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# ON THE $p$-LENGTH AND THE WIELANDT SERIES OF A FINITE $p$-SOLUBLE GROUP

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**Abstract**

The Wielandt subgroup of a group $G$, denoted by $\omega(G)$, is the intersection of the normalisers of all subnormal subgroups of $G$. The terms of the Wielandt series of $G$ are defined, inductively, by putting $\omega_0(G) = 1$ and $\omega_{i+1}(G)/\omega_i(G) = \omega(G/\omega_i(G))$. In this paper, we investigate the relations between the $p$-length of a $p$-soluble finite group and the Wielandt series of its Sylow $p$-subgroups. Some recent results are improved.

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## 1. Introduction

All groups considered in this paper are finite. Let $p$ be a prime and $P$ a $p$-group. For convenience, we denote

$$\Omega_k(P) = \langle x \in P : x^{p^k} = 1 \rangle \quad \text{and} \quad \Omega(P) = \begin{cases} \Omega_1(P) & \text{if } p \text{ is odd,} \\ \Omega_2(P) & \text{if } p = 2. \end{cases}$$

The Wielandt subgroup $\omega(G)$ of a group $G$ is defined to be the intersection of the normalisers of all subnormal subgroups of $G$ (see [10]). The terms of the Wielandt series of $G$ are defined, inductively, by putting $\omega_0(G) = 1$ and $\omega_{i+1}(G)/\omega_i(G) = \omega(G/\omega_i(G))$. If, for some integer $n$, $\omega_n(G) = G$, then we say that $G$ has a finite Wielandt length, and define the Wielandt length of $G$, denoted by $wl(G)$, to be the minimal $n$ such that $\omega_n(G) = G$.

Let $P$ be a $p$-group for some prime $p$. Recall that the terms of the upper central series of $P$ are defined, inductively, by putting $Z_0(P) = 1$ and $Z_{i+1}(P)/Z_i(P) = Z(P/Z_i(P))$. The nilpotent class of $P$, denoted by $c(P)$, is defined to be the minimal $n$ such that $Z_n(P) = P$. It is clear that for any nonnegative integer $i$, $Z_i(P) \leq \omega_i(P)$. Hence, the Wielandt length of $P$ is less than or equal to the nilpotent class of $P$.

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In [5], Hall and Higman obtained bounds for the $p$-length of a finite $p$-soluble group $G$ in terms of the structure of a Sylow $p$-subgroup of $G$. One nice result proved in this classical paper is that the $p$-length of $G$, denoted by $l_p(G)$, is bounded by the nilpotent class of $P$.

Theorem 1.1 [5]. *Let $p$ be a prime and let $P$ be a Sylow $p$-subgroup of a $p$-soluble group $G$. Then $l_p(G) \le c(P)$.*

More recently, González-Sánchez and Weigel [3] gave a sufficient condition for the $p$-length of a $p$-soluble group to be at most 1 for odd primes.

Theorem 1.2 [3, Theorem E]. *Let $p$ be an odd prime and let $P$ be a Sylow $p$-subgroup of a $p$-soluble group $G$. If $\Omega(P) \le Z_{p-2}(P)$, then the $p$-length of $G$ is at most* 1.

In [8], we proved the following theorem.

Theorem 1.3 [8, Corollary 4.1]. *Let $p$ be a prime and let $P$ be a Sylow $p$-subgroup of a $p$-soluble group $G$. Then $l_p(G) \le wl(P)$.*

Clearly, Theorem 1.3 has improved Theorem 1.1 by replacing '$l_p(G) \le c(P)$' in Theorem 1.1 with '$l_p(G) \le wl(P)$'. A natural question is whether Theorem 1.2 can be improved in a similar way; more precisely, can we weaken the condition '$\Omega(P) \le Z_{p-2}(P)$' in Theorem 1.2 to '$\Omega(P) \le \omega_{p-2}(P)$'?

In this paper, our first result will give an affirmative answer to this question. Unlike Theorem 1.2, we will also include the case $p = 2$. Moreover, unless $p$ is a Fermat prime and a Sylow 2-subgroup of $G$ is abelian, we only require $\Omega(P) \le \omega_{p-1}(P)$, instead of $\Omega(P) \le \omega_{p-2}(P)$, to prove that the $p$-length of $G$ is at most 1.

