APPROXIMATION THEOREMS THROUGHOUT REVERSE MATHEMATICS

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Abstract. Reverse Mathematics (RM) is a program in the foundations of mathematics where the aim is to find the minimal axioms needed to prove a given theorem of ordinary mathematics. Generally, the minimal axioms are equivalent to the theorem at hand, assuming a weak logical system called the base theory. Moreover, many theorems are either provable in the base theory or equivalent to one of four logical systems, together called the *Big Five*. For instance, the *Weierstrass approximation theorem*, i.e., that a continuous function can be approximated uniformly by a sequence of polynomials, has been classified in RM as being equivalent to *weak König's lemma*, the second Big Five system. In this paper, we study approximation theorems for *discontinuous* functions via Bernstein polynomials from the literature. We obtain many equivalences between the latter and weak König's lemma. We also show that slight variations of these approximation theorems fall far outside of the Big Five but fit in the recently developed RM of new 'big' systems, namely the uncountability of $\mathbb{R}$, the enumeration principle for countable sets, the pigeon-hole principle for measure, and the Baire category theorem.

§1. Introduction and preliminaries.

1.1. Aim and motivation. The aim of the program *Reverse Mathematics* (RM; see Section 1.2.1 for an introduction) is to find the minimal axioms needed to prove a given theorem of ordinary mathematics. Generally, the minimal axioms are equivalent to the theorem at hand, assuming a weak logical system called the base theory. The *Big Five phenomenon* is a central topic in RM, as follows.

> [...] we would still claim that the great majority of the theorems from classical mathematics are equivalent to one of the big five. This phenomenon is still quite striking. Though we have some sense of why this phenomenon occurs, we really do not have a clear explanation for it, let alone a strictly logical or mathematical reason for it. The way I view it, gaining a greater understanding of this phenomenon is currently one of the driving questions behind reverse mathematics. (see [57, p. 432])

A natural example is the equivalence between the Weierstrass approximation theorem and weak König's lemma from [83, IV.2.5]. In [68], Dag Normann and the author greatly extend the Big Five phenomenon by establishing numerous equivalences involving the *second-order* Big Five systems on one hand, and well-known *third-order* theorems from analysis about possibly discontinuous functions

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on the other hand, working in Kohlenbach's *higher-order* RM (see Section 1.2.1). Moreover, following [68, Section 2.8], slight variations/generalisations of these third-order theorems cannot be proved from the Big Five and much stronger systems. Nonetheless, an important message of [68] is just how similar second- and higher-order RM can be, as the latter reinforces the existing Big Five with many further examples.

In this paper, we develop the RM-study of approximation theorems, often involving *Bernstein polynomials* B_n, defined as follows:

$$B_n(f,x) := \sum_{k=0}^{n} f\big(k/n\big)\, p_{n,k}(x), \text{ where } p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}. \tag{1.1}$$

Weierstrass [92] and Borel [9, p. 80] already study polynomial approximations while Bernstein [6] provides the first explicit polynomials, namely B_n as above, and showed that for continuous $f : [0,1] \to \mathbb{R}$, we have for all $x \in [0,1]$ that

$$f(x) = \lim_{n\to\infty} B_n(f,x), \tag{1.2}$$

where the convergence is uniform. Approximation results for discontinuous functions are in [18, 33, 43, 71] and[1] take the form (1.2) for points of continuity and the form (1.3) if the left and right limits $f(x-)$ and $f(x+)$ exist at $x \in (0,1)$:

$$\tfrac{f(x+)+f(x-)}{2} = \lim_{n\to\infty} B_n(f,x). \tag{1.3}$$

We stress that for general $f : [0,1] \to \mathbb{R}$, (1.2) may hold at x where f is *dis*continuous, i.e., (1.2) is much weaker than continuity at x. For instance, Dirichlet's function $\mathbb{1}_{\mathbb{Q}}$ satisfies (1.2) for all rationals and is discontinuous everywhere.

In this paper, we develop the RM-study of approximation theorems based on (1.2) and (1.3) for regulated, semi-continuous, cliquish, and Riemann integrable functions. Scheeffer and Darboux study regulated functions in [20, 80] without naming this class, while Baire introduced semi-continuity in [4]. Hankel studies cliquish functions using an equivalent definition in [30]. Thus, there is plenty of historical motivation for our RM-study.

In particular, we show that many approximation theorems for discontinuous functions are equivalent to weak König's lemma in Theorem 2.3. We also show that slight variations fall far outside of the Big Five, but do yield equivalences for four new 'Big' systems, namely *the uncountability of* $\mathbb{R}$ [78], the *enumeration principle for countable sets* [64], the *Baire category theorem* [79], and the *pigeon-hole principle* for measure [79]. These new Big systems boast equivalences involving principles based on pointwise continuity and it is perhaps surprising that the same holds for the weaker condition (1.2). Moreover, the equivalences for these new Big systems are *robust* as follows:

> A system is *robust* if it is equivalent to small perturbations of itself.
> ([57, p. 432]; emphasis in original)

[1] Picard studies approximations of Riemann integrable $f : [0,1] \to \mathbb{R}$ in [71, p. 252] *related* to Berstein polynomials following [18, p. 64].

In particular, our equivalences generally go through for many variations of the function classes involved, with examples in Theorems 3.3 and 3.8. There is an apparent tension here with the earlier observation that slight variations in the theorems can greatly affect the logical strength required to prove them. We attempt to explain this phenomenon in Remark 3.11.

Finally, in Section 1.2.1, we provide an introduction to RM; we introduce essential higher-order concepts in Sections 1.2.2–1.2.4. We establish the equivalences for weak König's lemma in Section 2 and the equivalences for the 'new' big systems in Section 3.2–3.4.

1.2. Preliminaries and definitions. We briefly introduce RM in Section 1.2.1. We introduce some essential axioms (Section 1.2.2) and definitions (Section 1.2.3) needed in the below. We discuss the definition of countable set in higher-order RM in Section 1.2.4.

1.2.1. Reverse Mathematics. RM is a program in the foundations of mathematics initiated around 1975 by Friedman [27, 28] and developed extensively by Simpson and others [22, 82, 83]. The aim of RM is to identify the minimal axioms needed to prove theorems of ordinary, i.e., non-set theoretical, mathematics.

First of all, we refer to [84] for a basic introduction to RM and to [22, 82, 83] for an overview of RM. We expect basic familiarity with RM, in particular Kohlenbach's *higher-order* RM [46] essential to this paper, including the base theory RCA_0^ω, also introduced in Section 4.2. An extensive introduction can be found in, e.g., [61, 62, 65] and elsewhere.

Secondly, as to notations, we have chosen to include a brief introduction as a technical appendix, namely Section 4. All undefined notions may be found in the latter, while we do point out here that we shall sometimes use common notations from type theory. For instance, the natural numbers are type 0 objects, denoted n^0 or $n \in \mathbb{N}$. Similarly, elements of Baire space are type 1 objects, denoted $f \in \mathbb{N}^\mathbb{N}$ or f^1. Mappings from Baire space $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}$ are denoted $Y : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ or Y^2.

Thirdly, the main topic of this paper is the RM-study of real analysis, for which the following notations suffice. Both in RCA_0 and RCA_0^ω, real numbers are given by Cauchy sequences (see Definition 4.4 and [83, II.4.4]) and equality between reals '$=_\mathbb{R}$' has the same meaning in second- and higher-order RM. Functions on $\mathbb{R}$ are defined as mappings Φ from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}^\mathbb{N}$ that respect real equality, i.e., $x =_\mathbb{R} y \to \Phi(x) =_\mathbb{R} \Phi(y)$ for any $x, y \in \mathbb{R}$, which is also called *function extensionality*.

Fourth, experience bears out that the following fragment of the Axiom of Choice is often convenient when (first) proving equivalences. This principle can often be omitted by developing a more sophisticated alternative proof (see, e.g., [63]).

Principle 1.1 ($\mathsf{QF\text{-}AC}^{0,1}$). *For any Y^2, we have*:

$$(\forall n \in \mathbb{N})(\exists f \in \mathbb{N}^\mathbb{N})(Y(f, n) = 0) \to (\exists (f_n)_{n\in\mathbb{N}})(\forall n \in \mathbb{N})(Y(f_n, n) = 0). \quad (1.4)$$

As discussed in [46, Remark 3.13], this principle is not provable in ZF while $\mathsf{RCA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$ suffices to prove the local equivalence between (epsilon–delta) continuity and sequential continuity, which is also not provable in ZF.

Finally, the main difference between Friedman–Simpson and Kohlenbach's framework for RM is whether the language is restricted to *second-order* objects

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or if one allows *third-order* objects. An important message of [68] and this paper is that the second-order Big Five are equivalent to third-order theorems concerning possibly discontinuous functions, as is also clear from Theorems 2.3 and 2.4.

1.2.2. Some comprehension functionals. In second-order RM, the logical hardness of a theorem is measured via what fragment of the comprehension axiom (broadly construed) is needed for a proof. For this reason, we introduce some axioms and functionals related to *higher-order comprehension* in this section. We are mostly dealing with *conventional* comprehension here, i.e., only parameters over $\mathbb{N}$ and $\mathbb{N}^{\mathbb{N}}$ are allowed in formula classes like Π_k^1 and Σ_k^1.

First of all, the functional φ in $(\exists^2)$ is also *Kleene's quantifier* $\exists^2$ and is clearly discontinuous at $f = 11\dots$ in Cantor space:

$$(\exists\varphi^2 \leq_2 1)(\forall f^1)\big[(\exists n^0)(f(n) = 0) \leftrightarrow \varphi(f) = 0\big]. \qquad (\exists^2)$$

In fact, $(\exists^2)$ is equivalent to the existence of $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = 1$ if $x >_{\mathbb{R}} 0$, and 0 otherwise (see [46, Proposition 3.12]). Related to $(\exists^2)$, the functional μ^2 in (μ^2) is called *Feferman's* μ (see [3]) and may be found—with the same symbol—in Hilbert–Bernays' Grundlagen [34, Supplement IV]:

$$(\exists\mu^2)(\forall f^1)\big[(\exists n)(f(n) = 0) \to [f(\mu(f)) = 0 \wedge (\forall i < \mu(f))(f(i) \neq 0)] \qquad (\mu^2)$$
$$\wedge\, [(\forall n)(f(n) \neq 0) \to \mu(f) = 0]\big].$$

We have $(\exists^2) \leftrightarrow (\mu^2)$ over RCA_0^ω (see [46, Section 3]) and $\mathsf{ACA}_0^\omega \equiv \mathsf{RCA}_0^\omega + (\exists^2)$ proves the same sentences as ACA_0 by [35, Theorem 2.5].

Secondly, the functional S^2 in (S^2) is called *the Suslin functional* [46]:

$$(\exists \mathsf{S}^2 \leq_2 1)(\forall f^1)\big[(\exists g^1)(\forall n^0)(f(\overline{g}n) = 0) \leftrightarrow \mathsf{S}(f) = 0\big]. \qquad (\mathsf{S}^2)$$

The system $\Pi_1^1\text{-}\mathsf{CA}_0^\omega \equiv \mathsf{RCA}_0^\omega + (\mathsf{S}^2)$ proves the same Π_3^1-sentences as $\Pi_1^1\text{-}\mathsf{CA}_0$ by [75, Theorem 2.2]. By definition, the Suslin functional S^2 can decide whether a Σ_1^1-formula as in the left-hand side of (S^2) is true or false. We similarly define the functional S_k^2 which decides the truth or falsity of Σ_k^1-formulas from L_2; we also define the system $\Pi_k^1\text{-}\mathsf{CA}_0^\omega$ as $\mathsf{RCA}_0^\omega + (\mathsf{S}_k^2)$, where (S_k^2) expresses that S_k^2 exists. We note that the operators ν_n from [17, p. 129] are essentially S_n^2 strengthened to return a witness (if existent) to the Σ_n^1-formula at hand.

Thirdly, full second-order arithmetic Z_2 is readily derived from $\cup_k \Pi_k^1\text{-}\mathsf{CA}_0^\omega$, or from

$$(\exists E^3 \leq_3 1)(\forall Y^2)\big[(\exists f^1)(Y(f) = 0) \leftrightarrow E(Y) = 0\big], \qquad (\exists^3)$$

and we therefore define $\mathsf{Z}_2^\Omega \equiv \mathsf{RCA}_0^\omega + (\exists^3)$ and $\mathsf{Z}_2^\omega \equiv \cup_k \Pi_k^1\text{-}\mathsf{CA}_0^\omega$, which are conservative over Z_2 by [35, Corollary 2.6]. Despite this close connection, Z_2^ω and Z_2^Ω can behave quite differently, as discussed in, e.g., [61, Section 2.2] and Section 3.1. The functional from $(\exists^3)$ is also called '$\exists^3$', and we use the same convention for other functionals.

Finally, Kleene's quantifier $\exists^2$ plays a crucial role throughout higher-order RM. We recall that $(\exists^2)$ is equivalent to the existence of a discontinuous function on $\mathbb{R}$ (or $\mathbb{N}^{\mathbb{N}}$) by [46, Proposition 3.12], using so-called Grilliot's trick. Thus, $\neg(\exists^2)$ is equivalent to Brouwer's theorem, i.e., all functions on the reals (and Baire space) are continuous. We will often make use of the latter fact without explicitly pointing this out.

1.2.3. Some definitions. We introduce some required definitions, stemming from mainstream mathematics. We note that subsets of $\mathbb{R}$ are given by their characteristic functions as in Definition 1.2, well-known from measure and probability theory. We shall generally work over ACA_0^ω as some definitions make little sense over RCA_0^ω.

First of all, we make use the usual definition of (open) set, where $B(x, r)$ is the open ball with radius $r > 0$ centred at $x \in \mathbb{R}$. We note that our notion of 'measure zero' does not depend on (the existence of) the Lebesgue measure.

Definition 1.2 (Sets).

