finite-time Gerber–Shiu loss* for X. Let $Z_{\delta,w,T}$ be a family of finite-time Gerber–Shiu losses given by

$$Z_{\delta,w,T} := \left\{ Z^X_{\delta,w,T} | X \in \mathcal{M}_T(\mathbb{D}) \right\} \subset \mathcal{M}_T(\mathbb{R})$$

Then, the risk of asset X at time t can be measured by a map $\rho_t : Z_{\delta,w,T} \to \mathcal{M}_t(\overline{\mathbb{R}})$ defined as

$$\varrho_t(Z) = e^{\delta t}\mathbb{E}[Z|\mathcal{F}_t], \quad Z \in Z_{\delta,w,T}$$

Remark 4.2. Since $\varrho = (\varrho_t)_{t\geq 0}$ is a simple conditional expectation of the Gerber–Shiu loss, it is a DRM in the sense of Definition 4.1, and clearly coherent. Moreover, it also holds that

$$\varrho_s(\varrho_t(Z)) = \varrho_s(Z) \quad \text{for any } s < t \in [0, T]$$

which is known as "time consistency" for a DRM; see Cheridito & Kupper (2011) for details.

Remark 4.3. When X is a Markov process, the strong Markov property yields that

$$\varrho_t\left(Z^X_{\delta,w,T}\right) = e^{\delta t} \cdot \phi^X_0(X_t, T)$$

Hence, the computation of ρ_t is reduced to that of the finite-time Gerber–Shiu function, which is possible in principle (see Cojocaru *et al.*, 2014).

Example 4.1. Consider a risk management in [0, T]-term with the following risk measure ϱ:

$$\varrho\left(-X^u_{\tau\wedge T}\right) := \mathbb{E}\left[w\left(-X^u_{\tau\wedge T}\right)\right], \quad -^u_{\tau\wedge T} \in \mathcal{Z}_{0,(-x),T}$$

with $w(x) = 10x\mathbf{1}_{\{x>0\}} + 0.5x\mathbf{1}_{\{x\leq 0\}}$.

The spirit of this penalty function is the following: the company prepares the expected value of decuple of deficit at ruin by T in preparation for ruin; in the non-ruin case, the company can use half of the surplus. The company finally prepares those weighted expectations. This is also a risk measure in an actuarial sense. The conditional expectation

$$\varrho_t = \mathbb{E}[w(-X_{\tau_u\wedge T})|\mathcal{F}_t]$$

is a recalculated version of ρ under $\mathcal{F}_t$.

5. Application To Asset-and-Liability Management (ALM)

In a regulation framework, for example, Solvency II, insurance companies should meet not only *TPs* consisting of best estimates of obligations plus a risk margin, but also an SCR, or at least a *minimum capital requirement (MCR)*, which is required to absorb "unexpected risk" under the company's *going concern*.

From a going-concern view in a solvency regulation, ruin-related risk should be considered, so that companies can continue their business to maturity. For that purpose, Gerber–Shiu risk measures are suited to solvency evaluation. In this section, we propose a method for using Gerber–Shiu DRMs in an ALM context.

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5.1. Solvency and acceptability

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$, and let $X^x = \left(X_t^x\right)_{t\geq 0}$ be a *net asset process*, which is described as

$$X_t = A_t - L_t, \quad X_0 = x$$

where $A = (A_t)_{t\geq 0}$ and $L = (L_t)_{t\geq 0}$ are an asset and a liability process, respectively. Here, liability L means a TP process. Although the evaluation of the best estimate of obligations is also an important topic, it is outside the scope of this paper; thus, we assume that TP has been estimated appropriately. A simple example is given later.

In the Solvency II framework, insurance companies are required to keep a minimum capital level to prevent a bankruptcy, the MCR. For simplicity, we assume the following:

[MCR]: the MCR is given as a constant $d \geq 0$.

This means that a supervisor requires the insurance company to keep its risk margin greater than d any time. See Remark 5.1 for evaluation of the MCR.

