J. Aust. Math. Soc. **109** (2020), 217–229
doi:10.1017/S1446788719000090

CHARACTERIZATIONS OF QUASICONVEX AND PSEUDOCONVEX FUNCTIONS BY THEIR SECOND-ORDER REGULAR SUBDIFFERENTIALS

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(Received 3 August 2018; accepted 16 February 2019; first published online 30 April 2019)

Communicated by W. Moors

Dedicated to the memory of Jonathan Borwein

Abstract

We present the second-order necessary and sufficient conditions for quasiconvex and pseudoconvex functions in terms of their second-order regular subdifferentials.

2010 *Mathematics subject classification*: primary 26B25; secondary 49J40, 49J52, 49J53.

Keywords and phrases: quasiconvexity, second-order subdifferential, positive semidefinite property, second-order optimality conditions.

1. Introduction

Second-order subdifferentials and their application in the characterization of quasiconvexity and pseudoconvexity have been investigated in the literature. From classical analysis, we know that the second-order differential of a convex function $f : \mathbb{R}^n \to \mathbb{R}$, which is called the Hessian ($\nabla^2 f$), is positive semidefinite, and positive definiteness of the Hessian implies strict convexity of f. The notion of a generalized Hessian (second-order subdifferential) was introduced by Mordukhovich, as a coderivative of a subdifferential mapping [14]. Since then, the generalized Hessian has been used frequently as a strong tool for characterizing the various sorts and generalizations of convexity (see, for example, [5–7]). As a significant result in this area, it was shown in [6] that the following second-order condition, namely, the positive semidefinite property, is a characterization of the convexity of a C^1 function f, defined on a Hilbert space X, that is,

$$\langle z, u\rangle \geq 0, \quad \forall u \in X \quad \text{and} \quad z \in \partial^2 f(x, y)(u) \quad \text{with } (x, y) \in \text{ gph } \partial f.$$

Second-order subdifferentials, which are utilized extensively in optimization problems, especially for characterizing the tilt-stable local minimum in optimization problems, were introduced by Poliquin and Rockafellar [19] (see [10–12, 16, 17]).

The structure of our paper is as follows. Section 2 includes some basic definitions that are required in the subsequent work, and Section 3 contains the second-order necessary and sufficient conditions for approximately convex functions (and semismooth functions on $\mathbb{R}^n$) to be quasiconvex or pseudoconvex.

2. Basic definitions

In this section, we recall some basic definitions that will be needed in the subsequent work. For more details, see [15, 16, 21]. Throughout this paper, X is a Banach space endowed with a norm $\|.\|$, X^* is its dual space, X^{**} is its second dual space and $\langle .,. \rangle$ is the dual pairing between X and X^*.

Let $F : X \rightrightarrows Y$ be a set-valued mapping between Banach spaces. The effective domain and graph of F are

$$\operatorname{dom} F := \{x \in X : F(x) \neq \emptyset\} \quad \text{and} \quad \operatorname{gph} F := \{(x, y) \in X \times Y : y \in F(x)\}.$$

Given $\varepsilon \geq 0$ and $\Omega \subseteq X$, the $\varepsilon-$*normals* to Ω at $\bar{x}$ are defined by

$$\widehat{N}_\varepsilon(\bar{x}; \Omega) := \left\{x^* \in X^* \mid \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq \varepsilon\right\},$$

where the symbol $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. When $\varepsilon = 0$, the set $\widehat{N}_0(\bar{x}, \Omega) = \widehat{N}(\bar{x}, \Omega)$ is called the regular normal cone, or prenormal cone, to Ω at $\bar{x}$.

Definition 2.1. The regular coderivative of F at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is

$$\widehat{D}^* F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* : (x^*, -y^*) \in \widehat{N}((\bar{x}, \bar{y}), \operatorname{gph} F)\} \quad \forall y^* \in Y^*.$$

Let $f : X \to \bar{\mathbb{R}} = [-\infty, +\infty]$ be an extended real-valued function. We define

$$\operatorname{dom} f := \{x \in X : |f(x)| < \infty\} \quad \text{and} \quad \operatorname{epi}(f) := \{(x, \mu) \in (X \times \mathbb{R}) : \mu \geq f(x)\}.$$

The presubdifferential or regular subdifferential of f at $\bar{x} \in \operatorname{dom} f$ is

$$\widehat{\partial} f(\bar{x}) := \{x^* \in X^* : (x^*, -1) \in \widehat{N}((\bar{x}, \bar{y}), \operatorname{epi} f)\}.$$

For $\bar{x} \notin \operatorname{dom} f$, we put $\widehat{\partial} f(\bar{x}) = \emptyset$.

