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On the Dirichlet problem with several volume constraints on the level sets

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We consider minimization problems involving the Dirichlet integral under an arbitrary number of volume constraints on the level sets and a generalized boundary condition. More precisely, given a bounded open domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, we study the problem of minimizing $\int_\Omega |\nabla u|^2$ among all those functions $u \in H^1$ that simultaneously satisfy n-dimensional measure constraints on the level sets of the kind $|\{u = l_i\}| = \alpha_i$, $i = 1, \dots, k$, and a generalized boundary condition $u \in \mathcal{K}$. Here, $\mathcal{K}$ is a closed convex subset of H^1 such that $\mathcal{K} + H^1_0 = \mathcal{K}$; the invariance of $\mathcal{K}$ under H^1_0 provides that the condition $u \in \mathcal{K}$ actually depends only on the trace of u along $\partial\Omega$.

By a penalization approach, we prove the existence of minimizers and their Hölder continuity, generalizing previous results that are not applicable when a boundary condition is prescribed.

Finally, in the case of just two volume constraints, we investigate the Γ-convergence of the above (rescaled) functionals when the total measure of the two prescribed level sets tends to saturate the whole domain Ω. It turns out that the resulting Γ-limit functional can be split into two distinct parts: the perimeter of the interface due to the Dirichlet energy that concentrates along the jump, and a boundary integral term due to the constraint $u \in \mathcal{K}$. In the particular case where $\mathcal{K} = H^1$ (i.e. when no boundary condition is prescribed), the boundary term vanishes and we recover a previous result due to Ambrosio *et al*.

1. Introduction

The paper deals with the following problem, whose motivation can be found in a model of equilibrium interface between immiscible fluids (see [4,7] for more details). Given a domain (connected bounded open set) $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary and a finite set of pairs of real numbers $\{(l_j, \alpha_j)\}_{j=1}^k$ satisfying

$$l_1 < l_2 < \dots < l_k, \quad \alpha_j > 0 \quad \text{and} \quad \sum_{j=1}^k \alpha_j < |\Omega|, \tag{1.1}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω, let Π denote the class of all functions $u \in H^1(\Omega)$ such that, for every $j = 1, \dots, k$, the Lebesgue measure of the level set

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$\{u = l_j\}$ is equal to α_j. We consider the problem of finding a function $u \in H^1(\Omega)$ that minimizes the Dirichlet integral, among all those functions that simultaneously belong to Π and satisfy certain boundary conditions along $\partial\Omega$. In fact, we allow boundary conditions of a rather general form, which can be written as $u \in \mathcal{K}$, where $\mathcal{K} \subseteq H^1(\Omega)$ is a non-empty closed convex set satisfying $\mathcal{K} + H^1_0(\Omega) = \mathcal{K}$ (the invariance of $\mathcal{K}$ with respect to H^1_0 provides that the constraint $u \in \mathcal{K}$ is a real 'boundary condition', since only the trace of u along $\partial\Omega$ is involved).

In short, the minimization problem we are going to consider reads as follows:

$$\left.\begin{aligned} &\inf\biggl\{\int_\Omega |\nabla u|^2 \,\mathrm{d}x : u \in \Pi \cap \mathcal{K}\biggr\}, \\ &\qquad\qquad \text{where } \Pi := \{u \in H^1(\Omega) : |\{u = l_j\}| = \alpha_j,\ j = 1, \dots, k\} \\ &\qquad\qquad \text{and } \mathcal{K} \subseteq H^1(\Omega) \text{ is convex, closed and } \mathcal{K} + H^1_0(\Omega) = \mathcal{K}. \end{aligned}\right\} \tag{M}$$

We remark that such a setting includes quite a broad range of boundary constraints. Here are some particular cases that frequently arise in the applications.

(i) Prescribed values on the boundary. Let $S \subseteq \partial\Omega$ be a smooth portion of the boundary and $g \in H^{1/2}(S)$. Then $\mathcal{K} := \{u \in H^1(\Omega) : u = g \text{ over } S\}$.

(ii) Boundary obstacle. Let S be as above and $g_1, g_2 \in H^{1/2}(S)$ with $g_1 \leqslant g_2$. Then $\mathcal{K} := \{u \in H^1(\Omega) : g_1 \leqslant u \leqslant g_2 \text{ over } S\}$.

(iii) Integral condition on the boundary. In this case,

$$\mathcal{K} := \biggl\{u \in H^1(\Omega) : \int_{\partial\Omega} f(x, u(x)) \,\mathrm{d}\mathcal{H}^{n-1}(x) \leqslant 0\biggr\},$$

where f is a given Borel function such that $f(x, \cdot)$ is convex, and $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure.

Note also that even the case without any boundary constraint is modelled by our setting, if we take $\mathcal{K} := H^1(\Omega)$.

Free-boundary problems of the kind (M) have already been considered in the case of only one volume constraint (i.e. $k = 1$ in (1.1)). For instance, in [2] and [1], only the measure of the level set $\{u = 0\}$ is prescribed, and a non-negative boundary value is given along $\partial\Omega$ (or a smooth portion of it). This particular case is related to the problem of finding a set having minimum Newtonian capacity and prescribed Lebesgue measure. In [2], existence and Lipschitz continuity of the solutions have been proved as well as regularity (outside a small singular set) of the free boundary $\partial\{u > 0\}$, by means of measure-theoretic techniques: however, the global subharmonicity of the minimizers, which cannot be expected in the general case we are considering, plays an important role in the approach presented in [1,2].

The case with several volume constraints (i.e. $k \geqslant 2$ in (1.1)) was first considered in [4], where existence of the minimizers was proved for vector-valued u and for functionals more general than the Dirichlet integral, but without any boundary condition (i.e. in the case where $\mathcal{K} = H^1(\Omega)$). The existence results from [4], however, rely on the assumption that the vectors $\{l_j\}_{j=1}^k$ are extreme point of a convex set, which, in the scalar case (i.e. when u is real valued), turns into a severe restriction, since it does not allow one to deal with the case of more than two volume constraints.

Finally, existence and Hölder continuity of the minimizers in the case of an arbitrary number of volume constraints were first proved in [8], when the function u is real valued. We point out that the techniques adopted in [8] to prove the existence of solutions cannot be used to solve (M), since the competitor functions therein constructed do not preserve the boundary values (hence they may fail to be admissible functions for (M)).

The main difficulty in studying (M) is that, although the functional is coercive and semicontinuous, the class of functions Π is not closed in any reasonable topology. More precisely, on trying to apply the direct method of the calculus of variations, it is natural to work with the strong topology of $L^2(\Omega)$ in view of the coercivity of the Dirichlet integral; however, if $u_\nu \to u$ in $L^2(\Omega)$ (or even uniformly) as $\nu \to \infty$, then all we can say about the level sets of u (see [4]) is that $|\{u = l_j\}| \geqslant \limsup_\nu |\{u_\nu = l_j\}|$, so that u may not belong to Π, even if all the u_ν do. Therefore, the direct methods provide the existence minimizers only for the following 'relaxed' problem:

$$\left.\begin{aligned}&\inf\left\{\int_\Omega |\nabla u|^2\,\mathrm{d}x : u \in \Pi' \cap \mathcal{K}\right\},\\ &\qquad\text{where } \Pi' := \{u \in H^1(\Omega) : |\{u = l_j\}| \geqslant \alpha_j,\ j = 1, \dots, k\}.\end{aligned}\right\} \tag{M$'$}$$

Note that here Π' is closed with respect to the strong topology of $L^2(\Omega)$ (see [4]). If a solution u to (M$'$) is continuous, then its level sets $\{u = l_j\}$ are disjoint closed sets (with respect to Ω), and one could use some localization technique to show that, in fact, such a u solves (M) as well (i.e. that $u \in \Pi$). Unfortunately, we have no *a priori* information on the continuity of the solutions to (M$'$), and at this stage this seems to be a difficult task.

In this paper we show the existence and Hölder continuity of solutions to (M), simultaneously allowing an arbitrary number of volume constraints and a rather wide range of boundary conditions (see theorem 2.1). In particular, our existence results generalize those contained in [8] (which can be easily recovered dropping the boundary condition, i.e. letting $\mathcal{K} = H^1(\Omega)$).

This existence result seems to be new even for the case of just one volume constraint (i.e. $k = 1$) plus a Dirichlet boundary condition, since, unlike the existence result from [2], it is valid without any restriction on the boundary data, except for the assumption that the constant function $u \equiv l_1$ should not belong to $\mathcal{K}$ (on the other hand, if this assumption is violated, then (M) has no solution at all, as we shall see in § 2).

We further show that every solution to (M$'$) solves (M). Moreover, in theorem 3.1 we characterized (M$'$) as the relaxation (in the strong topology of $L^2(\Omega)$) of the original problem (M).

Finally, in § 4, we study the Γ-convergence of our problems (suitably rescaled) in the case of only two volume constraints, say $|\{u = 0\}| = \alpha_\nu$ and $|\{u = 1\}| = \beta_\nu$, and a boundary condition $u \in \mathcal{K}$ as above, when $\alpha_\nu \to \alpha$ and $\beta_\nu \to \beta$, with $\alpha + \beta = |\Omega|$ (this means that a minimizer of the limit problem is forced to be the characteristic function 1_E of a set $E \subset \Omega$, satisfying the measure constraint $|E| = \beta$).

In [4], it was proved that, in the case of no boundary condition (i.e. when $\mathcal{K} = H^1(\Omega)$), then the Γ-limit energy of an admissible configuration $u = 1_E$ (with $|E| = \beta$) is given by $P(E,\Omega)^2$, i.e. the square of the perimeter of the set E in Ω. Roughly speaking, this means that the volume energy of a sequence of minimizers u_ν, converging to 1_E in $L^2(\Omega)$, tends to concentrate as surface energy along the boundary of E in Ω. In our case, that is, when a boundary condition $u_\nu \in \mathcal{K}$ is forced to hold, we can prove (theorem 4.1) that an additional term appears in the limit energy. More precisely, if $u = 1_E$ is an admissible configuration for the Γ-limit problem (i.e. $1_E \in BV(\Omega)$ and $|E| = \beta$), then the energy of u is given by

$$\left(P(E,\Omega) + \inf_{g\in\mathcal{K}} \int_{\partial\Omega} |g - 1_E| \,\mathrm{d}\mathcal{H}^{n-1}\right)^2.$$

In the above formula, both g and 1_E restricted to $\partial\Omega$ are to be meant in the sense of the trace (of a Sobolev and of a BV function, respectively), and hence they are regarded as elements of $L^1(\partial\Omega,\mathcal{H}^{n-1})$. The presence of the boundary integral term (in addition to the perimeter term already found in [4]) is due to the boundary condition $u_\nu \in \mathcal{K}$. We observe that if $\mathcal{K} \subseteq H^1(\Omega)$ is regarded as a convex subset of $L^1(\partial\Omega,\mathcal{H}^{n-1})$ by means of the trace operator, then the limit energy can be rewritten as

$$(P(E,\Omega) + \operatorname{dist}(1_E,\mathcal{K}))^2,$$

where the distance is computed in the metric of $L^1(\partial\Omega,\mathcal{H}^{n-1})$. We finally remark that the Γ-convergence result of [4] can be obtained as a particular case of theorem 4.1, letting $\mathcal{K} = H^1(\Omega)$. Indeed, in this case, the 'distance' term in the above expression vanishes, since the traces along $\partial\Omega$ of H^1 functions are a dense subspace of $L^1(\partial\Omega,\mathcal{H}^{n-1})$.