If the net asset decreases below d, then the supervisor enforces that the company add extra cash to the position. This case is given by

$$\tau_u := \inf\left\{t > 0 \middle| X_t^x < d\right\} = \inf\left\{t > 0 \middle| X_t^u < 0\right\}, \quad u := x - d$$

which is the time of ruin for net asset X^{x-d}. For solvency at time t, the company should have at least $X_t^x \geq d$. Of course, it is necessary that

$$u > 0$$

To evaluate a risk of a company's position, we use a dynamic Gerber–Shiu risk measure $GS^{\epsilon}_{\cdot,T} = \left(GS^{\epsilon}_{t,T}\right)_{t\geq 0}$, where $T > 0$ is the maturity of the business:

$$GS^{\epsilon}_{t,T} : \mathcal{M}_T(\mathbb{D}) \to \mathcal{M}_t(\overline{\mathbb{R}})$$

Since the Gerber–Shiu risk measure is constructed to prevent the ruin of the company up to the corresponding maturity, we remark that this is a kind of risk measure from a going-concern perspective. If $GS^{\epsilon}_{t,T}(\widetilde{X}) < 0$, the company can absorb a "ruin risk" in $[t, T]$, even if they use cash $-GS^{\epsilon}_{t,T}(\widetilde{X})$ at time t. The degree of going concern is determined by the value of ϵ. Thus, we define solvency as follows, in a manner similar to Wüthrich & Merz (2013) (see also Artzner & Eisele, 2010).

Definition 5.1. Under the MCR condition, an insurance company with a net asset process X^x is said to be solvent from a going-concern perspective at time $t \geq 0$ if the following two conditions are satisfied:

(i) (run-off view) $X_t^x \geq d$;

(ii) (going-concern view) $GS^{\epsilon}_{t,T}\left(\widetilde{X}^x\right) \leq 0$.

Moreover, we say the company is sufficiently solvent from a going-concern view at time $t \geq 0$ if it further satisfies that

(iii) $GS^{\epsilon}_{t,T}\left(\widetilde{X}^u\right) \leq 0$ with $u := x - d$.

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Condition (i) means the company can survive at time t if it cashes out all liabilities at time t. Unless (i) is satisfied, the supervisor intervenes and the finite-time ruin probability is given by $\mathbb{P}(\tau_u \leq T)$. Although this condition is similar to the *accounting condition* in Wüthrich & Merz (2013) (Definition 9.15), the difference is that we consider an MCR.

Condition (ii) corresponds to the *acceptability condition* in Wüthrich & Merz (2013) and Artzner & Eisele (2010), which were also introduced from a run-off view. However, since we use a Gerber–Shiu risk measure up to maturity, condition (ii) is interpreted as a criterion from the going-concern perspective. Note that condition (iii) is more severe than (ii), since

$$GS^{\epsilon}_{t,T}\left(\widetilde{X}^u\right) \geq GS^{\epsilon}_{t,T}\left(\widetilde{X}^x\right) \text{ a.s.}$$

for any x, $d \geq 0$.

Definition 5.2. Suppose that an insurance company with the net asset X^x is solvent from a going-concern view at time t. Then, the SCR at time t, SCR_t, should be defined by

$$SCR_t := X^x_t + GS^{\epsilon}_{t,T}\left(\widetilde{X}^u\right) \quad \text{a.s.,} \quad u := x - d$$

Example 5.1. Consider a simple measure based on the finite-time ruin probability

$$\rho_t := GS^{\epsilon}_{t,T}\left(\widetilde{X}^u\right) = \inf\left\{z \geq 0 \middle| \mathbb{P}\left(\tau_{X^u_t + z} \leq T - t\right) < \epsilon\right\} \tag{5.1}$$

which is the case in which the penalty function $w(x, y) = \mathbf{1}_{\{y<0\}}$ is chosen. It follows from the definition that

$$\mathbb{P}\left(\tau_{X^u_t + \rho_t} \leq T - t\right) \leq \epsilon$$

which implies that $X^u_t + \rho_t \geq 0$ a.s. Then, it follows that

$$SCR_t = X^x_t + \rho_t = X^u_t + \rho_t + d \geq d \quad \text{a.s.}$$

That is, the SCR is always greater than the MCR.