Definition 2.2 [3]. Let $f : X \to \mathbb{R}$ be a locally Lipschitz function. The Clarke directional derivative of f at x in direction u is defined as

$$f^\circ(x, u) := \limsup_{t \to 0^+, y \to x} \frac{f(y + tu) - f(y)}{t},$$

and the Clarke subdifferential of f at x is defined as

$$\partial_c f(x) := \{x^* \in X^* : \langle x^*, v\rangle \leq f^\circ(x, u), \forall v \in X\}.$$

A locally Lipschitz function is said to be directionally Clarke regular (d-regular) at x if, for every $u \in X$, the Clarke directional derivative of f at x in direction u coincides with $d^- f(x, u)$, where

$$d^- f(x, u) := \liminf_{t \to 0^+} \frac{f(x + tu) - f(x)}{t}.$$

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It is not difficult to see that $\partial_c f(x)$ is a convex and w*-compact subset of X^*. Also, $\partial_c f : X \rightrightarrows X^*$ is norm-w* upper semicontinuous and

$$f^\circ(x,u) = \max\{\langle x^*, u\rangle : x^* \in \partial_c f(x)\}.$$

Definition 2.3 [15]. Let $f : X \longrightarrow \bar{\mathbb{R}}$ be a function with a finite value at $\bar{x}$. For any $\bar{y} \in \widehat{\partial} f(\bar{x})$, the map $\widehat{\partial}^2 f(\bar{x},\bar{y}) : X^{**} \rightrightarrows X^*$ with the values

$$\widehat{\partial}^2 f(\bar{x},\bar{y})(u) := (\widehat{D}^* \widehat{\partial} f)(\bar{x},\bar{y})(u) \quad \forall u \in X^{**},$$

is said to be the regular second-order subdifferential of f at $\bar{x}$ relative to $\bar{y}$.

3. Second-order characterization for quasiconvexity and pseudoconvexity of semismooth functions

Characterizations of twice-differentiable quasiconvex and pseudoconvex functions $f : C \subseteq \mathbb{R}^n \to \mathbb{R}$, where ∇f is locally Lipschitz, have been extended by Crouziex and Ferland (1996) [8].

Definition 3.1. A set-valued mapping $F : X \rightrightarrows X^*$ is said to be:

(i) quasimonotone if, for every $x, y \in X$ and $x^* \in F(x), y^* \in F(y)$,

$$\langle x^*, y - x\rangle > 0 \Longrightarrow \langle y^*, y - x\rangle \geq 0;$$

(ii) pseudomonotone if, for every $x, y \in X$ and $x^* \in F(x), y^* \in F(y)$,

$$\langle x^*, y - x\rangle > 0 \Longrightarrow \langle y^*, y - x\rangle > 0;$$

(iii) submonotone at $x_0 \in X$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_1, x_2 \in B(x_0, \delta) \cap \mathrm{dom} F$ and $y_1 \in F(x_1), y_2 \in F(x_2)$,

$$\langle y_2 - y_1, x_2 - x_1\rangle \geq -\varepsilon \|x_2 - x_1\|;$$

(iv) semisubmonotone at $x_0 \in X$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_1 \in B(x_0, \delta) \cap \mathrm{dom} F$ and $y_0 \in F(x_0), y_1 \in F(x_1)$,

$$\langle y_1 - y_0, x_1 - x_0\rangle \geq -\varepsilon \|x_1 - x_0\|;$$

(v) submonotone (respectively, semisubmonotone) on X if it is submonotone (respectively, semisubmonotone) at all $x \in X$.

The quasimonotonicity for single-valued mappings is defined analogously as in Definition 3.1.

Definition 3.2. A single-valued mapping $F : X \to X^*$ is said to be quasimonotone if, for any $x, y \in X$,

$$\langle F(x), y - x\rangle > 0 \Longrightarrow \langle F(y), y - x\rangle \geq 0.$$

In particular, $g : \mathbb{R} \to \mathbb{R}$ is quasimonotone if, for any $t_1, t_2 \in \mathbb{R}$,

$$g(t_2)(t_1 - t_2) > 0 \Longrightarrow g(t_1)(t_1 - t_2) \geq 0.$$

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Definition 3.3. Consider a convex subset C of a normed linear space X.

(i) A function $f : C \to \mathbb{R}$ is said to be quasiconvex on C if, for every $x, y \in C$ and $t \in]0, 1[$,

$$f(x + t(y - x)) \leq \max\{f(x), f(y)\},$$

or, equivalently, if its level sets are convex, that is, for every $\alpha \in \mathbb{R}$, $\mathrm{Lev}_\alpha f =: \{x \in C : f(x) \leq \alpha\}$ is convex.

(ii) A function $f : C \to \mathbb{R}$ is said to be pseudoconvex on C if, for every $x, y \in C$, $x \neq y$ and $x^* \in \widehat{\partial} f(x)$,

$$\langle x^*, y - x\rangle \geq 0 \Longrightarrow f(y) \geq f(x).$$

(iii) A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be lower-C^1 [23] if, for each $\bar{x} \in \mathbb{R}^n$, there exists a neighborhood V of $\bar{x}$ such that f has the representation

$$f(x) = \max_{t \in T} f_t(x),$$

where the functions f_t are of class C^1 on V, the index set T is compact and $f_t(x)$ and $\nabla f_t(x)$ are jointly continuous on $(t, x) \in T \times V$.

(iv) Let X be a real Banach space and let U be a nonempty open subset of X. We say that $f : U \to \mathbb{R}$ is approximately convex at $x_0 \in U$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $x, y \in B(x_0, \delta)$ and $t \in [0, 1]$,

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + \varepsilon t(1 - t)\|x - y\|.$$

We say that f is approximately convex on U if it is approximately convex at all $x_0 \in U$.