2. Existence and regularity of solutions

Before stating the existence result, let us observe that there is just one case where problem (M) has no solution. This is the case of only one volume constraint (i.e. $k = 1$ in (1.1)), when the set $\mathcal{K}$ contains the constant function $u \equiv l_1$. Indeed, in this case, we fix a compact set $K \subset \Omega$ such that $|K| = \alpha_1$ and we let, for $\varepsilon > 0$,

$$u_\varepsilon := l_1 + \varepsilon \operatorname{dist}(x, K \cup \partial\Omega).$$

It is clear that $u_\varepsilon \in \mathcal{K} \cap \Pi$ and

$$\int_\Omega |\nabla u_\varepsilon|^2 \,\mathrm{d}x = \varepsilon^2(|\Omega| - \alpha_1),$$

by well-known properties of the distance function. Since $\varepsilon > 0$ is arbitrary, we see that the infimum in (M) is zero, which can be attained only by constant functions. On the other hand, constant functions are not admissible since $\alpha_1 < |\Omega|$ according to (1.1), hence (M) has no solution in this case.

In all the other cases, we can claim the existence of solutions to (M).

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded connected set with Lipschitz boundary. Fix an integer $k \geqslant 1$ and let α_j, l_j satisfy (1.1). Assume $\mathcal{K}$ is a closed convex* *subset of $H^1(\Omega)$ satisfying $\mathcal{K} + H_0^1(\Omega) = \mathcal{K}$. Moreover, if $k = 1$, we further assume that the constant function $u \equiv l_1$ does not belong to $\mathcal{K}$. Then problem (M) has a solution. Furthermore, every minimizer u of (M) is locally Hölder continuous in Ω, for every exponent $\theta \in (0, 1)$. More precisely, the continuity estimate*

$$|u(x) - u(y)| \leqslant C_K|x - y| \log \frac{C_K}{|x - y|} \quad \forall x, y \in K \tag{2.1}$$

holds for every compact set $K \subset \Omega$, where C_K is a constant depending on K.

REMARK 2.2. We remark that the continuity estimate (2.1) implies that u is (locally) Hölder continuous for every exponent $\theta \in (0, 1)$. In fact, we were not able to prove that the minimizers of (M) are locally Lipschitz continuous. The local Lipschitz continuity has so far been established only in the case of one prescribed level [2] and for the case of two prescribed levels, but without any boundary constraint, in [8].

In order to prove theorem 2.1, we rely on a technique that develops the one used in [8] for the problem without boundary constraints. We consider an auxiliary penalized functional depending on a parameter $\lambda > 0$, defined by

$$F_\lambda(u) := \int_\Omega |\nabla u|^2 \, \mathrm{d}x + P_\lambda(u), \tag{2.2}$$

where

$$P_\lambda(u) := \lambda \sum_{j=1}^{k} (\alpha_j - |\{u = l_j\}|\,)^+ \tag{2.3}$$

and $x^+ := \max\{x, 0\}$. We will show that for each $\lambda > 0$ the functional F_λ attains a minimum over the set of functions $u \in H^1(\Omega)$ that satisfy $u \in \mathcal{K}$, and that every minimizers is Hölder continuous. Next we will show that for sufficiently large λ, the minimizers of F_λ are minimizers of (M) as well, thus proving our claim.

We call F_λ the 'penalized functional', whereas $P_\lambda(u)$ will be called the 'penalization' of u. We observe that, according to (2.3), a level set $\{u = l_i\}$ is penalized if (and only if) its measure is less than the prescribed value α_i.

We remark for further use that there exists a constant $\mu > 0$ independent of λ such that

$$\inf_{u \in \mathcal{K}} F_\lambda(u) \leqslant \mu. \tag{2.4}$$

To see this, note that

$$\inf_{u \in \mathcal{K}} F_\lambda(u) \leqslant \inf_{u \in \mathcal{K} \cap \Pi} F_\lambda(u) = \inf_{u \in \mathcal{K} \cap \Pi} \int_\Omega |\nabla u|^2 \, \mathrm{d}x,$$

since $P_\lambda(u) = 0$ whenever $u \in \Pi$. Therefore, it is enough to let

$$\mu := \inf \int_\Omega |\nabla u|^2 \, \mathrm{d}x,$$

where the infimum is over $u \in \Pi \cap \mathcal{K}$, to obtain (2.4).

Finally, we observe that, under our assumptions on $\mathcal{K}$ and Π, it is easy to check that $\Pi \cap \mathcal{K}$ is never empty (recall that $\mathcal{K} + H_0^1(\Omega) = \mathcal{K}$).

Proposition 2.3. *Under the assumptions of theorem 2.1, suppose that* $\lambda > \max_j(2\mu+2)/\alpha_j$. *Then the functional* F_λ *admits a minimum over* $H^1(\Omega)\cap\mathcal{K}$. *If* u *minimizes* F_λ*, then it is a continuous function satisfying (2.1). Finally, the level sets of* u *satisfy*

$$\tfrac{1}{2}\alpha_j \leqslant |\{u = l_j\}| \leqslant \alpha_j, \quad j = 1,\dots,k, \tag{2.5}$$

while the Euler-type equation

$$\int_\Omega \nabla u\cdot\nabla(\psi f(u))\,\mathrm{d}x = 0 \tag{2.6}$$

holds for all $\psi\in W_0^{1,\infty}(\Omega)$ *and for all Lipschitz continuous* $f:\mathbb{R}\to\mathbb{R}$ *satisfying* $f(l_j) = 0$ *for every* $j = 1,\dots,k$.

Proof.

Step 1. We prove the existence of a minimizer to F_λ. Let $u_\nu\in\mathcal{K}$ be a minimizing sequence for F_λ. Using (2.4), we can assume that

$$\sup_\nu\biggl(\lambda(\alpha_1 - |\{u_\nu = l_1\}|\,) + \int_\Omega |\nabla u_\nu|^2\biggr) \leqslant \sup_\nu F_\lambda(u_\nu) \leqslant \mu + 1.$$

Then $\{\nabla u_\nu\}$ are bounded in $L^2(\Omega)$ and, due to the assumption on λ, one has $u_\nu = l_1$ on a set of measure not less than $\frac{1}{2}\alpha_1$.

From a well-known Poincaré-type inequality applied to $u_\nu - l_1$ (see [9, theorem 4.4.2]), we obtain that also $\{u_\nu\}$ is bounded in $L^2(\Omega)$. Hence, passing to a subsequence (not relabelled), there exists $u\in H^1(\Omega)$ such that $u_\nu\to u$ weakly in $H^1(\Omega)$, strongly in $L^2(\Omega)$ and pointwise almost everywhere. Furthermore, $u\in\mathcal{K}$, since $\mathcal{K}$ is convex and strongly closed (hence weakly closed) in $H^1(\Omega)$. Then u is a minimum for F_λ over $\mathcal{K}$, since this functional is lower semicontinuous (note that the penalization P_λ is lower semicontinuous, since it is the sum of non-increasing functions of the measures of some level sets).

Step 2. Now we show the validity of (2.5), for any minimizer u of F_λ over $\mathcal{K}$. Indeed, we have, from (2.4),

$$\lambda(\alpha_j - |\{u = l_j\}|\,) \leqslant P_\lambda(u) \leqslant F_\lambda(u) \leqslant \mu < \mu + 1, \quad j = 1,\dots,k,$$

which proves the first inequality in (2.5).

The second inequality in (2.5) is achieved immediately once one shows the continuity of the minimizer. In fact, suppose that u is continuous while this inequality is not true, i.e. $|E_j| > \alpha_j$ for some j, where $E_j := \{u = l_j\}$. We note first that $|E_j| < |\Omega|$. Indeed, if $k > 1$, then the first inequality in (2.5) implies that $|\{u = l_i\}| > 0$ for every i, in particular for some $i\neq j$. On the other hand, if $k = 1$ and $|E_j| = |\Omega|$, then $u\equiv l_1$ on Ω, thus violating that $u\in\mathcal{K}$ (recall the assumptions of theorem 2.1, when $k = 1$). Now observe that E_j is closed in Ω in view of the continuity of u. We can choose a point $x_0\in\Omega\cap\partial E_j$ such that $|B(x_0,r)\cap E_j| > 0$ for all sufficiently small $r > 0$ (the existence of such x_0 follows from elementary measure-theoretic arguments). In particular, since $u(x_0) = l_j$ and u is continuous, there is $r > 0$ such that $B_r := B(x_0,r)\subset\Omega$ and

$$0 < |B_r\cap E_j| < |B_r| \leqslant |E_j| - \alpha_j \quad\text{and}\quad |B_r\cap\{u = l_i\}| = 0, \quad i\neq j. \tag{2.7}$$

Then, if $v \in H^1(\Omega)$ is harmonic in B_r and coincides with u outside B_r, we have $|\{v = l_j\}| \geqslant |E_j| - |B_r| \geqslant \alpha_j$, while $|\{v = l_i\}| = |\{u = l_i\}|$ when $i \neq j$. Then $P_\lambda(v) = P_\lambda(u)$, whereas

$$\int_\Omega |\nabla v|^2 \, \mathrm{d}x < \int_\Omega |\nabla u|^2 \, \mathrm{d}x,$$

since u is not harmonic inside B_r, by virtue of (2.7) (note that a non-constant function harmonic in a ball has every level set of null measure, due to analiticity), thus violating the minimality of u. This proves that $|\{u = l_j\}| \leqslant \alpha_j$, provided u is continuous.

To conclude the proof of the second inequality of (2.5), we will show the continuity of minimizer by proving (2.1). This is achieved by comparison with harmonic functions, as follows. Let B_r be an open ball such that $\bar{B}_r \subset \Omega$, and let v_r be the harmonic function on B_r that coincides with u on ∂B_r. Replacing u with v_r inside B_r, the Dirichlet integral on Ω decreases by

$$\int_{B_r} |\nabla u|^2 \, \mathrm{d}x - \int_{B_r} |\nabla v_r|^2 \, \mathrm{d}x = \int_{B_r} |\nabla (u - v_r)|^2 \, \mathrm{d}x.$$

On the other hand, since this variation affects the values of u only inside B_r, the penalization cannot increase more than $\lambda |B_r|$, hence we obtain

$$\int_{B_r} |\nabla (u - v_r)|^2 \, \mathrm{d}x \leqslant \lambda \omega_n r^n,$$

where ω_n stands for the Lebesgue measure of the unit ball in $\mathbb{R}^n$, since u is a minimizer. Then (2.1) follows from the arbitrariness of B_r, reasoning as in theorem 2.1 of [3].

STEP 3. Finally we prove (2.6). Let ψ and f be as in proposition 2.3. Letting $u_\varepsilon := u + \varepsilon \psi f(u)$ for $\varepsilon \in \mathbb{R}$, one observes that $u_\varepsilon - u \in H^1_0(\Omega)$, hence $u_\varepsilon \in \mathcal{K}$. Moreover, since $f(l_j) = 0$, we have

$$\{u = l_j\} \subset \{u_\varepsilon = l_j\}, \quad j = 1, \dots, k,$$

hence $P_\lambda(u_\varepsilon) \leqslant P_\lambda(u)$. Therefore, since u minimizes F_λ, we have

$$\int_\Omega |\nabla u|^2 \, \mathrm{d}x \leqslant \int_\Omega |\nabla u_\varepsilon|^2 \, \mathrm{d}x$$

for every $\varepsilon \in \mathbb{R}$, from which (2.6) easily follows. □

The following statement, combined with proposition 2.3, will be the crucial step in the proof of theorem 2.1.