Theorem A. Let $p$ be a prime and let $P$ be a Sylow $p$-subgroup of a $p$-soluble group $G$. Suppose that $\Omega(P) \le \omega_n(P)$, where $n = p - 2$ if $p$ is a Fermat prime and a Sylow 2-subgroup of $G$ is not abelian, and $n = p - 1$ otherwise. Then the $p$-length of $G$ is at most 1.

Using Theorem A, we can prove the following results as applications.

Theorem B. Let $p$ be a prime and let $P$ be a Sylow $p$-subgroup of a group $G$. Suppose that $\Omega(P) \le \omega_{p-1}(P)$. Then $G$ is $p$-nilpotent if $N_G(P)$ is $p$-nilpotent.

Corollary 1.4 [3, Theorem D]. *Let $p$ be an odd prime and let $P$ be a Sylow $p$-subgroup of a group $G$. Suppose that $\Omega(P) \le Z_{p-1}(P)$. Then $G$ is $p$-nilpotent if $N_G(P)$ is $p$-nilpotent.*

As another application of Theorem A, we can improve Theorem 1.3 by giving a better bound for the $p$-length of a finite $p$-soluble group $G$ in terms of the Wielandt length of a Sylow $p$-subgroup of $G$:

Theorem C. Let $p$ be a prime and let $P$ be a Sylow $p$-subgroup of a $p$-soluble group $G$. Then $l_p(G) \le \max\{1, wl(P) - (p-3)\}$. Moreover, unless $p$ is a Fermat prime and a Sylow 2-subgroup of $G$ is not abelian, then $l_p(G) \le \max\{1, wl(P) - (p-2)\}$.

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## 2. Preliminaries

The following theorem plays a crucial role in the proof of Theorem A.

THEOREM 2.1 [5, Theorem B]. *Let $H$ be a $p$-soluble linear group over a field of characteristic $p$, with no normal $p$-subgroup greater than 1. If $g$ is an element of order $p^m$ in $H$, then the minimal equation of $g$ is $(x-1)^r = 0$, where $r = p^m$, unless there is an integer $m_0$, not greater than $m$, such that $p^{m_0} - 1$ is a power of a prime $q$ for which the Sylow $q$-subgroups of $H$ are not abelian, in which case, if $m_0$ is the least such integer, then $p^{m-m_0}(p^{m_0} - 1) \le r \le p^m$.*

We now give some properties of the Wielandt series of finite groups. The first one follows immediately from the definition.

LEMMA 2.2. *Let $i$ be a nonnegative integer. Let $K$ be a subnormal subgroup of a group $G$. Then $\omega_{i+1}(G) \le N_G(K\omega_i(G))$. In particular, if $G$ is a nilpotent group, then $\omega_{i+1}(G) \le N_G(H\omega_i(G))$ for any subgroup $H$ of $G$.*

LEMMA 2.3. *Let $p$ be a prime and $P$ a $p$-group. Let $M$ be a subgroup of $P$ and $N$ a normal subgroup of $P$. Then:*

(i) $M \cap \omega(P) \le \omega(M)$;

(ii) $\omega(P)N/N \le \omega(P/N)$.

PROOF. (i) Let $x$ be any element of $\omega(P) \cap M$. Let $K$ be any subnormal subgroup of $M$. Clearly, $K$ is also a subnormal subgroup of $P$ since $P$ is a $p$-group. It follows that $x \in \omega(P) \cap M \le N_P(K) \cap M = N_M(K)$. Hence, $x \in \omega(M)$ and $M \cap \omega(P) \le \omega(M)$.

(ii) Let $x$ be any element of $\omega(P)$. Let $K/N$ be any subnormal subgroup of $P/N$. Clearly, $K$ is a subnormal subgroup of $P$. It follows that $x \in N_P(K)$ and thus $xN \in N_{P/N}(K/N)$. Hence, $xN \in \omega(P/N)$ and $\omega(P)N/N \le \omega(P/N)$. □

LEMMA 2.4. *Let $p$ be a prime and $P$ a $p$-group. Let $M$ be a subgroup of $P$ and $N$ a normal subgroup of $P$. Then, for any nonnegative integer $i$, we have:*

(i) $M \cap \omega_i(P) \le \omega_i(M)$;

(ii) $\omega_i(P)N/N \le \omega_i(P/N)$.

*In particular, the Wielandt length of any subgroup of $P$ and the Wielandt length of any factor group of $P$ are not greater than the Wielandt length of $P$.*

PROOF. This lemma follows from Lemma 2.3 and [9, Proposition 2.4]. □

The following are some basic properties of the $p$-length of a $p$-soluble group.