(v) A function $f : X \to \mathbb{R}$ is said to be semismooth at $x_0 \in X$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $x \in B(x_0, \delta)$ and $t \in]0, 1[$,

$$f(tx + (1 - t)x_0) \leq tf(x) + (1 - t)f(x_0) + \varepsilon t(1 - t)\|x - x_0\|.$$

Remark 3.4. The class of approximately convex functions was introduced by Ngai *et al.* [18]. The notion of semismooth functions was presented by Aussel *et al.* as a generalization of approximately convex functions [2]. The class of subsmooth (respectively, semisubsmooth) sets was introduced by Aussel *et al.* [2] as an epigraphic characterization of approximately convex (respectively, semismooth) functions. These sets are the generalizations of convex sets and prox-regular sets presented by Poliquin *et al.* [20]. The aforementioned concepts are studied comprehensively in [2], and the following main results were obtained in real Banach spaces.

$$f \text{ is approximately convex } \Longleftrightarrow \partial f \text{ is submonotone } \Longleftrightarrow \text{epi } f \text{ is subsmooth.}$$

Similar implications are obtained in finite-dimensional spaces for semismooth, d-regular functions.

It has been proved by Daniilidis and Georgiev [9] that, in finite-dimensional spaces, the class of locally Lipschitz approximately convex functions coincides with the class of lower-C^1 functions.

We need to use the concept of marginal functions, as defined below.

DEFINITION 3.5 [15]. For $F : X \rightrightarrows X^*$ and $u \in X$, we associate the marginal function as

$$f_u(x) := \inf\{\langle y, u\rangle : y \in F(x)\}$$

and the minimum set (or marginal map) as

$$M_u(x) := \{y \in F(x) : f_u(x) = \langle y, u\rangle\} \quad \forall x \in X.$$

Also, when X is a reflexive Banach space, for every $u, \bar{x} \in X$, we define the marginal function $g_{u,\bar{x}}$ on $\mathbb{R}$ as

$$g_{u,\bar{x}}(t) := f_u(\bar{x} + tu) = \inf\{\langle y, u\rangle : y \in F(\bar{x} + tu)\}.$$

DEFINITION 3.6 [4]. Assume that X and Y are Hausdorff topological vector spaces and $F : X \rightrightarrows Y$ is a set-valued mapping. F is said to be upper semicontinuous at $x \in \operatorname{dom} F$ if, for all open $W \subseteq Y$ such that $F(x) \subseteq W$, there exists a neighborhood U of x such that $F(\bar{x}) \subseteq W$ for all $\bar{x} \in U$.

In what follows, we assume that X is a separable reflexive Banach space and that $F(x)$ is a w*-compact subset of X^* for any $x \in X$.

LEMMA 3.7. *Assume that $F : X \rightrightarrows X^*$ is a norm $-$ w^* upper semicontinuous mapping. Then $g_{u,\bar{x}}$ is lower semicontinuous for any $u, \bar{x} \in X$.*

PROOF. The proof is trivial and can be deduced from the definition. □

LEMMA 3.8 [15]. *Let $F : X \rightrightarrows X^*$ be a set-valued map with f_u as its marginal function. Then $\widehat{\partial} f_u(\bar{x}) \subseteq \widehat{D}^* F(\bar{x}, \bar{y})(u)$ for any $\bar{y} \in M_u(\bar{x})$.*

DEFINITION 3.9 [8]. Assume that f is a twice differentiable function on an open convex subset C of $\mathbb{R}^n$. We define the following properties.

(sdp): $x \in C, \langle \nabla f(x), h\rangle = 0 \Longrightarrow \langle \nabla^2 f(x)h, h\rangle \geq 0$.

(cq): $x, x - h \in C, \nabla f(x) = 0, \langle \nabla^2 f(x)h, h\rangle = 0, f(x - h) < f(x) \Longrightarrow$ for every $\widehat{t} > 0$ there exists $t \in]0, \widehat{t}]$ so that $f(x + th) \geq f(x)$.

(cp): $x \in C, \nabla f(x) = 0 \Longrightarrow f$ has a local minimum at x.

THEOREM 3.10 [8]. *Assume that f is a twice differentiable function defined on the open convex set C and that ∇f is locally Lipschitz on C. Then:*

(i) *f is quasiconvex on C if and only if the two conditions (sdp) and (cq) hold; and*
(ii) *f is pseudoconvex if and only if the two conditions (sdp) and (cp) hold.*

Luc and Schaible [13] extended those results for $C^{1,1}$ functions (functions with locally Lipschitz differential). First, they defined $D_+F(x, u)$ and $D_-F(x, u)$ for the locally Lipschitz mappings, as follows (where $\partial F(x)$ is designated for the Clarke subdifferential of F).

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DEFINITION 3.11 [13]. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a locally Lipschitz mapping. For every $u \in \mathbb{R}^n$, define

$$D_+F(x,u) := \sup\{\langle u, Au\rangle : A \in \partial F(x)\},$$
$$D_-F(x,u) := \inf\{\langle u, Au\rangle : A \in \partial F(x)\}.$$

The following theorem is the main result of [13] which gives a characterization for quasimonotone mappings. An analogous result also is given for the pseudomonotone mappings.

THEOREM 3.12 [13]. *Suppose that S is a nonempty convex open subset of $\mathbb{R}^n$. The locally Lipschitz map $F : S \longrightarrow \mathbb{R}^n$ is quasimonotone on S if and only if the following conditions hold for every $x \in S, u \in \mathbb{R}^n$.*

(i) $\langle F(x), u\rangle = 0 \Longrightarrow D_+F(x,u) \geq 0$.
(ii) $\langle F(x), u\rangle = 0, 0 \in \{\langle u, Au\rangle : A \in \partial F(x)\}$ *and* $\langle F(x + \bar{t}u), u\rangle \geq 0$ *for some* $\bar{t} < 0$. *Then there exists* $\widehat{t} > 0$ *such that* $\langle F(x + tu), u\rangle \geq 0$ *for all* $t \in [0, \widehat{t}]$.