Remark 5.1. Note that the SCR is evaluated to be larger than the MCR because $X^x_t \geq d$. In Solvency II, this is required so that the MCR does not fall below 25% nor exceed 45% of the SCR. If we should determine the MCR initially, it should be that

$$0.25 \cdot SCR_0 \leq d \leq 0.45 \cdot SCR_0 \tag{5.2}$$

See Example 5.2 for more practical computation.

Furthermore, another MCR criterion calculates the requirement as the minimum capital to ensure 85% 1-year survival probability at the initial time. In this case, we can apply measure (5.1). For example

$$d = x + GS^{0.15}_{0,1}\left(\widetilde{X}^x\right) = \inf\{z \geq 0 | \mathbb{P}(\tau_{x+z} \leq 1) < 0.15\} > 0 \quad \text{a.s.}$$

which gives a constant MCR.

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Example 5.2. Suppose that an insurance net asset process X^x is modelled as

$$X_t^x = x + ct - S_t + \sigma W_t$$

where $c > 0$ is a premium rate; S_t is the aggregate insurance liabilities of the form $S_t = \sum_{i=1}^{N_t} U_i$, in which the U_is are i.i.d. positive random sequences with $\mathbb{E}[U_i] = \mu$ and N a Poisson process with intensity λ; and σW a standard Brownian motion representing an asset/liability process of a financial investment with $\sigma^2 > 0$.

Let $T = \infty$ for simplicity. Per Tsai & Willmot (2002), we have the following Cramér-type approximation under suitable integrability conditions:

$$\phi_0^X(u, \infty) \sim \frac{\lambda C(\rho, \gamma) + w(0, 0)(\rho + \gamma)\sigma^2/2}{\lambda \int_0^\infty x e^{\gamma x} F_U(dx) - c + \sigma^2 \gamma} e^{-\gamma u}, \quad u \to \infty$$

where F_U is the distribution of U_i; $\rho \geq 0$ and $-\gamma < 0$ are solutions to the generalised Lundberg equation

$$c\gamma - \frac{\sigma^2}{2}\gamma^2 - \lambda(m_U(\gamma) - 1) = \delta$$

and where

$$C(\rho, \gamma) = \int_0^\infty (e^{\gamma x} - e^{-\rho x}) \int_x^\infty w(x, y - x) F_U(dy) dx$$

Considering a DRM given in the previous example with $T = \infty$:

$$GS_{t,\infty}^{\epsilon}\left(\widetilde{X}^u\right) = \inf\{z \geq 0 \,|\, \phi_0^X(X_t^u + z, \infty) < \epsilon\} \quad \text{on } \{\tau > t\}$$

we have that, on $\{\tau > t\}$

$$GS_{t,\infty}^{\epsilon}\left(\widetilde{X}^u\right) \sim \inf\left\{ z \geq 0 \,\middle|\, \frac{\lambda C(\rho, \gamma) + w(0, 0)(\rho + \gamma)\sigma^2/2}{\lambda \int_0^\infty x e^{\gamma x} F_U(dx) - c + \sigma^2 \gamma} e^{-\gamma\left(z + X_t^u\right)} < \epsilon \right\}, \quad u \to \infty$$

That is,

$$SCR_t \sim s_\epsilon(\rho, \gamma) + d, \quad x \to \infty \tag{5.3}$$

where

$$s_\epsilon(\rho, \gamma) := \frac{1}{\gamma} \log \frac{\lambda C(\rho, \gamma) + w(0, 0)(\rho + \gamma)\sigma^2/2}{\epsilon\left(\lambda \int_0^\infty x e^{\gamma x} F_U(dx) - c + \sigma^2 \gamma\right)}$$

If the MCR (at time t) is determined after criterion (5.2), it should satisfy that

$$0.25(s_\epsilon(\rho, \gamma) + d) \leq d \leq 0.45(s_\epsilon(\rho, \gamma) + d)$$

Equivalently

$$\frac{1}{3} s_\epsilon(\rho, \gamma) \leq d \leq \frac{9}{11} s_\epsilon(\rho, \gamma)$$

Example 5.3. The same approximation is possible for a more general case, in which X^u is a Lévy process starting at u and $GS_{t,\infty}^{\epsilon}\left(\widetilde{X}^u\right)$ is the more general Gerber–Shiu function, since we have the Cramér approximation of the Gerber–Shiu function (see e.g. Feng & Shimizu, 2013).

