Bull. Aust. Math. Soc. (First published online 2025), page 1 of 11*
doi:10.1017/S0004972725000061
*Provisional—final page numbers to be inserted when paper edition is published

ON MARCH'S CRITERION FOR TRANSIENCE ON ROTATIONALLY SYMMETRIC MANIFOLDS

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(Received 22 October 2024; accepted 26 December 2024)

Abstract

We show that March's criterion for the existence of a bounded nonconstant harmonic function on a weak model (that is, $\mathbb{R}^n$ with a rotationally symmetric metric) is also a necessary and sufficient condition for the solvability of the Dirichlet problem at infinity on a family of metrics that generalise metrics with rotational symmetry on $\mathbb{R}^n$. When the Dirichlet problem at infinity is not solvable, we prove some quantitative estimates on how fast a nonconstant harmonic function must grow.

2020 *Mathematics subject classification*: primary 31C12; secondary 31C05, 53C21.

Keywords and phrases: Dirichlet problem at infinity, bounded harmonic functions, model Riemannian space.

1. Introduction

March [10] gave a criterion for the transience of a rotationally symmetric Riemannian manifold, that is, for a metric on $\mathbb{R}^n$ of the form

$$g = dr^2 + \phi^2(r) g_{\mathbb{S}^{n-1}},$$

with ϕ a smooth function such that $\phi > 0$ for $r > 0$, $\phi(0) = 0$ and $\phi'(0) = 1$, and where $g_{\mathbb{S}^{n-1}}$ is the standard round metric on $\mathbb{S}^{n-1}$; $M_g = (\mathbb{R}^n, g)$ is called a weak model. March proved that M_g supports bounded nonconstant harmonic functions if and only if

$$\int_1^\infty \phi^{n-3}(\sigma) \int_\sigma^\infty \phi^{1-n}(\tau)\, d\tau\, d\sigma < \infty. \tag{1.1}$$

The behaviour of the Brownian motion on a manifold and the existence of bounded nontrivial harmonic functions are closely related (see the excellent survey [8]).

We go beyond the result of March and show that for metrics of the form

$$g = dr^2 + \phi^2(r) g_\omega, \tag{1.2}$$

where g_ω is *any* metric on $\mathbb{S}^{n-1}$, March's criterion (1.1) implies the solvability on M of the Dirichlet problem at infinity. It is important to clarify that g is defined in $\mathbb{R}^n \setminus \{0\}$

and it can only be smoothly extended to the origin when $g_\omega = g_{\mathbb{S}^{n-1}}$ if it is also assumed that all derivatives of even order vanish at $r = 0$. Because g might not be smoothly extendable to the origin, our definition of harmonicity in $(\mathbb{R}^n, g)$ has the following nuance (see also [6, Section 1]): we say that u is harmonic if $u \in C(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\})$, and in $\mathbb{R}^n \setminus \{0\}$, it satisfies

$$\Delta_g u = 0, \quad \text{where } \Delta_g = \frac{\partial^2}{\partial r^2} + (n-1)\frac{\phi'}{\phi}\frac{\partial}{\partial r} + \frac{1}{\phi^2}\Delta_{g_\omega}.$$

Thus, we establish conditions on a class of metrics that generalise the case of weak models, not necessarily rotationally symmetric, for which the Dirichlet problem at infinity is solvable and, as a bonus, we give some quantitative results. That is, we shall prove the following result.

THEOREM 1.1. *Let $M = (\mathbb{R}^n, g)$, $n \geq 3$, with g of the form (1.2).*

(i) *For any continuous data $f \in C(\mathbb{S}^{n-1})$, if*

$$\int_1^\infty \phi^{n-3}(\sigma) \int_\sigma^\infty \phi^{1-n}(\tau)\, d\tau\, d\sigma < \infty,$$

then the Dirichlet problem at infinity is uniquely solvable on M.