Now we give a characterization for the locally Lipschitz approximately convex quasiconvex functions by their second-order subdifferentials in the sense of regular. For this purpose, it is sufficient to characterize the quasimonotonicity of set-valued mappings ($F : X \rightrightarrows X^*$) in terms of their subdifferentials, since the relationship between the generalized convexity of a function and generalized monotonicity of its subdifferential has been established. First, we extend Definition 3.11 for the set-valued mappings in terms of regular coderivatives.

DEFINITION 3.13. Let $F : X \rightrightarrows X^*$ be a set-valued mapping and let X be a Banach space. For every $\bar{x} \in X$ and $u \in X^{**}$, define

$$\widehat{D}_+F(\bar{x},u) := \sup\{\langle z, u\rangle : z \in \widehat{D}^*F(x,y)(u), x \to \bar{x}, y \to \bar{y}, y \in F(x)\},$$
$$\widehat{D}_-F(\bar{x},u) := \inf\{\langle z, u\rangle : z \in \widehat{D}^*F(x,y)(u), x \to \bar{x}, y \to \bar{y}, y \in F(x)\}.$$

LEMMA 3.14. *If $F : X \rightrightarrows X^*$ is a quasimonotone map, then $g_{u,\bar{x}} : \mathbb{R} \longrightarrow \mathbb{R}$ is quasimonotone for every $u, \bar{x} \in X$. Moreover, suppose either F is single valued or semisubmonotone on X. Then the converse holds.*

PROOF. Suppose that F is quasimonotone, $\bar{x}, u \in X$ are arbitrary and $g_{u,\bar{x}}$ is the associated marginal function in the sense of Definition 3.5. Assume, in addition, that $g_{u,\bar{x}}(t_2)(t_1 - t_2) > 0$ for arbitrary $t_1, t_2 \in \mathbb{R}$. We can find some $y_2 \in F(\bar{x} + t_2u)$ such that $g_{u,\bar{x}}(t_2) = \langle y_2, u\rangle$, which means that

$$\langle y_2, (\bar{x} + t_1u) - (\bar{x} + t_2u)\rangle = \langle y_2, u\rangle(t_1 - t_2) > 0.$$

Now, by quasimonotonicity of F, for any $y_1 \in F(\bar{x} + t_1u)$, we conclude that

$$\langle y_1, (\bar{x} + t_1u) - (\bar{x} + t_2u)\rangle \geq 0,$$

which implies that $g_{u,\bar{x}}(t_1)(t_1 - t_2) \geq 0$ and $g_{u,\bar{x}}$ is quasimonotone.

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For the converse, suppose, on the contrary, that there exist $x_1, x_2, y_1 \in F(x_1)$ and $y_2 \in F(x_2)$ such that $\langle y_2, x_1 - x_2\rangle > 0$ and $\langle y_1, x_1 - x_2\rangle < 0$. Let $\bar{x} := x_1$ and $u := x_2 - x_1$. We divide the rest of the proof into the following two cases.

Case 1. F is single valued. Then $g_{u,\bar{x}}(0) > 0$ and $g_{u,\bar{x}}(1) < 0$, which contradicts the quasimonotonicity of g_{u,x_1}.

Case 2. Suppose that F is semisubmonotone and $\langle y_1, u\rangle = \gamma$ for some $\gamma > 0$. Set $\varepsilon := \gamma/2$. By semisubmonotonicity of F at x_1, we can find some $\delta > 0$ such that

$$\langle y - y_1, x - x_1\rangle \geq -\varepsilon\|x - x_1\| \quad \forall x \in B(x_1, \delta) \quad \text{and} \quad y \in F(x).$$

Thus, for $0 < t < \min\{\delta, 1\}$ and $x := x_1 + t(x_2 - x_1)$, we conclude that

$$\langle y, u\rangle = \langle y, x_2 - x_1\rangle \geq \langle y_1, u\rangle - \varepsilon\|u\| = \frac{\gamma}{2} \quad \forall y \in F(x_1 + tu).$$

Therefore, from the definition of marginal functions, $g_{u,x_1}(t) = \gamma/2 > 0$, which contradicts the quasimonotonicity of g_{u,x_1}. □

Now we give a necessary and sufficient condition for quasimonotonicity of the semisubmonotone upper semicontinuous mappings, by their regular coderivative.

THEOREM 3.15. *Let $F : X \rightrightarrows X^*$ be a norm-w* upper semicontinuous mapping. If F is quasimonotone, then the following two conditions hold for every $\bar{x}, u \in X$.*

(i) *$f_u(\bar{x}) = 0$ implies that $\widehat{D}_+F(\bar{x}, u) \geqslant 0$.*
(ii) *$f_u(\bar{x}) = 0$, $\widehat{D}_+F(\bar{x}, u) \geqslant 0, \widehat{D}_-F(\bar{x}, u) \leq 0$ and $\langle y_{\bar{t}}, u\rangle > 0$ (for some $\bar{t} < 0$ and $y_{\bar{t}} \in F(\bar{x} + \bar{t}u)$) implies that there exists $\widehat{t} > 0$ such that $\langle y_t, u\rangle \geq 0$ for every $t \in [0, \widehat{t}]$ and $y_t \in F(\bar{x} + tu)$.*

Moreover, if F is semisubmonotone and (i) and (ii) hold, then F is quasimonotone.

