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Exploring the relation between Intuitionistic BI and Boolean BI: an unexpected embedding

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The logic of Bunched Implications, through both its intuitionistic version (BI) and one of its classical versions, called Boolean BI (BBI), serves as a logical basis to spatial or separation logic frameworks. In BI, the logical implication is interpreted intuitionistically whereas it is generally interpreted classically in spatial or separation logics, as in BBI. In this paper, we aim to give some new insights into the semantic relations between BI and BBI. Then we propose a sound and complete syntactic constraints based framework for the Kripke semantics of both BI and BBI, a sound labelled tableau proof system for BBI, and a representation theorem relating the syntactic models of BI to those of BBI. Finally, we deduce as our main, and unexpected, result, a sound and faithful embedding of BI into BBI.

1. Introduction

Spatial logics for dynamic processes, static spatial logics for trees or processes, context logic, separation logic, resource and processes logic all share a common core language for expressing logical properties. This is the language of BI, the Logic of Bunched Implications (Pym 1999; Pym 2002) mixing multiplicative connectives $*$, $-\!\!*$ with additive connectives $\wedge, \vee, \rightarrow$ and its Kripke sharing interpretation of the multiplicative connectives:

$$m \Vdash A * B \quad \text{iff} \quad \text{there exist } a, b \text{ such that } a \circ b \triangleright m \text{ and } a \Vdash A \text{ and } b \Vdash B\,.$$

The ternary relation $- \circ - \triangleright -$ has different interpretations depending on the semantic framework used: process composition/interaction (for $\circ$) and structural congruence (for $\triangleright$) in spatial logics (Cardelli and Gordon 2000; Caires and Lozes 2006); finite edge-labelled tree/process composition and structural equivalence for static spatial logics (Calcagno *et al.* 2005; Lozes 2004); contexts composition and structural equivalence for context logic (Calcagno *et al.* 2007); disjoint heap union and equality (or inclusion) for separation logic (Ishtiaq and O'Hearn 2001), denoted SL here; and the product of resource and process composition for SCRP/MBI (Pym and Tofts 2006).

Even if they are based on the same language together with the sharing interpretation, these logics differ because the underlying models do not have the same properties: the set of valid formulae differ from one logic to the other, some have decidable model-checking and others do not, and so on. Moreover, nearly all of these logics use a classically or pointwise defined Kripke semantics for the additive implication →, thereby favouring Boolean BI (BBI), whereas BI has an intuitionistically defined additive implication. It should be noted, however, that there was an attempt to model intuitionistic implication in Ishtiaq and O'Hearn (2001), but the authors quickly dismissed it as a particular case of the classical implication: the reason for this could be viewed as a rather restrictive choice for the order relation, map inclusion in this case. In this paper, we aim to give some new insights into the relation between BI and BBI in the general case.

The model theoretic properties of these logics, based on BBI, have been widely studied: in particular, decidability (Calcagno *et al.* 2005) and undecidability (Caires and Lozes 2006), quantifier and adjunct elimination (Lozes 2004), and expressivity (Brochenin *et al.* 2008). However, even though the proof theory of BI has been extensively explored with both natural deduction and sequent style proof systems (Pym 2002) and labelled tableau proof systems (Galmiche and Méry 2003; Galmiche *et al.* 2005), the proof theory of BBI is either missing or heavily based on model-checking methods, as in Calcagno *et al.* (2005).

The relations between BI and BBI are often misunderstood. Whereas classical logic CL (the additive fragment of BBI) can be faithfully embedded into intuitionistic logic IL (the additive fragment of BI) by the Gödel translation, for example, this result has no known extension when linear operators are added. Moreover, in this paper we show that the reverse is true: it is possible to faithfully embed BI into BBI. This result suggests that proof and counter-model search in BBI is certainly no easier than in BI and might in fact be much more difficult.

Our approach to BI and BBI and their relations is not model oriented. We aim to study the formulae of BI/BBI that hold in *all* of these models, propose proof-systems to prove or refute these formulae and compare the provability relation of BI and BBI. We consider BI and BBI to be defined by the abstract Kripke semantics, namely *partially ordered partial monoids* for BI and *partial monoids* for BBI. We use the models generated by the syntactic constraints occurring in tableau proof-search. Soundness and completeness of our tableau systems ensure that the sub-class of models generated by syntactic constraints is complete with respect to the abstract Kripke semantics and thus, these syntactic constraints and their solutions encompass the semantic properties of these logics.