LEMMA 2.5 [6, page 689, Hilfssatz 6.4]. *Let $G$ be a $p$-soluble group.*

(i) *If $N \trianglelefteq G$, then $l_p(G/N) \le l_p(G)$.*

(ii) *If $U \le G$, then $l_p(U) \le l_p(G)$.*

(iii) *If $N_1$ and $N_2$ are two normal subgroups of $G$, then*

$$l_p(G/(N_1 \cap N_2)) = \max\{l_p(G/N_1), l_p(G/N_2)\}.$$

(iv) $l_p(G/\Phi(G)) = l_p(G)$.

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Lemma 2.6. *Let $G$ be a $p$-soluble group with $p$-length at most 1 and $P$ be a Sylow $p$-subgroup of $G$. If $N_G(P)$ is $p$-nilpotent, then $G$ is $p$-nilpotent.*

Proof. Since $l_p(G) \leq 1$, $G = N_G(P\,\mathrm{O}_{p'}(G)) = N_G(P)\,\mathrm{O}_{p'}(G)$. It follows that $G/\,\mathrm{O}_{p'}(G) = (N_G(P)\,\mathrm{O}_{p'}(G))/\,\mathrm{O}_{p'}(G)$ is $p$-nilpotent and thus $G$ is $p$-nilpotent. □

## 3. Proof of theorems

Proof of Theorem A. Suppose that this theorem is false and let $G$ be a counterexample of minimal order. Let $\mathcal{F}$ be the class of all $p$-soluble groups with $p$-length at most 1. From Lemma 2.5, we know that $\mathcal{F}$ is a saturated formation. Let $G^{\mathcal{F}}$ be the $\mathcal{F}$-residual of $G$ and let $K = G^{\mathcal{F}}\Phi(G)$. Then $G^{\mathcal{F}} \nleq \Phi(G)$ since $G \notin \mathcal{F}$ and $\mathcal{F}$ is a saturated formation. Hence, $K > \Phi(G)$. In the following, we will derive a contradiction through several steps.

*Step 1.* $\mathrm{O}_{p'}(G) = 1$.

Suppose that $\mathrm{O}_{p'}(G) \neq 1$. Clearly, $G/\,\mathrm{O}_{p'}(G)$ satisfies the hypotheses of this theorem. Hence, the minimal choice of $G$ implies that the $p$-length of $G/\,\mathrm{O}_{p'}(G)$ is at most 1. It then follows that the $p$-length of $G$ is at most 1, which contradicts the choice of $G$.

*Step 2.* For any proper subgroup $H$ of $G$, we have $H \in \mathcal{F}$.

Let $H$ be a proper subgroup of $G$ and let $P_1$ be a Sylow $p$-subgroup of $H$. Without loss of generality, we may assume that $P_1 \leq P$. Since $\Omega(P_1) \leq \Omega(P) \leq \omega_n(P)$, by Lemma 2.4 we have $\Omega(P_1) \leq \omega_n(P) \cap P_1 \leq \omega_n(P_1)$. Hence, $H$ satisfies the hypotheses of this theorem and the minimal choice of $G$ implies that $H \in \mathcal{F}$.

*Step 3.* $K/\Phi(G)$ is the unique minimal normal subgroup of $G/\Phi(G)$.

This follows from step 2 and [1, Theorem 1].

*Step 4.* $K/\Phi(G)$ is a $p$-group and $G^{\mathcal{F}} \leq \Omega(P)$.

Since $G$ is $p$-soluble and $K/\Phi(G)$ is a minimal normal subgroup of $G/\Phi(G)$, $K/\Phi(G)$ is either a $p$-group or a $p'$-group. If $K/\Phi(G)$ is a $p'$-group, then $K/\Phi(G)$ is $p$-nilpotent and $K$ is not a $p$-group. It follows that $K$ is a $p$-nilpotent normal subgroup of $G$ and $\mathrm{O}_{p'}(K) \neq 1$, which contradicts step 1. Hence, $K/\Phi(G)$ is a $p$-group.

Since $\mathrm{O}_{p'}(G) = 1$ by step 1, $\Phi(G)$ is a $p$-group. It follows that $K$ is a $p$-group. Since $K > \Phi(G)$, $G$ has a maximal subgroup $L$ such that $G = KL$. By step 2, $L \in \mathcal{F}$. It follows that $G^{\mathcal{F}} \leq \Omega(P)$ by [1, Proposition 1].