PROOF. We first show that quasimonotonicity of F implies (i) and (ii). We can consider that $\|u\| = 1$ (when this is not the case, we can divide u to $\|u\| \neq 0$). Suppose that $g_{u,\bar{x}}(0) = \inf\{\langle y, u\rangle : y \in F(\bar{x})\} = 0$. We show that $\widehat{D}_+F(\bar{x}, u) \geqslant 0$.

First, suppose that $g_{u,\bar{x}}(t_0) > 0$ for some $t_0 < 0$. By Lemma 3.14, $g_{u,\bar{x}}$ is a quasimonotone function from $\mathbb{R}$ into $\mathbb{R}$ for any $\bar{x}$ and u. Thus, we have $g_{u,\bar{x}}(t) \geq 0$ for any $t > t_0$. So we conclude that $0 = x^* \in \widehat{\partial} g_{u,\bar{x}}(0)$. By using the approximate chain rule [22, Theorem 9.2.9] for $g_{u,\bar{x}}$ as the composition of f_u and $h(t) = \bar{x} + tu$, for every $\varepsilon > 0$ and $x^* \in \widehat{\partial} g_u(0)$, we can find $t_0 \in (-\varepsilon/2, \varepsilon/2)$, $x_0 \in B_X(\bar{x}, \varepsilon/2)$, $y^* \in \widehat{\partial} f_u(x_0)$, $z^* \in B_{X^*}(y^*, \varepsilon/2)$ and $\bar{t} = \langle z^*, u\rangle = \widehat{\partial}\langle z^*, h\rangle(t_0)$ such that $|x^* - \bar{t}| < \varepsilon$. Therefore, we conclude that

$$\langle y^*, u\rangle = \langle y^* - z^*, u\rangle + \langle z^*, u\rangle \geq -\frac{\varepsilon}{2} + x^* - \varepsilon \geq -\frac{3}{2}\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\langle y^*, u\rangle \geq 0$. By Lemma 3.8, $y^* \in \widehat{\partial} f_u(x_0) \subseteq \widehat{D}^*F(x_0, y_0)(u)$ for some $y_0 \in M_u(x_0)$, which implies that $\widehat{D}_+F(\bar{x}, u) \geqslant 0$.

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Now suppose that $g_{u,\bar{x}}(t) \leq g_{u,\bar{x}}(0) = 0$ for any $t < 0$. Then $g_{u,\bar{x}}$ is continuous from the left at zero, since it is lower semicontinuous. So there exists some $x^* \in \partial g_{u,\bar{x}}(0)$ for which $x^* \geq 0$, and, by the definition of a limiting subdifferential, we can find a sequence (s_n) that converges to zero and $x_n^* \in \widehat{\partial} g_{u,\bar{x}}(s_n)$ such that $x_n^* \to x^*$.

By using the approximate chain rule [22, Theorem 9.2.9] for $g_{u,\bar{x}}$ as the composition of two functions f_u and $h(t) = \bar{x} + tu$, for every $\varepsilon > 0$ and $x_n^* \in \widehat{\partial} g_{u,\bar{x}}(s_n)$, we can find $t_n \in]s_n - (\varepsilon/2), s_n + \varepsilon/2[$, $x_n \in B_X(\bar{x} + s_n u, \varepsilon/2)$, $y_n^* \in \widehat{\partial} f_u(x_n)$, $z_n^* \in B_{X^*}(y_n^*, \varepsilon/2)$ and $\bar{t}_n = \langle z_n^*, u\rangle = \widehat{\partial}\langle z_n^*, h\rangle(t_n)$ such that $|x_n^* - \bar{t}_n| < \varepsilon$.

So, for any $\varepsilon > 0$ and sufficiently large n, we conclude that

$$-\varepsilon + x_n^* - \frac{\varepsilon}{2} < \bar{t}_n - \frac{\varepsilon}{2} < \langle y_n^*, u\rangle < \bar{t}_n + \frac{\varepsilon}{2} < x_n^* + \varepsilon + \frac{\varepsilon}{2},$$

which implies that $|\langle y_n^*, u\rangle - x_n^*| < \frac{3}{2}\varepsilon$. So we have $\langle y_n^*, u\rangle \to x^*$. Since $F(x_n)$ is w*-compact, by Lemma 3.8, $y_n^* \in \widehat{\partial} f_u(x_n) \subseteq \widehat{D}^* F(x_n, y_n)(u)$ for some $y_n \in M_u(x_n)$. The upper semicontinuity of F implies that (y_n) converges to some $\bar{y} \in F(\bar{x})$, and hence $\widehat{D}_+ F(\bar{x}, u) \geqslant 0$.

If (ii) does not hold, then there exist $\bar{t} < 0$ and $t_0 > 0$, $y_{t_0} \in F(\bar{x} + t_0 u)$ and $y_{\bar{t}} \in F(x + \bar{t}u)$ such that $\langle y_{\bar{t}}, u\rangle > 0$ and $\langle y_{t_0}, u\rangle < 0$. So

$$\langle y_{t_0}, \bar{x} + t_0 u - (\bar{x} + \bar{t}u)\rangle = \langle y_{t_0}, (t_0 - \bar{t})u\rangle < 0$$

and

$$\langle y_{\bar{t}}, \bar{x} + t_0 u - (\bar{x} + \bar{t}u)\rangle = \langle y_{\bar{t}}, (t_0 - \bar{t})u\rangle > 0,$$

which contradicts the quasimonotonicity of F. It should be noted that the other assumptions in (ii) are not used in the proof of necessity and are applied only for the sufficiency.