In this paper we will show the following results:

— a sound and complete syntactic constraints based framework for Kripke semantics of both BI and BBI;
— a sound labelled tableau proof system for BBI;
— a representation theorem linking the syntactic models of BI to those of BBI; and, as a consequence,
— a faithful embedding of BI into BBI.

We also discuss some expressivity properties of BI that can be deduced from our results.

As for the potential consequences and later developments of this work (which are more fully described in the conclusion of this paper), we can list the following items:

— a sound and complete proof and counter-model search method for BI based on partial monoidal constraints as opposed to the existing resource graph method (Galmiche *et al.* 2005);

— a concrete and complete class of separation logic style models for BI based on the distinction between observable and unobservable resources;
— hopefully, a characterisation of the full class of BBI-generated constraints and explicit forms for constraint extensions;
— further expressivity properties for BBI as well as for BI.

We will now describe the contents of the sections leading to the result that the function $G \mapsto (\mathsf{I} \wedge \mathsf{H}) \rightarrow G^\circ$ constitutes a sound and faithful embedding of BI into BBI. Here:

— The BBI-formula G° is an image of the BI-formula G defined by (linear) structural induction using two spare logical variables L and K.
— H is some given fixed BBI-formula depending only on the logical variables L and K.
— I is the multiplicative unit of BBI.

In Section 2, we recall the monoidal Kripke semantics of BI and BBI. We point out different semantic frameworks for interpreting the monoidal relation like spatial logics, separation logics or abstract monoidal Kripke semantics. We also stress the difference between the intuitionistic and classical interpretation of the additive implication $\rightarrow$, and the properties required for those models.

In Section 3, we describe a common framework for dealing with the Kripke semantics of BI and BBI. This framework is based on particular binary relations between words (which are in fact multisets in this paper) expressed by sets of constraints of the form $m-n$ where m and n are two words. An atomic constraint $m-n$ is the syntactic expression of a semantic relation between the words/labels m and n that should hold in all the interpretations of m and n that satisfy the constraint $m-n$. From a finite or infinite set of (atomic) constraints, we generate particular relations by closure. These particular relations are themselves the 'least' models of the syntactic constraints from which they originate. These (closed) relations are called *partial monoidal orders* (PMOs) for BI and *partial monoidal equivalences* (PMEs) for BBI. They are characterised as being closed under some particular deduction rules. We introduce a Kripke interpretation within this PMO/PME framework and prove that the corresponding semantics is equivalent to the abstract Kripke semantics of BI and BBI, respectively: up to some quotient by an equivalence relation, the PMO $\sqsubseteq$ is in fact a partially ordered partial monoid and the PME $\sim$ is a partial monoid. Then we show how to build PMOs and PMEs by closure from arbitrary sets of constraints and derive some properties linking sets of constraints and their closures, such as, for example, a compactness property or the way in which the constraints involving the empty word ϵ behave in the closure[†].

In Section 4, we present a link between PMOs (models of BI) and PMEs (models of BBI). Indeed, we describe a map $\sim\ \mapsto\ \sqsubseteq_{\sim}^{L,K}$ that associates a PMO $\sqsubseteq_{\sim}^{L,K}$ to any PME $\sim$, given some alphabets L and K. The idea of the map is that the words of $L^\star$ are Kripke-interpreted in both BI and BBI, whereas the words of $K^\star$ are only Kripke-interpreted in BBI. Thus, we say that the words of $K^\star$ are unobservable by BI. The relation $m \sqsubseteq_{\sim}^{L,K} n$ holds whenever $m, n \in L^\star$ belong to the observable words of $L^\star$ and are

[†] The empty word ϵ plays a particular role because it is the only word that can be squared or erased *a priori*.

equivalent up to some unobservable word $\delta \in K^\star$, that is, $\delta m \sim n$ holds. We show how this idea extends the intuitionistic interpretation of the implication $\rightarrow$ by heap inclusion in intuitionistic Separation Logic SL (Ishtiaq and O'Hearn 2001). To any formula G of BI, we associate a formula G° of BBI such that the Kripke semantics of G in the model defined by $\sqsubseteq_\sim^{L,K}$ is equivalent to the Kripke semantics of G° in the model defined by $\sim$.