(ii) *If there is an $\eta > 0$ such that $\phi'(r) \geq \eta$ outside of a compact subset of M and*

$$\int_1^\infty \phi^{n-3}(\sigma) \int_\sigma^\infty \phi^{1-n}(\tau)\, d\tau\, d\sigma = \infty,$$

then there is a constant $c > 0$ such that a nonconstant harmonic function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ must satisfy

$$\max_{r \leq R} |u(\omega, r)| \gtrsim \exp\Big(c \int_1^R \phi^{n-3}(\sigma) \int_\sigma^R \frac{1}{\phi^{n-1}(\tau)}\, d\tau\, d\sigma\Big). \tag{1.3}$$

The constant c depends on η and λ_1, the first eigenvalue of the Laplacian Δ_{g_ω} on $(\mathbb{S}^{n-1}, g_\omega)$. (For example, $c = \lambda_1^2/(2\lambda_1 + 4)$ if the radial curvature of the cone is everywhere nonpositive.) In particular, the Dirichlet problem at infinity is not solvable in M.

(iii) *If $\phi'(r) \leq \beta$, for a positive constant β, then any bounded harmonic function must be constant. In fact, there is a constant $l := l(\beta, \lambda_1, n) > 0$ such that a nonconstant harmonic function u must satisfy*

$$\max_{r \leq R} |u(\omega, r)| \gtrsim r^l. \tag{1.4}$$

Some observations are in place here. The hypotheses of the second part of the theorem are satisfied whenever outside a compact set of M, $\phi' > 0$ and the radial curvature $-\phi''/\phi$ is nonpositive, or just when the radial curvature is nonpositive. Regarding condition (iii), the hypothesis on ϕ holds as soon as the radial curvature is nonnegative. In this case (when $g_\omega = g_{\mathbb{S}^{n-1}}$), Liouville's theorem holds, as proved by Yau for metrics of nonnegative Ricci curvature. However, the reader can readily check that if $\phi(r) = \frac{1}{2}(r + \sin r)$, the radial curvature has no definite sign, and in this case, condition (iii) still applies. A proof of condition (iii) is presented in [4], and we have included it here because it gives a nice complement to condition (ii).

The study of the Dirichlet problem at infinity has a history of many deep and beautiful results (see, for instance, [1–3, 5, 11, 12, 14]). In particular, Theorem 1.1 was proved by Hsu [9] in the case where $g_w = g_{\mathbb{S}^{n-1}}$, the standard round metric on the sphere, for Cartan–Hadamard manifolds. However, it is not clear that his methods, which are probabilistic in nature, extend to arbitrary metrics on $\mathbb{S}^{n-1}$, as he uses the symmetry of the heat kernel in the case of a rotationally invariant metric. Our proof is elementary and relies on separation of variables, estimating solutions to a Ricatti equation and some Fourier analysis. We not only generalise Hsu's result but also provide an alternative to the probabilistic methods employed by him. Also, our estimate on the minimal growth of a nonconstant harmonic function (when the Dirichlet problem at infinity is not solvable) seems to be new.

Following March's arguments, Theorem 1.1 implies an almost sharp restriction on the curvature of a Cartan–Hadamard manifold whose metric is of the form (1.2) and where the Dirichlet problem at infinity is solvable (see [10, Theorem 2]).

Corollary 1.2. *Let $M = (\mathbb{R}^n, g)$ with a metric of the form (1.2), such that ϕ' is eventually nonnegative, and let $k(r) = -\phi''/\phi(r)$ be its radial curvature, with $k(r) \leq 0$ outside a compact subset of $\mathbb{R}^n$. Let $c_2 = 1$ and $c_n = \frac{1}{2}$ for $n \geq 3$. If $k(r) \leq -c/(r^2 \log r)$ outside a compact set for some $c > c_n$, then the Dirichlet problem at infinity is solvable. If $k(r) \geq -c/(r^2 \log r)$ for some $c < c_n$ outside a compact set, then the Dirichlet problem at infinity is not solvable.*

In March's original result (and in Hsu's result), it is required that the radial curvature $k(r)$ is nonpositive: in such a case, $\phi' > 0$ holds automatically everywhere.

The layout of this paper is as follows. In Section 2, we define what we understand by solving the Dirichlet problem at infinity; in Section 3, we prove our main result.