For the converse, suppose, on the contrary, that F is not quasimonotone. So, by Lemma 3.14,

$$g_{u,x_1}(t) := \inf\{\langle y, u\rangle : y \in F(x_1 + tu)\}$$

is not quasimonotone for some $x_1, u \in X$. Without loss of generality, we can assume that $g_{u,x_1}(0) \geq 0$ and $g_{u,x_1}(1) < 0$. In other words, for $x_2 = u + x_1$, some $y_2 \in F(x_2)$ and any $y_1 \in F(x_1)$, we have $\langle y_2, x_1 - x_2\rangle > 0$ and $\langle y_1, x_1 - x_2\rangle < 0$. Let

$$t_0 := \sup\{t : g_{u,x_1}(t) \geq 0, 0 \leqslant t < 1\},$$

and

$$x_0 := x_1 + t_0(x_2 - x_1).$$

We show that $g_{u,x_1}(t_0) = 0$:

First, suppose that $g_{u,x_1}(t_0) = \gamma > 0$. Thus $\langle y_0, y\rangle = \gamma$ for some $y_0 \in F(x_0)$. By the semisubmonotonicity of F at x_0, for $\varepsilon := \gamma/2$, there exists some $\delta > 0$ such that

$$\langle y - y_0, x - x_0\rangle \geq -\varepsilon\|x - x_0\| \quad \forall x \in B(x_0, \delta) \quad \text{and} \quad y \in F(x).$$

Let $x := x_0 + t(x_2 - x_0)$ for $0 < t < \min\{1 - t_0, \delta\}$. Similarly to the proof of Lemma 3.14, we conclude that $\langle y, u\rangle > \gamma/2 > 0$ for every $y \in F(x)$, which contradicts the definition of t_0.

On the other hand, suppose that $g_{u,x_1}(t_0) = \langle y_0, u\rangle = -\gamma < 0$ for some $y_0 \in F(x_0)$. Thus, by the definition of t_0, we can find a sequence (t_n) that converges increasingly to t_0 such that $g_{u,x_1}(t_n) = \langle y_n, u\rangle \geq 0$, where $y_n \in F(x_1 + t_n(x_2 - x_1))$. But, by the definition of semisubmonotonicity, for sufficiently large n,

$$\langle y_n - y_0, (t_n - t_0)u\rangle \geq -\frac{1}{n}|t_n - t_0|\|u\|.$$

This means that, for sufficiently large n,

$$\langle y_n, u\rangle \leq \langle y_0, u\rangle + \frac{\|u\|}{n} = -\gamma + \frac{\|u\|}{n},$$

which is a contradiction, since $\langle y_n, u\rangle \geq 0$ for each n.

Therefore, (i) implies that $\widehat{D}_+F(x_0, u) \geq 0$. Now we show that $\widehat{D}_-F(x_0, u) < 0$.

We have $g_{u,x_1}(t_0) = 0$ and $g_{u,x_1}(t) \leq 0$ for each $t \in \,]t_0, 1]$. Moreover, note that g_{u,x_1} is continuous from the right at t_0, since it is lower semicontinuous, and $t_0 \neq 1$. Therefore, we can find some $x^* \in \partial g_u(t_0)$ such that $x^* \leq 0$. So there exist two sequences, (s_n) and (x_n^*), such that $s_n \to t_0$ and $x_n^* \to x^*$, where $x_n^* \in \widehat{\partial} g_{u,x_1}(s_n)$.

Now, by using the approximate chain rule [22, Theorem 9.2.9], for $g_{u,x_1}(s_n) = f_u(x_1 + s_n u)$, we can find $s_n' \in (t_0 - \varepsilon/2, t_0 + \varepsilon/2)$, $x_n \in B_X(x_1 + s_n u, \varepsilon/2)$, $y_n^* \in \widehat{\partial} f_u(x_n)$, $z_n^* \in B_{X^*}(y_n^*, \varepsilon/2)$ and $\bar{t}_n = \langle z_n^*, u\rangle = \widehat{\partial}\langle z_n^*, h\rangle(s_n')$ such that $|x_n^* - \bar{t}_n| < \varepsilon$. Therefore, we conclude that

$$x_n^* + \varepsilon > \bar{t}_n = \langle z_n^*, u\rangle = \langle z_n^* - y_n^*, u\rangle + \langle y_n^*, u\rangle > -\frac{\varepsilon}{2} + \langle y_n^*, u\rangle,$$

which implies that $\langle y_n^*, u\rangle < x_n^* + \frac{3}{2}\varepsilon$. Thus, $\limsup_n \langle y_n^*, u\rangle \leq 0$. By Lemma 3.8, $y_n^* \in \widehat{\partial} f_u(x_n) \subseteq \widehat{D}^* F(x_n, y_n)(u)$ for some $y_n \in M_u(x_n)$. Hence, the upper semicontinuity of F implies that (y_n) converges to some $y_0 \in F(x_0)$ and

$$\widehat{D}_-F(x_0, u) = \inf\{\langle z, u\rangle : z \in \widehat{D}^* F(x, y)(u), x \to x_0, y \to y_0, y_0 \in F(x_0)\} \leq 0.$$

Letting $\bar{t} = -t_0 < 0$, we have $\langle y_1, u\rangle > 0$ for $y_1 \in F(x_1) = F(x_0 + \bar{t}u)$. So all of the conditions of (ii) hold. Therefore, by (ii), there exists $\widehat{t} > 0$ such that, for every $t \in [0, \widehat{t}[$ and $y \in F(x_0 + tu)$, we have $\langle y, u\rangle \geq 0$. This shows that $g_{u,x_1}(t_0 + t) \geq 0$, which contradicts the definition of t_0 and the proof of sufficiency is complete. □

We present the following second-order characterization for locally Lipschitz approximately convex functions, defined on X. Note that, in the following result, for each u, we denote the marginal function of $F = \partial_c f$ by φ_u.