5.2. Financial and insurance liabilities

Let $\mathbb{D}^d[0, T]$ be a space of d-dimensional càdlàg functions $x = (x_t)_{t \in [0, T]}$ with a suitable metric (e.g. a uniform norm). Let $\mathcal{B}_t^d := \sigma(x : x_s, s \leq t)$ and $\mathbb{B}^d := \left(\mathcal{B}_t^d\right)_{t \in [0, T]}$. We split the filtration $\mathbb{F}$ into two filtrations: $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{T}_t$, where $\mathbb{G} := (\mathcal{G}_t)_{t\in[0,T]}$ is financial information and $\mathbb{T} := (\mathcal{T}_t)_{t\in[0,T]}$ is insurance information. We suppose that G_t and T_t are independent under P and that $\mathcal{F} = \mathcal{G} \vee \mathcal{T}$, where $G = \vee_{t\geq 0} G_t$ and $T = \vee_{t\geq 0} T_t$.

Let us consider an insurance company asset, $R = (R_t)_{t\geq 0}$, of the form

$$R_t = x + P_t + Y_t(F) - S_t - S_t^{'}(F), \quad t \in [0, T]$$

where x is an initial asset of the insurance company, and the other parameters are defined as follows:

- $P : \Omega \to \mathbb{D}[0, T]$ is $\mathbb{T}$-adapted and represents the premium income process.
- $F : \Omega \to \mathbb{D}^d[0, T]$ is $\mathbb{G}$-adapted and represents the value of a financial portfolio.
- $Y : \Omega\times \mathbb{D}^d[0, T] \to \mathbb{D}[0, T]$ is $\mathbb{G} \otimes \mathbb{B}^d$-adapted with $Y_0(F) = 0$ and represents an aggregate financial gain-and-loss due to F; that is, the company has a latent profit at t if $Y_t(F) < 0$ and has a latent loss if $Y_t(F) < 0$.
- $S : \Omega \to \mathbb{D}[0, T]$ is $\mathbb{T}$-adapted and represents insurance aggregate claims and other technical variables.
- $S' : \Omega\times \mathbb{D}^d[0, T] \to \mathbb{D}[0, T]$ is $\mathbb{F} \otimes \mathbb{B}^d$-adapted. Given $x \in \mathbb{D}^d[0, T]$, $S'(x) : \Omega \to \mathbb{D}[0, T]$ is $\mathbb{T}$-adapted. $S'(F)$ represents insurance obligations due to financial variable F, for example, payments for equity-linked insurance or variable annuities.

Note that the aggregate loss up to time t is given by

$$l_t := -Y_t(F) + S_t + S_t^{'}(F)$$

By the best estimate of the future loss l_T at time t, say $R_t(l_T)$, the net asset process is given by

$$X_t^x = x + P_t - R_t(l_T) \tag{5.4}$$

Example 5.4. We give some examples of the aggregate loss process $l = (l_T)_{t\in[0,\ T]}$:

- $S_t = \sum_{i=1}^{N_t} U_i + Z$, where N is a claim number process and U_is are claims. Moreover, we regard Z as an operational risk, which is $\mathcal{T}_T$-measurable.
- Consider a variable annuity with a guaranteed minimum maturity benefit (GMMB) with minimum guarantee G and maturity T. Suppose the company sells a GMMB with premium F_0 to a person who is x years old, which is invested to a stock $F = (F_t)_{t\geq 0}$.