In Section 5, we introduce the labels and constraints based tableau proof system TBI, which is sound and complete for BI. We also define a tableau proof system for BBI called TBBI and prove its soundness with respect to BBI (completeness also holds, but for reasons explained later, we will not prove this in this paper). Elementary PMOs are generated from finite sequences of constraints of the form $ab - m$, $am - b$, $m - b$ or $\epsilon - m$, where a and b are new letters and m is already defined by previous constraints. The sequence can be infinite for simple PMOs. We prove that the constraints occurring in TBI-tableaux are elementary PMOs for finite tableaux and simple PMOs for infinite tableaux branches, so simple PMOs form a complete sub-class of models for BI. Hence, every invalid formula of BI has a counter-model that is a simple PMO.

In Section 6, we define the notions of an elementary and simple PME as generated by sequences of constraints of the form $ab - m$, $am - b$, $m - b$ or $\epsilon - b$, where a and b are new letters and m is already defined. We show that the map $\sim \mapsto \sqsubseteq_\sim^{L,K}$ is surjective on simple PMOs, its 'reverse map' being described as an algorithm transforming a simple PMO $\sqsubseteq$ into a simple PME $\sim$ such that $\sqsubseteq = \sqsubseteq_\sim^{L,K}$. The validity of this algorithm can be considered as the main technical result of this paper and is based on the notion of an elementary representation, which is basically a PMO/PME pair $(\sqsubseteq, \sim)$ that verifies some specific conditions including $\sqsubseteq = \sqsubseteq_\sim^{L,K}$. The proof of the validity of the algorithm requires us to establish some properties about free PME extensions, including, for example, the extension of the PME $\sim$ with the constraint $ab - m$ where a and b are new letters and m is already defined. We also need to prove that simple PMEs have no square, that is, for a simple PME $\sim$, the relation $mm \sim mm$ only holds when $m \sim \epsilon$.

In Section 7, we describe a tableau transformation algorithm of a TBI-proof of some BI-formula G into a TBBI-proof of the formula $(\mathsf{I} \wedge \mathsf{H}) \rightarrow G^\circ$, hence establishing that the map $G \mapsto (\mathsf{I} \wedge \mathsf{H}) \rightarrow G^\circ$ is a sound embedding of BI into BBI. As tableau proofs proceed by branch expansion, we show how to map any TBI-branch expansion into a combination of TBBI-branch expansions. The soundness of the transformation is based on the properties of elementary representations. Starting from a closed TBI-tableau (that is, a proof) for G, the resulting TBBI-tableau for $(\mathsf{I} \wedge \mathsf{H}) \rightarrow G^\circ$ is not necessarily closed, but it is pseudo-closed, and we later show that pseudo-closed tableaux can be expanded into closed tableaux in TBBI.

In Section 8, we show that the formula $(\mathsf{I} \wedge \mathsf{H}) \rightarrow G^\circ$ is BBI-invalid whenever G is BI-invalid by counter-model transformation. The result is based on the fact that if a BI-formula G is invalid, it has a BI-counter-model based on a simple PMO $\sqsubseteq$, the class of simple PMOs being complete for BI. As $\sqsubseteq$ is a simple PMO, there exists a PME $\sim$ such that $\sqsubseteq = \sqsubseteq_\sim^{L,K}$, and we prove that $\sim$ provides a BBI-counter-model to $(\mathsf{I} \wedge \mathsf{H}) \rightarrow G^\circ$. We present this counter-model transformation using the example of the intuitionistically invalid formula $X \vee (X \rightarrow \bot)$ (excluded middle).

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In Section 9, we introduce some basic applications of our results to the expressivity of BI. For example, the property $\epsilon \sqsubseteq m$ is trivially expressed by the logical constant I. On the other hand, neither the property $m \sqsubseteq \epsilon$ nor the property $mm \sqsubseteq mm$ can be expressed by formulae of BI.