Corollary 3.16. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz function. If f is quasiconvex, then the following assertions hold for every $\bar{x}, u \in X$.*

(i) $\varphi_u(\bar{x}) = \inf\{\langle y, u\rangle : y \in \partial_c f(\bar{x})\} = 0$ *implies that* $\widehat{D}_+\partial_c f(\bar{x}, u) \geqslant 0$.

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(ii) *$\varphi_u(\bar{x}) = 0$, $\widehat{D}_+\partial_c f(\bar{x}, u) \geq 0$, $\widehat{D}_-\partial_c f(\bar{x}, u) \leq 0$ and $\langle y_{\bar{t}}, u\rangle > 0$ (for some $\bar{t} < 0$ and $y_{\bar{t}} \in \partial_c f(\bar{x} + \bar{t}u)$) implies that there exists $\widehat{t} > 0$ such that $\langle y_t, u\rangle \geq 0$ for every $t \in [0, \widehat{t}]$ and $y_t \in \partial_c f(\bar{x} + tu)$.*

Moreover, consider the following statements:

(iii) *f is approximately convex;*
(iv) *$X = \mathbb{R}^n$ and f is semismooth and d-regular;*
(v) *$X = \mathbb{R}^n$ and f is lower-C^1.*

If (i) and (ii) and one of the conditions (iii), (iv) and (v) hold, then f is quasiconvex.

Proof. A lower semicontinuous function is quasiconvex if and only if its Clarke subdifferential is quasimonotone [1, Theorem 4.1]. On the other hand, the Clarke subdifferential mapping is norm to w* upper semicontinuous. So it suffices to apply Theorem 3.15 for $F = \partial_c f$. □

Example 3.17. Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ as

$$f(x) = \begin{cases} -x^2 & x < 0, \\ -x^2 + x & 0 < x \leq \frac{1}{2}, \\ \frac{1}{4} & x > \frac{1}{2}. \end{cases}$$

The Clarke subdifferential mapping of f can be calculated easily by

$$\partial_c f(x) = F(x) = \begin{cases} -2x & x \leq 0, \\ \{[0, 1]\} & x = 0, \\ -2x + 1 & 0 < x < \frac{1}{2}, \\ 0 & x > \frac{1}{2}. \end{cases}$$

We show that the conditions (i) and (ii) of Corollary 3.16 hold. For (i), it suffices to check that $\widehat{D}_+\partial_c f(\bar{x}, u) \geqslant 0$ for $\bar{x} = 0$ and $\bar{x} = \frac{1}{2}$, since, for another points, the condition holds trivially. So we calculate the regular normal cone of gph $\partial_c f$ at $(0, y) \in$ gph $\partial_c f$ as

$$\widehat{N}((0, y), \text{gph } \partial_c f) = \begin{cases} \{(v_1, v_2) : v_2 \leq 0, v_2 \leq \frac{1}{2}v_1\} & y = 0, \\ \mathbb{R} \times \{0\} & 0 < y < 1, \\ \{(v_1, v_2) : v_2 \geq 0, v_2 \geq \frac{1}{2}v_1\} & y = 1. \end{cases}$$

We show that $\widehat{D}_+\partial_c f(\bar{x}, u) \geqslant 0$. For $u \geq 0$,

$$\sup\{\langle z, u\rangle : z \in \widehat{D}^*\partial_c f(0, 0)(u)\} \geq 0.$$

Also, for $u < 0$,

$$\sup\{\langle z, u\rangle : z \in \widehat{D}^*\partial_c(0, 1)(u)\} \geq 0.$$

Then, for each $u \in \mathbb{R}$, we conclude that $\widehat{D}_+\partial_c f(0, u) \geqslant 0$.

For $\bar{x} = \frac{1}{2}$, it suffices to note that, for any $x > \frac{1}{2}$,

$$\widehat{N}((x, 0), \text{gph } \partial_c f) = \{(0, v) : v \in \mathbb{R}\},$$

which means that $0 \in \{\langle z, u\rangle : z \in \widehat{D}^*\partial_c(x, y)(u), x \to \frac{1}{2}, y \to 0\}$ and $\widehat{D}_+\partial_c f(\frac{1}{2}, u) \geqslant 0$.

Assertion (ii) holds trivially at $\bar{x} = 0$. For $u \geq 0$, we have $\langle y_t, u\rangle > 0$ for sufficiently small $t > 0$. Also, for $u < 0$ and every $t \in \mathbb{R}$, we have $\langle y_t, u\rangle \leq 0$.