2. Preliminaries

To define what we mean by the Dirichlet problem being solvable at infinity, we represent $\mathbb{R}^n$ as a cone over the sphere $\mathbb{S}^{n-1}$, that is,

$$\mathbb{R}^n \sim [0, \infty) \times \mathbb{S}^{n-1}/(\{0\} \times \mathbb{S}^{n-1}).$$

The equivalence class of the origin (the pole) will be denoted by O, that is,

$$O = \{0\} \times \mathbb{S}^{n-1}/(\{0\} \times \mathbb{S}^{n-1}).$$

We can compactify $\mathbb{R}^n$ by defining

$$\overline{\mathbb{R}^n} = [0, \infty] \times \mathbb{S}^{n-1}/(\{0\} \times \mathbb{S}^{n-1}),$$

where $[0, \infty]$ is a compactification of $[0, \infty)$.

We say that the Dirichlet problem on $M = (\mathbb{R}^n, g)$ is solvable at infinity for boundary data $f \in C(\mathbb{S}^{n-1}_\infty)$ if there is a harmonic function u on M, that is, $\Delta_g u = 0$ on $\mathbb{R}^n \setminus \{0\}$, which extends continuously to a function $\overline{u} : \overline{\mathbb{R}^n} \longrightarrow \mathbb{R}$ and such that $\overline{u}(\infty, \cdot) = f(\cdot)$.

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Our definition of solvability at infinity coincides with that of Choi [5] on Cartan–Hadamard manifolds. Indeed, for a Cartan–Hadamard manifold, the sphere at infinity given by the Eberlein–O'Neill compactification is obtained by adding at infinity the equivalence classes of geodesic rays starting at the pole (a point around which the metric can be written as in (1.2)), where two geodesic rays are equivalent if $\limsup_{t\to\infty} d(\gamma_1(t), \gamma_2(t)) < \infty$, and then endowing this set with the cone topology (see [5]). If we identify the pole of the manifold with the vertex of the cone (or pole, that is, the equivalence class of $\{0\} \times \mathbb{S}^{n-1}$), the set of equivalence classes of geodesics can be identified with the unit sphere on T_pM, where p is a pole of the manifold, which in turn implies that it can be identified with $\mathbb{S}^{n-1}_\infty = \{\infty\} \times \mathbb{S}^{n-1}$.

3. Proof of the main result

3.1. Proof of part (i). In this section, we will show that

$$J = \int_1^\infty \phi^{n-3}(\sigma) \int_\sigma^\infty \phi^{1-n}(\tau)\, d\tau\, d\sigma < \infty$$

implies the solvability of the Dirichlet problem at infinity. (In Section 3.2, we will show nonsolvability in the case where $J = \infty$.)

We will follow the approach implemented in [7], so we use separation of variables to find solutions to the equation

$$\Delta_g u = 0 \quad \text{on } \mathbb{R}^n \setminus \{0\},$$

which extend continuously to $\overline{\mathbb{R}^n}$ and satisfy some given data at infinity. With this in mind, we let $f_{m,k}$, $k = 0, 1, 2, \ldots, k_m$, be eigenfunctions of the mth eigenvalue, λ_m^2 ($\lambda_m \geq 0$, $m = 0, 1, 2, \ldots$), of the Laplacian Δ_{g_ω} on $(\mathbb{S}^{n-1}, g_\omega)$, such that the set $\{f_{m,k}\}_{m,k}$ is an orthonormal basis for $L^2(\mathbb{S}^{n-1})$ with respect to the inner product induced by the metric g_ω. If φ_m is such that

$$\Delta_g(\varphi_m f_{m,k}) = 0,$$

then φ_m must satisfy the radial Laplacian equation

$$\varphi_m'' + (n-1)\frac{\phi'}{\phi}\varphi_m' - \frac{\lambda_m^2}{\phi^2}\varphi_m = 0. \tag{3.1}$$

It is proved in [7] that we may assume $\varphi_0 = 1$ and that φ_m for $m > 0$ has the form

$$\varphi_m(r) = r^l \alpha(r),$$

where $l = \frac{1}{2}(-(n-2) + \sqrt{(n-2)^2 + 4\lambda_m^2}) > 0$ satisfies the indicial equation

$$l(l-1) + (n-1)l - \lambda_m^2 = 0$$

and α is a smooth function. Also, it is shown that φ_m can be chosen so that it is nondecreasing and thus, $\varphi_m \geq 0$. Therefore, as argued in [7], to show solvability of the Dirichlet problem at infinity, it suffices to show that the φ_m are bounded.