- A subset $A \subset \mathbb{R}$ is given by its characteristic function $F_A : \mathbb{R} \to \{0, 1\}$, i.e., we write $x \in A$ for $F_A(x) = 1$, for any $x \in \mathbb{R}$.
- A subset $O \subset \mathbb{R}$ is *open* in case $x \in O$ implies that there is $k \in \mathbb{N}$ such that $B(x, \frac{1}{2^k}) \subset O$.
- A subset $O \subset \mathbb{R}$ is *RM-open* in case there are sequences of reals $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ such that $O = \cup_{n\in\mathbb{N}}(a_n, b_n)$.
- A subset $C \subset \mathbb{R}$ is *closed* if the complement $\mathbb{R} \setminus C$ is open.
- A subset $C \subset \mathbb{R}$ is *RM-closed* if the complement $\mathbb{R} \setminus C$ is RM-open.
- A set $A \subset \mathbb{R}$ is *enumerable* if there is a sequence of reals that includes all elements of A.
- A set $A \subset \mathbb{R}$ is *countable* if there is $Y : \mathbb{R} \to \mathbb{N}$ that is injective on A, i.e.,

$$(\forall x, y \in A)(Y(x) =_0 Y(y) \to x =_{\mathbb{R}} y).$$

- A set $A \subset \mathbb{R}$ is *measure zero* if for any $\varepsilon > 0$ there is a sequence of open intervals $(I_n)_{n\in\mathbb{N}}$ such that $\cup_{n\in\mathbb{N}} I_n$ covers A and $\varepsilon > \sum_{n=0}^{\infty} |I_n|$.
- A set $A \subset \mathbb{R}$ is *dense* in $B \subset \mathbb{R}$ if for $k \in \mathbb{N}, b \in B$, there is $a \in A$ with $|a - b| < \frac{1}{2^k}$.
- A set $A \subset \mathbb{R}$ is *nowhere dense* in $B \subset \mathbb{R}$ if A is not dense in any open sub-interval of B.

As discussed in Section 1.2.4, the study of regulated functions already gives rise to open sets that do not come with additional representation beyond the second item in Definition 1.2. We will often assume $(\exists^2)$ from Section 1.2.2 to guarantee that basic objects like the unit interval are sets in the sense of Definition 1.2.

Secondly, we study the following notions, many of which are well-known and hark back to the days of Baire, Darboux, Dini, Jordan, Hankel, and Volterra [4, 5, 20, 21, 30, 31, 41, 90]. We use 'sup' and other operators in the 'virtual' or 'comparative' way of second-order RM (see, e.g., [83, X.1] or [15]). In this way, a formula of the form 'sup $A > a$' or '$x \in \overline{S}$' makes sense as shorthand[2] for a formula in the language of all finite types, even when sup A or the closure $\overline{S}$ need not exist in RCA_0^ω.

[2]For instance, 'sup $A > a$' simply abbreviates $(\exists x \in A)(x > a)$, while '$x \in \overline{S}$' means that there is a sequence of elements in S converging to x.

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Definition 1.3. For $f : [0,1] \to \mathbb{R}$, we have the following definitions:

- f is *upper semi-continuous* at $x_0 \in [0,1]$ if $f(x_0) \geq_{\mathbb{R}} \limsup_{x\to x_0} f(x)$.
- f is *lower semi-continuous* at $x_0 \in [0,1]$ if $f(x_0) \leq_{\mathbb{R}} \liminf_{x\to x_0} f(x)$.
- f *has bounded variation* on $[0,1]$ if there is $k_0 \in \mathbb{N}$ such that $k_0 \geq \sum_{i=0}^{n} |f(x_i) - f(x_{i+1})|$ for any partition $x_0 = 0 < x_1 < \cdots < x_{n-1} < x_n = 1$.
- f is *regulated* if for every x_0 in the domain, the 'left' and 'right' limit $f(x_0-) = \lim_{x\to x_0-} f(x)$ and $f(x_0+) = \lim_{x\to x_0+} f(x)$ exist.
- f is *cadlag* if it is regulated and $f(x) = f(x+)$ for $x \in [0,1)$.
- f is a U_0-function ([1, 29, 40, 54]) if it is regulated and for all $x \in (0,1)$:

$$\min(f(x+), f(x-)) \leq f(x) \leq \max(f(x+), f(x-)). \tag{1.5}$$

- f is *Baire* 1 if it is the pointwise limit of a sequence of continuous functions.
- f is *Baire* 1* if[3] there is a sequence of closed sets $(C_n)_{n\in\mathbb{N}}$ such $[0,1] = \cup_{n\in\mathbb{N}} C_n$ and $f_{\restriction C_m}$ is continuous for all $m \in \mathbb{N}$.
- f is *quasi-continuous* at $x_0 \in [0,1]$ if for $\varepsilon > 0$ and any open neighbourhood U of x_0, there is non-empty open $G \subset U$ with $(\forall x \in G)(|f(x_0) - f(x)| < \varepsilon)$.
- f is *cliquish* at $x_0 \in [0,1]$ if for $\varepsilon > 0$ and any open neighbourhood U of x_0, there is a non-empty open $G \subset U$ with $(\forall x, y \in G)(|f(x) - f(y)| < \varepsilon)$.
- f is *pointwise discontinuous* if for any $x \in [0,1], k \in \mathbb{N}$ there is $y \in B(x, \frac{1}{2^k})$ such that f is continuous at y.
- f is *locally bounded* if for any $x \in [0,1]$, there is $N \in \mathbb{N}$ such that $(\forall y \in B(x, \frac{1}{2^N}) \cap [0,1])(|f(y)| \leq N)$.

As to notations, a common abbreviation is 'usco', 'lsco', and 'BV' for the first three items. Cliquishness and pointwise discontinuity on the reals are equivalent, the non-trivial part of which was already observed by Dini [21, Section 63]. Moreover, if a function has a certain weak continuity property at all reals in $[0,1]$ (or its intended domain), we say that the function has that property. The fundamental theorem about BV-functions was proved already by Jordan in [41, p. 229].

Theorem 1.4 (Jordan decomposition theorem). *A BV-function $f : [0,1] \to \mathbb{R}$ is the difference of two non-decreasing functions $g, h : [0,1] \to \mathbb{R}$.*

Theorem 1.4 has been studied in RM via second-order codes [60]. For a BV-function $f : [0,1] \to \mathbb{R}$, the *total variation* is defined as

$$V_0^1(f) := \sup_{0\leq x_0<\cdots<x_n\leq 1} \sum_{i=0}^{n} |f(x_i) - f(x_{i+1})|. \tag{1.6}$$

Thirdly, the following sets are often crucial in proofs relating to discontinuous functions, as can be observed in, e.g., [1, Theorem 0.36].

Definition 1.5. The sets C_f and D_f (if they exist) respectively gather the points where $f : \mathbb{R} \to \mathbb{R}$ is continuous and discontinuous.

One problem with the sets C_f, D_f is that the definition of continuity involves quantifiers over $\mathbb{R}$. In general, deciding whether a given $\mathbb{R} \to \mathbb{R}$-function is

[3] The notion of Baire 1* goes back to [23] and equivalent definitions may be found in [44]. In particular, Baire 1* is equivalent to the Jayne-Rogers notion of *piecewise continuity* from [39].

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continuous at a given real, is as hard as $\exists^3$ from Section 1.2.2. For these reasons, the sets C_f, D_f only exist in strong systems. A solution is discussed in just below.

In this section, we introduce *oscillation functions* and provide some motivation for their use. We have previously studied usco, Baire 1, Riemann integrable, and cliquish functions using oscillation functions (see [77, 79]). As will become clear, such functions are generally necessary for our RM-study.

Fourth, the study of regulated functions in [64, 66, 67, 78] is really only possible thanks to the associated left- and right limits (see Definition 1.3) *and* the fact that the latter are computable in $\exists^2$. Indeed, for regulated $f : \mathbb{R} \to \mathbb{R}$, the formula

$$f \textit{ is continuous at a given real } x \in \mathbb{R} \tag{C}$$

involves quantifiers over $\mathbb{R}$ but is equivalent to the *arithmetical* formula $f(x+) = f(x) = f(x-)$. In this light, we can define the set D_f—using only $\exists^2$—and proceed with the usual (textbook) proofs. An analogous approach, namely the study of usco, Baire 1, Riemann integrable, and cliquish functions, was used in [77, 79]. To this end, we used *oscillation functions* as in Definition 1.6. We note that Riemann, Ascoli, and Hankel already considered the notion of oscillation in the study of Riemann integration [2, 30, 72], i.e., there is ample historical precedent.

Definition 1.6 (Oscillation functions). For any $f : \mathbb{R} \to \mathbb{R}$, the associated *oscillation functions* are defined as follows: $\mathsf{osc}_f([a, b]) := \sup_{x\in[a,b]} f(x) - \inf_{x\in[a,b]} f(x)$ and $\mathsf{osc}_f(x) := \lim_{k\to\infty} \mathsf{osc}_f(B(x, \frac{1}{2^k}))$.

We stress that $\mathsf{osc}_f : \mathbb{R} \to \mathbb{R}$ is *only*[4] a third-order object, as clearly indicated by its type. Now, our main interest in Definition 1.6 is that (C) is equivalent to the *arithmetical* formula $\mathsf{osc}_f(x) = 0$, assuming the latter function is given. Hence, in the presence of $\mathsf{osc}_f : \mathbb{R} \to \mathbb{R}$ and $\exists^2$, we can define D_f and proceed with the usual (textbook) proofs, which is the approach we *often* took in [77, 79]. Indeed, one can generally avoid the use of oscillation functions for usco functions.

1.2.4. On countable sets. In this section, we discuss the correct definition of countable set for higher-order RM, where we hasten to add that 'correct' is only meant to express 'yields many equivalences over weak systems like the base theory'.

First of all, the correct choice of definition for mathematical notions is crucial to the development of RM, as can be gleaned from the following quote.

> Under the old definition [of real number in [81]], it would be consistent with RCA_0 that there exists a sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ such that $(x_n + \pi)_{n\in\mathbb{N}}$ is not a sequence of real numbers. We thank Ian Richards for pointing out this defect of the old definition. Our new definition [of real number in [16]], given above, is adopted in order to remove this defect. All of the arguments and results of [81] remain correct under the new definition. [16, p. 129]

In short, the early definition of 'real number' from [81] was not suitable for the development of RM, highlighting the importance of the right choice of definition.

[4]To be absolutely clear, the notation 'osc_f' and the appearance of f therein in particular, is purely symbolic, i.e., we do not make use of the fourth-order object $\lambda f.\mathsf{osc}_f$ in this paper.

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Similar considerations exist for the definition of continuous function in constructive mathematics (see [14, 91]), i.e., this situation is not unique to RM.

Secondly, we focus on identifying the correct definition for 'countable set'. Now, going back to Hankel [30], the sets C_f and D_f of (dis)continuity points of $f : [0,1] \to \mathbb{R}$ play a central role in real analysis as is clear from, e.g., the Vitali–Lebesgue theorem which expresses Riemann integrability in terms of C_f. For regulated $f : [0,1] \to \mathbb{R}$, the set of discontinuity points satisfies $D_f = \cup_{k\in\mathbb{N}} D_k$ for

$$D_k := \{x \in [0,1] : |f(x) - f(x+)| > \tfrac{1}{2^k} \vee |f(x) - f(x-)| > \tfrac{1}{2^k}\}, \qquad (1.7)$$

where D_k is finite via a standard compactness argument. In this way, D_f is countable but we are unable to construct an injection (let alone a bijection) from the former to $\mathbb{N}$ in, e.g., Z_2^ω. Hence, we readily encounter countable sets 'in the wild', namely D_f for regulated f, for which the set-theoretic definition based on injections and bijections can apparently not be established in weak logical systems. Similarly, while D_k is closed, Z_2^ω does not prove the existence of an RM-code for D_k. In this way, theorems about countable (or RM-closed) sets *in the sense of Definition 1.2* cannot be applied to D_f or D_k in fairly strong systems like Z_2^ω and the development of the RM of real analysis therefore seemingly falters.

The previous negative surprise notwithstanding, (1.7) also provides the solution to our problem: we namely have $D_f = \cup_{k\in\mathbb{N}} D_k$, i.e., the set D_f is

the union over $\mathbb{N}$ *of finite sets.*

Moreover, this property of D_f *can* be established in a (rather) weak logical system. Thus, we arrive at Definition 1.7 which yields *many* equivalences involving the statement *the unit interval is not height-countable* (see Section 3.2 and [78]) on one hand, and basic properties of regulated functions on the other hand.

DEFINITION 1.7. A set $A \subset \mathbb{R}$ is *height-countable* if there is a *height* function $H : \mathbb{R} \to \mathbb{N}$ for A, i.e., for all $n \in \mathbb{N}$, $A_n := \{x \in A : H(x) < n\}$ is finite.

Height functions can be found in the modern literature [37, 47, 56, 73, 89], but also go back to Borel and Drach circa 1895 (see [10–12]) Definition 1.7 amounts to 'union over $\mathbb{N}$ of finite sets', as is readily shown in ACA_0^ω.

Finally, the observations regarding countable sets also apply *mutatis mutandis* to finite sets. Indeed, finite as each D_n from (1.7) may be, we are unable to construct an injection to a finite subset of $\mathbb{N}$, even assuming Z_2^ω. By contrast, one readily[5] shows that D_n from (1.7) is finite as in Definition 1.8.

DEFINITION 1.8 (Finite set). Any $X \subset \mathbb{R}$ is *finite* if there is $N \in \mathbb{N}$ such that for any finite sequence $(x_0, \dots, x_N)$ of distinct reals, there is $i \le N$ such that $x_i \notin X$.

The number N from Definition 1.8 is called a *size bound* for the finite set $X \subset \mathbb{R}$. Analogous to countable sets, the RM-study of regulated functions should be based on Definition 1.8 and *not* on the set-theoretic definition based on injections/bijections to finite subsets of $\mathbb{N}$ or similar constructs.

[5]The standard compactness argument that shows that D_k is finite as in Definition 1.8 goes through in $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$ by (the proof of) Theorem 3.3.