  $$S_t^{'}(F) = e^{-r(T-t)}(G - F_t)_+ \mathbf{1}_{\{\xi_x < T\}} - \int_0^T e^{-rs} \cdot m \cdot F_s\, ds;$$

  $$Y(F)_t = F_t - F_0$$

  where ξ_x is the death time of the insured person and m is the rider fee rate. We further suppose that F satisfies the following stochastic differential equation:

  $$dF_t = a_t dt + b_t dW_t$$

  where a,b are some stochastic processes, and W is a Wiener process.
- $\mathcal{G}_t = \sigma(W_u : u \leq t),\ \mathcal{T}_t = \sigma(S_u : u \leq t) \vee \sigma(Z_u : u \leq t) \vee \sigma(\xi_x)$

To obtain liability $R_t(l_T)$, we consider the following:

- There is a risk-free asset process $B = (B_t)_{t\geq 0}$ (e.g. a bank accounting), which is $\mathbb{G}$-adapted and referred to as a *numérarie*.

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- There exists a probability measure $\mathbb{P}^*$ that is equivalent to $\mathbb{P}$ on $\mathcal{G}$ with density process ζ:

$$\left.\frac{d\mathbb{P}^*}{d\mathbb{P}}\right|_{\mathcal{G}_t} = \zeta_t, \quad \zeta_0 = 1, \quad t \in [0, T] \tag{5.5}$$

Note that ζ is $\mathbb{G}$-martingale under $\mathbb{P}$. We assume that $\mathbb{P}^*$ is a *risk-neutral measure*, under which the discounted financial asset $B_t^{-1}F_t$ is $(\mathbb{P}^*, \mathbb{G})$-martingale.

- There exists a probability measure $\mathbb{P}^\dagger$ that is equivalent to $\mathbb{P}$ on T such that

$$\mathbb{E}^\dagger[S_T|T_t] - \mathbb{E}[S_T|T_t] > 0, \quad t \in [0, T] \tag{5.6}$$

We assume that the insurance premium is calculated under $\mathbb{P}^\dagger$; that is, expectation $\mathbb{E}^\dagger[S_T]$ is an *(actuarial) premium calculation principle*. Condition (5.6) implies the *net profit condition*. We call this $\mathbb{P}^\dagger$ a *(risk-adjusted) distortion probability* or an *insurance technical probability*.

For a stochastic process Z, let

$$\widehat{Z}_{t,T} := \mathcal{B}_t \int_t^T \mathcal{B}_s^{-1} dZ_s$$

If Z is a loss process, $\widehat{Z}_{t,T}$ is a discounted aggregate loss in $[t, T]$ evaluated at time t. Then, a market-consistent present value of liability $\mathcal{R}_t(l_T)$ should be evaluated as

$$\mathcal{R}_t(l_T) := -\left\{Y_t(F) + \mathbb{E}^*\left[\widehat{Y}_{t,T}(F)|\mathcal{G}_t\right]\right\} + \left\{S_t + \mathbb{E}^\dagger\left[\widehat{S}_{t,T}|T_t\right]\right\} + \left\{S_t'(F) + \mathbb{E}^\dagger\left[\widehat{S'}_{t,T}(F) \mid \mathcal{F}_t\right]\right\} \tag{5.7}$$

which is the TP plus a risk margin.

5.3. A simple example of ALM in insurance

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which a Wiener process W and a compound Poisson process S are equipped, and suppose that W and S are independent. We denote by $\mathcal{G}_t = \sigma(W_u : u \le t)$, $\mathcal{T}_t = \sigma(S_u : u \le t)$. For notation in the previous section, we assume the following:

- $S_t = \sum_{i=1}^{N_t} U_i$, where $N_t \sim Po(\lambda t)$ is a number process of insurance claims, and U_is are claim sizes, which are i.i.d. with mean μ.
- $S'(F) \equiv 0$ for simplicity.
- $Y_t(F) = F_t - F_0$ for a stock price F. Suppose that

$$F_t = F_0 + \int_0^t b_u\, du + \int_0^t \sigma_u\, dW_u$$

  where b, σ are $\mathbb{G}$-adapted processes with $\sigma_t > 0$ a.s. for any $t \in [0, T]$.

- Premium income is given by $P_t = (1+\theta)\lambda\mu t$, where $\theta > 0$ is a safety loading.
- $B_t = e^{rt}$, $r > 0$, is a risk-free asset.