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Indeed, let us sketch why this is so (for details, we refer the reader to [7]). Once we have proved that the φ_m are bounded, we may without loss of generality assume that $0 \leq \varphi_m \leq 1$ and that $\lim_{r\to\infty} \varphi_m(r) = 1$. Given $f \in C(\mathbb{S}^{n-1}_\infty)$, we can expand it in a Fourier series,

$$f(\omega) = \sum_{m,k} c_{m,k} f_{m,k}(\omega).$$

If f is smooth, a theorem of Peetre [13] guarantees that this series converges absolutely and uniformly to f (not just in $L^2(\mathbb{S}^{n-1})$ as it always does). Thus, a harmonic function solving the Dirichlet problem at infinity with boundary data f is given by

$$u(r, \omega) = \sum_m \varphi_m(r) \sum_k c_{m,k} f_{m,k}(\omega).$$

We call u a *harmonic extension* of f. If f is just continuous, an approximation argument using smooth functions gives the existence of a harmonic extension for f. Uniqueness follows from the maximum principle (see [7]).

In the case of $f \in L^2$, again, as proved in [7], the boundedness of φ_m implies solvability. All we are left to do to show solvability is the boundedness of the φ_m, which we do next. (Uniqueness, in the case of continuous f, follows from the maximum principle.)

LEMMA 3.1. *Given a solution φ_m to the radial Laplacian (3.1), there are constants $A, B > 0$ such that the bound*

$$\varphi_m(s) \leq B \exp\bigg(\int_1^s \frac{\lambda_m^2}{\phi^{n-1}(\tau)}\bigg(A + \int_1^\tau \phi^{n-3}(\sigma)\, d\sigma\bigg)\, d\tau\bigg)$$

holds for all $s \geq 1$.

PROOF. Write

$$\varphi_m(s) = B \exp\bigg(\int_1^s \frac{\lambda_m^2}{\phi^{n-1}(\tau)} x_m(\tau)\, d\tau\bigg) \tag{3.2}$$

for $s \geq 1$, with $B = \varphi_m(1)$ and $x_m(t)$ a smooth function. From the equation satisfied by φ_m, we arrive at an equation for x_m, that is,

$$x_m'(s) + \frac{\lambda_m^2}{\phi^{n-1}(s)} x_m^2(s) = \phi^{n-3}(s), \tag{3.3}$$

which leads to the inequality

$$x_m'(s) \leq \phi^{n-3}(s).$$

This yields the estimate

$$x_m(s) \leq A + \int_1^s \phi^{n-3}(\sigma)\, d\sigma$$

for an appropriately chosen constant $A > 0$, and the lemma follows. □

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From Lemma 3.1, it follows that if

$$\int_1^\infty \frac{1}{\phi^{n-1}(\tau)}\,d\tau + \int_1^\infty \frac{1}{\phi^{n-1}(\tau)} \int_1^\tau \phi^{n-3}(\sigma)\,d\sigma\,d\tau < \infty,$$

then φ_m is bounded. The second integral can be rewritten as

$$\int_1^\infty \phi^{n-3}(\sigma) \int_\sigma^\infty \phi^{1-n}(\tau)\,d\tau\,d\sigma.$$

Since the inner integral must converge for almost all τ, the convergence of the second integral implies the convergence of the integral

$$\int_1^\infty \frac{1}{\phi^{n-1}(\tau)}\,d\tau.$$

So we see that

$$\int_1^\infty \phi^{n-3}(\sigma) \int_\tau^\infty \phi^{1-n}(\tau)\,d\tau\,d\sigma < \infty$$

implies that φ_m is bounded, which, by the results in [7], gives the solvability of the Dirichlet problem at infinity.

3.2. Proof of part (ii). Next, we show that if

$$J = \int_1^\infty \phi^{n-3}(\sigma) \int_\sigma^\infty \phi^{1-n}(\tau)\,d\tau\,d\sigma = \infty$$

and there is an $\eta > 0$ such that $\phi'(r) \geq \eta$ outside of a compact set, then the Dirichlet problem at infinity is not solvable. The idea is to bound φ_m from below, and then use Plancherel's theorem to obtain estimate (1.3).