In conclusion, Definitions 1.7 and 1.8 provide the correct definition of countable set in that they give rise to many RM-equivalences involving basic properties of regulated functions, as established in [78]. By contrast, the set-theoretic definition involving injections and bijections, does not seem to seem suitable for the development of (higher-order) RM.

§2. Equivalences involving weak König's lemma. We establish equivalences between WKL_0 and approximation theorems for Bernstein polynomials for (dis)continuous functions. We sketch similar results for the other Big Five systems ACA_0 and ATR_0. The RM-study of Bernstein polynomial approximation of course hinges on the following set, definable in ACA_0^ω as the defining formula is arithmetical:

$$B_f := \{x \in [0,1] : f(x) = \lim_{n\to\infty} B_n(f,x)\}. \tag{2.1}$$

Similarly, the left and right limits of regulated functions can be found using $(\exists^2)$ (see [64, Section 3]), which is another reason why we shall often incorporate the latter in our base theory.

First of all, we establish the equivalence between WKL_0 and the Weierstrass approximation theorem via Bernstein polynomials ([6]), as the proof is instructive for that of Theorem 2.2 and essential to that of Theorems 2.3 and 2.4.

THEOREM 2.1 (RCA_0^ω). *The following are equivalent*:

- WKL_0.
- *For continuous* $f : [0,1] \to \mathbb{R}$ *and* $x \in (0,1)$, *we have*

$$f(x) = \lim_{n\to\infty} B_n(f,x), \tag{2.2}$$

where the convergence is uniform.

PROOF. First of all, it is well-known that RCA_0 can prove basic facts about basic objects like polynomials, as established in, e.g., [83, II.6]. Similarly, the following can be proved in RCA_0, for $p_{n,k}(x) := \binom{n}{k}x^k(1-x)^{n-k}$ and any $x \in [0,1], m \leq n$:

$$p_{m,n}(x) \geq 0 \text{ and } \textstyle\sum_{k\leq n}\left(x - \frac{k}{n}\right)^2 p_{n,k}(x) = \frac{x(1-x)}{n} \text{ and } \sum_{k\leq n} p_{n,k}(x) = 1 \tag{2.3}$$

using, e.g., binomial expansion and similar basic properties of polynomials.

Secondly, by [83, IV.2.5], WKL_0 is equivalent to the Weierstrass approximation theorem for *codes* of continuous functions. Every code for a continuous function on [0, 1] denotes a third-order (continuous) function by [68, Theorem 2.2], which can be established directly by applying $\mathsf{QF\text{-}AC}^{1,0}$ from RCA_0^ω to the formula expressing the totality of the code on [0, 1]. Hence, the upward implication follows.

Thirdly, for the remaining implication, WKL_0 implies that continuous functions are *uniformly* continuous on [0, 1], both for codes (see [83, IV.2.3]) and for third-order functions (see [68, Theorem 2.3]). Hence, fix continuous $f : [0,1] \to \mathbb{R}$ and $k_0 \in \mathbb{N}$ and let $N_0 \in \mathbb{N}$ be such that $(\forall x, y \in [0,1])(|x-y| < \frac{1}{2^{N_0}} \to |f(x) - f(y)| < \frac{1}{2^{k_0+1}})$. Clearly, f is bounded on [0, 1], say by $M_0 \in \mathbb{N}$. Now fix $x_0 \in (0,1)$, choose

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$n \geq 2^{2M_0+2N_0+k_0+1}$, and consider

$$
\begin{aligned}
|f(x_0) - B_n(f,x_0)| &= |f(x_0) - \textstyle\sum_{k=0}^{n} f\big(\tfrac{k}{n}\big) p_{n,k}(x_0)| \\
&= |\textstyle\sum_{k=0}^{n} \big(f(x_0) - f\big(\tfrac{k}{n}\big)\big) p_{n,k}(x_0)| \\
&\leq \textstyle\sum_{k=0}^{n} \big|f\big(\tfrac{k}{n}\big) - f(x_0)\big| p_{n,k}(x_0) \\
&= \textstyle\sum_{i=0,1} \sum_{k\in A_i} \big|f\big(\tfrac{k}{n}\big) - f(x_0)\big| p_{n,k}(x_0), \qquad (2.4)
\end{aligned}
$$

where $A_0 := \{k \leq n : |x_0 - \frac{k}{n}| \leq \frac{1}{2^{N_0}}\}$ and $A_1 := \{0, 1, \dots n\} \setminus A_0$, and where the second equality follows by the final formula in (2.3). The sets A_i $(i = 0, 1)$ exist in RCA_0 by *bounded comprehension* (see [83, X.4.4]). Now consider

$$\textstyle\sum_{k\in A_0} \big|f\big(\tfrac{k}{n}\big) - f(x_0)\big| p_{n,k}(x_0) \leq \frac{1}{2^{k_0+1}} \sum_{k\in A_0} p_{n,k}(x_0) \leq \frac{1}{2^{k_0+1}} \sum_{k\leq n} p_{n,k}(x_0) = \frac{1}{2^{k_0+1}},$$

where the final equality follows by (2.3). For the other sum in (2.4), we have

$$\textstyle\sum_{k\in A_1} \big|f\big(\tfrac{k}{n}\big) - f(x_0)\big| p_{n,k}(x_0) \leq 2M_0 \sum_{k\in A_1} \frac{(x_0-\frac{k}{n})^2}{1/2^{2N_0}} p_{n,k}(x_0) \leq 2M_0 \sum_{k\leq n} \frac{(x_0-\frac{k}{n})^2}{1/2^{2N_0}} p_{n,k}(x_0),$$

where $k \in A_1$ implies $\frac{(x_0-\frac{k}{n})^2}{1/2^{2N_0}} \geq 1$ by definition and where $p_{n,k}(x_0) \geq 0$ is also used. Now apply the second formula from (2.3) to the final formula in the previous centred equation to obtain

$$\textstyle\sum_{k\in A_1} \big|f\big(\tfrac{k}{n}\big) - f(x_0)\big| p_{n,k}(x_0) \leq 2M_0 2^{2N_0} \frac{x_0(1-x_0)}{n} \leq \frac{1}{2^{k_0+1}}.$$

Thus, we have obtained $|f(x_0) - B_n(f,x_0)| \leq \frac{1}{2^{k_0}}$ and uniform convergence. ⊣

Next, we wish to generalise the previous theorem to discontinuous functions, for which the following theorem from the literature (see, e.g., [33, Theorem 5.1] or [18, Section 4, p. 68]) is essential, namely to Theorems 2.3 and 2.4.

THEOREM 2.2 (ACA_0^ω). *For any $f : [0,1] \to [0,1]$ and $x \in (0,1)$ such that $f(x+)$ and $f(x-)$ exist*[6], *we have*

$$\frac{f(x+)+f(x-)}{2} = \lim_{n\to\infty} B_n(f,x). \qquad (2.5)$$

PROOF. First of all, the proof of the theorem is similar to that of Theorem 2.1, but more complicated as $f(x_0)$ in (2.4) is replaced by $\frac{f(x_0+)+f(x_0-)}{2}$ in (2.6). In particular, (2.6) involves more sums than (2.1), but the only 'new' part is to show that (2.11) becomes arbitrarily small. An elementary proof of this fact is tedious but straightforward. For this reason, we have provided a sketch with ample references.

Secondly, fix $k_0 \in \mathbb{N}$ and $x_0 \in \mathbb{R}$ such that the left and right limits $f(x_0-)$ and $f(x_0+)$ exist. By definition, there is $N_0 \in \mathbb{N}$ such that for any $y, z \in [0,1]$:

$$x_0 - \tfrac{1}{2^{N_0}} < z < x_0 < y < x_0 + \tfrac{1}{2^{N_0}} \to \big[|f(x_0+) - f(y)| < d\,\tfrac{1}{2^{k_0+1}} \wedge |f(x_0-) - f(z)| < d\,\tfrac{1}{2^{k_0+1}}\big].$$

[6] Using $(\exists^2)$, rational approximation yields $\Phi : [0,1] \to \mathbb{R}^2$ such that $\Phi(x) = (f(x+), f(x-))$ in case the limits exist at $x \in [0,1]$. In this way, '$\lambda x.f(x+)$' makes sense for regulated functions.

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Note that increasing N_0 does not change the previous property. Now use $(\exists^2)$ to define $A_0 := \{k \leq n : x_0 \leq k/n < x_0 + \frac{1}{2^{N_0}}\}$, $A_1 := \{k \leq n : x_0 - \frac{1}{2^{N_0}} \leq k/n < x_0\}$, and $A_2 := \{k \leq n : |x_0 - k/n| \geq \frac{1}{2^{N_0}}\}$ and consider:

$$\begin{aligned} B_n(f, x_0) - \tfrac{f(x_0+)+f(x_0-)}{2} &= \textstyle\sum_{k=0}^{n} \big(f\big(\tfrac{k}{n}\big) - \tfrac{f(x_0+)+f(x_0-)}{2}\big) p_{n,k}(x_0) \\ &= \textstyle\sum_{i=0}^{2} \sum_{k\in A_i} \big(f\big(\tfrac{k}{n}\big) - \tfrac{f(x_0+)+f(x_0-)}{2}\big) p_{n,k}(x_0), \end{aligned} \tag{2.6}$$

where the first equality follows by the final formula in (2.3) for $x = x_0$.

Thirdly, we note that $k \in A_2$ implies $\frac{(x_0 - \frac{k}{n})^2}{1/2^{2N_0}} \geq 1$ by definition, yielding

$$\textstyle\sum_{k\in A_2} p_{n,k}(x_0) \leq \sum_{k\in A_2} \frac{(x_0-\frac{k}{n})^2}{1/2^{2N_0}} p_{n,k}(x_0) \leq \sum_{k\leq n} \frac{(x_0-\frac{k}{n})^2}{1/2^{2N_0}} p_{n,k}(x_0), \tag{2.7}$$

where the second inequality holds since $p_{n,k}(x_0) \geq 0$ by (2.3). Now apply the second formula in (2.3) to (2.7) to obtain $\sum_{k\in A_2} p_{n,k}(x_0) \leq 2^{2N_0} \frac{x_0(1-x_0)}{n}$. Since f is assumed to be bounded on $[0, 1]$, we have for $n \geq 2^{2N_0+k_0+1}$:

$$\big|\textstyle\sum_{k\in A_2} \big(f\big(k/n\big) - \frac{f(x_0+)+f(x_0-)}{2}\big) p_{n,k}(x)\big| < \frac{1}{2^{k_0+1}}. \tag{2.8}$$

Fourth, we consider another sum from (2.6), namely the following:

$$\begin{aligned} &\textstyle\sum_{k\in A_0} \big(f\big(\tfrac{k}{n}\big) - \tfrac{f(x_0+)+f(x_0-)}{2}\big) p_{n,k}(x_0) \\ &= \textstyle\sum_{k\in A_0} (f(\tfrac{k}{n}) - f(x_0+)) p_{n,k}(x_0) + \tfrac{f(x_0+)-f(x_0-)}{2} \big(\sum_{k\in A_0} p_{n,k}(x_0)\big). \end{aligned} \tag{2.9}$$

By the choice of N_0, the first sum in (2.9) satisfies

$$\begin{aligned} |\textstyle\sum_{k\in A_0} (f(\tfrac{k}{n}) - f(x_0+)) p_{n,k}(x_0)| &\leq \textstyle\sum_{k\in A_0} |f(\tfrac{k}{n}) - f(x_0+)| p_{n,k}(x_0) \\ &\leq \tfrac{1}{2^{k_0+1}} \textstyle\sum_{k\in A_0} p_{k,n}(x_0) \leq \tfrac{1}{2^{k_0+1}}, \end{aligned} \tag{2.10}$$

where the final step in (2.10) follows from the final formula in (2.3). The same, namely a version of (2.9) and (2.10), holds *mutatis mutandis* for A_1. Thus, consider (2.11) which consists of the second sum of (2.9) and the analogous sum for A_1:

$$\tfrac{f(x_0+)-f(x_0-)}{2} \big(\textstyle\sum_{k\in A_0} p_{n,k}(x_0) - \sum_{k\in A_1} p_{n,k}(x_0)\big). \tag{2.11}$$

Now, $p_{n,k}(x)$ is well-known as the *binomial distribution* and the former's properties are usually established in probability theory via conceptual results like the *central limit theorem*. In particular, for large enough N_0 and associated n, the sums $\sum_{k\in A_i} p_{n,k}(x_0)$ for $i = 0, 1$ from (2.11) can be shown to be arbitrarily close to $\frac{1}{2}$. In light of (2.11), a proof (in ACA_0^ω) of this limiting behaviour of $\sum_{k\in A_i} p_{n,k}(x_0)$ (for $i = 0, 1$) establishes (2.5) and the theorem.

Finally, an *elementary* proof of the limit behaviour of $\sum_{k\in A_i} p_{n,k}(x_0)$ $(i = 0, 1)$ proceeds along the following lines, based on the *de Moivre–Laplace theorem* which is apparently a predecessor to the central limit theorem.

- First of all, *Stirling's formula* provides approximations to the factorial $n!$. There are *many* elementary proofs of this formula (see, e.g., [13, 19, 24, 38]).

- Secondly, the de Moivre–Laplace theorem states approximations to the sum $\sum_{k=k_1}^{k_2} p_{n,k}(x)$ in terms of the Gaussian integral $\int e^{\frac{-t^2}{2}} dt$ ([70]). There are elementary proofs of this approximation using (only) Stirling's formula, namely [25, Chapter VII, Section 3, p. 182] or [19, Theorem 6, p. 228].
- Thirdly, applying the previous for $\sum_{k\in A_i} p_{n,k}(x_0)$ $(i = 0, 1)$, one observes that the only non-explicit term in the latter is $\int_0^{+\infty} e^{\frac{-t^2}{2}} dt = \frac{\sqrt{2\pi}}{2}$; the former sum is therefore arbitrarily close to $\frac{1}{2}$ for N_0 and n large enough.