2. Sharing interpretation and monoidal Kripke semantics for BI and BBI

The logics BI and BBI are syntactically defined by the following grammar (Pym 2002), where Var is a set of propositional variables and X ranges over Var:

$$\begin{array}{lll} \mathsf{BI} & : & A, B ::= X \mid \bot \mid \top \mid A \wedge B \mid A \vee B \mid A \rightarrow B \mid \mathsf{I} \mid A * B \mid A -\!\!* B \\ \mathsf{BBI} & : & A, B ::= X \mid \bot \mid \top \mid A \wedge B \mid A \vee B \mid A \rightarrow B \mid \neg A \mid \mathsf{I} \mid A * B \mid A -\!\!* B\,. \end{array}$$

2.1. *The monoidal Kripke semantics of* BI *and* BBI

Before we introduce the Kripke semantics of BI and BBI, we recall the general semantic framework under which the Kripke interpretation is going to be defined.

Definition 2.1. A *partial monoid* is a triple $(\mathcal{M}, \circ, \mathsf{e})$ where $\mathsf{e} \in \mathcal{M}$ and $\circ : \mathcal{M} \times \mathcal{M} \rightharpoonup \mathcal{M}$ is a partial map for which the following conditions hold:

1. $\forall a \in \mathcal{M}, \mathsf{e} \circ a\!\downarrow \wedge\, \mathsf{e} \circ a = a$ (identity).
2. $\forall a, b \in \mathcal{M}, a \circ b\!\downarrow \Rightarrow b \circ a\!\downarrow \wedge\, a \circ b = b \circ a$ (commutativity).
3. $\forall a, b, c \in \mathcal{M}, a \circ (b \circ c)\!\downarrow \Rightarrow (a \circ b) \circ c\!\downarrow \wedge\, a \circ (b \circ c) = (a \circ b) \circ c$ (associativity).

We write $x \circ y\!\downarrow$ when the composition of x and y by $\circ$ is defined. Note that $a \circ (b \circ c)$ can only be defined if $b \circ c$ is itself defined, hence $a \circ (b \circ c)\!\downarrow$ implies $b \circ c\!\downarrow$. We also assume that the meta-logical connectives $\Rightarrow$, $\wedge$ and $\vee$ will not be confused with the logical connectives of BI/BBI, even though the conjunction and disjunction have the same denotation.

A binary relation $\triangleright \subseteq \mathcal{M} \times \mathcal{M}$ is a *partial order* if it is reflexive, antisymmetric and transitive. To give a Kripke interpretation to the formulae of BI (respectively, BBI), we start from a given structure $(\mathcal{M}, \circ, \mathsf{e}, \triangleright)$ where $(\mathcal{M}, \circ, \mathsf{e})$ is a partial monoid of *resources*, $\triangleright$ is a partial order on $\mathcal{M}$ (respectively, the identity relation $\triangleright \equiv =$ on $\mathcal{M}$) such that composition is *monotonic*:

$$\forall k, x, y \in \mathcal{M},\ (k \circ y\!\downarrow \wedge\, x \triangleright y) \Rightarrow (k \circ x\!\downarrow \wedge\, k \circ x \triangleright k \circ y)\,.$$

The structure $(\mathcal{M}, \circ, \mathsf{e}, \triangleright)$ is thus a *partially ordered partial monoid* for the case of BI formulae and a *partial monoid* for the case of BBI formulae. The Kripke interpretation on the set of logical variables Var is given by a *forcing relation* $\Vdash\, \subseteq \mathcal{M} \times \mathsf{Var}$ that verifies the *monotonicity condition*:

$$\forall X \in \mathsf{Var}, \forall m, n \in \mathcal{M},\ (m \triangleright n \wedge m \Vdash X) \Rightarrow n \Vdash X\,.$$

For BBI, both the monotonicity of composition and the monotonicity condition trivially hold because the order $\triangleright$ is the identity $=$.

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$$
\begin{array}{llllll}
m \Vdash \mathsf{I} & \text{iff} & \mathsf{e} \rhd m & m \Vdash \neg A & \text{iff} & m \nVdash A \\
m \Vdash \bot & \text{iff} & \text{never} & m \Vdash A \vee B & \text{iff} & m \Vdash A \text{ or } m \Vdash B \\
m \Vdash \top & \text{iff} & \text{always} & m \Vdash A \wedge B & \text{iff} & m \Vdash A \text{ and } m \Vdash B
\end{array}
$$

$$
\begin{array}{lll}
m \Vdash A \rightarrow B & \text{iff} & \forall x \in \mathscr{M},\ (m \rhd x \text{ and } x \Vdash A) \Rightarrow x \Vdash B \\
m \Vdash A * B & \text{iff} & \exists x, y \in \mathscr{M},\ x \circ y\!\downarrow \text{ and } x \circ y \rhd m \text{ and } x \Vdash A \text{ and } y \Vdash B \\
m \Vdash A \mathrel{-\!\!*} B & \text{iff} & \forall x, y \in \mathscr{M},\ (x \circ m\!\downarrow \text{ and } x \circ m \rhd y \text{ and } x \Vdash A) \Rightarrow y \Vdash B
\end{array}
$$

Fig. 1. The Kripke semantics of BI and BBI.