*Step 5.* $G$ has a maximal subgroup $M$ such that $G/\Phi(G) = (K/\Phi(G)) \rtimes (M/\Phi(G))$. Moreover, $M/\Phi(G)$ is not a $p'$-group.

Since $K/\Phi(G)$ is a soluble minimal normal subgroup of $G/\Phi(G)$ by step 4 and $\Phi(G/\Phi(G)) = 1$, $G$ has a maximal subgroup $M$ such that $G/\Phi(G) = (K/\Phi(G)) \rtimes (M/\Phi(G))$. If $M/\Phi(G)$ is a $p'$-group, then $K/\Phi(G)$ is the normal Sylow $p$-subgroup of $G/\Phi(G)$. It then follows that $G$ is $p$-closed, which contradicts the choice of $G$.

*Step 6.* $\Phi(G) = C_M(K/\Phi(G))$ and thus $M/\Phi(G) = M/C_M(K/\Phi(G))$ can be regarded as a linear group over a field of characteristic $p$ through the conjugation action of $M/\Phi(G)$

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on $K/\Phi(G)$. If $g$ is an element of $M/\Phi(G)$ of order $p$, then the minimal equation of $g$ is $(x-1)^r = 0$, where $r = p$, unless $p$ is a Fermat prime and a Sylow 2-subgroup of $G$ is not abelian, in which case $p-1 \le r \le p$.

Clearly, $\Phi(G) \le C_M(K/\Phi(G))$. On the other hand, $(C_M(K/\Phi(G))/\Phi(G)) \trianglelefteq G/\Phi(G)$ and $(C_M(K/\Phi(G))/\Phi(G)) \cap K/\Phi(G) = 1$. Therefore, $C_M(K/\Phi(G))/\Phi(G) = 1$ since $K/\Phi(G)$ is the unique minimal normal subgroup of $G/\Phi(G)$ by step 3. It follows that $C_M(K/\Phi(G)) \le \Phi(G)$ and thus $\Phi(G) = C_M(K/\Phi(G))$.

Since $G$ is $p$-soluble, $M/\Phi(G)$ is also $p$-soluble. Since $G/\Phi(G) = (K/\Phi(G)) \rtimes (M/\Phi(G))$ and $K/\Phi(G)$ is a soluble minimal normal subgroup of $G/\Phi(G)$, $M/\Phi(G)$ acts irreducibly on $K/\Phi(G)$. Clearly, $M/\Phi(G) = M/C_M(K/\Phi(G))$ acts faithfully on $K/\Phi(G)$. It then follows from [2, Ch. A, Lemma 13.6] that $O_p(K/\Phi(G)) = 1$.

Let $g$ be an element of $M/\Phi(G)$ of order $p$. By Theorem 2.1, the minimal equation of $g$ is $(x-1)^r = 0$, where $r = p$, unless $p-1$ is a power of a prime $q$ for which a Sylow $q$-subgroup of $M/\Phi(G)$ is not abelian, in which case $p-1 \le r \le p$. Suppose that $p-1$ is a power of a prime $q$ for which a Sylow $q$-subgroup of $M/\Phi(G)$ is not abelian. Then $p$ is odd and $p-1$ is even. It then follows that in this case we have $q = 2$, $p$ is a Fermat prime and a Sylow 2-subgroup of $G$ is not abelian.

*Step 7.* We have a contradiction.

Write $\overline{K} = K/\Phi(G)$, $\overline{M} = M/\Phi(G)$ and $\overline{P} = P/\Phi(G)$. By step 4 and the hypotheses of this theorem, $G^{\mathcal{F}} \le \Omega(P) \le \omega_n(P)$, where $n = p-2$ if $p$ is a Fermat prime and a Sylow 2-subgroup of $G$ is not abelian, and $n = p-1$ otherwise. It then follows from Lemma 2.4 that $\overline{K} = K/\Phi(G) = (G^{\mathcal{F}}\Phi(G))/\Phi(G) \le (\omega_n(P)\Phi(G))/\Phi(G) \le \omega_n(P/\Phi(G)) = \omega_n(\overline{P})$. Since $\overline{M} = M/\Phi(G)$ is not a $p'$-group by step 5, we can pick an element $g$ of $\overline{M}$ of order $p$.