Let us estimate φ_m. Again, we write φ_m as in (3.2) for $s \geq 1$ and x_m a smooth function. We should make an important observation. Since the φ_m are nondecreasing (by the arguments in [7]), it follows that $x_m(t) \geq 0$. This fact will be used below. Since x_m satisfies (3.3), for any $s \geq 1$, we must have either

$$x_m'(s) > \tfrac{1}{4}\phi^{n-3}(s) \tag{3.4}$$

or

$$\frac{\lambda_m^2}{\phi^{n-1}(s)} x_m^2(s) > \frac{1}{4}\phi^{n-3}(s). \tag{3.5}$$

By hypothesis, there are $\eta > 0$ and $r_0 > 0$ such that $\phi'(r) \geq \eta$ for $r \geq r_0$ (in the case of a Cartan–Hadamard manifold, we can take $\eta = 1$). Without loss of generality, we will assume that $r_0 = 1$. We shall prove that these two conditions imply that for $\overline{\eta} = \min\{1, \eta\}$,

$$x_m(t) \geq \frac{\overline{\eta}}{2\lambda_m + 4} \int_1^t \phi^{n-3}(s)\,ds. \tag{3.6}$$

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Let

$$I = \{q \geq 1 : \text{ (3.6) holds for } t \in [1, q]\}.$$

It is clear that $1 \in I$ since

$$x_m(1) \geq \frac{\overline{\eta}}{2\lambda_m + 4} \int_1^1 \phi^{n-3}(s)\, ds = 0.$$

Define $t_0 = \sup I$ and let us show that $t_0 = \infty$. To this end, assume that $t_0 < \infty$. By continuity, it is clear that $t_0 \in I$. To reach a contradiction, we will show that for $\rho > 0$ small enough, $[t_0, t_0 + \rho) \subset I$.

First, we estimate

$$\phi^{n-2}(t) = \phi^{n-2}(1) + (n-3) \int_1^t \phi^{n-3}(s)\phi'(s)\, ds \geq \eta \int_1^t \phi^{n-3}(s)\, ds.$$

Let $\rho > 0$ be small enough so that either (3.4) or (3.5) holds. If (3.4) holds, then we can estimate $x_m(t)$ for $t \in [t_0, t_0 + \rho)$ by

$$x_m(t) \geq x_m(t_0) + \frac{1}{4} \int_{t_0}^t \phi^{n-3}(s)\, ds \geq \frac{\overline{\eta}}{2\lambda_m + 4} \int_1^t \phi^{n-3}(s)\, ds.$$

If (3.5) holds (here, we use the observation made above that $x_m \geq 0$), then

$$x_m(t) \geq \frac{1}{2\lambda_m + 4}\phi^{n-2}(t) \geq \frac{n-2}{2\lambda_m + 4} \int_1^t \phi^{n-3}(s)\phi'(s)\, ds \ \geq \frac{\eta}{2\lambda_m + 4} \int_1^t \phi^{n-3}(s)\, ds,$$

and, hence, (3.6) holds up to $t < t_0 + \rho$, and t_0, if finite, cannot be the supremum of I. Thus, we have proved that (3.6) holds, and then we can estimate

$$\begin{aligned}
\varphi_m(s) &\geq \exp\biggl(\frac{\lambda_m^2 \overline{\eta}}{2\lambda_m + 4} \int_1^s \frac{1}{\phi^{n-1}(\tau)} \int_1^\tau \phi^{n-3}(\sigma)\, d\sigma\, d\tau\biggr) \\
&= \exp\biggl(\frac{\lambda_m^2 \overline{\eta}}{2\lambda_m + 4} \int_1^s \phi^{n-3}(\sigma) \int_\sigma^s \frac{1}{\phi^{n-1}(\tau)}\, d\tau\, d\sigma\biggr) \\
&\geq \exp\biggl(\frac{\lambda_1^2 \overline{\eta}}{2\lambda_1 + 4} \int_1^s \phi^{n-3}(\sigma) \int_\sigma^s \frac{1}{\phi^{n-1}(\tau)}\, d\tau\, d\sigma\biggr).
\end{aligned}$$