The Kripke interpretation is inductively extended to the compound formulae of BI (respectively, BBI) by the equations of Figure 1. We may write $\Vdash_{\circ,\rhd}$ to denote this forcing relation extended to the whole BI (respectively, BBI).

Definition 2.2. Given a Kripke structure $(\mathscr{M}, \circ, \mathsf{e}, \rhd, \Vdash)$, a formula F of BI (respectively, BBI) is *valid* in $\mathscr{M}$ if $\mathsf{e} \Vdash_{\circ,\rhd} F$ (respectively, $\forall m \in \mathscr{M}$, $m \Vdash_{\circ,\rhd} F$) and we say that $\mathscr{M}$ is a BI-*model* (respectively, BBI-*model*) of F. If F is not valid, then it is *invalid* and $\mathscr{M}$ is a BI-*counter-model* if $\mathsf{e} \nVdash_{\circ,\rhd} F$ (respectively, BBI-*counter-model* if $m \nVdash_{\circ,\rhd} F$ for some $m \in \mathscr{M}$).

Within this framework, it is possible to prove (see Pym (2002), for example) that the monotonicity condition extends to any formula of BI (respectively, BBI) and that the logical rules of the natural deduction proof theory of BI and BBI (Pym 2002) are sound with respect to this Kripke interpretation. Moreover, if $\rhd$ is symmetric (which is a property fulfilled when $\rhd$ is the identity relation, as required for BBI) it is possible to show that the logical implication $\rightarrow$ is in fact interpreted classically (or pointwise), that is, $m \Vdash A \rightarrow B$ if and only if $m \Vdash A \Rightarrow m \Vdash B$. This classical Kripke semantics for the additive implication $\rightarrow$ is one of the reasons why BI is sometimes called *intuitionistic* BI whereas BBI is called *Boolean* BI. Without the linear connectives $*$ and $\mathrel{-\!\!*}$, the distinction between intuitionistic and classical Kripke interpretations for the connective $\rightarrow$ gives rise to propositional intuitionistic logic IL and propositional classical logic CL, which are related but also have huge differences. On the one hand, it is possible to embed CL faithfully into IL using the well-known Gödel translation. On the other hand, the decision of validity in IL is PSPACE-complete (Statman 1979), whereas it is coNP-complete for CL, leading to the very unlikely existence of some (low complexity) reverse faithful embedding of IL into CL. This paper aims to establish a new relation between BI and BBI, which may be viewed naively as, respectively, IL and CL enriched with the linear connectives $*$ and $\mathrel{-\!\!*}$. We claim that the embedding relation we establish tells us that this naive view is not very accurate.

2.2. *Overview of various instances of the monoidal Kripke semantics*

In order to present some of the different existing frameworks for interpretation of BI and BBI, we single out the sharing interpretation of the $*$ operator:

$$m \Vdash A * B \quad \text{iff} \quad \exists x, y \in \mathscr{M},\ x \circ y\!\downarrow \wedge\, x \circ y \rhd m \wedge x \Vdash A \wedge y \Vdash B\,.$$

$A * B$ is forced at m if there is composition $a \circ b$ somehow related to m by $a \circ b \triangleright m$ such that A is forced at a and B is forced at b. So the semantics of $A * B$ depends on the particular interpretation we provide for the composition $\circ$ and the relation $\triangleright$. Hence, the language of BI/BBI gives rise to different logics (where universally valid formulae differ) depending on how $(\circ, \triangleright)$ are interpreted:

— What we call BI is the logic defined by abstract Kripke models, that is, $\mathscr{M}$ can be any partially ordered partial monoid. There exists a proof system for this logic (Galmiche *et al.* 2005). Similarly, what we call the logic BBI is defined by partial monoids without further restrictions, and we will present a proof system for it in Section 5.