It is a tedious but straightforward verification that the above proofs can be formalised in ACA_0^ω (and likely RCA_0). Alternatively, one establishes basic properties of the Γ-function (see [13]) and derives the de Moivre–Laplace theorem [70]. ⊣

The final part of the previous proof involving $\sum_{k\in A_i} p_{n,k}(x_0)$, amounts to the special case of the theorem for (a version of) of the Heaviside function as follows:

$$H(x) := \begin{cases} 1, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Building on the previous theorem, we can now develop the RM-study of approximation theorem for discontinuous functions.

THEOREM 2.3 (RCA_0^ω). *The following are equivalent, where* $I \equiv [0, 1]$.

(a) WKL_0.
(b) *For any function* $f : [0, 1] \to \mathbb{R}$, f *is continuous on* $(0, 1)$ *if and only if the equation* (2.2) *holds uniformly on* $(0, 1)$.
(c) *For any cadlag* $f : I \to \mathbb{R}$ *and* $x \in (0, 1)$, *we have*

$$\tfrac{f(x+)+f(x-)}{2} = \lim_{n\to\infty} B_n(f, x), \tag{2.12}$$

with uniform convergence if $C_f = I$.
(d) *For any* U_0*-function* $f : I \to \mathbb{R}$ *and any* $x \in (0, 1)$, (2.12) *holds, with uniform convergence if* $C_f = I$.

If we additionally assume $\mathsf{QF}\text{-}\mathsf{AC}^{0,1}$, *the following are equivalent to* WKL_0.

(e) *For any regulated* $f : I \to \mathbb{R}$ *and* $x \in (0, 1)$, (2.12) *holds, with uniform convergence if* $C_f = I$.
(f) *For any locally bounded* $f : I \to \mathbb{R}$ *and any* $x \in (0, 1)$ *such that* $f(x+)$ *and* $f(x-)$ *exist*, (2.12) *holds, with uniform convergence if* $C_f = I$.
(g) *The previous item restricted to any function class containing* $C([0, 1])$.

PROOF. First of all, to derive item (b) from WKL_0, let $f : [0, 1] \to \mathbb{R}$ be such that (2.12) holds uniformly. Now fix $k_0 \in \mathbb{N}$ and let $N_0 \in \mathbb{N}$ be such that for $n \geq N_0$ and $x \in (0, 1)$, $|B_n(f, x) - f(x)| < \frac{1}{2^{k_0}}$. Then $B_{N_0}(f, x)$ is uniformly continuous, say with modulus h ([83, IV.2.9]). Fix $x, y \in (0, 1)$ with $|x - y| < \frac{1}{2^{h(k_0)}}$ and consider

$$|f(x) - f(y)| \leq |f(x) - B_{N_0}(f, x)| + |B_{N_0}(f, x) - B_{N_0}(f, y)| \\ + |B_{N_0}(f, y) - f(y)| \leq \frac{3}{2^{k_0}}.$$

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To show that item (a) follows from items (b)–(g), note that each of the latter implies the second item from Theorem 2.1, i.e., WKL_0 follows.

Secondly, to show that item (a) implies items (b)–(g), we invoke the law of excluded middle as in $(\exists^2) \vee \neg(\exists^2)$. In case $\neg(\exists^2)$, all functions on the reals are continuous by [46, Proposition 3.12]. In this case, WKL_0 implies the other items from the theorem by Theorem 2.1. In case $(\exists^2)$ holds, items (b)–(g) follow by Theorem 2.2, assuming we can provide an upper bound to the functions at hand. To this end, if $f : [0,1] \to \mathbb{R}$ is unbounded, apply $\mathsf{QF\text{-}AC}^{0,1}$ to the formula

$$(\forall n \in \mathbb{N})(\exists x \in [0,1])(|f(x)| > n), \tag{2.13}$$

yielding a sequence $(x_n)_{n\in\mathbb{N}}$ such that $|f(x_n)| > n$ for all $n \in \mathbb{N}$; the latter sequence has a convergent sub-sequence, say with limit $y \in [0,1]$, by sequential compactness [83, III.2]. Clearly, f is not locally bounded at y and hence also not regulated. We note that for unbounded cadlag or U_0-functions, (2.13) reduces to $(\forall n \in \mathbb{N})(\exists q \in [0,1] \cap \mathbb{Q})(|f(q)| > n)$, i.e., we can use $\mathsf{QF\text{-}AC}^{0,0}$ (included in RCA_0^ω). ⊣

The 'excluded middle trick' from the previous proof should be used sparingly as some of our mathematical notions do not make much[7] sense in RCA_0^ω. Using suitable modulus functions (for the definition of $f(x+)$ and $f(x-)$), one could express uniform convergence of (2.12) for regulated functions; this does not seem to yield very elegant results, however.

Next, we briefly treat equivalences involving ACA_0^ω (and ATR_0) and approximation theorems via Bernstein polynomials. Note that the functions g, h in the second item of Theorem 2.4 can be discontinuous and recall the set from (2.1). For the final item, there are *many* function classes between BV and regulated, like the functions of bounded Waterman variation (see [1]).

THEOREM 2.4 (RCA_0^ω). *The following are equivalent to* ACA_0 :

(a) (*Jordan*) *For continuous* $f : [0,1] \to \mathbb{R}$ *in* BV, *there are continuous and non-decreasing* $g, h : [0,1] \to \mathbb{R}$ *such that* $f = g - h$.
(b) *For continuous* $f : [0,1] \to \mathbb{R}$ *in* BV, *there are non-decreasing* $g, h : [0,1] \to \mathbb{R}$ *such that* $f = g - h$ *and* $B_g = B_h = [0,1]$.
(c) *For continuous* $f : [0,1] \to \mathbb{R}$ *in* BV, *there are non-decreasing* $g, h : [0,1] \to \mathbb{R}$ *such that* $f = g - h$ *and* B_g *and* B_h *are dense in* $[0,1]$.
(d) *For continuous* $f : [0,1] \to \mathbb{R}$ *in* BV, *there are non-decreasing* $g, h : [0,1] \to \mathbb{R}$ *such that* $f = g - h$ *and* B_g *and* B_h *have measure* 1.
(e) *For continuous* $f : [0,1] \to \mathbb{R}$ *in* BV, *there are non-decreasing* $g, h : [0,1] \to \mathbb{R}$ *such that* $f = g - h$ *and* B_g *and* B_h *are non-enumerable.*

PROOF. The equivalence between ACA_0 and item (a) is proved in [60] *for RM-codes*. Now, RM-codes for continuous functions denote third-order functions by [68, Theorem 2.2], working in RCA_0^ω. Moreover, a continuous and non-decreasing function on $[0,1]$ is determined by the function values on $[0,1] \cap \mathbb{Q}$, i.e., one readily obtains an RM-code for such functions in RCA_0^ω. In this way, item (a) is equivalent to ACA_0 as well. Item (b) follows from item (a) by Theorem 2.2. To show that item

[7]Following Definition 1.2, the unit interval is not a set in RCA_0^ω; a more refined framework for the study of open sets may be found in [69].

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(b) implies item (a), invoke the law of excluded middle as in $(\exists^2) \vee \neg(\exists^2)$. In case $(\exists^2)$ holds, ACA_0 and item (a) is immediate. In case $\neg(\exists^2)$ holds, all functions on $\mathbb{R}$ are continuous [46, Proposition 3.12] and item (a) trivially follows from item (b). The other items are treated in (exactly) the same way. $\dashv$

We could also generalise Theorem 2.4 using *pseudo-monotonicity* [1, Definition 1.14], originally introduced by Josephy [42] as the largest class such that composition with BV maps to BV. The RM of ATR_0 as in [68, Theorem 2.25] includes the Jordan decomposition theorem restricted to BV-functions with an *arithmetical* graph. One readily shows that ATR_0 is equivalent to the third-to-fifth items in Theorem 2.4 with 'continuous' removed and restricted to 'arithmetical $f : [0,1] \to \mathbb{R}$'.

Next, we establish an equivalence for $(\exists^2)$ involving Bernstein polynomials. We note that 'splittings' as in Theorem 2.5 are rare in second-order RM, but not in higher-order RM, as studied in detail in [76]. One can prove that WKL_0 in the theorem cannot be replaced by *weak weak König's lemma* (see, e.g., [83, X.1]).

THEOREM 2.5 (RCA_0^ω). *The following are equivalent to* $(\exists^2)$:

(a) *There is* $f : [0,1] \to [0,1]$ *and* $x \in (0,1)$ *such that* $f(x) \neq \lim_{n\to\infty} B_n(f,x)$.
(b) WKL_0 (*or* ACA_0) *plus*: *there is* $f : [0,1] \to \mathbb{R}$ *and* $x \in (0,1)$ *such that* $f(x) \neq \lim_{n\to\infty} B_n(f,x)$.
(c) WKL_0 (*or* ACA_0) *plus*: *there is* $f : [0,1] \to \mathbb{R}$ *that is not bounded* (*or*: *not Riemann integrable*, *or*: *not uniformly continuous*).

We cannot remove WKL_0 *from the second or third item*.

PROOF. That $(\exists^2)$ implies items (a) and (b) follows by applying Theorem 2.2 to the (suitably modified) Heaviside function. Item (c) follows from $(\exists^2)$ by considering, e.g., Dirichlet's function $\mathbb{1}_{\mathbb{Q}}$ for the Riemann integrable case. Now let $f : [0,1] \to [0,1]$ be as in item (a) and use Theorem 2.2 to conclude that f is discontinuous; then $(\exists^2)$ follows by [46, Proposition 3.12].

Next, assume item (b) and suppose $f : [0,1] \to \mathbb{R}$ as in the latter is continuous on $[0,1]$. Then WKL_0 implies that f is bounded [68, Theorem 2.8]; now use Theorem 2.2 to obtain a contradiction, i.e., f must be discontinuous and $(\exists^2)$ follows as before. That item (c) implies $(\exists^2)$ follows in the same way. Finally, observe that by Theorem 2.1, $\mathsf{RCA}_0^\omega + \neg\mathsf{WKL}_0$ proves: *there is* $f : [0,1] \to \mathbb{R}$ *and* $x \in (0,1)$ *such that* $f(x) \neq \lim_{n\to\infty} B_n(f,x)$. Since $\neg\mathsf{WKL}_0 \to \neg(\exists^2)$, the final sentence follows and we are done. $\dashv$

Finally, the previous results are not unique: [8, Theorem 2] expresses that for $f : [-1,1] \to \mathbb{R}$ of bounded variation and $x_0 \in (-1,1)$, the lim sup and lim inf of the Hermite–Fejér polynomial $H_n(f,x)$ is some (explicit) term involving $f(x_0+)$ and $f(x_0-)$; this term reduces to $f(x)$ in case f is continuous at x. The proof of this convergence result is moreover lengthy but straightforward, i.e., readily formalised in ACA_0^ω. Perhaps surprisingly, the general case essentially reduces to the particular case of the Heaviside function, like in Theorem 2.2. A more complicated, but conceptually similar, approximation result may be found in [51]. Moreover, the *Bohman–Korovkin theorem* [55] suggests near-endless variations of Theorem 2.3.

§3. Equivalences involving new Big systems.

3.1. Introduction. In the below sections, we establish equivalences between the new Big systems from [64, 78, 79] and properties of Bernstein polynomials for (dis)continuous functions, as sketched in Section 1.1.

- The *uncountability of* $\mathbb{R}$ is equivalent to the statement that for regulated functions, the Bernstein polynomials converge to the function value for at least one real (Section 3.2).
- The *enumeration principle* enum for countable sets is equivalent to the statement that for regulated functions, the Bernstein polynomials converge to the function value for all reals outside of a given sequence (Section 3.3).
- The *pigeon-hole principle for measure* is equivalent to the statement that for Riemann integrable functions, the Bernstein polynomials converge to the function value *almost everywhere* (Section 3.4).
- The *Baire category theorem* is equivalent to the statement that for semi-continuous functions, the Bernstein polynomials converge to the function value for at least one real (Section 3.5).

The second item provides interesting insights into the coding practice of RM: while the Banach space $C([0,1])$ can be given a code in $\mathsf{RCA}_0^\omega + \mathsf{WKL}$, the relatively powerful principle enum is required to code the Banach space of regulated functions by item (c) of Theorem 3.8.

Now, by the results in [64, 65], the relatively strong system $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}AC}^{0,1}$ cannot prove the uncountability of $\mathbb{R}$ formulated as follows

$$\mathsf{NIN}_{[0,1]}: \text{ there is no injection from } [0,1] \text{ to } \mathbb{N},$$

where Z_2^ω proves the same second-order sentences as *second-order arithmetic* Z_2 (see [35] and Section 1.2.2); the system Z_2^Ω does prove $\mathsf{NIN}_{[0,1]}$, as many of the usual proofs of the uncountability of $\mathbb{R}$ show. The above-itemised principles, i.e., the Baire category theorem, the enumeration principle, and the pigeon-hole principle for measure, which are studied in Sections 3.3 and 3.4, all imply $\mathsf{NIN}_{[0,1]}$. In this way, certain approximation theorems are classified in the Big Five by Theorems 2.3 and 2.4, while slight variations or generalisations go far beyond the Big Five in light of Theorems 3.3, 3.8, 3.17, and 3.15, but are still equivalent to known principles, in accordance with the general theme of RM.

Finally, at least two of the above systems yield conservative extensions of ACA_0, where $\mathsf{PHP}_{[0,1]}$ expresses that for a sequence of closed sets of measure zero, the union also has measure zero (see Section 3.4).

THEOREM 3.1.

- *The system* $\mathsf{ACA}_0^\omega + \mathsf{NIN}_{[0,1]}$ *is* Π_2^1*-conservative over* ACA_0.
- *The system* $\mathsf{ACA}_0^\omega + \mathsf{PHP}_{[0,1]}$ *is* Π_2^1*-conservative over* ACA_0.