Since $\overline{K} \le \omega_n(\overline{P})$, we have $\overline{K} \le N_{\overline{P}}(\langle g\rangle\omega_{n-1}(\overline{P}))$ by Lemma 2.2. Hence,

$$[\overline{K}, \langle g\rangle\omega_{n-1}(\overline{P})] \le \langle g\rangle\omega_{n-1}(\overline{P}). \tag{3.1}$$

Let $i$ be an arbitrary nonnegative integer. By Lemma 2.2, we have $\omega_{i+1}(\overline{P}) \le N_{\overline{P}}(\langle g\rangle\omega_i(\overline{P}))$. Clearly, $\langle g\rangle \le N_{\overline{P}}(\langle g\rangle\omega_i(\overline{P}))$. Therefore, $\langle g\rangle\omega_{i+1}(\overline{P}) \le N_{\overline{P}}(\langle g\rangle\omega_i(\overline{P}))$ and it follows that

$$[\langle g\rangle\omega_{i+1}(\overline{P}), \langle g\rangle\omega_i(\overline{P})] \le \langle g\rangle\omega_i(\overline{P}). \tag{3.2}$$

From (3.1) and (3.2),

$$\begin{aligned}
&[\ldots[[[\overline{K}, \langle g\rangle\omega_{n-1}(\overline{P})], \langle g\rangle\omega_{n-2}(\overline{P})], \langle g\rangle\omega_{n-3}(\overline{P})], \ldots, \langle g\rangle\omega_0(\overline{P})]\\
&\quad\le [\ldots[[\langle g\rangle\omega_{n-1}(\overline{P}), \langle g\rangle\omega_{n-2}(\overline{P})], \langle g\rangle\omega_{n-3}(\overline{P})], \ldots, \langle g\rangle\omega_0(\overline{P})]\\
&\quad\le [\ldots[\langle g\rangle\omega_{n-2}(\overline{P}), \langle g\rangle\omega_{n-3}(\overline{P})], \ldots, \langle g\rangle\omega_0(\overline{P})]\\
&\quad\ \ \vdots\\
&\quad\le \langle g\rangle\omega_0(\overline{P}) = \langle g\rangle.
\end{aligned} \tag{3.3}$$

On the other hand, since $\overline{K} \trianglelefteq \overline{P}$,

$$[\ldots[[[\overline{K}, \langle g\rangle\omega_{n-1}(\overline{P})], \langle g\rangle\omega_{n-2}(\overline{P})], \langle g\rangle\omega_{n-3}(\overline{P})], \ldots, \langle g\rangle\omega_0(\overline{P})] \le \overline{K}. \tag{3.4}$$

Combining (3.3) and (3.4), we know that for any element $k \in \overline{K}$,

$$\begin{aligned}
&[\ldots[[[k, \underbrace{g], g], g], \ldots, g}_{n}]\\
&\quad \in [\ldots[[[\overline{K}, \underbrace{\langle g\rangle], \langle g\rangle], \langle g\rangle], \ldots, \langle g\rangle}_{n}]\\
&\quad \le [\ldots[[[\overline{K}, \langle g\rangle\omega_{n-1}(\overline{P})], \langle g\rangle\omega_{n-2}(\overline{P})], \langle g\rangle\omega_{n-3}(\overline{P})], \ldots, \langle g\rangle\omega_{0}(\overline{P})]\\
&\quad \le \langle g\rangle \cap \overline{K} \le \overline{M} \cap \overline{K} = 1. \qquad (3.5)
\end{aligned}$$

If we regard $g$ as a linear transformation over a field of characteristic $p$, through the conjugation action of $g$ on $\overline{K}$, then from (3.5) and [7, Ch. IX, Lemma 1.8] we have $(g-1)^n = 0$, where $n = p-2$ if $p$ is a Fermat prime and a Sylow 2-subgroup of $G$ is not abelian, and $n = p-1$ otherwise. This contradicts step 6. □

Proof of Theorem B. Suppose that this theorem is false and let $G$ be a counterexample of minimal order. From the minimal choice of $G$, it is easy to see that $O_{p'}(G) = 1$.