— If we restrict models so that $\mathscr{M}$ is the set of *heaps* where heaps are partially and finitely defined functions $\mathsf{Loc} \rightharpoonup_{fin} \mathsf{Val}$ mapping locations to values, and $\circ \equiv (\cdot)$ is the disjoint union of heaps (undefined when domains overlap), we obtain *Separation Logic* SL (Ishtiaq and O'Hearn 2001). The relation $\triangleright \equiv =$ is interpreted as identity in SL giving rise to models of BBI formulae. Starting from SL, if $\triangleright \equiv \subseteq$ is interpreted as partial map inclusion instead of identity, we obtain models of BI-formulae (Ishtiaq and O'Hearn 2001) and the corresponding logic is called intuitionistic SL.

— If $\mathscr{M}$ is the set of *finite unordered resource trees* and $\circ$ is the composition of resource trees, we obtain *resource tree logic* (Biri and Galmiche 2007), which can be viewed as models of BI or BBI.

The logics arising from these different interpretations are not necessarily identical: indeed, weakening $A * B \to A$ is universally valid in intuitionistic SL, whereas it has an obvious counter-model in partially ordered partial monoids, and hence in the version of BI we favour in this paper. The study of the faithful embeddings between some of these logics and some of their sub-logics has already provided results:

— The modal translation of intuitionistic SL into classical SL provides a faithful embedding. As suggested in Ishtiaq and O'Hearn (2001), this embedding is not so surprising because map inclusion $\triangleright \equiv \subseteq$ is a very restrictive interpretation for the relation $\triangleright$.

— A faithful embedding of the modal logic S4, and hence intuitionistic logic IL, into BBI has also been established, see Galmiche and Larchey (2006), where BBI is given a *non-deterministic* monoidal semantics. The same argument applies *as is* with partial monoidal semantics for BBI (instead of non-deterministic monoidal semantics), so the embedding of S4 into BBI is also faithful with the (partial and deterministic) interpretation of BBI we favour in this paper.

— The well-known Gödel translation provides a faithful embedding of classical propositional logic CL into intuitionistic logic IL. However, as far as we are aware, there is no extension of this translation providing a faithful embedding of BBI into BI.

In this paper we will establish the existence of a faithful embedding of BI into BBI in the context of their general abstract semantics, namely, partially ordered partial monoids for BI and partial monoids for BBI. By faithful embedding, we mean a map that preserves both validity and invalidity. Even though a faithful embedding from intuitionistic SL into classical SL already exists (Ishtiaq and O'Hearn 2001), that embedding is based on a restrictive interpretation of $\triangleright$, which, arguably, gives rise to a denatured BI where weakening is valid (that is, the formula $A * B \to A$ is valid in intuitionistic SL). Of course, this is not the case in BI. So our faithful embedding of BI into BBI is a much more unexpected result.

3. A complete semantics for BI and BBI based on words and constraints

We will now introduce a framework of labels and constraints to establish an original semantic relation between the abstract models of BI and BBI. This framework is useful because it provides a unifying view of both the models of BI and BBI and the proofs in BI and BBI. The main idea here is that restricting monoidal composition to word combination does not alter validity in the Kripke interpretation of BI (respectively, BBI).

3.1. *Words, constraints,* PMO*s and* PME*s*

Let L be a (potentially infinite) alphabet of letters. We consider the set of words $L^\star$ in which the order of letters is not taken into account, that is, we consider *words as finite multisets of letters*. The composition of words is denoted *multiplicatively* and the *empty word* is denoted ϵ.

We use $x \prec y$ to denote the fact that x is a *subword* of y, that is, when there exists a word k such that $kx = y$. If $x \prec y$, there is only one k such that $xk = y$ and it is denoted by y/x, hence $y = x(y/x)$. The *(carrier) alphabet of a word m* is the set of letters of which it is composed: $\mathsf{A}_m = \{l \in L \mid l \prec m\}$. We may view the alphabet L or any of its subsets $X \subseteq L$ as a subset $X \subset L^\star$, that is, we identify letters and one-letter words.

Definition 3.1. Let L be an alphabet. A *constraint* is an ordered pair (m, n) of words in $L^\star \times L^\star$ and denoted $m \multimap n$.