PROPOSITION 2.4. *There exists a positive number* $\bar{\lambda} > 0$*, depending only on* Ω*,* $\{\alpha_j\}$ *and* h*, where*

$$h := \begin{cases} \min_j \{l_{j+1} - l_j\}, & k > 1, \\ 1, & k = 1, \end{cases}$$

such that whenever $\lambda > \bar{\lambda}$ *and* $u \in H^1(\Omega)$ *minimizes* F_λ *over* $\mathcal{K}$*, then* $u \in \Pi$*.*

The proof will be obtained by contradiction and will require several technical constructions.

Proof. Choose a connected smooth open set Ω_0 satisfying $\bar{\Omega}_0 \subset \Omega$,

$$|\Omega \setminus \Omega_0| \leqslant \tfrac{1}{4}\alpha_j \quad \text{and} \quad |\Omega_0| > \alpha_j, \quad j = 1, \dots, k, \tag{2.8}$$

and a function $\psi \in W_0^{1,\infty}(\Omega)$ such that $0 \leqslant \psi \leqslant 1$ and $\psi = 1$ in Ω_0 (observe that Ω_0 and ψ can be chosen independently of λ). From now on, we say that some quantity is *universal* if it does not depend on λ.

We can suppose that λ satisfies the assumption of proposition 2.3. Let u be a minimizer of F_λ over $\mathcal{K}$, and assume that $u \notin \Pi$. In view of (2.5), this means that

$$\tfrac{1}{2}\alpha_i \leqslant |\{u = l_i\}| < \alpha_i \tag{2.9}$$

(note the last strict inequality) for some $i \in \{1, \dots, k\}$. We will show that this assumption leads to a universal upper bound for λ, thus proving our claim.

STEP 1. Observe that, without loss of generality, we may assume $l_i = 0$ (it suffices to replace $\mathcal{K}$ by $\mathcal{K} - l_i$, l_j by $l_j - l_i$ and u by $u - l_i$). From (2.8) and (2.9), we deduce that

$$|\Omega_0 \cap \{u = 0\}| \geqslant \tfrac{1}{4}\alpha_i \quad \text{and} \quad |\Omega_0 \cap \{u \neq 0\}| \geqslant |\Omega_0| - \alpha_i > 0. \tag{2.10}$$

For $\delta > 0$, define the sets

$$\Sigma_\delta^+ := \{|u| \geqslant \delta\}, \qquad \Sigma_\delta^- := \{0 < |u| < \delta\}, \qquad D_\delta := \Omega_0 \cap \Sigma_\delta^-.$$

We will show the existence of universal constants C_2, $C_3 > 0$ such that, for all sufficiently small $\delta > 0$, there holds

$$\lambda |D_\delta| \leqslant C_2 \delta, \tag{2.11}$$

and

$$|D_\delta| \geqslant \delta / C_3. \tag{2.12}$$

Hence, combining the two above estimates one obtains $\lambda \leqslant C_4$ for some universal constant C_4, concluding our proof. It remains to establish (2.11) and (2.12), which we will do in the following steps.

STEP 2. We prove the estimate (2.11), showing the existence of functions $w_\delta \in H_0^1(\Omega)$ such that the inequalities

$$\lambda |D_\delta| \leqslant \int_\Omega |\nabla w_\delta|^2 \, \mathrm{d}x \tag{2.13}$$

and

$$\int_\Omega |\nabla w_\delta|^2 \, \mathrm{d}x \leqslant C_2 \delta \tag{2.14}$$

hold for all sufficiently small $\delta > 0$. Since

$$\lim_{\delta \downarrow 0} |D_\delta| = 0 \quad \text{and} \quad \lim_{\delta \downarrow 0} |\Omega_0 \cap \Sigma_\delta^+| = |\Omega_0 \cap \{u \neq 0\}|,$$

by (2.9) and (2.10), we can find δ (depending on u) such that $0 < \delta < \frac{1}{2}h$ and

$$|\{u = 0\}| + |D_\delta| < \alpha_i \quad \text{and} \quad |\Omega_0 \cap \Sigma_\delta^+| \geqslant \tfrac{1}{2}(|\Omega_0| - \alpha_i). \tag{2.15}$$

We fix δ with the above properties and we define the piecewise affine function

$$f_\delta(t) = \begin{cases} -t & \text{if } |t| < \delta, \\ \operatorname{sign}(t)\delta\dfrac{|t| - h}{h - \delta} & \text{if } \delta \leqslant |t| < h, \\ 0 & \text{if } |t| \geqslant h. \end{cases}$$

Let us define $w_\delta(x) := \psi(x) f_\delta(u(x))$ and observe that, in view of (2.6), one has

$$\int_\Omega \nabla u \cdot \nabla w_\delta \, \mathrm{d}x = 0. \tag{2.16}$$

Note that $u + w_\delta \in \mathcal{K}$, since $u \in \mathcal{K}$, $w_\delta \in H_0^1(\Omega)$ and $\mathcal{K} + H_0^1 = \mathcal{K}$ by assumption. Since u minimizes F_λ over $\mathcal{K}$, one has

$$P_\lambda(u) - P_\lambda(u + w_\delta) \leqslant \int_\Omega |\nabla(u + w_\delta)|^2 - |\nabla u|^2 \, \mathrm{d}x = \int_\Omega |\nabla w_\delta|^2 \, \mathrm{d}x, \tag{2.17}$$

where (2.16) has been used. From the definition of w_δ and D_δ, it immediately follows that

$$D_\delta \sqcup \{u = 0\} \subset \{u + w_\delta = 0\}, \quad \{u = l_j\} \subset \{u + w_\delta = l_j\} \quad \text{when } j \neq i, \tag{2.18}$$

where $\sqcup$ denotes disjoint union. Therefore, using the relationship

$$|\{u + w_\delta = 0\}| \geqslant |\{u = 0\}| + |D_\delta|$$

and (2.15), one obtains

$$\begin{aligned} P_\lambda(u) - P_\lambda(u + w_\delta) &\geqslant \lambda((\alpha_i - |\{u = l\}|)^+ - (\alpha_i - |\{u + w_\delta = l\}|)^+) \\ &\geqslant \lambda((\alpha_i - |\{u = 0\}|) - (\alpha_i - |\{u = 0\}| - |D_\delta|)^+) \\ &= \lambda |D_\delta|. \end{aligned}$$

Then (2.13) follows on combining (2.17) with the last inequality.

To prove (2.14), observe that

$$|\nabla w_\delta|^2 \leqslant 2 f_\delta(u)^2 |\nabla \psi|^2 + 2\psi^2 f_\delta'(u)^2 |\nabla u|^2.$$

Since $|f_\delta| \leqslant \delta$, $\psi^2 \leqslant \psi \leqslant 1$ and $\delta < \frac{1}{2}h$, using (2.4) we find

$$\begin{aligned} \int_\Omega |\nabla w_\delta|^2 \, \mathrm{d}x &\leqslant \delta^2 C_\psi + 2\int_{\Sigma_\delta^-} \psi^2 |\nabla u|^2 \, \mathrm{d}x + \frac{2\delta^2}{(h - \delta)^2} \int_{\Sigma_\delta^+} \psi^2 |\nabla u|^2 \, \mathrm{d}x \\ &\leqslant \delta^2 C_\psi + 2\int_{\Sigma_\delta^-} \psi |\nabla u|^2 \, \mathrm{d}x + \frac{8\delta^2}{h^2} \mu, \end{aligned}$$

hence (2.14) follows from

$$\int_{\Sigma_\delta^-} \psi |\nabla u|^2 \, \mathrm{d}x \leqslant C_1 \delta \tag{2.19}$$

for some universal constant $C_1 > 0$. To prove (2.19), observe that (2.16) can be written as

$$\int_{\Sigma_\delta^-} \psi|\nabla u|^2 \,\mathrm{d}x = \int_\Omega f_\delta(u)\nabla u\cdot\nabla\psi \,\mathrm{d}x + \frac{\delta}{h-\delta}\int_{\Sigma_\delta^+} \psi|\nabla u|^2 \,\mathrm{d}x, \tag{2.20}$$

because $\nabla u = 0$ a.e. on $\{u = 0\}$. Since $|f_\delta| \leqslant \delta$, $\delta < \frac{1}{2}h$ and $0 \leqslant \psi \leqslant 1$, from (2.20), using (2.4) and the Hölder inequality, we deduce

$$\int_{\Sigma_\delta^-} \psi|\nabla u|^2 \,\mathrm{d}x \leqslant \delta\sqrt{\mu}C_\psi + \frac{\delta}{h-\delta}\int_{\Sigma_\delta^+} |\nabla u|^2 \leqslant \delta\sqrt{\mu}C_\psi + \delta\frac{2\mu}{h},$$

which proves (2.19).

STEP 3. Now we prove (2.12). For this purpose, let $A_t := \{|u| < t\}\cap \Omega_0$ for $t \in (0,\delta)$ and small δ, as in the previous step. Since

$$\{u = 0\}\cap \Omega_0 \subseteq A_t \quad \text{and} \quad \Sigma_\delta^+ \cap \Omega_0 \subseteq \Omega_0 \setminus A_t \quad \text{for every } t \in (0,\delta),$$

it is clear that the first inequality in (2.10) and the second one in (2.15) provide universal lower bounds for the Lebesgue measure of A_t and $\Omega_0 \setminus A_t$, respectively. Then, from the isoperimetric inequality relative to Ω_0, we obtain a universal lower bound for the perimeter of A_t in Ω_0, namely

$$\frac{1}{C} \leqslant \mathcal{H}^{n-1}(\partial A_t \cap \Omega_0) \leqslant \mathcal{H}^{n-1}(\{|u| = t\}\cap \Omega_0) \quad \text{for } t \in (0,\delta),$$

the latter inequality following from the continuity of u, where $C > 0$ is a universal constant. Integrating the last inequality with respect to t over $(0,\delta)$ and using the coarea formula, we obtain

$$\begin{aligned} \frac{\delta}{C} &\leqslant \int_0^\delta \mathcal{H}^{n-1}(\{|u| = t\}\cap \Omega_0)\,\mathrm{d}t \\ &= \int_{D_\delta} |\nabla u| \,\mathrm{d}x \\ &= \int_{D_\delta} \sqrt{\psi}|\nabla u| \,\mathrm{d}x \\ &\leqslant |D_\delta|^{1/2}\biggl(\int_{D_\delta} \psi|\nabla u|^2 \,\mathrm{d}x\biggr)^{1/2} \\ &\leqslant |D_\delta|^{1/2}\biggl(\int_{\Sigma_\delta^-} \psi|\nabla u|^2 \,\mathrm{d}x\biggr)^{1/2}. \end{aligned}$$

Then squaring and using (2.19) we obtain (2.12). □

Proof of theorem 2.1. Choose $\lambda > \bar\lambda$, where $\bar\lambda$ is given by proposition 2.4, and consider a minimizer u of F_λ over $\mathcal{K}$ (whose existence follows from proposition 2.3). From proposition 2.4, we know that $u \in \Pi$, i.e. u satisfies the volume constraints of problem (M). Note that this implies $P_\lambda(u) = 0$, hence

$$F_\lambda(u) = \int_\Omega |\nabla u|^2 \,\mathrm{d}x.$$

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Therefore, if v is any function in $\mathcal{K} \cap \varPi$, we have

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x = F_\lambda(u) \leqslant F_\lambda(v) = \int_\Omega |\nabla v|^2 \,\mathrm{d}x + P_\lambda(v) = \int_\Omega |\nabla v|^2 \,\mathrm{d}x,$$

since, in particular, $v \in \varPi$, hence $P_\lambda(v) = 0$. Since $v \in \mathcal{K} \cap \varPi$ is arbitrary, we see that u is a solution to (M), and the continuity of u follows from proposition 2.3. Finally, it is easy to check that any other solution to (M) is *a fortiori* a minimizer of F_λ over $\mathcal{K}$, hence the continuity of any solution to (M) (as claimed in theorem 2.1) follows from proposition 2.3 as well. □

The following corollary relating the minima for the problems (M) and (M′) is now almost immediate.