PROOF. Kreuzer shows in [50] that $\mathsf{ACA}_0^\omega + (\boldsymbol{\lambda})$ is Π_2^1-conservative over ACA_0, where $(\boldsymbol{\lambda})$ expresses the existence of the Lebesgue measure $\boldsymbol{\lambda}^3$ as a fourth-order functional on $2^{\mathbb{N}}$ and $[0,1]$. To derive $\mathsf{NIN}_{[0,1]}$ in the former system, let $Y:[0,1]\to\mathbb{N}$ be an injection and derive a contradiction from the sub-additivity of $\boldsymbol{\lambda}$ as follows:

$$\boldsymbol{\lambda}([0,1]) = \boldsymbol{\lambda}(\cup_{n\in\mathbb{N}} E_n) \le \textstyle\sum_{n\in\mathbb{N}} \boldsymbol{\lambda}(E_n) = 0,$$

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where $E_n := \{x \in [0,1] : Y(x) = n\}$ is at most a singleton, i.e., $\boldsymbol{\lambda}(E_n) = 0$. The system $\mathsf{ACA}_0^\omega + (\boldsymbol{\lambda})$ trivially proves $\mathsf{PHP}_{[0,1]}$. ⊣

3.2. The uncountability of the reals. In this section, we establish equivalences between approximation theorems involving Bernstein polynomials and the uncountability of the reals. Our results also improve the base theory used in [78].

First of all, we will study the uncountability of the reals embodied by the following principle, motivated by the observations in Section 1.2.4. As discussed in Section 3.1, $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}AC}^{0,1}$ does not[8] prove $\mathsf{NIN}_{\mathsf{alt}}$, and the same for the equivalent approximation theorems in Theorem 3.3.

PRINCIPLE 3.2 ($\mathsf{NIN}_{\mathsf{alt}}$). *The unit interval is not height-countable.*

This principle was first introduced in [78] where *many* equivalences are established, mainly for basic properties of regulated functions and related classes.

Secondly, we have the following theorem involving equivalences for $\mathsf{NIN}_{\mathsf{alt}}$. Item (d) is defined using B_f from (2.1) and constitutes a variation of Volterra's early theorem from [90] that there is no $f : \mathbb{R} \to \mathbb{R}$ satisfying $C_f = \mathbb{Q}$. In light of the first equivalence, the restriction in item (c) is non-trivial.

THEOREM 3.3 ($\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$). *The following are equivalent.*

(a) *The uncountability of* $\mathbb{R}$ *as in* $\mathsf{NIN}_{\mathsf{alt}}$.
(b) *For regulated* $f : [0,1] \to \mathbb{R}$, *there is* $x \in (0,1) \setminus \mathbb{Q}$ *where* f *is continuous.*
(c) *For regulated and pointwise discontinuous* $f : [0,1] \to \mathbb{R}$, *there is* $x \in (0,1) \setminus \mathbb{Q}$ *where* f *is continuous.*
(d) *(Volterra) There is no regulated* $f : [0,1] \to \mathbb{R}$, *such that* $B_f = \mathbb{Q}$.
(e) *For regulated* $f : [0,1] \to \mathbb{R}$, *there is* $x \in (0,1)$ *where* f *is continuous.*
(f) *For regulated* $f : [0,1] \to \mathbb{R}$, *there is* $x \in (0,1)$ *where* $f(x) = \lim_{n\to\infty} B_n(f,x)$.
(g) *For regulated* $f : [0,1] \to \mathbb{R}$, *the set* B_f *is not (height-)countable.*

PROOF. First of all, (a)→(b) is proved by contraposition as follows: let $f : [0,1] \to \mathbb{R}$ be regulated and discontinuous on $[0,1] \setminus \mathbb{Q}$. In particular, $[0,1] \setminus \mathbb{Q} = D_f = \cup_{k\in\mathbb{N}} D_k$ where D_k is as in (1.7). To show that D_k is finite, suppose it is not, i.e., for any $N \in \mathbb{N}$, there are $x_0, \dots x_N \in D_k$. Use $(\exists^2)$ and $\mathsf{QF\text{-}AC}^{0,1}$ to obtain a sequence $(x_n)_{n\in\mathbb{N}}$ in D_k. Since sequential compactness follows from ACA_0 [83, III.2.2], the latter sequence has a convergent sub-sequence, say with limit $y \in [0,1]$. Then either on the left or on the right of y, one finds infinitely many elements of the sequence. Then either $f(y+)$ or $f(y-)$ does not exist, a contradiction, and D_k is finite.

To establish $\neg\mathsf{NIN}_{\mathsf{alt}}$, let $(q_m)_{m\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}$ without repetitions and define a height function $H : [0,1] \to \mathbb{N}$ for $[0,1]$ as follows: $H(x) = n$ in case $x \in D_n$ and n is the least such number, $H(x) = m$ in case $x = q_m$ otherwise. By contraposition and using $\mathsf{QF\text{-}AC}^{0,1}$, one proves that the union of two finite sets is finite, implying that $D_k \cup \{q_0, \dots, q_k\}$ is finite. A direct proof in ACA_0^ω is also possible as the second set has an (obvious) enumeration. Hence, $[0,1]$ is height countable, as required for $\neg\mathsf{NIN}_{\mathsf{alt}}$.

[8] Note that an injection is a special kind of height function, i.e., $\mathsf{NIN}_{\mathsf{alt}}$ implies that there is no injection from $[0,1]$ to $\mathbb{N}$.

Secondly, (b)→(d) is immediate by Theorem 2.2 and the fact that $f(x+) = f(x-) = f(x)$ in case $x \in C_f$. Now assume item (d) and suppose $\mathsf{NIN}_{\mathsf{alt}}$ is false, i.e., $H : [0,1] \to \mathbb{N}$ is a height function for $[0,1]$. Define $g : [0,1] \to [0,1]$ using $(\exists^2)$:

$$g(x) := \begin{cases} \frac{1}{2^{H(x)}} & x \notin \mathbb{Q}, \\ 0 & x \in \mathbb{Q}. \end{cases}$$

To show that $g(x+) = g(x-) = 0$ for any $x \in (0,1)$, consider

$$(\forall k \in \mathbb{N})(\exists N \in \mathbb{N})(\forall y \in (x - \tfrac{1}{2^N}, x))(|g(y)| < \tfrac{1}{2^k}). \tag{3.1}$$

Suppose (3.1) is false, i.e., there is $k_0 \in \mathbb{N}$ such that $(\forall N \in \mathbb{N})(\exists y \in (x - \frac{1}{2^N}, x))(|f(y)| \geq \frac{1}{2^k})$. Modulo the coding of real numbers, apply $\mathsf{QF\text{-}AC}^{0,1}$ to obtain a sequence of reals $(y_n)_{n\in\mathbb{N}}$ such that

$$(\forall N \in \mathbb{N})(y_N \in (x - \tfrac{1}{2^N}, x) \wedge |g(y_N)| \geq \tfrac{1}{2^{k_0}}). \tag{3.2}$$

Using μ^2, we can guarantee that each y_n is unique using the (limited) primitive recursion[9] available in RCA_0^ω. Now, by definition, the set $C := \{x \in [0,1] : H(x) \leq k_0\}$ is finite, say with upper bound $K_0 \in \mathbb{N}$. Apply (3.2) for $N = K_0 + 2$ and note that $y_0, \dots, y_{K_0+2}$ are in C by the definition of g, a contradiction. Hence, (3.1) is correct, implying that g is regulated. Now apply item (d) to obtain $x_0 \in (0,1) \setminus \mathbb{Q}$ such that $g(x_0) = \lim_{n\to\infty} B_n(g, x_0)$. However, $g(x_0) > 0$ and $B_n(g, x_0) = 0$, a contradiction, and $\mathsf{NIN}_{\mathsf{alt}}$ must hold. The equivalence involving item (c) is immediate as g is continuous at every rational point in $[0,1]$, i.e., pointwise discontinuous.

Thirdly, we immediately have (b)→(e)→(f) by Theorem 2.2, and to show that item (f) implies $\mathsf{NIN}_{\mathsf{alt}}$, we again proceed by contraposition. To this end, let $H : \mathbb{R} \to \mathbb{N}$ be a height function for $[0,1]$ and define $h(x) := \frac{1}{2^{H(x)+1}}$. We now prove prove $\lim_{n\to\infty} B_n(h, x) = 0$ for all $x \in [0,1]$. As Bernstein polynomials only invoke rational function values, $B_n(h, x) = B_n(\tilde{h}, x)$ for all $x \in [0,1]$ and $n \in \mathbb{N}$, where

$$\tilde{h}(x) := \begin{cases} 0 & \text{if } x \notin \mathbb{Q}, \\ \frac{1}{2^{H(x)+1}} & \text{otherwise}, \end{cases} \tag{3.3}$$

which is essentially a version of Thomae's function [68, 86]. Similar to g above, one verifies that $\tilde{h}$ is continuous on $[0,1] \setminus \mathbb{Q}$ using the fact that H is a height function. By Theorem 2.2, we have $0 = \tilde{h}(x) = \lim_{n\to\infty} B_n(\tilde{h}, x)$ for $x \in (0,1) \setminus \mathbb{Q}$. By the uniform continuity of the polynomials $p_{k,n}(x)$ on $[0,1]$, we also have $\lim_{n\to\infty} B_n(\tilde{h}, q) = 0$ for $q \in [0,1] \cap \mathbb{Q}$. As a result, we obtain $\lim_{n\to\infty} B_n(h, x) = 0$ for all $x \in [0,1]$, which implies the negations of items (e) and (f), as $h(x) > 0$ for all $x \in [0,1]$. Clearly, item (g) implies item (f) while the former readily follows from $\mathsf{NIN}_{\mathsf{alt}}$. Indeed, if item (g) is false for some regulated $f : [0,1] \to \mathbb{R}$, then $C_f \subset B_f = \cup_{n\in\mathbb{N}} E_n$ by Theorem 2.2 for a sequence of finite sets $(E_n)_{n\in\mathbb{N}}$. Since $D_f = \cup_{n\in\mathbb{N}} D_n$, $[0,1]$ is height-countable as $D_n \cup E_n$ is finite by the first paragraph of the proof. ⊣

[9] Define $h(z) := (\mu m)(z < x - \frac{1}{2^m})$, $H(0) := y_0$ and $H(n+1) = y_{h(H(n))}$.

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We note that the base theory in the previous theorem is much weaker than in [78]. Indeed, in the latter, the base theory also contained various axioms governing finite sets, all provable from the induction axiom. It seems that the latter can always be replaced by proofs by contradiction involving $\mathsf{QF\text{-}AC}^{0,1}$.

It is interesting that we can study items (b) and (d) without using (basic) real analysis, while items (e) and (f) seem to (really) require the study of the Bernstein approximation of (3.3). Theorem 3.3 goes through for 'regulated' replaced by 'bounded variation' if we assume, e.g., a small fragment of the classical function hierarchy, namely that a regulated function has bounded Waterman variation [1].

Next, we show that we can weaken the conclusion of item (e) in Theorem 3.3 to the weak continuity notions from Definition 3.4; these are a considerable improvement over, e.g., quasi-continuity in [78] and go back over a hundred years, namely to Young [93] and Blumberg [7].

Definition 3.4 (Weak continuity). For $f : [0,1] \to \mathbb{R}$, we have that:

- f is *almost continuous* (Husain, see [7, 36]) at $x \in [0,1]$ if for any open $G \subset \mathbb{R}$ containing $f(x)$, the set $\overline{f^{-1}(G)}$ is a neighbourhood of x.
- f has the *Young condition* at $x \in [0,1]$ if there are sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ on the left and right of x with the latter as limit and $\lim_{n\to\infty} f(x_n) = f(x) = \lim_{n\to\infty} f(y_n)$.

With these definitions in place, we now have the following corollary to Theorem 3.3, which establishes a nice degree of robustness for the RM of $\mathsf{NIN}_{\mathsf{alt}}$.

Corollary 3.5. *Theorem 3.3 holds if we replace 'continuity' in item (e) by 'almost continuity' or 'the Young condition'.*

Proof. Assuming $\mathsf{NIN}_{\mathsf{alt}}$, the function $h : [0,1] \to \mathbb{R}$ from the proof of Theorem 3.3 does not satisfy either weak continuity notion anywhere. Indeed, the sequences from the Young condition are immediately seen to violate the fact that H from the proof of Theorem 3.3 is a height-function for [0, 1]. Similarly, take any $x_0 \in [0,1]$ and note that for $k_0 \in \mathbb{N}$ such that $\frac{1}{2^{k_0}} < h(x_0)$, the set $f^{-1}(B(x_0, \frac{1}{2^{k_0}}))$ is finite, as H is a height-function for [0, 1]. ⊣

3.3. The enumeration principle. In this section, we establish equivalences between approximation theorems involving Bernstein polynomials and the enumeration principle for height-countable sets as in Principle 3.6. Our results significantly improve the base theory used in [64] and provide the 'definitive' RM of Jordan's decomposition theorem [41], especially in light of the new connection to Helly's selection theorem as in Principle 3.7.

First of all, we will study Principle 3.6, motivated by Section 1.2.4.

Principle 3.6 (enum). *A height-countable set in* [0, 1] *can be enumerated.*

We stress that textbooks (see, e.g., [1, p. 28] and [74, p. 97]) generally only prove that certain sets are (height) countable, i.e., no enumeration is provided, while the latter is readily assumed in other places, i.e., enum is implicit in the mathematical mainstream. By the results in [64], $\mathsf{ACA}_0^\omega + \mathsf{enum}$ proves ATR_0 and $\Pi_1^1\text{-}\mathsf{CA}_0^\omega + \mathsf{enum}$ proves $\Pi_2^1\text{-}\mathsf{CA}_0$, i.e., enum is rather 'explosive', in contrast to $\mathsf{NIN}_{[0,1]}$ by Theorem 3.1.

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A variation of enum for countable sets, called cocode_0, is studied in [64] where many equivalences are established; the base theories used in [64] are however not always elegant. Now, most proofs in [64] go through *mutatis mutandis* for 'countable' replaced by 'height-countable'; the latter's central role was not known during the writing of [64]. We provide some examples in Theorem 3.8, including new results for Bernstein approximation. We also need the following principle which is a contraposed version of *Helly's selection theorem* [32, 58]; the RM of the latter for *codes* of BV-functions is studied in, e.g., [49].