We represent *binary relations* $R \subseteq L^\star \times L^\star$ between words of $L^\star$ as *set of constraints* through the logical equivalence $x \; R \; y$ if and only if $x \multimap y \in R$. We view constraints as syntactic objects, whereas relations between words can be viewed as either syntactic or semantic. When $\mathscr{C} = \{\ldots, x_i \multimap y_i, \ldots\}$ represents a finite or infinite collection of individual constraints, it is viewed as a syntactic notion, and we write $x \multimap y \in \mathscr{C}$, for example. When R represents a relation between words, it is viewed as a semantic notion and we write $x \; R \; y$ instead. But the very natures of $\mathscr{C}$ and R are the same, that of a set of constraints. So we will use both terminologies for the same objects throughout this paper depending on whether we interpret them more as syntactic or semantic objects. We will consider particular sets of constraints closed under some deduction rules and their corresponding relations. Because closed relations/closed sets of constraints can themselves be viewed as models, they will usually be considered as semantic rather than syntactic objects.

Definition 3.2. Let L be an alphabet. The *language* of a binary relation $R \subseteq L^\star \times L^\star$, denoted $\mathscr{L}^R$, is defined by $\mathscr{L}^R = \{x \in L^\star \mid \exists m, n \in L^\star \text{ s.t. } xm \; R \; n \vee m \; R \; xn\}$. The *carrier alphabet* of R, denoted A_R, is defined by $\mathsf{A}_R = \bigcup\{\mathsf{A}_m \cup \mathsf{A}_n \mid m \; R \; n\}$.

A word $m \in L^\star$ is said to be *defined in R* if $m \in \mathscr{L}^R$ and is *undefined* otherwise. A letter $l \in L$ is *new to R* if $l \notin \mathsf{A}_R$. The language $\mathscr{L}^R$ is downward closed with respect to the

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PMOs	PMEs	PMOs & PMEs	
$\dfrac{x \mathrel{-\!\!\bullet\!\!-} y}{x \mathrel{-\!\!\bullet\!\!-} x}\langle l\rangle$	$\dfrac{x \mathrel{-\!\!\bullet\!\!-} y}{y \mathrel{-\!\!\bullet\!\!-} x}\langle s\rangle$	$\dfrac{}{\epsilon \mathrel{-\!\!\bullet\!\!-} \epsilon}\langle \epsilon\rangle$	$\dfrac{ky \mathrel{-\!\!\bullet\!\!-} ky \qquad x \mathrel{-\!\!\bullet\!\!-} y}{kx \mathrel{-\!\!\bullet\!\!-} ky}\langle c\rangle$
$\dfrac{x \mathrel{-\!\!\bullet\!\!-} y}{y \mathrel{-\!\!\bullet\!\!-} y}\langle r\rangle$		$\dfrac{xy \mathrel{-\!\!\bullet\!\!-} xy}{x \mathrel{-\!\!\bullet\!\!-} x}\langle d\rangle$	$\dfrac{x \mathrel{-\!\!\bullet\!\!-} y \qquad y \mathrel{-\!\!\bullet\!\!-} z}{x \mathrel{-\!\!\bullet\!\!-} z}\langle t\rangle$

Table 1. *Rules for the definition of* PMOs *and* PMEs.

subword order $\prec$. The inclusion $\mathscr{L}^R \subseteq \mathsf{A}_R^\star$ and the identity $\mathsf{A}_R = \mathscr{L}^R \cap L$ hold. If R_1 and R_2 are two relations such that $R_1 \subseteq R_2$, then the inclusions $\mathsf{A}_{R_1} \subseteq \mathsf{A}_{R_2}$ and $\mathscr{L}^{R_1} \subseteq \mathscr{L}^{R_2}$ hold. We will now define the particular sets of constraints/relations we are interested in.

Definition 3.3 (PMO/PME). Consider the rules of Table 1. A *partial monoidal order* (PMO) $\sqsubseteq$ over the alphabet L is a binary relation $\sqsubseteq \subseteq L^\star \times L^\star$ that is closed under the rules $\langle \epsilon, l, r, d, c, t\rangle$. A *partial monoidal equivalence* (PME) $\sim$ over the alphabet L is a binary relation $\sim \subseteq L^\star \times L^\star$ that is closed under the rules $\langle \epsilon, s, d, c, t\rangle$.

Proposition 3.4. Rules $\langle l\rangle$ and $\langle r\rangle$ can be derived from rules $\langle s\rangle$ and $\langle t\rangle$. Hence any PME is also a PMO.