COROLLARY 2.5. *Under the same assumptions as in theorem 2.1, the relaxed problem (M′) has a solution. Moreover, every solution of (M′) solves (M) as well, and, in particular, it is locally Hölder continuous.*

Proof. The existence of solutions to (M′) easily follows from the direct methods of the calculus of variations. If u' solves (M′), then $P_\lambda(u') = 0$ (since, in particular, $u' \in \varPi'$). On the other hand, due to proposition 2.4, any minimizer u of F_λ over $\mathcal{K}$ satisfies $u \in \varPi \subset \varPi'$ and $P_\lambda(u) = 0$, provided $\lambda > \bar{\lambda}$. By the minimality of u', we have

$$\int_\Omega |\nabla u'|^2 \,\mathrm{d}x \leqslant \int_\Omega |\nabla u|^2 \,\mathrm{d}x,$$

i.e. u' also minimizes F_λ over $\mathcal{K}$ for large λ, therefore it also solves (M) and it is locally Hölder continuous. □

3. Relaxation

So far we have referred to (M′) as to the ‘relaxed’ problem (this terminology was also used in [4] for the same problem with no boundary condition). In fact, it turns out that (M′) is exacly the relaxation of (M) in the strong topology of $L^2(\Omega)$. Namely, consider the functionals F and F' defined over $L^2(\Omega)$ by

$$F(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 \,\mathrm{d}x, & u \in \mathcal{K} \cap \varPi, \\ +\infty, & \text{otherwise} \end{cases}$$

and

$$F'(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 \,\mathrm{d}x, & u \in \mathcal{K} \cap \varPi', \\ +\infty, & \text{otherwise.} \end{cases}$$

The following assertion is valid.

THEOREM 3.1. *F' is the relaxation of F in the strong topology of $L^2(\Omega)$.*

Proof. Clearly, whenever $\{u_\nu\} \subset \mathcal{K} \cap \Pi$ is such that $u_\nu \to u$ in $L^2(\Omega)$, then $u \in \Pi'$. Furthermore, we may assume that, up to choosing a subsequence (not relabelled),

$$\sup_\nu \int_\Omega |\nabla u_\nu|^2 \, dx < +\infty.$$

Hence, again up to a subsequence, we may assume that $u_\nu \rightharpoonup u$ weakly in $H^1(\Omega)$, and therefore $u \in \mathcal{K}$, since $\mathcal{K}$ is closed and convex. Then $F'(u) \leqslant \liminf_\nu F(u_\nu)$ follows from the lower semicontinuity of the Dirichlet integral.

It remains now to prove that, given $u \in L^2(\Omega)$, one can find a sequence $\{u_\nu\} \subset \mathcal{K}$ satisfying $u_\nu \to u$ and $F'(u) = \lim_\nu F(u_\nu)$. Clearly, we can suppose that $u \in \mathcal{K} \cap \Pi'$, and that $u \notin \Pi$. Consider all j such that $|\{u = l_j\}| > \alpha_j$, find for each such j a compact set $K_j \subset \{u = l_j\}$ satisfying $|K_j| = \alpha_j$, and let K denote the union of these compact sets. Set $\varphi(x) := \operatorname{dist}(x, K \cup \partial\Omega)$. Clearly, $\varphi \in H_0^1(\Omega)$. Now fix a $\delta > 0$ such that $\delta < \min_j\{|l_{j+1} - l_j|\}$, and define for $\varepsilon > 0$ the function

$$u_\varepsilon(x) := (1 - \varphi(x))u(x) + \varphi(x) f_\varepsilon(u(x)),$$

where

$$f_\varepsilon(y) := \begin{cases} l_j - \delta + \dfrac{\varepsilon + \delta}{\delta}(y - l_j + \delta), & l_j - \delta \leqslant y < l_j,\ |\{u = l_j\}| > \alpha_j, \\ l_j + \varepsilon + \dfrac{\delta - \varepsilon}{\delta}(y - l_j), & l_j \leqslant y < l_j + \delta,\ |\{u = l_j\}| > \alpha_j, \\ y, & \text{otherwise.} \end{cases}$$

It is easy to see that $u_\varepsilon \in \mathcal{K}$ for all $\varepsilon > 0$ and $u_\varepsilon \to u$ in $H^1(\Omega)$ as $\varepsilon \to 0$. Note that whenever $|\{u = l_j\}| = \alpha_j$, one has $\{u_\varepsilon = l_j\} = \{u = l_j\}$. However, if $|\{u = l_j\}| > \alpha_j$, then $K_j = \{u = l_j\} \cap \{\varphi = 0\} \subset \{u_\varepsilon = l_j\}$. Furthermore, $|\{u_\varepsilon = l_j\} \setminus K_j| \neq 0$ may hold only for at most countably many values of $\varepsilon \in (0, \delta)$. In fact, the sets $\{u_\varepsilon = l_j\} \setminus K_j = \{u_\varepsilon = l_j\} \cap \{\varphi \neq 0\}$ are pairwise disjoint for all $\varepsilon \in (0, \delta)$, since, for these values of ε, the function f_ε is injective. Now we are able to conclude the proof by choosing $\{\varepsilon_\nu\} \subset (0, \delta)$ to satisfy $\varepsilon_\nu \to 0$ and $|\{u_{\varepsilon_\nu} = l_j\} \setminus K_j| = 0$ for each ν. Indeed, letting $u_\nu := u_{\varepsilon_\nu} \in \mathcal{K} \cap \Pi$ provides the desired sequence. ☐

4. Asymptotics and Γ-convergence

In this section we consider the variational convergence of problem (M) in the case of only two volume constraints, i.e. when $k = 2$. By translating and scaling, without loss of generality, we may assume that $l_1 = 0$ and $l_2 = 1$.

More precisely, we are interested in the behaviour of the solutions to the problems

$$\inf\left\{ \int_\Omega |\nabla u|^2 \, dx : u \in \mathcal{K},\ |\{u = 0\}| = \alpha_\nu,\ |\{u = 1\}| = \beta_\nu \right\} \qquad (\mathrm{M}_\nu)$$

for positive values of α_ν and β_ν, satisfying $\alpha_\nu + \beta_\nu < |\Omega|$, when $\nu \to \infty$, assuming $\alpha_\nu \to \alpha$ and $\beta_\nu \to \beta$ with $\alpha, \beta > 0$ and $\alpha + \beta = |\Omega|$. We find out that, at least when Ω is smooth, the solutions to the problems (M_ν), up to a subsequence, tend in $L^2(\Omega)$ to that of the minimization problem

$$\inf\left\{\left(\inf_{g\in\mathcal{K}}\int_{\partial\Omega}|g-1_E|\,\mathrm{d}\mathcal{H}^{n-1}+P(E,\Omega)\right)^2 : E\subset\Omega,\ |E|=\beta\right\}, \tag{M$_\infty$}$$

where $P(E,\Omega)$ denotes the perimeter of a Caccioppoli set E relative to Ω (see [9]). Also, the minimum values of the properly rescaled functionals of the problems (M$_\nu$) tend to that of (M$_\infty$). To be more precise, the following result holds (we refer the reader to [5] for more details on Γ-convergence).

THEOREM 4.1. *Let $\Omega\subset\mathbb{R}^n$ be a connected bounded open set with C^2 boundary and let $\{\alpha_\nu\}$, $\{\beta_\nu\}$ be two sequences of positive number satisfying*

$$\alpha_\nu+\beta_\nu<|\Omega|\quad\forall\nu,\qquad\lim_{\nu\to\infty}\alpha_\nu=\alpha>0,\qquad\lim_{\nu\to\infty}\beta_\nu=\beta>0,\qquad\alpha+\beta=|\Omega|.$$

Let $\mathcal{K}\subseteq H^1(\Omega)$ be a closed convex set satisfying $\mathcal{K}+H^1_0(\Omega)=\mathcal{K}$, and consider the sequence of functionals $\{F_\nu\}$ defined over $L^2(\Omega)$ by

$$F_\nu(u):=\begin{cases} m_\nu\displaystyle\int_\Omega|\nabla u|^2\,\mathrm{d}x & \text{if } u\in\mathcal{K}, |\{u=0\}|=\alpha_\nu \text{ and } |\{u=1\}|=\beta_\nu,\\ +\infty & \text{otherwise,}\end{cases}$$

where $m_\nu:=|\Omega|-\alpha_\nu-\beta_\nu$ is a scale factor. Then the functionals $\{F_\nu\}$ Γ-converge (with respect to the strong topology of $L^2(\Omega)$) to the functional F defined over $L^2(\Omega)$ by

$$F(u):=\begin{cases}\left(\displaystyle\inf_{g\in\mathcal{K}}\int_{\partial\Omega}|g-1_E|\,\mathrm{d}\mathcal{H}^{n-1}+P(E,\Omega)\right)^2 & \\ \qquad\qquad \text{if } u=1_E\in BV(\Omega) \text{ for some } E\subset\Omega \text{ satisfying } |E|=\beta, & \\ +\infty \qquad \text{otherwise.}\end{cases}$$

The proof of this theorem relies on several lemmas, which are stated and proved later on in this section.

Proof. We have to show that $G^+(u)\leqslant F(u)\leqslant G^-(u)$, where G^+ and G^- denote, respectively, the upper and lower Γ-limit of F_ν (see [5]).