This estimate shows that $\varphi_m(s) \to \infty$ as $s \to \infty$ for $m \neq 0$, under the assumption that $J = \infty$. Using this, we will show that the Dirichlet problem at infinity cannot be solved. In fact, we shall show that any bounded harmonic function must be constant. So, let $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a harmonic function. For $R > 0$ arbitrary, we let B_R be the ball of radius R centred at the origin and

$$u_R(\omega) = u|_{\partial B_R}.$$

Since u_R is smooth, we can expand it as the Fourier series

$$u_R(\omega) = \sum_m \sum_k c_{m,k,R} f_{m,k}(\omega),$$

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where $f_{m,k}$ is an eigenfunction of the eigenvalue λ_m^2 of the Laplacian Δ_ω of $\mathbb{S}^{n-1}$. From now on, we shall assume that $\{f_{m,k}\}$ is an orthonormal basis for $L^2(\mathbb{S}^{n-1})$.

The harmonic extension of u_R to B_R, and thus, by uniqueness, u on B_R (see [7, Theorem 3]), is given by

$$u(r,\omega) = \sum_m \frac{\varphi_m(r)}{\varphi_m(R)} \sum_k c_{m,k,R} f_{m,k}(\omega). \tag{3.7}$$

Indeed, the proof is exactly the same as that given in [7] for the Dirichlet problem at infinity. First, we must show that the sum converges in the L^2-sense for every $r \le R$. Here, we need the fact that the φ_m can be chosen to be nonnegative and *nondecreasing*: that this can be done is shown in [7]. Therefore, $0 \le \varphi_m(r)/\varphi_m(R) \le 1$ for $0 \le r \le R$ and, from having convergence at R, our assertion follows. Next, we must show that the boundary condition at ∂M_R is met as $r \to R$, and for this, we also ask the reader to consult [7]. From now on, we shall assume, without loss of generality, that $c_{0,0} = 0$.

Thus far, we have shown that in M_R, u can be written as given in (3.7). We shall prove that u can be represented, in the whole of M, as

$$\sum_m \frac{\varphi_m(r)}{\varphi_m(1)} \sum_k c_{m,k,1} f_{m,k}(\omega), \tag{3.8}$$

where

$$\sum_{m,k} c_{m,k,1} f_{m,k}(\omega)$$

is the Fourier expansion of u restricted to ∂M_1. First, observe that by taking $R = 1$ in (3.7), we obtain (3.8). Now consider (3.7) at any fixed $R > 1$, evaluate at $r = 1$ and then compare with (3.8) when evaluated at $r = 1$. By uniqueness,

$$c_{m,k,R} = \frac{\varphi_m(R)}{\varphi_m(1)} c_{m,k,1}.$$

However, then it follows that (3.8) represents u for any $R \ge 1$ and, as this representation is already valid for $R < 1$, our claim is proved.

Now, we can use Plancherel's theorem to compute

$$\|u\|^2_{L^2(\partial B_R)} = \sum_m \left(\frac{\varphi_m(R)}{\varphi_m(1)}\right)^2 \sum_k |c_{m,k,1}|^2 \phi^{n-1}(R).$$

If we assume that $u(\omega, R) = O(1)$, then

$$\sum_m \left(\frac{\varphi_m(R)}{\varphi_m(1)}\right)^2 \sum_k |c_{m,k,1}|^2 \phi^{n-1}(R) = O(1)\phi^{n-1}(R),$$

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that is,

$$\sum_m \Big(\frac{\varphi_m(R)}{\varphi_m(1)}\Big)^2 \sum_k |c_{m,k,1}|^2 = O(1),$$

which contradicts the fact that $\varphi_m(R) \to \infty$ as $R \to \infty$.

Next, we derive the estimate (1.3). We have

$$\begin{aligned}\max_{r\le R} |u(r,\omega)|^2 &= \max_{r=R} |u(r,\omega)|^2 \quad \text{(by the maximum principle)}\\ &\ge \sum_m \Big(\frac{\varphi_m(R)}{\varphi_m(1)}\Big)^2 \sum_k |c_{m,k,1}|^2\\ &\gtrsim \exp\Big(\frac{2\lambda_1^2\overline{\eta}}{2\lambda_1+4}\int_1^s \phi^{n-3}(\sigma)\int_\sigma^s \frac{1}{\phi^{n-1}(\tau)}\,d\tau\,d\sigma\Big)\end{aligned}$$

and the estimate follows.