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When the function has more than one variable, the second-order calculations and, more specially, the calculation of $\widehat{D}^*F(\bar{x},\bar{y})(u)$ for any $\bar{y}\in\partial_c f(\bar{x})=F(\bar{x})$ with $0=\inf\{\langle y,u\rangle : y\in\partial_c f(\bar{x})\}$ are more complicated. But we should calculate $\widehat{D}_+F(\bar{x},u)$ and it suffices to find some $z\in\widehat{D}^*F(\bar{x},\bar{y})(u)$ with $\langle z,u\rangle\geq 0$ to guarantee condition (i) of Corollary 3.16.

Example 3.18. Consider the function $f : S=\{x : \|x\|<\frac{1}{2}\}\subseteq\mathbb{R}^2\to\mathbb{R}$ defined as

$$f(x_1,x_2)=f(x)=-\|x\|^2+\|x\|.$$

It is easy to see that f is continuously differentiable on $S\setminus\{(0,0)\}$. Also, the Clarke subdifferential at $(0,0)$ is

$$\partial_c f((0,0))=\{x^*=(x_1^*,x_2^*)\in\mathbb{R}^2 : \|x^*\|\leq 1\}.$$

For every $0\neq u\in\mathbb{R}^2$, we have $\inf\{\langle y,u\rangle : y\in\partial_c f((0,0))\}<0$, since the closed unit ball is a balanced subset of $\mathbb{R}^2$. Therefore, clearly, (i) holds.

For (ii), suppose that $u\neq(0,0)$ is arbitrary. Then an easy calculation implies that

$$\langle\nabla f(tu),u\rangle=(u_1^2+u_2^2)\Big(-2t+\frac{1}{\sqrt{u_1^2+u_2^2}}\Big)\geq 0$$

for every $t\in[0,\widehat{t}]$ with $\widehat{t}:=2(u_1^2+u_2^2)^{-1/2}$, which means that (ii) holds.

Similarly to the above results, we can find a characterization for pseudomonotone mappings and pseudoconvex functions. Note that every pseudomonotone mapping is quasimonotone, and therefore conditions (i) and (ii) of the above results hold in this case. But, for the converse, condition (ii) can be replaced by a weaker condition. In this way, the proof is similar to the quasimonotone case.

Theorem 3.19. *Let $F : X\rightrightarrows X^*$ be a norm-w* upper semicontinuous mapping. If F is pseudomonotone, then the following two conditions hold for every $\bar{x},u\in X$.*

(i) *$f_u(\bar{x})=0$ implies that $\widehat{D}_+F(\bar{x},u)\geqslant 0$.*
(ii) *$f_u(\bar{x})=0$, $\widehat{D}_+F(\bar{x},u)\geqslant 0$ and $\widehat{D}_-F(\bar{x},u)\leq 0$ implies that there exists $\widehat{t}>0$ such that $\langle y_t,u\rangle\geq 0$ for every $t\in[0,\widehat{t}]$ and $y_t\in F(\bar{x}+tu)$.*

Moreover, F is pseudomonotone if (i) and (ii) hold, and F is semisubmonotone on X.

Proof. Suppose that F is pseudomonotone. Then F is quasimonotone, and (i) holds by Theorem 3.15. If (ii) does not hold, then we can find some $t>0$ and $y_t\in F(\bar{x}+tu)$ such that $\langle y_t,u\rangle<0$, which means that $\langle y_t,\bar{x}+tu-\bar{x}\rangle<0$. But, since $f_u(\bar{x})=0$, we have $\langle\bar{y},\bar{x}+tu-\bar{x}\rangle=0$ for some $\bar{y}\in F(\bar{x})$, which contradicts the pseudomonotonicity of F.

For the converse, suppose, on the contrary, that F is not pseudomonotone. Therefore we can find $x_1,x_2\in X$ and $y_1\in F(x_1), y_2\in F(x_2)$ such that $\langle y_2,x_1-x_2\rangle>0$ and $\langle y_1,x_1-x_2\rangle\leq 0$. Now it suffices to define g_{u,x_1}, t_0 and x_0, similarly to the proof of sufficiency in Theorem 3.15, and follow its proof to get a contradiction. □

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Corollary 3.20. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz function. If f is pseudoconvex, then the following assertions hold for every $\bar{x}, u \in X$.*

(i) $\varphi_u(\bar{x}) = \inf\{\langle y, u\rangle : y \in \partial_c f(\bar{x})\} = 0$ *implies that* $\widehat{D}_+\partial_c f(\bar{x}, u) \geqslant 0$.

(ii) $\varphi_u(\bar{x}) = 0$, $\widehat{D}_+\partial_c f(\bar{x}, u) \geq 0$ *and* $\widehat{D}_-\partial_c f(\bar{x}, u) \leq 0$ *implies that there exists* $\widehat{t} > 0$ *such that* $\langle y_t, u\rangle \geq 0$ *for every* $t \in [0, \widehat{t}]$ *and* $y_t \in \partial_c f(\bar{x} + tu)$.

Moreover, consider the following statements:

(iii) *f is approximately convex;*

(iv) *$X = \mathbb{R}^n$ and f is semismooth and d-regular;*

(v) *$X = \mathbb{R}^n$ and f is lower-C^1.*

If (i) and (ii) and one of the conditions (iii), (iv) or (v) hold, then f is pseudoconvex.

Proof. It suffices to apply Theorem 3.19 for $F = \partial_c f$, which is norm to w* upper semicontinuous and is pseudomonotone if and only if f is pseudoconvex. □

Acknowledgement

The authors are grateful to the Associate Editor and the reviewer for valuable remarks.

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