STEP 1. We first show $F(u)\leqslant G^-(u)$, that is, $u_\nu\to u$ in $L^2(\Omega)$ implies $F(u)\leqslant\liminf_\nu F_\nu(u_\nu)$. Clearly, we can suppose that $F_\nu(u_\nu)<+\infty$ for every ν and that $\liminf_\nu F_\nu(u_\nu)<+\infty$, which implies $|\{u_\nu=0\}|=\alpha_\nu$ and $|\{u_\nu=1\}|=\beta_\nu$, hence

$$|\{u=0\}|\geqslant\limsup|\{u_\nu=0\}|=\alpha,\qquad|\{u=1\}|\geqslant\limsup|\{u_\nu=1\}|=\beta.$$

Since $\alpha+\beta=|\Omega|$, this means $u=1_E$ for some measurable $E\subset\Omega$ with $|E|=\beta$. Then, using the Hölder inequality, we obtain

$$\begin{aligned}\liminf_\nu\sqrt{F_\nu(u_\nu)}&=\liminf_\nu m_\nu^{1/2}\left(\int_\Omega|\nabla u_\nu|^2\,\mathrm{d}x\right)^{1/2}\\ &\geqslant\liminf_\nu\int_{\Omega\setminus\{u\in\{0,1\}\}}|\nabla u_\nu|\,\mathrm{d}x\\ &=\liminf_\nu|Du_\nu|(\Omega).\end{aligned}\tag{4.1}$$

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Since the first term is finite and $u_\nu \to u = 1_E$ in $L^2(\Omega)$ (hence in L^1), this means that $u = 1_E \in BV(\Omega)$, i.e. $|Du|(\Omega) = P(E,\Omega)$ is finite. Therefore, combining (4.1) with lemma 4.2 below, we have

$$\begin{aligned}\liminf_\nu \sqrt{F_\nu(u_\nu)} &\geqslant P(E,\Omega) + \liminf_\nu \int_{\partial\Omega} |u_\nu - 1_E| \,\mathrm{d}\mathcal{H}^{n-1} \\ &\geqslant P(E,\Omega) + \inf_{g\in\mathcal{K}} \int_{\partial\Omega} |g - 1_E| \,\mathrm{d}\mathcal{H}^{n-1} \\ &= \sqrt{F(u)},\end{aligned}$$

where in the last inequality we used $u_\nu \in \mathcal{K}$. Hence, squaring, we prove our claim.

STEP 2. Now we prove $G^+(u) \leqslant F(u)$ for arbitrary $u \in L^2(\Omega)$. It is obviously enough to suppose $u = 1_E$ for some Caccioppoli set $E \subset \Omega$ satisfying $|E| = \beta$ (otherwise $F(u) = +\infty$ and the claim is trivial). According to lemma 4.3 from [4], one can find a sequence of bounded open sets $\{D_m\}$ in $\mathbb{R}^n$, each having smooth boundary, and such that, letting $E_m := D_m \cap \Omega$, one has $|E_m| = \beta$ and also

$$1_{E_m} \to 1_E \quad \text{in } L^2(\Omega) \quad \text{and} \quad \mathcal{H}^{n-1}(\partial D_m \cap \bar{\Omega}) \to P(E,\Omega) \quad \text{as } m \to \infty. \tag{4.2}$$

We claim that, due to the above conditions, we have

$$\mathcal{H}^{n-1}(\partial D_m \cap \partial\Omega) \to 0 \quad \text{and} \quad 1_{E_m} \to 1_E \quad \text{in } L^1(\partial\Omega, \mathcal{H}^{n-1}) \quad \text{as } m \to \infty. \tag{4.3}$$

Indeed, since $\mathcal{H}^{n-1}$ restricted to ∂D_m is a Borel measure, we have

$$\begin{aligned}\mathcal{H}^{n-1}(\partial D_m \cap \bar{\Omega}) &= \mathcal{H}^{n-1}(\partial D_m \cap \Omega) + \mathcal{H}^{n-1}(\partial D_m \cap \partial\Omega) \\ &= P(E_m,\Omega) + \mathcal{H}^{n-1}(\partial D_m \cap \partial\Omega).\end{aligned}$$

Taking the lim inf of both sides and using (4.2), we obtain

$$\begin{aligned}P(E,\Omega) &= \liminf_m (P(E_m,\Omega) + \mathcal{H}^{n-1}(\partial D_m \cap \partial\Omega)) \\ &\geqslant \liminf_m P(E_m,\Omega) + \liminf_m \mathcal{H}^{n-1}(\partial D_m \cap \partial\Omega) \\ &\geqslant P(E,\Omega) + \liminf_m \mathcal{H}^{n-1}(\partial D_m \cap \partial\Omega),\end{aligned}$$

which yields the first half of (4.3), since the last inequalities remain true if we consider an arbitrary subsequence $\{D_{m_j}\}$. Finally, the second half of (4.3) follows from theorem 2.11 in [6].

Recalling that G^+ is *a fortiori* lower semicontinuous and applying lemma 4.3 below (with E_m and D_m in place of E and D) to estimate $G^+(1_{E_m})$, we obtain

$$\begin{aligned}G^+(1_E) &\leqslant \liminf_m G^+(1_{E_m}) \\ &\leqslant \liminf_m \biggl(\frac{(1+\varepsilon)}{\theta} \int_{\partial\Omega} \gamma \,\mathrm{d}\mathcal{H}^{n-1} \int_{\partial\Omega} \frac{|g - 1_{E_m}|^2}{\gamma - \varepsilon} \,\mathrm{d}\mathcal{H}^{n-1} \\ &\qquad + C(1 + (\mathcal{H}^{n-1}(\partial D_m \cap \bar{\Omega}))^2)\mathcal{H}^{n-1}(\partial D_m \cap \partial\Omega) \\ &\qquad + \frac{1}{1-\theta}(\mathcal{H}^{n-1}(D_m \cap \bar{\Omega}))^2 \biggr),\end{aligned}$$

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where $\varepsilon, \theta \in (0,1)$ and $g \in \mathcal{K}$, $\gamma \in C^1(\partial\Omega)$ are arbitrary, provided that $\min_{\partial\Omega}\gamma > \varepsilon$. Now, recalling (4.3), we see that the last lim inf is in fact a limit, and the last estimate simplifies to

$$G^+(1_E) \leqslant \frac{(1+\varepsilon)}{\theta}\int_{\partial\Omega}\gamma\,\mathrm{d}\mathcal{H}^{n-1}\int_{\partial\Omega}\frac{|g-1_E|^2}{\gamma-\varepsilon}\,\mathrm{d}\mathcal{H}^{n-1} + \frac{1}{1-\theta}P(E,\Omega)^2. \tag{4.4}$$

Since this estimate holds for arbitrary smooth positive $\gamma > \varepsilon$, one can, in particular, choose $\gamma = 2\varepsilon + \gamma_j$ in (4.4), where $\{\gamma_j\}$ are positive and converge to $|g - 1_E|$ ($g \in \mathcal{K}$ being fixed for the moment) both in $L^1(\partial\Omega, \mathcal{H}^{n-1})$ and $\mathcal{H}^{n-1}$ a.e. on $\partial\Omega$. Taking first the limit as $j \to \infty$ (using dominated convergence), and then letting $\varepsilon \downarrow 0$, one obtains

$$G^+(1_E) \leqslant \frac{1}{\theta}\biggl(\int_{\partial\Omega}|g-1_E|\,\mathrm{d}\mathcal{H}^{n-1}\biggr)^2 + \frac{1}{1-\theta}P(E,\Omega)^2.$$

Now, since also $\theta \in (0,1)$ is arbitrary, one can choose the value of θ that minimizes the right-hand side, thus obtaining

$$G^+(1_E) \leqslant \biggl(\int_{\partial\Omega}|g-1_E|\,\mathrm{d}\mathcal{H}^{n-1} + P(E,\Omega)\biggr)^2.$$

Finally, the arbitrariness of $g \in \mathcal{K}$ allows one to take the infimum of the right-hand side over $g \in \mathcal{K}$, which yields $G^+(1_E) \leqslant F(1_E)$. □

In the above proof we used the following lemmas.

LEMMA 4.2. *Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set with C^2 boundary. If $\{u_\nu\} \subset BV(\Omega)$ and $u_\nu \to u$ in $L^1(\Omega)$, then*

$$\liminf_{\nu}|Du_\nu|(\Omega) \geqslant |Du|(\Omega) + \liminf_{\nu}\int_{\partial\Omega}|u_\nu - u|\,\mathrm{d}\mathcal{H}^{n-1}. \tag{4.5}$$

Proof. Consider the signed distance function

$$d(x) := \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega) - \operatorname{dist}(x, \Omega).$$

For small $t > 0$, let Γ_t denote the manifold $\{d_\Omega = t\}$, and let $\Omega_t \subset \Omega$ denote the open set $\{0 < d_\Omega < t\}$. It is well known (see [6]) that, for some small $\varepsilon_0 > 0$, the function d is as smooth as the boundary of Ω (C^2 in our case) in the domain $\{-\varepsilon_0 \leqslant d \leqslant \varepsilon_0\}$. If $0 < s < \varepsilon_0$ and f is smooth enough (say $W^{1,1}(\Omega)$), the divergence theorem applied to the vector field $f\nabla d$ yields

$$\int_{\Gamma_0} f\,\mathrm{d}\mathcal{H}^{n-1} = \int_{\Gamma_s} f\,\mathrm{d}\mathcal{H}^{n-1} - \int_{\Omega_s}(\nabla f\cdot\nabla d + f\Delta d)\,\mathrm{d}x.$$

Therefore, if $0 < s \leqslant t < \varepsilon_0$, we have

$$\int_{\Gamma_0} f\,\mathrm{d}\mathcal{H}^{n-1} \leqslant \int_{\Gamma_s}|f|\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Omega_t}|\nabla f|\,\mathrm{d}x + C\int_{\Omega_t}|f|\,\mathrm{d}x,$$

where $C > 0$ depends on Ω. Integrating the above inequality with respect to s over $(0,t)$ and dividing by t, we obtain

$$\int_{\Gamma_0} f\,\mathrm{d}\mathcal{H}^{n-1} \leqslant \biggl(C + \frac{1}{t}\biggr)\int_{\Omega_t}|f|\,\mathrm{d}x + \int_{\Omega_t}|\nabla f|\,\mathrm{d}x.$$

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Then if $f \in BV(\Omega_t)$, by a density argument (see [6, remark 2.12]), we have

$$\int_{\Gamma_0} f \, \mathrm{d}\mathcal{H}^{n-1} \leqslant \left(C + \frac{1}{t}\right) \int_{\Omega_t} |f| \, \mathrm{d}x + |Df|(\Omega_t).$$

In particular, choosing $f = |u_\nu - u|$, we obtain

$$\int_{\Gamma_0} |u_\nu - u| \, \mathrm{d}\mathcal{H}^{n-1} \leqslant \left(C + \frac{1}{t}\right) \int_{\Omega_t} |u_\nu - u| \, \mathrm{d}x + |Du_\nu|(\Omega_t) + |Du|(\Omega_t).$$

Let E be the set of those $t \in (0, \varepsilon_0)$ such that $|Du_\nu|(\Gamma_t) = 0$ for every $\nu \in \mathbb{N}$. It is well known that $(0, \varepsilon_0) \setminus E$ is at most countable, in particular, E contains arbitrarily small $t > 0$. If we choose such $t \in E$, adding $|Du_\nu|(\Omega \setminus \Omega_t)$ to both sides of the last inequality we obtain

$$|Du_\nu|(\Omega \setminus \Omega_t) + \int_{\Gamma_0} |u_\nu - u| \, \mathrm{d}\mathcal{H}^{n-1} \leqslant \left(C + \frac{1}{t}\right) \int_{\Omega_t} |u_\nu - u| \, \mathrm{d}x + |Du_\nu|(\Omega) + |Du|(\Omega_t).$$