The risk-neutral measure $\mathbb{P}^*$ on $(\Omega, \mathcal{G})$ exists as the following: Let

$$\left.\frac{d\mathbb{P}^*}{d\mathbb{P}}\right|_{\mathcal{G}_t} = \exp\left(-\int_0^t \vartheta_s\, dW_s - \frac{1}{2}\int_0^t \vartheta_s^2 ds\right) \quad \text{with } \vartheta_t = \frac{b_t - rF_t}{\sigma_t}$$

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Under some regularity conditions, for example, $\mathbb{E}\left[exp\left(\frac{1}{2}\int_0^T \vartheta_s^2 ds\right)\right] < \infty$, it follows by the Girsanov theorem that

$$W_t^* = \int_0^t \vartheta_s ds + W_t$$

is a $(\mathbb{P}^*, \mathbb{G})$-Brownian motion. From Ito's formula,

$$B_t^{-1} F_t = F_0 + \int_0^t e^{-ru} \sigma_u dW_u^*$$

which is $(\mathbb{P}^*, \mathbb{G})$-martingale; that is, $\mathbb{P}^*$ is the equivalent martingale measure. We assume this regularity throughout this section. Then, the liability for $Y(F)$ is evaluated as follows.

Lemma 5.1. For $Y_t(F) = F_t - F_0$, it holds that

$$\mathbb{E}^*\left[\widehat{Y}_{t,T}(F)|\mathcal{G}_t\right] = \left(1 - e^{-r(T-t)}\right) F_t$$

Proof: Note that

$$\begin{aligned}\widehat{Y}_{t,T}(F) &= e^{rt}\int_t^T e^{-ru} dF_u \\ &= re^{rt}\int_t^T e^{-ru} F_u\, du + e^{rt}\int_t^T e^{-ru}\sigma_u dW_u^*\end{aligned}$$

Since the last term is $(\mathbb{P}^*, \mathbb{G})$-martingale, we see that

$$\mathbb{E}^*[\widehat{Y}_{t,T}(F)|\mathcal{G}_t] = re^{rt}\int_t^T e^{-ru}\mathbb{E}^*[F_u|\mathcal{G}_t] du \tag{5.8}$$

Set $G_u := \mathbb{E}^*[F_u \,|\, \mathcal{G}_t]$ $(u \geq t)$ for fixed t. Then, it follows from a martingale property that

$$\begin{aligned}G_u &= F_0 + \int_0^t rF_s ds + r\int_t^u G_s ds + \int_0^t \sigma_u dW_u^* \\ &= F_t + r\int_t^u G_s ds \qquad \text{a.s.}\end{aligned}$$

Hence, we have that $G_u = e^{r(u-t)} F_t$ $(u \geq t)$, which yields the consequence from (5.8). □

Although there are many possibilities in choosing $\mathbb{P}^\dagger$ to evaluate insurance liabilities, we assume that

$$\mathbb{E}^\dagger[S_t] = \mathbb{E}[(1+\beta)S_t]$$

for some $\beta > 0$ to simplify the argument here.

Setting $S_t^* = S_t - \lambda\mu t$, it is easy to see that S^* is $(\mathbb{P}^\dagger, \mathbb{T})$-martingale. Then

$$\begin{aligned}\mathbb{E}^\dagger\left[\widehat{S}_{t,T}|\mathcal{T}_t\right] &= \mathbb{E}^\dagger\left[e^{rt}\int_t^T e^{-ru} dS_u^* + \lambda\mu e^{rt}\int_t^T e^{-ru}\, du\middle|\mathcal{T}_t\right] \\ &= \frac{\lambda\mu}{r}(1+\beta)\left(1 - e^{-r(T-t)}\right)\end{aligned}$$

Consequently, we have liability estimates from (5.7) as follows:

$$R_t(l_T) = -\left\{\left[1 + \left(1 - e^{-r(T-t)}\right)\right] F_t - F_0\right\} + \left\{S_t + \frac{\lambda\mu}{r}(1+\beta)\left(1 - e^{-r(T-t)}\right)\right\}$$