PRINCIPLE 3.7 (Helly). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of $[0,1]\to[0,1]$-functions in BV with pointwise limit $f:[0,1]\to[0,1]$ which is not in BV. Then there is unbounded $g\in\mathbb{N}^{\mathbb{N}}$ such that $g(n)\le V_0^1(f_n)\le g(n)+1$ for all $n\in\mathbb{N}$.*

Intuitively, Helly is a rather weak statement that significantly simplifies the RM-study of enum and NIN_{alt}.

Next, we establish Theorem 3.8 where we note that the base theory is weaker than in [64]. Indeed, in the latter, the base theory also contains various axioms governing countable sets, mostly provable from the induction axiom.

THEOREM 3.8 ($\text{ACA}_0^\omega+\text{QF-AC}^{0,1}$). *The following are equivalent.*

(a) *The enumeration principle* enum.
(b) *For regulated $f:[0,1]\to\mathbb{R}$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ enumerating D_f.*
(c) *For regulated $f:[0,1]\to\mathbb{R}$ and $p,q\in[0,1]\cap\mathbb{Q}$, $\sup_{x\in[p,q]}f(x)$ exists*[10].
(d) *For regulated and pointwise discontinuous $f:[0,1]\to\mathbb{R}$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ enumerating D_f.*
(e) *For regulated $f:[0,1]\to\mathbb{R}$, there is $(x_n)_{n\in\mathbb{N}}$ enumerating $[0,1]\setminus B_f$.*
(f) *(Jordan) For $f:\mathbb{R}\to\mathbb{R}$ which is in $BV([0,a])$ for all $a>0$, there are monotone $g,h:\mathbb{R}\to\mathbb{R}$ such that $f(x)=g(x)-h(x)$ for $x\ge 0$.*
(g) *The combination of:*
    (f.1) *Helly's selection theorem as in Principle 3.7.*
    (f.2) *(Jordan) For $f:[0,1]\to\mathbb{R}$ in BV, there are non-decreasing $g,h:\mathbb{R}\to\mathbb{R}$ such that $f(x)=g(x)-h(x)$ for $x\in[0,1]$.*

PROOF. The implication (a)→(b) follows by noting that D_k as in (1.7) is finite, as established in the proof of Theorem 3.3, and applying enum. The implication (b)→(e) follows from Theorem 2.2. The implication (b)→(c) is immediate as we can replace the supremum over p,q in $\sup_{x\in[p,q]}f(x)$ by a supremum over $[p,q]\cap(\mathbb{Q}\cup D_f)$.

For the implication (e)→(a), let A be height-countable, i.e., there is $H:\mathbb{R}\to\mathbb{N}$ such that $A_n:=\{x\in[0,1]:H(x)<n\}$ is finite. Note that we can use μ^2 to enumerate $A\cap\mathbb{Q}$, i.e., we may assume the latter is empty. Now define $g:[0,1]$ as

$$g(x):=\begin{cases}\frac{1}{2^{n+1}} & \text{if } x\in A \text{ and } n \text{ is the least natural such that } H(x)<n,\\ 0 & \text{if } x\notin A.\end{cases} \tag{3.4}$$

[10]To be absolutely clear, we assume the existence of a 'supremum operator' $\Phi:\mathbb{Q}^2\to\mathbb{R}$ such that $\Phi(p,q)=\sup_{x\in[p,q]}f(x)$ for all $p,q\in[0,1]\cap\mathbb{Q}$. For Baire 1 functions, this kind of operator exists in ACA_0^ω by [68, Section 2], even for irrational intervals.

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The function g satisfies $g(x+) = g(x-) = 0$ for $x \in (0, 1)$, which is proved in exactly the same way as in the proof of Theorem 3.3. Let $(x_n)_{n\in\mathbb{N}}$ be the sequence provided by item (e) and note that by Theorem 2.2, $(\forall n \in \mathbb{N})(x \neq x_n)$ implies $g(x) = \frac{g(x+)+g(x-)}{2} = 0$ for any $x \in (0, 1)$. Hence, $(x_n)_{n\in\mathbb{N}}$ includes all elements of A, and $(\exists^2)$ can remove all elements not in A, as required for enum. Item (b) trivially implies item (d), while g from (3.4) is continuous on the rationals and therefore pointwise discontinuous, i.e., item (d) also implies item (a). To show that item (c) implies enum, apply the former to g as in (3.4). In particular, one enumerates D_g using the usual interval-halving technique.

For the implication (a)→(f), let f be as in the former and define $D_{n,k}$ as

$$D_{n,k} := \{x \in [0, n] : |f(x) - f(x+)| > \tfrac{1}{2^k} \vee |f(x) - f(x-)| > \tfrac{1}{2^k}\},$$

where we note that BV-functions are regulated (using $\textsf{QF-AC}^{0,1}$) by [65, Theorem 3.33]. As for D_k in (1.7), this set is finite and $D_f = \cup_{n,k\in\mathbb{N}} D_{n,k}$ can be enumerated thanks to enum. Now consider the *variation function* defined as

$$V_0^y(f) := \sup_{0\leq x_0<\cdots<x_n\leq y} \sum_{i=0}^{n} |f(x_i) - f(x_{i+1})|, \tag{3.5}$$

where the supremum is over all partitions of $[0, y]$. Since we have an enumeration of D_f, (3.5) can be defined using $\exists^2$ by restricting the supremum to $\mathbb{Q}$ and this enumeration. By definition, $g(x) := \lambda x.V_0^y(f)$ is non-decreasing and the same for $h(x) := g(x) - f(x)$. For the reverse implication, let A be height-countable, i.e., there is $H : \mathbb{R} \to \mathbb{N}$ such that $A_n := \{x \in [0, 1] : H(x) < n\}$ is finite. Then let $f : \mathbb{R} \to \mathbb{R}$ be the indicator function of A_n on $[0, n]$, which is BV on $[0, n]$ by [65, Theorem 3.33]. Now apply item (f) and note that $(\exists^2)$ can enumerate D_g for monotone g by [65, Theorem 3.33], i.e., enum now follows.

To establish item (g) using enum, sub-item (f.2) is a special case of item (f). For sub-item (f.1), consider Helly's selection theorem, usually formulated as follows.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of BV-functions such that f_n and $V_0^1(f_n)$ are uniformly bounded. Then there is a sub-sequence $(f_{n_k})_{k\in\mathbb{N}}$ with pointwise limit $f \in BV$.

To establish the centred statement, Helly's original proof from [32, p. 287] or [58, p. 222] goes through in $\textsf{ACA}_0^\omega + \textsf{enum}$ as follows: for a sequence $(f_n)_{n\in\mathbb{N}}$ in BV as above, one uses enum to obtain sequences of monotone functions $(g_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ such that $f_n = g_n - h_n$. Then $(g_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ have convergent sub-sequences with limits g and h that are monotone, which is (even) provable in $\textsf{ACA}_0^\omega$. Essentially by definition, $f = g - h$ is then the limit of the associated sub-sequence of $(f_n)_{n\in\mathbb{N}}$. To obtain sub-item (f.1), use the contraposition of the centred statement, i.e., if the limit function f is not in BV, then $(\forall N \in \mathbb{N})(\exists n)(V_0^1(f_n) \geq N)$. As in the previous paragraph, $V_0^1(f_n)$ can be defined using $\exists^2$, given enum. The function $g \in \mathbb{N}^\mathbb{N}$ from the conclusion of sub-item (f.1) is therefore readily obtained (using $\textsf{QF-AC}^{0,0}$ and $\exists^2$).

For the remaining implication (g)→(a), let A be height-countable, i.e., there is $H : \mathbb{R} \to \mathbb{N}$ such that $A_n := \{x \in [0, 1] : H(x) < n\}$ is finite. As above, we may assume $A \cap \mathbb{Q} = \emptyset$ as $\exists^2$ can enumerate the rationals in A. Define $f_n(x) := \mathbb{1}_{A_n}(x)$, which is BV by [65, Theorem 3.33] and note $\lim_{n\to\infty} f_n = f$ where $f := \mathbb{1}_A$. If $f \in BV$, apply sub-item (f.2) and recall that $(\exists^2)$ can enumerate D_g for monotone

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g by [65, Theorem 3.33], i.e., enum now follows. If $f \notin BV$, let $g_0 \in \mathbb{N}^{\mathbb{N}}$ be the function provided by sub-item (f.1) and note that $V_0^1(f_n) \leq g_0(n) + 1$ implies that A_n is finite *and* has size bound $g_0(n) + 2$. Now let $\tilde{g} : [0, 1] \to \mathbb{R}$ be (3.4) with '$\frac{1}{2^n}$' replaced by '$\frac{1}{2^n(g_0(n)+2)}$'. Clearly, for any partition of $[0, 1]$, we have $\sum_{i=0}^n |\tilde{g}(x_i) - \tilde{g}(x_{i+1})| \leq \sum_{i=0}^n \frac{1}{2^i} \leq 2$, i.e., $\tilde{g}$ is in BV. Applying sub-item (f.2) to $\tilde{g}$, enum follows as before, and we are done. ⊣

Regarding item (e) in Theorem 3.8, basic examples show that $C_f \neq B_f$, while the first equivalence in Theorem 3.3 shows that the restriction in item (d) is non-trivial. Item (c) expresses that the Banach space of regulated functions requires enum, in contrast to, e.g., the Banach space of continuous functions ([83, IV.2.13]). Moreover, the use of monotone functions in Theorem 3.8 can be replaced by weaker conditions, as follows.

COROLLARY 3.9. *One can replace 'monotone' in items* (f) *or* (g) *by*:

- U_0*-function, or*:
- *regulated* $f : [0, 1] \to \mathbb{R}$ *such that for all* $x \in (0, 1)$, *we have*

$$|f(x) - \lim_{n\to\infty} B_n(f, x)| \leq |\tfrac{f(x+)-f(x-)}{2}|. \tag{3.6}$$

PROOF. By [68, Theorem 2.16], $(\exists^2)$ suffices to enumerate the discontinuity points for functions satisfying the items in the corollary. ⊣

Next, Helly's theorem as in Principle 3.7 is useful in the RM of the uncountability of $\mathbb{R}$, as follows.

THEOREM 3.10 $(\mathsf{ACA}_0^\omega + \mathsf{Helly})$. *The following are equivalent.*

(a) *The uncountability of* $\mathbb{R}$ *as in* $\mathsf{NIN}_{\mathsf{alt}}$.
(b) (*Volterra*) *There is no* $f : [0, 1] \to \mathbb{R}$ *in* BV, *such that* $B_f = \mathbb{Q}$.
(c) *For* $f : [0, 1] \to \mathbb{R}$ *in* BV, *there is* $x \in (0, 1)$ *where f is continuous.*
(d) *For* $f : [0, 1] \to \mathbb{R}$ *in* BV, *there is* $x \in (0, 1)$ *where* $f(x) = \lim_{n\to\infty} B_n(f, x)$.
(e) *For* $f : [0, 1] \to \mathbb{R}$ *in* BV, *there is* $x \in (0, 1)$ *such that* (1.5) *or* (3.6).

PROOF. That $\mathsf{NIN}_{\mathsf{alt}}$ implies the items (b)–(e) follows as in the proof of Theorem 3.3, namely by considering D_k from (1.7). For $f \in BV$ with variation bounded by 1, this set has at most 2^n elements, as each element of D_n contributes at least $\frac{1}{2^n}$ to the total variation, by definition. Hence, D_k is finite *without* the use of $\mathsf{QF\text{-}AC}^{0,1}$. Now, $\mathsf{NIN}_{\mathsf{alt}}$ implies that $D_f = \cup_{k\in\mathbb{N}} D_k$ is not all of $[0, 1]$, i.e., $C_f \neq \emptyset$ and the other items follow. To derive $\mathsf{NIN}_{\mathsf{alt}}$ from items (b)–(e), let $H : [0, 1] \to \mathbb{N}$ be a height-function for $[0, 1]$ and consider the finite sets $A_n = \{x \in [0, 1] : H(x) < n\}$ and $B_n = A_n \setminus \mathbb{Q}$. As above, $\mathbb{1}_{B_n}$ is in BV but $\lim_{n\to\infty} \mathbb{1}_{B_n} = \mathbb{1}_{\mathbb{R}\setminus\mathbb{Q}}$ is not in BV. Let $g_0 \in \mathbb{N}^{\mathbb{N}}$ be the function provided by Helly's selection theorem and let $g_1 \in \mathbb{N}^{\mathbb{N}}$ be such that $|A_n \cap \mathbb{Q}| \leq g_1(n)$ for all $n \in \mathbb{N}$, which is readily defined using $\exists^2$. By definition, $|A_n| \leq g_0(n) + g_1(n) + 1$ for all $n \in \mathbb{N}$ and define

$$g(x) := \tfrac{1}{2^{H(x)+1}} \tfrac{1}{(g_0(H(x))+g_1(H(x)+1)+1)}.$$

The latter is in BV (total variation at most 1) and is totally discontinuous, i.e., item (c) is false, as required. The other implications follow in the same way. ⊣

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We believe that Helly's selection theorem as in Principle 3.7 is weak and in particular does not imply $\mathsf{NIN}_{[0,1]}$.

Finally, we attempt to explain why seemingly related mathematical notions behave so differently in higher-order RM.

REMARK 3.11 (Second-order-ish functions). As discussed in detail in [68], quasi-continuous and cliquish $[0,1] \to [0,1]$ functions are intimately related from the pov of real analysis. Nonetheless, $\mathsf{RCA}_0^\omega + \mathsf{WKL}_0$ proves that the former have a supremum while $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}AC}^{0,1}$ cannot prove the existence of a supremum for the latter. A similar observation can be made for many pairs of function classes, including cadlag and regulated functions.

The crucial observation here is that for quasi-continuous and cadlag functions, the function value $f(x)$ for *any* $x \in [0,1]$ is determined if we know $f(q)$ for any $q \in [0,1] \cap \mathbb{Q}$, provably in ACA_0^ω. We refer to such function classes as *second-order-ish* as their definition comes with an obvious second-order approximation device. By contrast, cliquish and regulated functions are not determined in this way, i.e., they apparently lack the latter device.