Taking the lim inf of both sides as $\nu \to \infty$ and recalling that $u_\nu \to u$ in $L^1(\Omega)$, we obtain

$$\begin{aligned}
|Du|(\Omega \setminus \Omega_t) + \liminf_\nu \int_{\Gamma_0} |u_\nu - u| \, \mathrm{d}\mathcal{H}^{n-1} \qquad & \\
\leqslant \liminf_\nu |Du_\nu|(\Omega \setminus \Omega_t) + \liminf_\nu \int_{\Gamma_0} |u_\nu - u| \, \mathrm{d}\mathcal{H}^{n-1} & \\
\leqslant \liminf_\nu \left(|Du_\nu|(\Omega \setminus \Omega_t) + \int_{\Gamma_0} |u_\nu - u| \, \mathrm{d}\mathcal{H}^{n-1} \right) & \\
\leqslant |Du|(\Omega_t) + \liminf_\nu |Du_\nu|(\Omega) &
\end{aligned}$$

for every fixed $t \in E$. Finally, taking the limit as $E \ni t \to 0$, we obtain (4.5), since $\Omega_t \downarrow \emptyset$. □

LEMMA 4.3. *Under the assumptions of theorem 4.1, let $E := D \cap \Omega$, where $D \subset \mathbb{R}^n$ is an open bounded set with smooth boundary, and suppose that $|E| = \beta$. Choose two numbers $\varepsilon, \theta \in (0, 1)$ and any two functions, $g \in \mathcal{K}$ and $\gamma \in C^1(\partial\Omega)$, satisfying* $\min_{\partial\Omega} \gamma > \varepsilon$. *Then*

$$\begin{aligned}
G^+(1_E) \leqslant \frac{(1+\varepsilon)}{\theta} \int_{\partial\Omega} \gamma \, \mathrm{d}\mathcal{H}^{n-1} \int_{\partial\Omega} \frac{|g - 1_E|^2}{\gamma - \varepsilon} \, \mathrm{d}\mathcal{H}^{n-1} & \\
+ C(1 + (\mathcal{H}^{n-1}(\partial D \cap \bar{\Omega}))^2)\mathcal{H}^{n-1}(\partial D \cap \partial\Omega) & \\
+ \frac{1}{1-\theta}(\mathcal{H}^{n-1}(D \cap \bar{\Omega}))^2 & \qquad (4.6)
\end{aligned}$$

for some positive constant $C = C(\Omega, \gamma, \varepsilon, \theta)$, where G^+ is the upper Γ-limit (in the $L^2(\Omega)$ topology) of the sequence of functionals $\{F_\nu\}$.

The rest of this section is devoted to the proof of lemma 4.3, which consists of three main steps. It also relies on some lemmas, which are stated and proved at the end of this section (except for lemma 4.4, which is proved in the appendix).

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Proof of lemma 4.3.

STEP 1. Recall that, due to the smoothness of $\partial\Omega$, there is a tubular neighbourhood $A_0 \supset \partial\Omega$ diffeomorphic to $\partial\Omega \times (-\delta_0, \delta_0)$ for some small $\delta_0 > 0$ by means of the diffeomorphism

$$(y,t) \in \partial\Omega \times (-\delta_0, \delta_0) \mapsto x := y + t\bar{n}(y) \in A_0,$$
$$x \in A_0 \mapsto (p(x), d_\Omega(x)) \in \partial\Omega \times (-\delta_0, \delta_0),$$

where $\bar{n}(y)$ is the inward normal at the point $y \in \partial\Omega$, d_Ω is the signed distance function (positive in Ω),

$$d_\Omega(x) := \begin{cases} \operatorname{dist}(x, \partial\Omega), & x \in \Omega, \\ -\operatorname{dist}(x, \partial\Omega), & x \notin \Omega, \end{cases}$$

and $p : A_0 \to \partial\Omega$ is the projection onto the boundary given by

$$p(x) := x - d_\Omega(x)\nabla d_\Omega(x).$$

Note also that $\bar{n}(p(x)) = \nabla d_\Omega(x)$ for all $x \in A_0$.

Now let the functions g, γ and the numbers θ, ε be as in the statement of lemma 4.3. Letting $\gamma_{\max} := \max_{\partial\Omega} \gamma$, consider the family of annuli

$$T_\delta := \{x \in \Omega : 0 < d_\Omega(x) < \delta\gamma(p(x))\} \quad \text{for } 0 < \delta < \delta_0/\gamma_{\max},$$

so that $T_\delta \subset A_0$. It is clear that $|T_\delta| \downarrow 0$ as $\delta \downarrow 0$, and, moreover, the function $\delta \mapsto |T_\delta|$ is continuous and strictly increasing in a neighbourhood of 0^+. Since $m_\nu \to 0$ by assumption, there exists a sequence of positive numbers $\delta_\nu \to 0$ such that

$$|T_{\delta_\nu}| = \theta m_\nu \quad \text{for large } \nu. \tag{4.7}$$

Note that the above relation defines δ_ν uniquely, at least for ν large enough. We fix such a sequence $\{\delta_\nu\}$ and we define the regions

$$A_\nu := \{x \in \Omega : 0 < d_\Omega(x) < \delta_\nu\varepsilon\},$$
$$B_\nu := \{x \in \Omega : \delta_\nu\varepsilon < d_\Omega(x) < \delta_\nu\gamma(p(x))\},$$
$$D_\nu := \Omega \setminus (A_\nu \sqcup \bar{B}_\nu).$$

Note that $|\partial B_\nu| = 0$, since γ is smooth, hence

$$T_{\delta_\nu} = A_\nu \sqcup B_\nu \quad \text{and} \quad \Omega = T_{\delta_\nu} \sqcup D_\nu \quad \text{(up to null measure sets)}. \tag{4.8}$$

Consider the maps $q_\nu^- : A_0 \to \bar{B}_\nu \cap \bar{D}_\nu$ and $q_\nu^+ : A_0 \to \bar{B}_\nu \cap \bar{A}_\nu$ defined by

$$q_\nu^-(x) := p(x) + \delta_\nu\gamma(p(x))\nabla d_\Omega(x),$$
$$q_\nu^+(x) := p(x) + \varepsilon\delta_\nu\nabla d_\Omega(x).$$

Due to the smoothness of d_Ω and γ, for large ν, we have

$$\|Dp(x)\| + \|Dq_\nu^-(x)\| + \|Dq_\nu^+(x)\| \leqslant C = C(\Omega, \gamma), \tag{4.9}$$

where $\|\cdot\|$ denotes some matrix norm. We will construct a sequence of functions $\{u_\nu\}\subset H^1(\Omega)$ satisfying $u=g$ on $\partial\Omega$ (hence $u_\nu\in\mathcal{K}$, in view of $\mathcal{K}+H_0^1(\Omega)=\mathcal{K}$),

$$|\{u_\nu=0\}|=\alpha_\nu,\qquad |\{u_\nu=1\}|=\beta_\nu, \tag{4.10}$$

and such that $u_\nu\to 1_E$ in $L^2(\Omega)$,

$$\begin{aligned}\limsup_\nu m_\nu\int_{T_{\delta_\nu}}|\nabla u_\nu|^2\,\mathrm{d}x \leqslant \frac{1+\varepsilon}{\theta}\int_{\partial\Omega}\gamma\,\mathrm{d}\mathcal{H}^{n-1}\int_{\partial\Omega}\frac{|g-1_E|^2}{(\gamma-\varepsilon)}\,\mathrm{d}\mathcal{H}^{n-1}&\\ +C(1+(\mathcal{H}^{n-1}(\partial D\cap\bar{\Omega}))^2)\mathcal{H}^{n-1}(\partial D\cap\partial\Omega)&\end{aligned} \tag{4.11}$$

for some positive constant $C=C(\Omega,\gamma,\varepsilon,\theta)$, and also

$$\limsup_\nu(1-\theta)m_\nu\int_{D_\nu}|\nabla u_\nu|^2\,\mathrm{d}x\leqslant(\mathcal{H}^{n-1}(D\cap\bar{\Omega}))^2. \tag{4.12}$$

Then (4.6) will follow, from the definition of G^+, on combining (4.11) and (4.12).

Step 2. Now we construct the functions u_ν over D_ν, and we prove (4.12). Recall that $E=D\cap\Omega$, with $D\subset\mathbb{R}^n$ open, bounded and smooth, and $|E|=\beta$. Let d_D stand for the signed distance function from the boundary ∂D, negative in D,

$$d_D(x):=\begin{cases}-\operatorname{dist}(x,\partial D), & x\in D,\\ \operatorname{dist}(x,\partial D), & x\notin D.\end{cases}$$

Due to the smoothness of ∂D, there exists $\sigma>0$ such that d_D is smooth in the open set $\{|d_D|<\sigma\}$. Since $m_\nu\to 0$ and $D_\nu\uparrow\Omega$, for ν large enough, we have

$$|\{x\in D_\nu:|d_D(x)|<\sigma\}|>|D_\nu|-\alpha_\nu-\beta_\nu=(1-\theta)m_\nu, \tag{4.13}$$

the last equality being a consequence of (4.8) and (4.7). Hence there exist numbers $\lambda_\nu,\mu_\nu\in(-\sigma,\sigma)$ such that $\lambda_\nu<\mu_\nu$ and

$$|\{x\in D_\nu:d_D(x)\leqslant\lambda_\nu\}|=\beta_\nu,\qquad |\{x\in D_\nu:d_D(x)\geqslant\mu_\nu\}|=\alpha_\nu, \tag{4.14}$$

which, combined with (4.13), imply

$$(1-\theta)m_\nu=|\{x\in D_\nu:\lambda_\nu<d_D(x)<\mu_\nu\}|. \tag{4.15}$$

The relationship

$$w_\nu(x):=\begin{cases}1, & d_D(x)\leqslant\lambda_\nu,\\ \dfrac{\mu_\nu-d_D(x)}{\mu_\nu-\lambda_\nu}, & \lambda_\nu<d_D(x)<\mu_\nu,\\ 0, & d_D(x)\geqslant\mu_\nu,\end{cases}$$

defines $w_\nu\in\operatorname{Lip}(\mathbb{R}^n)$, such that $|\nabla w_\nu|$ equals $1/(\mu_\mu-\lambda_\nu)$ over $\{x\in D_\nu:\lambda_\nu<d_D(x)<\mu_\nu\}$ and vanishes elsewhere. We define $u_\nu(x):=w_\nu(x)$ for $x\in D_\nu$. To

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show (4.12), we estimate using the co-area formula and recalling (4.15),

$$\begin{aligned}(1-\theta)m_\nu \int_{D_\nu} |\nabla u_\nu|^2\,\mathrm{d}x &= (1-\theta)m_\nu \int_{\{x\in D_\nu:\lambda_\nu<d_D(x)<\mu_\nu\}} |\nabla w_\nu|^2\,\mathrm{d}x \\ &= \frac{(1-\theta)m_\nu}{(\mu_\nu-\lambda_\nu)^2}|\{x\in D_\nu:\lambda_\nu<d_D(x)<\mu_\nu\}| \\ &= \bigg(\frac{1}{\mu_\nu-\lambda_\nu}\int_{\lambda_\nu}^{\mu_\nu}\mathcal{H}^{n-1}(\{d_D=t\}\cap D_\nu)\,\mathrm{d}t\bigg)^2 \\ &\leqslant \bigg(\frac{1}{\mu_\nu-\lambda_\nu}\int_{\lambda_\nu}^{\mu_\nu}\mathcal{H}^{n-1}(\{d_D=t\}\cap \Omega)\,\mathrm{d}t\bigg)^2. \end{aligned}\tag{4.16}$$

It is easy to see that $\lambda_\nu, \mu_\nu \to 0$ as $\nu\to\infty$, hence, reasoning as in [4], it suffices to observe that

$$\limsup_{t\to 0^+}\mathcal{H}^{n-1}(\{d_D=t\}\cap \Omega)\,\mathrm{d}t \leqslant \mathcal{H}^{n-1}(\partial D\cap\bar{\Omega}),\tag{4.17}$$

to conclude that (4.12) is valid. Note that (4.15), (4.16) and (4.17) also imply

$$\limsup_{\nu}\frac{(1-\theta)m_\nu}{\mu_\nu-\lambda_\nu}\leqslant \mathcal{H}^{n-1}(\partial D\cap\bar{\Omega}),$$

which, combined with (4.28) of lemma 4.4, yields

$$\limsup_{\nu}\frac{\delta_\nu}{\mu_\nu-\lambda_\nu}\leqslant C(\gamma,\theta)\mathcal{H}^{n-1}(\partial D\cap\bar{\Omega}).\tag{4.18}$$

This estimate will be used in the next step of the proof.