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and net asset process X is given by

$$X_t = x + \lambda\mu(1+\theta)t - S_t - \frac{\lambda\mu}{r}(1+\beta)\left(1 - e^{-r(T-t)}\right) + \left[2 - e^{-r(T-t)}\right]F_t - F_0$$

When r is sufficiently small, using

$$1 - e^{-r(T-t)} = r(T-t) + o(r), \quad r \to 0$$

we have

$$X_t^x = x - \lambda\mu(1+\beta)T + \lambda\mu(2+\theta+\beta)t - S_t + Y_t(F) + o(1), \quad r \to 0 \tag{5.9}$$

Example 5.5. Consider a simple investment model, such as

$$Y_t(F) = F_t - F_0 = \alpha t + \sigma W_t, \quad \sigma > 0$$

Then, net asset X^x in (5.9) is approximately a diffusion perturbation model in classical ruin theory. In this case, we can approximate SCR_t for $T=\infty$ as in Example 5.2, replacing x with $x_T := x - \lambda\mu(1+\beta)T$ and

$$c = \lambda\mu(2+\theta+\beta+\alpha)$$

That is

$$SCR_t = X_t^x + GS_{t,\infty}^{\epsilon}\left(\widetilde{X}^u\right)$$
$$\sim \frac{1}{\gamma}\log\frac{\lambda C(\rho,\gamma) + w(0,0)(\rho+\gamma)\sigma^2/2}{\epsilon\left(\lambda\int_0^\infty x e^{\gamma x}\mathcal{F}_U(dx) - c + \sigma^2\gamma\right)} + d, \quad x_T \to \infty$$

where positive constants ρ and $-\gamma$ are solutions to

$$\lambda\mu(2+\theta+\beta+\alpha)\gamma - \frac{\sigma^2}{2}\gamma^2 - \lambda(m_U(\gamma) - 1) = \delta$$

In practice, the unknown quantities λ, μ, m_U and γ (the adjustment coefficient) are estimable from insurance claim data, and the volatility, σ^2, is also estimable from stock price data. Then, we can statistically estimate the DRM SCR_t given in (5.3). This simple model could be a benchmark for practical risk management.

6. Concluding Remarks

We propose a *Gerber–Shiu risk measure* to capture the insolvency risk of an insurance company with asset process X. The risk measure would be natural in the sense that it evaluates risks in the vicinity of the time of ruin, τ, such as $X_{\tau-}$ and X_τ, which would reflect the company's risks more accurately than the ruin probability alone. We also give a mathematical justification of the risk measure in accordance with a widely used modern mathematical definition of risk measure. Although similar attempts have been made by several authors, for example, Cojocaru *et al.* (2014) and Mitric & Trufin (2015), our contribution is that we reformulate it as a map on a functional space in which the "risk process" are ordered by *Gerber–Shiu risk*. Thus, we can intuitively understand the Gerber–Shiu risk measure as the usual mathematical risk measure for a company's asset process.

Moreover, we extend the concept to a dynamic version to realise the market-consistent (time-to-time) evaluation of a company's solvency risk. This dynamic version can evaluate the ruin risk for any moment in the future, and we can apply this risk measure to define the solvency of the company from a *going-concern* viewpoint (see section 5.1), which is an important aspect in certain recent solvency regulation frameworks.

In section 5.2, we presented just an idea of an ALM model. Although the examples are simple and ad hoc, it is important to note that we can explicitly compute a "solvency margin" in a classical setting of insurance surplus. For example, the solvency capital requirement (SCR_t) given in Example 5.5 has the clear meaning that it is the minimum extra capital to be added at time t in order to keep the Gerber–Shiu risk below $\epsilon > 0$. The meaning of "Gerber–Shiu risk" can be changed by selecting the penalty function w appropriately. This measure can be used as a benchmark to determine the capital required for claims reserves in practice.

Indeed, there are several problems to overcome for the practical use of this measure, for example, the numerical computation or statistical estimation of the finite-time Gerber–Shiu function, among others. These problems are important issues for the future.

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