With the gift of hindsight, properties of second-order-ish function classes can be established in RCA_0^ω extended with the Big Five systems, which is one of the main observations of [68]. By contrast, basic properties of non-second-order-ish functions can often not be proved in Z_2^ω or even $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}AC}^{0,1}$. Thus, our equivalences seems to be robust as long as we stay within the second-order-ish function classes, or dually: within the non-second-order-ish ones.

3.4. Pigeon hole principle for measure spaces. We introduce Tao's pigeonhole principle for measure spaces from [85] and obtain equivalences involving the approximation theorem for Riemann integrable functions via Bernstein polynomials. The latter is studied in [18] where it is claimed this result goes back to Picard [71].

First of all, the pigeonhole principle as in $\mathsf{PHP}_{[0,1]}$ is studied in [79], with a number of equivalences for basic properties of Riemann integrable functions.

PRINCIPLE 3.12 ($\mathsf{PHP}_{[0,1]}$). *If $(X_n)_{n\in\mathbb{N}}$ is an increasing sequence of measure zero and closed sets of reals in $[0,1]$, then $\bigcup_{n\in\mathbb{N}} X_n$ has measure zero.*

By the main result of [88], not all nowhere dense measure zero sets are the countable union of measure zero closed sets, i.e., $\mathsf{PHP}_{[0,1]}$ does not generate 'all' measure zero sets.

Secondly, fragments of the induction axiom are sometimes used in an essential way in second-order RM (see, e.g., [59]). An important role of induction is to provide 'finite comprehension' (see [83, X.4.4]). As in [79], we need the following fragment of finite comprehension, provable from the induction axiom.

PRINCIPLE 3.13 ($\mathsf{IND}_\mathbb{R}$). *For $F : (\mathbb{R} \times \mathbb{N}) \to \mathbb{N}, k \in \mathbb{N}$, there is $X \subset \mathbb{N}$ such that*

$$(\forall n \leq k)\big[(\exists x \in \mathbb{R})(F(x,n) = 0) \leftrightarrow n \in X\big].$$

In particular, the following rather important result seems to require $\mathsf{IND}_\mathbb{R}$.

THEOREM 3.14 ($\mathsf{ACA}_0^\omega + \mathsf{IND}_\mathbb{R}$). *For Riemann integrable $f : [0,1] \to \mathbb{R}$ with oscillation function $\mathsf{osc}_f : [0,1] \to \mathbb{R}$, the set $D_k := \{x \in [0,1] : \mathsf{osc}_f(x) \geq \frac{1}{2^k}\}$ is measure zero for any fixed $k \in \mathbb{N}$.*

PROOF. The theorem follows from [79, Corollary 3.4]. A sketch of the proof of the latter is as follows: let $f : [0,1] \to \mathbb{R}$ be Riemann integrable with oscillation function $\mathsf{osc}_f : [0,1] \to \mathbb{R}$, such that $D_{k_0} := \{x \in [0,1] : \mathsf{osc}_f(x) \geq \frac{1}{2^{k_0}}\}$ has measure $\varepsilon > 0$ for some fixed $k_0 \in \mathbb{N}$. For a partition P given as $0 = x_0, t_0, x_1, t_1, x_1, \dots, x_{k-1}, t_k, x_k = 1$ with small enough mesh, one obtains two partitions P', P'': as follows: in case $[x_i, x_{i+1}] \cap D_{k_0} \neq \emptyset$, replace t_i by respectively t_i' and t_i'' such that $|f(t_i') - f(t_i'')| \geq \frac{1}{2^{k_0}}$; these reals exist by definition of D_{k_0}. By definition, $S(f, P')$ and $S(f, P'')$ are at least $\varepsilon/2^{k_0+1}$ apart, i.e., f is not Riemann integrable, a contradiction. The definition of P', P'' can be formalised using $\mathsf{IND}_\mathbb{R}$. ⊣

Thirdly, we establish Theorem 3.15 where we recall the set B_f from (2.1).

THEOREM 3.15 ($\mathsf{ACA}_0^\omega + \mathsf{IND}_\mathbb{R} + \mathsf{QF\text{-}AC}^{0,1}$). *The following are equivalent.*

(a) *The pigeonhole principle for measure spaces as in* $\mathsf{PHP}_{[0,1]}$.
(b) (*Vitali–Lebesgue*) *For Riemann integrable* $f : [0,1] \to \mathbb{R}$ *with an oscillation function, the set* C_f *has measure* 1.
(c) *For Riemann integrable* $f : [0,1] \to \mathbb{R}$ *with an oscillation function, the set* B_f *has measure* 1.
(d) *For Riemann integrable* $f : [0,1] \to \mathbb{R}$ *with an oscillation function, the set of all* $x \in (0,1)$ *such that* (1.5) (*or* (3.6)) *has measure* 1.
(e) *For Riemann integrable and pointwise discontinuous* $f : [0,1] \to \mathbb{R}$ *with an oscillation function, the set* B_f *has measure* 1.
(f) *For Riemann integrable lsco* $f : [0,1] \to \mathbb{R}$, *the set* B_f *has measure* 1.

We can replace 'usco' by 'cliquish with an oscillation function' in the above.

PROOF. First of all, assume $\mathsf{PHP}_{[0,1]}$ and let $f : [0,1] \to \mathbb{R}$ be as in item (b) of the theorem. By Theorem 3.14, each $D_k := \{x \in [0,1] : \mathsf{osc}_f(x) \geq \frac{1}{2^k}\}$ has measure zero. By [79, Theorem 1.17], osc_f is usco and hence D_k is closed; both facts essentially follow by definition as well. Then $\mathsf{PHP}_{[0,1]}$ implies that $D_f = \cup_k D_k$ has measure zero, yielding item (b). By Theorem 2.2, B_f has measure one, i.e., item (c) follows, and the same for items (d) and (e).

For item (f), we proceed in essentially the same way: D_f exists in $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$ by [79, Theorem 2.4], and we just need to define some kind of 'replacement' set for D_k. To this end, let $f : [0,1] \to \mathbb{R}$ be lsco and consider:

$$f(x) \leq q \wedge (\forall N \in \mathbb{N})(\exists z \in B(x, \tfrac{1}{2^N}) \cap \mathbb{Q})(f(z) > q + \tfrac{1}{2^l}). \tag{3.7}$$

Using $(\exists^2)$, let $E_{q,l}$ be the set of all $x \in [0,1]$ satisfying (3.7). Since f is lsco, (3.7) is-essentially by definition-equivalent to

the formula (3.7) *with* '$\cap\ \mathbb{Q}$' *omitted.*

Intuitively, the set $E_{q,l}$ provides a 'replacement' set for D_k. Indeed, since f is lsco, the set $E_{q,l}$ is closed. Moreover, we also have $D_f = \cup_{l \in \mathbb{N}, q \in \mathbb{Q}} E_{q,l}$, by the epsilon–delta definition of (local) continuity. That $E_{q,l}$ has measure zero is proved in the same way as for D_k in Theorem 3.14. Again, $\mathsf{PHP}_{[0,1]}$ implies that D_f has measure zero and Theorem 2.2 implies that B_f has measure one, as required for item (f).

To derive $\mathsf{PHP}_{[0,1]}$ from item (f) (or the items (b)–(e)), let $(X_n)_{n\in\mathbb{N}}$ be an increasing sequence of closed and measure zero sets. Since $\mathbb{Q}$ has an enumeration, if $\cup_{n\in\mathbb{N}} X_n \setminus$

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$\mathbb{Q}$ has measure zero, so does $\cup_{n\in\mathbb{N}}X_n$, say in ACA_0^ω. Hence, we may assume that $\mathbb{Q}\cap\cup_{n\in\mathbb{N}}X_n=\emptyset$. Now consider the following function, which is usco, cliquish, and its own oscillation function by [79, Theorems 1.16–1.18]:

$$h(x) := \begin{cases} 0 & x \notin \cup_{m\in\mathbb{N}}X_m, \\ \frac{1}{2^{n+1}} & x \in X_n \text{ and } n \text{ is the least such number,} \end{cases} \tag{3.8}$$

and which satisfies $B_n(h,x)=0$ for all $x\in[0,1]$. Now, h is Riemann integrable with integral equal to zero on $[0,1]$, which can be proved using the obvious[11] epsilon–delta proof. Moreover, since $\mathbb{Q}\cap\cup_{n\in\mathbb{N}}X_n=\emptyset$, h is continuous at any rational and therefore pointwise discontinuous, i.e., the equivalence for item (e) readily follows. By item (f), B_f has measure 1, implying that for almost all $x\in(0,1)$, we have $h(x)=\lim_{n\to\infty}B_n(h,x)=0$, i.e., $\cup_{n\in\mathbb{N}}X_n$ has measure zero, and $\mathsf{PHP}_{[0,1]}$ follows. Note that item (d) also guarantees that $h(x)=0$ almost everywhere, i.e., $\cup_{n\in\mathbb{N}}X_n$ has measure zero. For the final sentence of the theorem, recall that the function $h:[0,1]\to\mathbb{R}$ from (3.8) is cliquish. ⊣

In light of Theorem 3.14, Riemann integrable functions can *almost* be proved to be continuous ae. However, $\mathsf{PHP}_{[0,1]}$ is needed to establish that $D_f=\cup_{k\in\mathbb{N}}D_k$ has measure zero. The restriction in item (e) is non-trivial as is it consistent with Z_2^ω that there are Riemann integrable functions that are *totally discontinuous*.

As to generalisations of the previous theorem, there are a surprisingly large number of rather diverse equivalent definitions of 'Baire 1' on the reals [5, 48, 53], including *B-class-1-measurability* and *fragmentedness* by [48, Theorem 2.3] and [52, Section 34, paragraph VII]. One readily shows that h from (3.8) satisfies these definitions, say in ACA_0^ω, i.e., the previous theorem can be formulated for these notions. By contrast, the restriction of item (b) or (e) in Theorem 3.15 to (the standard definition of) Baire 1 functions, can be proved in ACA_0^ω by [79, Theorem 3.7]. In general, the function h from (3.8) is rather well-behaved and therefore included in many (lesser known than Baire 1) function classes.

3.5. Baire category theorem. We introduce the Baire category theorem and obtain equivalences involving the approximation theorem for usco and cliquish functions via Bernstein polynomials. The RM of $\mathsf{BCT}_{[0,1]}$ as in Principle 3.16 and $\mathsf{PHP}_{[0,1]}$ is often similar (see [79]), which is why we treat the former in less detail.

First of all, we shall study the Baire category theorem formulated as follows. We have established a substantial number of equivalences in [79] between this principle and basic properties of usco functions and related classes.

Principle 3.16 ($\mathsf{BCT}_{[0,1]}$). *If $(O_n)_{n\in\mathbb{N}}$ is a decreasing sequence of dense open sets of reals in* $[0,1]$, *then $\bigcap_{n\in\mathbb{N}}O_n$ is non-empty.*

We assume that $O_{n+1}\subseteq O_n$ for all $n\in\mathbb{N}$ to avoid the use of induction to prove that a finite intersection of open and dense sets is again open and dense.

Secondly, we have the following theorem where we recall the set B_f from (2.1).

[11] Fix $\varepsilon>0$ and $n_0\in\mathbb{N}$ such that $\frac{1}{2^{n_0}}<\varepsilon$. Then $\mathsf{ACA}_0^\omega+\mathsf{QF\text{-}AC}^{0,1}$ proves that for any partition P with mesh $<\frac{1}{2^{n_0+2}}$, the Riemann sum $S(h,P)$ is $<\varepsilon$ in absolute value. To this end, cover X_{n_0} by a sequence of intervals of total length at most $\frac{1}{2^{n_0+2}}$ and find a finite sub-covering.

THEOREM 3.17 (ACA_0^ω). *The following are equivalent.*

(a) *The Baire category theorem as in* $\mathsf{BCT}_{[0,1]}$.
(b) *For usco* $f : [0,1] \to \mathbb{R}$, *the set* C_f *is non-empty.*
(c) *For cliquish* $f : [0,1] \to \mathbb{R}$ *with an oscillation function, we have* $C_f \neq \emptyset$.
(d) *For usco* $f : [0,1] \to \mathbb{R}$, *the set* B_f *is non-empty.*
(e) *For cliquish* $f : [0,1] \to \mathbb{R}$ *with an oscillation function,* B_f *is non-empty.*

PROOF. For the implications (d)→(a) and (e)→ (a), let $(O_n)_{n\in\mathbb{N}}$ be a decreasing sequence of dense open sets of reals in [0, 1]. In case there is $q \in \mathbb{Q}$ with $q \in \cap_{n\in\mathbb{N}} O_n$, $\mathsf{BCT}_{[0,1]}$ follows. For the case where $\emptyset = \mathbb{Q} \cap (\cap_{n\in\mathbb{N}} O_n)$, we proceed as follows. For $X_n := [0,1] \setminus O_n$, consider h as in (3.8), which is usco, cliquish, and its own oscillation function by [79, Theorems 1.16–1.18]. Hence, the set B_h from (2.1) is non-empty given item (d) or (e). We now show that $\lim_{n\to\infty} B_n(h,x) = 0$ for all $x \in [0,1]$, finishing the proof. To this end, recall that $\mathbb{Q} \subset \cup_{n\in\mathbb{N}} X_n$ and define $Q_n := X_n \cap \mathbb{Q}$ where $\mathbb{Q} = \cup_{n\in\mathbb{N}} Q_n$. Now consider $\tilde{h} : [0,1] \to \mathbb{R}$ as follows:

$$\tilde{h}(x) := \begin{cases} 0 & x \notin \mathbb{Q}, \\ \frac{1}{2^{n+1}} & x \in Q_n \text{ and } n \text{ is the least such number,} \end{cases} \tag{3.9}$$

which is really a modification of Thomae's function (see [68, 86]). Then $\tilde{h}$ is continuous on $[0,1] \setminus \mathbb{Q}$ since each X_n is nowhere dense and closed. By Theorem 2.2, we have $0 = \tilde{h}(x) = \lim_{n\to\infty} B_n(\tilde{h}, x)$ for $x \in (0,1) \setminus \mathbb{Q}$. By the uniform continuity of the polynomials $p_{k,n}(x)$ on [0, 1], we also have $\lim_{n\to\infty} B_n(\tilde{h}, q) = 0$ for $q \in [0,1] \cap \mathbb{Q}$. Since Bernstein polynomials only invoke rational function values, we have $B_n(h,x) = B_n(\tilde{h}, x)$ for any $x \in [0,1]$ and $n \in \mathbb{N}$. Hence, we conclude $\lim_{n\to\infty} B_n(h,x) = 0$ for $x \in [0,1]$, as required.