STEP 3. We now extend the functions u_ν to the sets T_{δ_ν} as to satisfy the boundary condition, proving (4.11) and (4.10). We abbreviate $\hat{\gamma}(x) := \gamma(p(x)) - \varepsilon$ and define

$$u_\nu(x) := \begin{cases}\tilde{g}(x), & x\in A_\nu,\\ \tilde{g}(q_\nu^+(x)) + \dfrac{d(x)-\varepsilon\delta_\nu}{\delta_\nu\hat{\gamma}(x)}(w_\nu(q_\nu^-(x))-\tilde{g}(q_\nu^+(x))), & x\in B_\nu,\end{cases}\tag{4.19}$$

where $\tilde{g}\in H^1(\Omega)$ is the (unique) function satisfying $\Delta\tilde{g}=0$ in Ω and $\tilde{g}=g$ over $\partial\Omega$. Recalling that, by definition, $u_\nu = w_\nu$ over D_ν, we see that $u_\nu\in H^1(\Omega)\cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and that $u_\nu = g$ on $\partial\Omega$.

In order to prove (4.11), we now estimate the Dirichlet integral of u_ν on T_{δ_ν}. First we note that, since $\nabla\tilde{g}\in L^2(\Omega)$ and $A_\nu\downarrow\emptyset$, it is enough to estimate the Dirichlet integral of u_ν over B_ν. It is a matter of calculus to see that for a.e. $x\in B_\nu$,

$$\begin{aligned}\nabla u_\nu(x) &= (1-\tau(x))\nabla\tilde{g}(q_\nu^+(x))Dq_\nu^+(x) + \tau(x)\nabla w_\nu(q_\nu^-(x))Dq_\nu^-(x) \\ &\quad + (w_\nu(q_\nu^-(x)) - \tilde{g}(q_\nu^+(x)))\bigg(\frac{\nabla d_\Omega(x)}{\delta_\nu\hat{\gamma}(x)} - \frac{\tau(x)}{\hat{\gamma}(x)}\nabla\gamma(p(x)\bigg)Dp(x)),\end{aligned}$$

where, for brevity, we have set

$$\tau(x) := \frac{d_\Omega(x)-\varepsilon\delta_\nu}{\delta_\nu\hat{\gamma}(x)},\quad x\in B_\nu.$$

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Recalling (4.9) and noting that $|\tau(x)|$, $|\nabla d_\Omega|$ and $|w_\nu|$ are all majorized by 1, we obtain, for some $C = C(\Omega, \gamma, \varepsilon)$,

$$|\nabla u_\nu(x)| \leqslant C|\nabla \tilde{g}(q_\nu^+(x))| + C|\nabla w_\nu(q_\nu^-(x))| + \frac{|w_\nu(q_\nu^-(x)) - \tilde{g}(q_\nu^+(x))|}{\delta_\nu \hat{\gamma}(x)} + C(1 + |\tilde{g}(q_\nu^+(x))|).$$

Let $u := 1_E$ on $\bar{\Omega}$ (in the sense of the trace along $\partial\Omega$). Using the estimate

$$|w_\nu(q_\nu^-(x)) - \tilde{g}(q_\nu^+(x))| \leqslant |u(p(x)) - g(p(x))| + |g(p(x)) - \tilde{g}(q_\nu^+(x))| + |w_\nu(q_\nu^-(x)) - u(p(x))|$$

and the inequality $2ab \leqslant \varepsilon a^2 + b^2/\varepsilon$, we easily obtain

$$\begin{aligned} |\nabla u_\nu(x)|^2 \leqslant (1+\varepsilon)&\frac{|u(p(x)) - g(p(x))|^2}{\delta_\nu^2 \hat{\gamma}(x)^2} \\ &+ C\biggl(|\nabla w_\nu(q_\nu^-(x))|^2 + \frac{|w_\nu(q_\nu^-(x)) - u(p(x))|^2}{\delta_\nu^2}\biggr) \\ &+ C(1 + |\nabla \tilde{g}(q_\nu^+(x))|^2 + |\tilde{g}(q_\nu^+(x))|^2) + C\frac{|g(p(x)) - \tilde{g}(q_\nu^+(x))|^2}{\delta_\nu^2} \end{aligned} \tag{4.20}$$

for some other $C = C(\Omega, \gamma, \varepsilon)$. It remains to estimate the integral (multiplied by m_ν) of each of the above terms over B_ν, as $\nu \to \infty$. The integral of the last two terms tends to zero in view of lemma 4.5 (note that $1 + |\nabla \tilde{g}|^2 + |\tilde{g}|^2$ is subharmonic). Let us consider the second term. Fix $\rho > 0$, let $V_\rho \supset \partial D$ be an open set such that $\operatorname{dist}(\mathbb{R}^n \setminus V_\rho, \partial D) = \rho$ and then choose ν large enough such that

$$\delta_\nu \gamma_{\max} < \tfrac{1}{2}\rho, \qquad \max(|\lambda_\nu|, |\mu_\nu|) < \tfrac{1}{2}\rho. \tag{4.21}$$

Now suppose that $x \in B_\nu$ but $x \notin V_\rho$. We claim that

$$\operatorname{dist}(q_\nu^-(x), \partial D) > \tfrac{1}{2}\rho, \qquad \operatorname{dist}(p(x), \partial D) > \tfrac{1}{2}\rho. \tag{4.22}$$

To see this, it suffices to observe that

$$\rho \leqslant \operatorname{dist}(x, \partial D) \leqslant \operatorname{dist}(x, q_\nu^-(x)) + \operatorname{dist}(q_\nu^-(x), \partial D) \leqslant \delta_\nu \gamma_{\max} + \operatorname{dist}(q_\nu^-(x), \partial D),$$

and to recall (4.21), while the proof for $p(x)$ is similar. Recalling the definition of w_ν, from (4.21) and (4.22) we obtain

$$\nabla w_\nu(q_\nu^-(x)) = 0 \quad \text{if } \nu \text{ is large and } x \in B_\nu \setminus V_\rho. \tag{4.23}$$

Moreover, from (4.21), we have $w_\nu(q_\nu^-(x)) = u(q_\nu^-(x))$ and since (4.21), (4.22) imply $B_{\rho/2}(p(x)) \cap \partial D = \emptyset$, we obtain that u is constant (either 0 or 1) on $B_{h/2}(p(x)) \cap \Omega$. Since $q_\nu^-(x) \in B_{h/2}(p(x))$, from the definition of trace for a BV function, we obtain that $u(p(x)) = u(q_\nu^-(x))$, and hence one has

$$w_\nu(q_\nu^-(x)) = u(p(x)) \quad \text{if } \nu \text{ is large and } x \in B_\nu \setminus V_\rho. \tag{4.24}$$

Combining now (4.23) and (4.24), we obtain for large ν the estimate

$$\begin{aligned} m_\nu \int_{B_\nu} \biggl(|\nabla w_\nu(q_\nu^-(x))|^2 + \frac{|w_\nu(q_\nu^-(x)) - u(p(x))|^2}{\delta_\nu^2} \biggr) \,\mathrm{d}x \\ \leqslant \frac{m_\nu}{\delta_\nu^2} \int_{B_\nu} (\delta_\nu^2 |\nabla w_\nu(q_\nu^-(x))|^2 + |w_\nu(q_\nu^-(x)) - u(p(x))|^2) 1_{V_\rho}(x) \,\mathrm{d}x \\ \leqslant \frac{m_\nu}{\delta_\nu^2} \biggl(\frac{\delta_\nu^2}{|\lambda_\nu - \mu_\nu|^2} + 1 \biggr) \int_{B_\nu} 1_{V_\rho}(x) \,\mathrm{d}x. \end{aligned}$$

Therefore, in view of (4.28), (4.18) and (4.29), we have

$$\begin{aligned} \limsup_\nu m_\nu \int_{B_\nu} \biggl(|\nabla w_\nu(q_\nu^-(x))|^2 + \frac{|w_\nu(q_\nu^-(x)) - u(p(x))|^2}{\delta_\nu^2 \hat{\gamma}(x)^2} \biggr) \,\mathrm{d}x \\ \leqslant C(\Omega, \gamma, \varepsilon, \theta)(1 + (\mathcal{H}^{n-1}(\partial D \cap \bar{\Omega}))^2) \mathcal{H}^{n-1}(\partial \Omega \cap \bar{V}_\rho). \end{aligned} \tag{4.25}$$

Since $\rho > 0$ was arbitrary, letting $V_\rho \downarrow \partial D$, we obtain

$$\begin{aligned} \limsup_\nu m_\nu \int_{B_\nu} \biggl(|\nabla w_\nu(q_\nu^-(x))|^2 + \frac{|w_\nu(q_\nu^-(x)) - u(p(x))|^2}{\delta_\nu^2 \hat{\gamma}(x)^2} \biggr) \,\mathrm{d}x \\ \leqslant C(\Omega, \gamma, \varepsilon, \theta)(1 + (\mathcal{H}^{n-1}(\partial D \cap \bar{\Omega}))^2) \mathcal{H}^{n-1}(\partial \Omega \cap \partial D), \end{aligned}$$

which accounts for the last term in (4.11). Finally, the first term in (4.11) comes from the integral of the first summand in (4.20). Indeed, using (4.27) and (4.28) of lemma 4.4, we get

$$\begin{aligned} \limsup_\nu m_\nu \int_{B_\nu} \biggl(\frac{|u(p(x)) - g(p(x))|^2}{\delta_\nu^2 \hat{\gamma}(x)^2} \biggr) \,\mathrm{d}x \\ = \frac{1}{\theta} \biggl(\int_{\partial \Omega} \gamma \,\mathrm{d}\mathcal{H}^{n-1} \biggr) \int_{\partial \Omega} \frac{|u - g|^2}{(\gamma - \varepsilon)} \,\mathrm{d}\mathcal{H}^{n-1}, \end{aligned} \tag{4.26}$$

and hence (4.11) is completely proved.

Now we prove equations (4.10). Observe that (4.14) implies $|\{u_\nu = 0\} \cap D_\nu| = \alpha_\nu$ and $|\{u_\nu = 1\} \cap D_\nu| = \beta_\nu$; therefore, if $|\{u_\nu = 0\} \cap T_{\delta_\nu}| = |\{u_\nu = 1\} \cap T_{\delta_\nu}| = 0$, then (4.10) is satisfied. If this is not the case, then we can replace u_ν with $u_\nu + \varepsilon_\nu \operatorname{dist}(\cdot, \mathbb{R}^n \setminus T_{\delta_\nu})$ for some suitable sequence $\varepsilon_\nu \downarrow 0$, in order to remove this pathology (the details are left to the reader).