3.3. Proof of part (iii). Assume that for all $r > 0$, $\phi'(r) \le \beta$, for a positive constant β. Again, write φ_m as in (3.2) for $s \ge 1$ and $x_m(t)$ a smooth function. Let

$$\eta_m = \frac{1}{2}\Big(-\frac{\beta(n-2)}{\lambda_m} + \sqrt{\frac{\beta^2(n-2)^2}{\lambda_m^2}+4}\Big).$$

Since x_m satisfies (3.3), at any given s, either

$$\frac{\lambda_m^2}{\phi^{n-1}(s)}x_m^2(s) > \eta_m^2\phi^{n-3}(s),$$

that is (since $x_m \ge 0$),

$$x_m(s) \ge \frac{\eta_m}{\lambda_m}\phi^{n-2}, \tag{3.9}$$

or

$$x_m'(s) > (1-\eta_m^2)\phi^{n-3}(s). \tag{3.10}$$

Our choice of η_m guarantees

$$\frac{1-\eta_m^2}{\beta(n-2)} = \frac{\eta_m}{\lambda_m}.$$

We shall show the following. Assume that at time $t = t_0$, we have the estimate

$$x_m(t) \ge \frac{1-\eta_m^2}{\beta(n-2)}\phi^{n-2}(t). \tag{3.11}$$

Then, there is an $\epsilon > 0$ such that the same estimate is valid on $(t_0, t_0+\epsilon)$. To do so, choose $\epsilon > 0$ such that either (3.9) or (3.10) is valid. In the first case, (3.11) follows immediately. In the second case, we estimate $x_m(\tau)$ as follows. For $\tau \in (t_0, t_0+\epsilon)$,

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$$\begin{aligned}x_m(\tau) &= x_m(t_0) + \int_{t_0}^{\tau} \phi^{n-3}(s)\,ds \\ &\geq x_m(t_0) + (1-\eta_m^2)\int_{t_0}^{\tau} \phi^{n-3}(s)\frac{\phi'(s)}{\beta}\,ds \\ &= x_m(t_0) + \frac{1-\eta_m^2}{\beta(n-2)}(\phi^{n-2}(\tau) - \phi^{n-2}(t_0)) \\ &\geq \frac{1-\eta_m^2}{\beta(n-2)}\phi^{n-2}(\tau).\end{aligned}$$

By continuity, the estimate (3.11) holds on a closed set and this set is nonempty since the estimate is satisfied when $t = 0$ (again, we are using $x_m \geq 0$ and $\phi(0) = 0$). Therefore, the estimate holds for all $t \in [0, \infty)$ and we obtain

$$x_m(t) \geq \frac{\eta_m}{\lambda_m}\phi^{n-2}(t).$$

Thus, we can take

$$\eta_m = \frac{1}{2}\frac{\lambda_m^2}{\beta(n-2)}\left[\sqrt{1 + \frac{4\lambda_m^2}{\beta^2(n-2)^2}} - 1\right]$$

and hence,

$$x_m(t) \geq a_m\phi^{n-2}(t),$$

where

$$a_m = \frac{1}{2}\frac{\lambda_m}{\beta(n-2)}\left[\sqrt{1 + \frac{4\lambda_m^2}{\beta^2(n-2)^2}} - 1\right].$$

This shows that

$$\varphi_m(r) \gtrsim \exp\left(a_m \int_1^r \frac{1}{\phi(s)}\,ds\right)$$

and, since the a_m form an increasing sequence, we have the estimate

$$\varphi_m(r) \gtrsim \exp\left(a_1 \int_1^r \frac{1}{\phi(s)}\,ds\right).$$

From Plancherel's theorem, as we reasoned above, if u is bounded and if

$$\int_1^{\infty} \frac{1}{\phi(s)}\,ds = \infty,$$

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then u must be constant. However, if $\phi'(r) \leq \beta$, then $\phi \leq \beta r$. Consequently, the integral above is divergent and Liouville's theorem holds in this case. Again, an argument using Plancherel's theorem shows that any nonconstant harmonic u function must satisfy

$$\max_{r \leq R} |u(\omega, r)| \gtrsim \exp\left(a_1 \int_1^r \frac{1}{\phi(s)}\, ds\right) \geq r^{a_1/\beta},$$

which proves (1.4).

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https://doi.org/10.1017/S0004972725000061 Published online by Cambridge University Press