We claim that $G$ is $p$-soluble and thus by [4, Ch. 6, Theorem 3.2] we have $C_G(O_p(G)) \le O_p(G)$. Indeed, since $G$ is not $p$-nilpotent, by Frobenius' $p$-nilpotence theorem, $P$ has a nontrivial subgroup $S$ such that $N_G(S)$ is not $p$-nilpotent. On the other hand, $N_G(P)$ is $p$-nilpotent by hypothesis. Therefore, we can find a nontrivial proper subgroup $Y$ of $P$ such that $N_G(Y)$ is not $p$-nilpotent but, for every $p$-subgroup $T$ of $G$ with $Y < T$, $N_G(T)$ is $p$-nilpotent. Write $A = N_G(Y)$. Suppose that $A < G$ and let $P_1$ be a Sylow $p$-subgroup of $A$. Without loss of generality, we may assume that $P_1 \le P$. Since $Y < P$, $N_P(Y) > Y$. It follows that $Y < P_1$ and thus $N_G(P_1)$ is $p$-nilpotent. Hence, $N_A(P_1) = A \cap N_G(P_1)$ is $p$-nilpotent. By Lemma 2.4, $\Omega(P_1) \le P_1 \cap \Omega(P) \le P_1 \cap \omega_{p-1}(P) \le \omega_{p-1}(P_1)$. It then follows from the minimal choice of $G$ that $A$ is $p$-nilpotent, which contradicts the choice of $Y$. Hence, $A = N_G(Y) = G$ and $Y$ is a nontrivial normal $p$-subgroup of $G$. Now, by the choice of $Y$, we can see that for any $p$-subgroup $B/Y$ of $P/Y$, $N_{G/Y}(B/Y) = (N_G(B))/Y$ is $p$-nilpotent. It follows that $G/Y$ is $p$-nilpotent by Frobenius' $p$-nilpotence theorem and thus $G$ is $p$-soluble.

Clearly, $G$ is not a $p$-group. Let $q$ be a prime divisor of the order of $G$ such that $q \ne p$. Since $G$ is $p$-soluble, $G$ has a Sylow $q$-subgroup $Q$ such that $PQ$ is a subgroup of $G$ by [4, Ch. 6,Theorem 3.5]. Let $K = PQ$ and let $H/O_p(K)$ be a minimal normal subgroup of $K/O_p(K)$. Then $H \trianglelefteq K$ and $H/O_p(K)$ is an abelian $q$-group. Let $L = PH$. Then $L$ is a $(p, q)$-group whose Sylow $q$-subgroup is abelian. If $p$ is a Fermat prime, then $p \ne 2$ and thus a Sylow 2-subgroup of $L$ is abelian. Clearly, $P$ is a Sylow $p$-subgroup of $L$ and $\Omega(P) \le \omega_{p-1}(P)$ by assumption. Therefore, by Theorem A, the $p$-length of $L$ is at most 1. Since $N_L(P) = N_G(P) \cap L$ is $p$-nilpotent, $L$ is $p$-nilpotent by Lemma 2.6. On the other hand, we have $O_p(G) \le O_p(L)$ since $P \le L$. It follows that $O_{p'}(L) \le C_G(O_p(L)) \le C_G(O_p(G)) \le O_p(G)$ and thus $O_{p'}(L) = 1$. But then $L$ must be a $p$-group since $L$ is $p$-nilpotent. This contradicts the fact that $L$ is a $(p, q)$-group and the proof is complete. □

Proof of Theorem C. Suppose that this theorem is false and let $G$ be a counterexample of minimal order. By Theorem A, we may assume that $wl(P) - (p-3) > 1$ when $p$ is a Fermat prime and a Sylow 2-subgroup of $G$ is not abelian, and assume that $wl(P) - (p-2) > 1$ when $p$ is not a Fermat prime or a Sylow 2-subgroup of $G$ is abelian.