Finally, we have to prove that $u_\nu \to 1_E$ in $L^2(\Omega)$. From the way we have defined D_ν, we see that if we choose a compact set $K \subset \Omega$, then we have $K \subset D_\nu$ for ν large enough. Then, from the way we have defined u_ν over D_ν, it is immediate to check that $u_\nu \to 1_E$ in $L^2_{\mathrm{loc}}(\Omega)$, and convergence in $L^2(\Omega)$ follows from the uniform bound

$$|u_\nu| \leqslant |w_\nu| + |\tilde{g}| \leqslant 1 + |\tilde{g}| \quad \text{in } \Omega,$$

which is an immediate consequence of our construction. □

LEMMA 4.4. *For every non-negative Borel function $h : \partial \Omega \to \mathbb{R}$, one has*

$$\lim_\nu \frac{1}{\delta_\nu} \int_{B_\nu} h(p(x)) \,\mathrm{d}x = \int_{\partial \Omega} h(x)(\gamma(x) - \varepsilon) \,\mathrm{d}\mathcal{H}^{n-1}(x). \tag{4.27}$$

Moreover,

$$\lim_{\nu} \frac{m_\nu}{\delta_\nu} = \frac{1}{\theta} \int_{\partial\Omega} \gamma(x)\, \mathrm{d}\mathcal{H}^{n-1}(x), \tag{4.28}$$

and, for every open set $V \subset \mathbb{R}^n$, one has

$$\limsup_{\nu} \frac{1}{\delta_\nu} \int_{B_\nu} 1_V(x)\, \mathrm{d}x \leqslant C(\gamma)\mathcal{H}^{n-1}(\partial\Omega \cap \bar{V}). \tag{4.29}$$

The proof is given in the appendix.

Lemma 4.5. *Let $f \in L^1(\Omega)$ be non-negative and subharmonic. Then*

$$\lim_{\nu\to\infty} \int_{B_\nu} f(q_\nu^+(x))\, \mathrm{d}x = 0.$$

Moreover, if $\tilde{g} \in H^1(\Omega)$ and g is its trace on $\partial\Omega$, then

$$\lim_{\nu} \frac{m_\nu}{\delta_\nu^2} \int_{B_\nu} |g(p(x)) - \tilde{g}(q_\nu^+(x))|^2\, \mathrm{d}x = 0.$$

Proof. For every $x \in B_\nu$, the ball $B_{x,\nu}$ of radius $\varepsilon\delta_\nu$ and centre $q_\nu^+(x)$ is contained in Ω. Since f is subharmonic, we have

$$f(q_\nu^+(x)) \leqslant \frac{C}{(\delta_\nu)^n} \int_{B_{x,\nu}} f(z)\, \mathrm{d}z$$

for some $C = C(\varepsilon, n) > 0$. If $R_\nu := \{x : 0 < d_\Omega(x) < 2\delta_\nu\gamma_{\max}\}$ is the annulus of width $2\delta_\nu\gamma_{\max}$, then $B_\nu \subseteq R_\nu$ and $B_{x,\nu} \subset R_\nu$, hence

$$\int_{R_\nu} f(q_\nu^+(x))\, \mathrm{d}x \leqslant \frac{C}{(\delta_\nu)^n} \int_{R_\nu} \left(\int_{R_\nu} f(z) 1_{B_{x,\nu}}(z)\, \mathrm{d}z \right) \mathrm{d}x,$$

which, by Fubini's theorem, is equal to

$$\frac{C}{(\delta_\nu)^n} \int_{R_\nu} f(z) \left(\int_{R_\nu} 1_{B_{x,\nu}}(z)\, \mathrm{d}x \right) \mathrm{d}z.$$

Since the inner integral is $O((\delta_\nu)^n)$ uniformly with respect to $z \in R_\nu$, we conclude the proof observing that $R_\nu \downarrow \emptyset$ and recalling that $f \in L^1(\Omega)$.

Concerning the second claim of the lemma, since, by (4.28), m_ν/δ_ν is bounded and $B_\nu \subset R_\nu$, it suffices to prove

$$\lim_{\nu} \frac{1}{\delta_\nu} \int_{R_\nu} |g(p(x)) - \tilde{g}(q_\nu^+(x))|^2\, \mathrm{d}x = 0.$$

Recalling the definition of q_ν^+, we see that the last equation is immediate (from the definition of trace of a $W^{1,2}$ function) when Ω is a half-space and $\tilde{g}$ has compact support. Since $\partial\Omega$ is smooth, the general case can easily be obtained by means of a partition of unity, using arguments similar to those in ch. 2 of [6]. The details are left to the reader. □

5. Concluding remarks

Note that theorem 4.1 covers, as a particular case, the Γ-convergence result from [4] for the case with no prescribed boundary condition. Namely, in this case we have $\mathcal{K} = H^1(\Omega)$ and hence $F(1_E) = P(E, \Omega)^2$.

Another relevant particular case is given by the classical Dirichlet boundary condition $u = g \in H^{1/2}(\partial\Omega)$, i.e.

$$\mathcal{K} = \{u \in H^1(\Omega) : u = g \text{ over } \partial\Omega\}.$$

In this case, the Γ-limit is given by

$$F(1_E) = \left(\int_{\partial\Omega} |g - 1_E| \, \mathrm{d}\mathcal{H}^{n-1} + P(E, \Omega) \right)^2.$$

Finally, we remark that the proof of theorem 4.1 can be easily modified (in fact, reduced) to handle the easier case of exactly one volume constraint. Namely, we have the following result.

Corollary 5.1. *Let Ω and $\mathcal{K}$ be as in theorem 4.1, and let $\{\alpha_\nu\}$ be a sequence of numbers satisfying $0 < \alpha_\nu < |\Omega|$ and $\lim_{\nu\to\infty} \alpha_\nu = |\Omega|$. Then the sequence of functionals $\{F_\nu\}$ defined over $L^2(\Omega)$ by*

$$F_\nu(u) := \begin{cases} (|\Omega| - \alpha_\nu) \displaystyle\int_\Omega |\nabla u|^2 \, \mathrm{d}x & \text{if } u \in \mathcal{K} \text{ and } |\{u = 0\}| = \alpha_\nu, \\ +\infty & \text{otherwise,} \end{cases}$$

Γ-converge (with respect to the strong topology of $L^2(\Omega)$) to the functional F defined over $L^2(\Omega)$ by

$$F(u) := \begin{cases} \left(\displaystyle\inf_{g\in\mathcal{K}} \int_{\partial\Omega} |g| \, \mathrm{d}\mathcal{H}^{n-1} \right)^2 & \text{if } u = 0 \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

Appendix A.

Proof of lemma 4.4. To prove (4.27), consider the annuli

$$R_\nu := \{x : 0 < d_\Omega(x) < \delta_\nu\}$$

and the diffeomorphisms

$$\Psi_\nu : x \in R_\nu \mapsto p(x) + (\varepsilon\delta_\nu + d(x)(\gamma(p(x)) - \varepsilon))\nabla d(x) \in B_\nu.$$

Computing the derivative, from (4.9), we obtain

$$D\Psi_\nu(x) = I + (\gamma(p(x)) - 1 - \varepsilon)\nabla d_\Omega(x) \otimes \nabla d_\Omega(x) + O(\delta_\nu),$$

where $O(\delta_\nu)$ is uniform with respect to $x \in B_\nu$. Observing that $\nabla d_\Omega(x) \otimes \nabla d_\Omega(x)$ is a rank-one matrix with the non-trivial eigenvalue equal to one, we conclude that the Jacobian of Ψ_ν satisfies

$$|\det D\Psi(x)| = \gamma(p(x)) - \varepsilon + O(\delta_\nu).$$

Observing further that $p \circ \Psi_\nu = p$ on R_ν, using the change of variable $x = \Psi(z)$, we find

$$\lim_\nu \frac{1}{\delta_\nu} \int_{B_\nu} h(p(x))\,\mathrm{d}x = \lim_\nu \frac{1}{\delta_\nu} \int_{R_\nu} h(p(z))(\gamma(p(z)) - \varepsilon)\,\mathrm{d}z$$

which, by the co-area formula, equals

$$\lim_\nu \frac{1}{\delta_\nu} \int_0^{\delta_\nu} \biggl(\int_{\Gamma_t} h(p(z)\biggr)(\gamma(p(z)) - \varepsilon)\,\mathrm{d}\mathcal{H}^{n-1}(z))\,\mathrm{d}t,$$

where Γ_t is the manifold $\{z : d_\Omega(z) = t\}$ (note that $\Gamma_0 = \partial\Omega$). Since $x \mapsto z = x + t\nabla d_\Omega(x)$ is a diffeomorphism of Γ_0 onto Γ_t, the tangential Jacobian of which is equal to $1 + O(t)$, changing variable we find, for $t \in (0, \delta_\nu)$,

$$\int_{\Gamma_t} h(p(z))(\gamma(p(z)) - \varepsilon)\,\mathrm{d}\mathcal{H}^{n-1}(z) = O(t) + \int_{\Gamma_0} h(x)(\gamma(x) - \varepsilon)\,\mathrm{d}\mathcal{H}^{n-1}(x),$$

from which the claim easily follows. The proof of (4.28) is similar (recalling that $m_\nu = |T_{\delta_\nu}|/\theta$), and the details are left to the reader.

It remains to prove (4.29). Letting $R_\nu := \{0 < d_\Omega(x) < \delta_\nu\gamma_{\max}\}$, we obtain, by the co-area formula,

$$\frac{1}{\delta_\nu} \int_{B_\nu} 1_V(x)\,\mathrm{d}x \leqslant \frac{1}{\delta_\nu} \int_{R_\nu} 1_V(x)\,\mathrm{d}x = \frac{1}{\delta_\nu} \int_0^{\delta_\nu\gamma_{\max}} \biggl(\int_{\Gamma_t} 1_V(x)\,\mathrm{d}\mathcal{H}^{n-1}\biggr)\,\mathrm{d}t.$$

On the other hand, we have, for $t \in (0, \delta_\nu\gamma_{\max})$,

$$\int_{\Gamma_t} 1_V(x)\,\mathrm{d}\mathcal{H}^{n-1} = O(t) + \int_{\Gamma_0} 1_V(x + t\nabla d_\Omega(x))\,\mathrm{d}\mathcal{H}^{n-1},$$

whence, by Fatou's lemma (note that our functions are bounded above),

$$\begin{aligned}\limsup_{t\to 0^+} \int_{\Gamma_t} 1_V(x)\,\mathrm{d}\mathcal{H}^{n-1} &\leqslant \int_{\Gamma_0} \limsup_{t\to 0^+} 1_V(x + t\nabla d_\Omega(x))\,\mathrm{d}\mathcal{H}^{n-1} \\ &\leqslant \int_{\Gamma_0} 1_{\bar V}(x)\,\mathrm{d}\mathcal{H}^{n-1} \\ &= \mathcal{H}^{n-1}(\partial\Omega \cap \bar V),\end{aligned}$$

from which our claim easily follows. □

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