Finally, the implications (a)→(b)→(d) and (a)→(c)→(e) follow from [79, Theorem 2.3 and 2.16] and Theorem 2.2. For the first implication, one now defines a variation of $E_{q,l}$ based on (3.7) to replace D_k, while the third one follows by the textbook proof. ⊣

As also stated in [79], much to our own surprise, the 'counterexample' function from (3.8) has nice properties *that are provable in weak systems*, including the behaviour of the associated Bernstein polynomials.

Finally, we discuss certain restrictions of the above theorems.

REMARK 3.18 (Restrictions, trivial, and otherwise). It should not be a surprise that suitable restrictions of principles that imply $\mathsf{NIN}_{[0,1]}$, are again provable from the (second-order) Big Five. We discuss three examples pertaining to the above where two are perhaps surprising.

First of all, item (b) in Theorem 3.8 is *equivalent* to ATR_0 if we restrict to functions with an arithmetical or Σ_1^1-graph by [68, Section 2.6]. The results in the latter pertain to bounded variation functions but are readily adapted to regulated functions, which also only have countably many points of discontinuity. Thus, the same restriction of item (e) in Theorem 3.8 is equivalent to ATR_0 in the same way. Similar results hold for the restriction to quasi-continuous functions.

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Secondly, item (b) in Theorem 3.8 is provable[12] from ATR_0 (and extra induction) when restricted to Baire 1 functions by [68, Section 2.6]. By Theorem 2.2, the same holds for the restriction to Baire 1 functions of (e) in Theorem 3.8. However, this restriction does not seem to be a 'real' one as many a textbook tells us that

$$\textit{regulated functions are Baire 1 on the reals.} \tag{3.10}$$

Of course, (3.10) is true *but* $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}AC}^{0,1}$ cannot prove it by [68, Theorem 2.35]. In general, the usual picture of the classical hierarchy of function classes looks *very* different in weak logical systems and even in Z_2^ω. Indeed, by Theorem 3.3, it is consistent with $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}AC}^{0,1}$ that there are regulated functions that are *discontinuous everywhere*.

Thirdly, similar to the previous paragraph, item (b) of Theorem 3.17 restricted to Baire 1 functions is provable in ACA_0^ω by [79, Theorem 2.9]. Again, this restriction does not seem 'real' as it is well-known that

$$\textit{usco functions are Baire 1 on the reals,} \tag{3.11}$$

but $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}AC}^{0,1}$ cannot prove (3.11) by [79, Corollary 2.8]. Unfortunately, the second-order coding of usco and lsco functions from [26] is such that the Baire 1 representation of usco and lsco is 'baked into' the coding, in light of [26, Section 6]. It would therefore be more correct to refer to the representations from [26] as *codes for usco functions that are also Baire 1*. What is worse, (3.11) implies enum is therefore not an 'innocent' background assumption in RM, as $\mathsf{ACA}_0^\omega + \mathsf{enum}$ proves ATR_0 and $\Pi_1^1\text{-}\mathsf{CA}_0^\omega + \mathsf{enum}$ proves $\Pi_2^1\text{-}\mathsf{CA}_0$ (see Section 3.3).

§4. Appendix. Technical Appendix: introducing Reverse Mathematics.

We discuss the language of Reverse Mathematics (Section 4.1) and introduce—in full detail—Kohlenbach's base theory of *higher-order* Reverse Mathematics (Section 4.2). Some common notations may be found in Section 4.3.

4.1. Introduction.

We sketch some aspects of Kohlenbach's *higher-order* RM [46] essential to this paper, including the base theory RCA_0^ω (Definition 4.1).

First of all, in contrast to 'classical' RM based on *second-order arithmetic* Z_2, higher-order RM uses L_ω, the richer language of *higher-order arithmetic*. Indeed, while the former is restricted to natural numbers and sets of natural numbers, higher-order arithmetic can accommodate sets of sets of natural numbers, sets of sets of sets of natural numbers, et cetera. To formalise this idea, we introduce the collection of *all finite types* $\mathbf{T}$, defined by the two clauses:

$$\text{(i) } 0 \in \mathbf{T} \text{ and (ii) If } \sigma, \tau \in \mathbf{T} \text{ then } (\sigma \to \tau) \in \mathbf{T},$$

where 0 is the type of natural numbers, and $\sigma \to \tau$ is the type of mappings from objects of type σ to objects of type τ. In this way, $1 \equiv 0 \to 0$ is the type of functions from numbers to numbers, and $n + 1 \equiv n \to 0$. Viewing sets as given by characteristic functions, we note that Z_2 only includes objects of type 0 and 1.

[12]Here, ATR_0 is just an upper bound and better results can be found in [78, Section 3.5].

Secondly, the language L_ω includes variables $x^\rho, y^\rho, z^\rho, \dots$ of any finite type $\rho \in \mathbf{T}$. Types may be omitted when they can be inferred from context. The constants of L_ω include the type 0 objects $0, 1$ and $<_0, +_0, \times_0, =_0$ which are intended to have their usual meaning as operations on $\mathbb{N}$. Equality at higher types is defined in terms of '$=_0$' as follows: for any objects x^τ, y^τ, we have

$$[x =_\tau y] \equiv (\forall z_1^{\tau_1} \dots z_k^{\tau_k})[x z_1 \dots z_k =_0 y z_1 \dots z_k], \tag{4.1}$$

if the type τ is composed as $\tau \equiv (\tau_1 \to \dots \to \tau_k \to 0)$. Furthermore, L_ω also includes the *recursor constant* $\mathbf{R}_\sigma$ for any $\sigma \in \mathbf{T}$, which allows for iteration on type σ-objects as in the special case (4.2). Formulas and terms are defined as usual. One obtains the sub-language L_{n+2} by restricting the above type formation rule to produce only type $n + 1$ objects (and related types of similar complexity).

4.2. The base theory of higher-order Reverse Mathematics. We introduce Kohlenbach's base theory RCA_0^ω, first introduced in [46, Section 2].

Definition 4.1. The base theory RCA_0^ω consists of the following axioms.

(a) Basic axioms expressing that $0, 1, <_0, +_0, \times_0$ form an ordered semi-ring with equality $=_0$.
(b) Basic axioms defining the well-known Π and Σ combinators (aka K and S in [3]), which allow for the definition of *λ-abstraction*.
(c) The defining axiom of the recursor constant $\mathbf{R}_0$: for m^0 and f^1 :

$$\mathbf{R}_0(f, m, 0) := m \text{ and } \mathbf{R}_0(f, m, n + 1) := f(n, \mathbf{R}_0(f, m, n)). \tag{4.2}$$

(d) The *axiom of extensionality*: for all $\rho, \tau \in \mathbf{T}$, we have

$$(\forall x^\rho, y^\rho, \varphi^{\rho\to\tau})\big[x =_\rho y \to \varphi(x) =_\tau \varphi(y)\big]. \tag{$\mathrm{E}_{\rho,\tau}$}$$

(e) The induction axiom for quantifier-free formulas of L_ω.
(f) $\mathsf{QF\text{-}AC}^{1,0}$: the quantifier-free Axiom of Choice as in Definition 4.2.

Note that variables (of any finite type) are allowed in quantifier-free formulas of the language L_ω: only quantifiers are banned. Recursion as in (4.2) is called *primitive recursion*; the class of functionals obtained from $\mathbf{R}_\rho$ for all $\rho \in \mathbf{T}$ is called *Gödel's system T* of all (higher-order) primitive recursive functionals.

Definition 4.2. The axiom $\mathsf{QF\text{-}AC}$ consists of the following for all $\sigma, \tau \in \mathbf{T}$:

$$(\forall x^\sigma)(\exists y^\tau)A(x, y) \to (\exists Y^{\sigma\to\tau})(\forall x^\sigma)A(x, Y(x)), \tag{$\mathsf{QF\text{-}AC}^{\sigma,\tau}$}$$

for any quantifier-free formula A in the language of L_ω.

As discussed in [46, Section 2], RCA_0^ω and RCA_0 prove the same sentences 'up to language' as the latter is set-based and the former function-based. This conservation results is obtained via the so-called ECF-interpretation, which we now discuss.

Remark 4.3 (The ECF-interpretation). The (rather) technical definition of ECF may be found in [87, p. 138, Section 2.6]. Intuitively, the ECF-interpretation $[A]_{\mathsf{ECF}}$ of a formula $A \in \mathsf{L}_\omega$ is just A with all variables of type two and higher replaced by type one variables ranging over so-called 'associates' or 'RM-codes' (see [45, Section 4]); the latter are (countable) representations of continuous functionals. The ECF-interpretation connects RCA_0^ω and RCA_0 (see [46, Proposition 3.1]) in that if RCA_0^ω

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proves A, then RCA_0 proves $[A]_{\mathsf{ECF}}$, again 'up to language', as RCA_0 is formulated using sets, and $[A]_{\mathsf{ECF}}$ is formulated using types, i.e., using type zero and one objects.

In light of the widespread use of codes in RM and the common practise of identifying codes with the objects being coded, it is no exaggeration to refer to ECF as the *canonical* embedding of higher-order into second-order arithmetic.

4.3. Notations and the like. We introduce the usual notations for common mathematical notions, like real numbers, as also introduced in [46].

Definition 4.4 (Real numbers and related notions in RCA_0^ω).

(a) Natural numbers correspond to type zero objects, and we use 'n^0' and '$n \in \mathbb{N}$' interchangeably. Rational numbers are defined as signed quotients of natural numbers, and '$q \in \mathbb{Q}$' and '$<_{\mathbb{Q}}$' have their usual meaning.

(b) Real numbers are coded by fast-converging Cauchy sequences $q_{(\cdot)} : \mathbb{N} \to \mathbb{Q}$, i.e., such that $(\forall n^0, i^0)(|q_n - q_{n+i}| <_{\mathbb{Q}} \frac{1}{2^n})$. We use Kohlenbach's 'hat function' from [46, p. 289] to guarantee that every q^1 defines a real number.

(c) We write '$x \in \mathbb{R}$' to express that $x^1 := (q^1_{(\cdot)})$ represents a real as in the previous item and write $[x](k) := q_k$ for the kth approximation of x.

(d) Two reals x, y represented by $q_{(\cdot)}$ and $r_{(\cdot)}$ are *equal*, denoted $x =_{\mathbb{R}} y$, if $(\forall n^0)(|q_n - r_n| \leq 2^{-n+1})$. Inequality '$<_{\mathbb{R}}$' is defined similarly. We sometimes omit the subscript '$\mathbb{R}$' if it is clear from context.

(e) Functions $F : \mathbb{R} \to \mathbb{R}$ are represented by $\Phi^{1\to1}$ mapping equal reals to equal reals, i.e., extensionality as in $(\forall x, y \in \mathbb{R})(x =_{\mathbb{R}} y \to \Phi(x) =_{\mathbb{R}} \Phi(y))$.

(f) The relation '$x \leq_\tau y$' is defined as in (4.1) but with '$\leq_0$' instead of '$=_0$'. Binary sequences are denoted '$f^1, g^1 \leq_1 1$', but also '$f, g \in C$' or '$f, g \in 2^{\mathbb{N}}$'. Elements of Baire space are given by f^1, g^1, but also denoted '$f, g \in \mathbb{N}^{\mathbb{N}}$'.

(g) For a binary sequence f^1, the associated real in [0, 1] is $\mathbb{r}(f) := \sum_{n=0}^{\infty} \frac{f(n)}{2^{n+1}}$.

(h) Sets of type ρ objects $X^{\rho\to0}, Y^{\rho\to0}, \ldots$ are given by their characteristic functions $F_X^{\rho\to0} \leq_{\rho\to0} 1$, i.e., we write '$x \in X$' for $F_X(x) =_0 1$.

For completeness, we list the following notational convention for finite sequences.

Notation 4.5 (Finite sequences). The type for 'finite sequences of objects of type ρ' is denoted ρ^*, which we shall only use for $\rho = 0, 1$. Since the usual coding of pairs of numbers goes through in RCA_0^ω, we shall not always distinguish between 0 and 0^*. Similarly, we assume a fixed coding for finite sequences of type 1 and shall make use of the type '1^*'. In general, we do not always distinguish between 's^ρ' and '$\langle s^\rho \rangle$', where the former is 'the object s of type ρ', and the latter is 'the sequence of type ρ^* with only element s^ρ'. The empty sequence for the type ρ^* is denoted by '$\langle\rangle_\rho$', usually with the typing omitted.

Furthermore, we denote by '$|s| = n$' the length of the finite sequence $s^{\rho^*} = \langle s_0^\rho, s_1^\rho, \ldots, s_{n-1}^\rho \rangle$, where $|\langle\rangle| = 0$, i.e., the empty sequence has length zero. For sequences s^{ρ^*}, t^{ρ^*}, we denote by '$s * t$' the concatenation of s and t, i.e., $(s * t)(i) = s(i)$ for $i < |s|$ and $(s * t)(j) = t(|s| - j)$ for $|s| \leq j < |s| + |t|$. For a sequence s^{ρ^*}, we define $\overline{s}N := \langle s(0), s(1), \ldots, s(N-1) \rangle$ for $N^0 < |s|$. For a sequence $\alpha^{0\to\rho}$, we also write $\overline{\alpha}N = \langle \alpha(0), \alpha(1), \ldots, \alpha(N-1) \rangle$ for *any* N^0. By way of shorthand, $(\forall q^\rho \in Q^{\rho^*})A(q)$ abbreviates $(\forall i^0 < |Q|)A(Q(i))$, which is (equivalent to) quantifier-free if A is.

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