We argue that the $p$-length of any proper factor group of $G$ is less then the $p$-length of $G$. In particular, since $l_p(G/\Phi(G)) = l_p(G)$ and $l_p(G/\operatorname{O}_{p'}(G)) = l_p(G)$, we have $\Phi(G) = \operatorname{O}_{p'}(G) = 1$. Suppose that this is not true and let $L$ be a nontrivial normal subgroup of $G$ such that $l_p(G/L) = l_p(G)$. By Lemma 2.4, $wl(PL/L) = wl(P/(P \cap L)) \leq wl(P)$. First assume that $p$ is a Fermat prime and a Sylow 2-subgroup of $G$ is not abelian. If a Sylow 2-subgroup of $G/L$ is not abelian, then the minimal choice of $G$ implies that $l_p(G) = l_p(G/L) \leq \max\{1, wl(PL/L) - (p-3)\} \leq \max\{1, wl(P) - (p-3)\}$, which is a contradiction. If a Sylow 2-subgroup of $G/L$ is abelian, then the minimal choice of $G$ implies that $l_p(G) = l_p(G/L) \leq \max\{1, wl(PL/L) - (p-2)\} \leq \max\{1, wl(PL/L) - (p-3)\} \leq \max\{1, wl(P) - (p-3)\}$, which is a contradiction. Now assume that $p$ is a Fermat prime or a Sylow 2-subgroup of $G$ is abelian. Then either $p$ is a Fermat prime or a Sylow 2-subgroup of $G/L$ is abelian and thus the minimal choice of $G$ implies that $l_p(G) = l_p(G/L) \leq \max\{1, wl(PL/L) - (p-2)\} \leq \max\{1, wl(P) - (p-2)\}$, which is again a contradiction.

Let $N$ be a minimal normal subgroup $G$. Then $N \leq \operatorname{O}_p(G)$ since $G$ is $p$-soluble and $\operatorname{O}_{p'}(G) = 1$. Suppose that $G$ has another minimal normal subgroup, say $N_1$. Without loss of generality, we may assume that $l_p(G/N) \geq l_p(G/N_1)$. Then by Lemma 2.5 $l_p(G) = l_p(G/(N \cap N_1)) = \max\{l_p(G/N), l_p(G/N_1)\} = l_p(G/N)$, which contradicts the conclusion of the above paragraph. Hence, $N$ is the unique minimal normal subgroup of $G$.

Since $\Phi(G) = 1$, $\operatorname{O}_p(G)$ is a direct product of minimal normal subgroups of $G$. It follows that $N = \operatorname{O}_p(G)$ and thus $C_G(N) = N$ by [4, Ch. 6, Theorem 3.2]. Also, from $\Phi(G) = 1$, we know that $G$ has a maximal subgroup $M$ such that $G = [N]M$.

Let $P_1 = M \cap P$. Then $N \cap P_1 \leq N \cap M = 1$ and thus $P_1 \cong P/N$. We now prove that $\omega(P) \cap P_1 = 1$. Indeed, suppose that $\omega(P) \cap P_1 \neq 1$ and pick an element $x \in \omega(P) \cap P_1$ of order $p$. Let $y$ be any element of $N$. Since $x \in \omega(P)$, $\langle x \rangle \langle y \rangle$ is a subgroup of $P$. Clearly, $\langle x \rangle \langle y \rangle$ is abelian since $|\langle x \rangle \langle y \rangle| \leq |\langle x \rangle|\,|\langle y \rangle| = p^2$. Hence, $[x, y] = 1$. It follows that $x \in C_G(N) \cap P_1 \leq N \cap P_1 = 1$, which is a contradiction.

Since $\omega(P) \cap P_1 = 1$, $wl(P_1) = wl((P_1\omega(P))/\omega(P))$. By Lemma 2.4, we have $wl((P_1\omega(P))/\omega(P)) \leq wl((P/\omega(P))$. Hence, $wl(P_1) \leq wl((P/\omega(P)) = wl(P) - 1$.

We are now ready to derive a contradiction. Since $\operatorname{O}_{p'}(G) = 1$, we have $N = \operatorname{O}_p(G) = \operatorname{O}_{p',p}(G)$ and thus $l_p(G/N) = l_p(G) - 1$. First assume that $p$ is a Fermat prime and a Sylow 2-subgroup of $G$ is not abelian. Then $p \neq 2$ and a Sylow 2-subgroup of $G/N$ is not abelian. From the minimal choice of $G$, $(l_p(G) - 1) = l_p(G/N) \leq \max\{1, wl(P/N) - (p-3)\}$. On the other hand, since $wl(P/N) = wl(P_1) \leq wl(P) - 1$, and $wl(P) - (p-3) > 1$ by the assumptions in the first paragraph of the proof, we have $\max\{1, wl(P/N) - (p-3)\} \leq \max\{1, wl(P) - (p-3)\} - 1$. It then follows that $(l_p(G) - 1) \leq (\max\{1, wl(P) - (p-3)\} - 1)$ and thus $l_p(G) \leq wl(P) - (p-3)$, which contradicts the choice of $G$. Similarly, we can derive a contradiction when $p$ is not a Fermat prime or a Sylow 2-subgroup of $G$ is abelian. The proof of this theorem is complete. □

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