Proof. We have the following two deduction trees:

$$\dfrac{x \mathrel{-\!\!\bullet\!\!-} y \qquad \dfrac{x \mathrel{-\!\!\bullet\!\!-} y}{y \mathrel{-\!\!\bullet\!\!-} x}\langle s\rangle}{x \mathrel{-\!\!\bullet\!\!-} x}\langle t\rangle \qquad\qquad \dfrac{\dfrac{x \mathrel{-\!\!\bullet\!\!-} y}{y \mathrel{-\!\!\bullet\!\!-} x}\langle s\rangle \qquad x \mathrel{-\!\!\bullet\!\!-} y}{y \mathrel{-\!\!\bullet\!\!-} y}\langle t\rangle$$

□

We will now informally discuss the meaning of the rules in Table 1. Let R be either a PMO $R = \sqsubseteq$ or a PME $R = \sim$. Proposition 3.6 below will show that a word m is defined in R if and only if $m\ R\ m$ holds. Thus, rule $\langle d\rangle$ ensures that subwords of defined words are defined. Rule $\langle \epsilon\rangle$ ensures that ϵ is always defined (even when nothing else is). Rules $\langle l, r\rangle$ ensure that words that are related to other words by R are defined. Rule $\langle t\rangle$ ensures that R is transitive and rule $\langle s\rangle$ that it is symmetric. Rule $\langle c\rangle$ states that word composition should be monotonic with respect to the relation R.

We will now give some derived rules, which will be more suitable for proving properties of PMOs/PMEs throughout this paper.

Proposition 3.5. The following rules $\langle p_l, p_r, e_l\rangle$ can be derived from rules $\langle l, r, d, c, t\rangle$, and rule $\langle e_r\rangle$ can be derived from rules $\langle s, d, c, t\rangle$. Hence PMOs and PMEs are closed under rules $\langle p_l, p_r, e_l\rangle$ and PMEs are closed under rule $\langle e_r\rangle$.

$$\dfrac{kx \mathrel{-\!\!\bullet\!\!-} y}{x \mathrel{-\!\!\bullet\!\!-} x}\langle p_l\rangle \qquad \dfrac{x \mathrel{-\!\!\bullet\!\!-} ky}{y \mathrel{-\!\!\bullet\!\!-} y}\langle p_r\rangle \qquad \dfrac{x \mathrel{-\!\!\bullet\!\!-} y \qquad yk \mathrel{-\!\!\bullet\!\!-} m}{xk \mathrel{-\!\!\bullet\!\!-} m}\langle e_l\rangle \qquad \dfrac{x \mathrel{-\!\!\bullet\!\!-} y \qquad m \mathrel{-\!\!\bullet\!\!-} yk}{m \mathrel{-\!\!\bullet\!\!-} xk}\langle e_r\rangle$$

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Proof. Rule $\langle p_l \rangle$ (respectively, $\langle p_r \rangle$) is a trivial combination of rules $\langle l, d \rangle$ (respectively, $\langle r, d \rangle$). For rules $\langle e_l \rangle$ and $\langle e_r \rangle$, we have the following two deduction trees:

$$
\dfrac{\dfrac{\dfrac{yk \mathrel{-\!\!\cdot\!\!-} m}{yk \mathrel{-\!\!\cdot\!\!-} yk}\,\langle l \rangle \qquad x \mathrel{-\!\!\cdot\!\!-} y}{xk \mathrel{-\!\!\cdot\!\!-} yk}\,\langle c \rangle \qquad yk \mathrel{-\!\!\cdot\!\!-} m}{xk \mathrel{-\!\!\cdot\!\!-} m}\,\langle t \rangle
\qquad\qquad
\dfrac{\dfrac{x \mathrel{-\!\!\cdot\!\!-} y \qquad \dfrac{m \mathrel{-\!\!\cdot\!\!-} yk}{yk \mathrel{-\!\!\cdot\!\!-} m}\,\langle s \rangle}{xk \mathrel{-\!\!\cdot\!\!-} m}\,\langle e_l \rangle}{m \mathrel{-\!\!\cdot\!\!-} xk}\,\langle s \rangle
$$

□

Rule $\langle p_l \rangle$ (respectively, $\langle p_r \rangle$) is a left (respectively, right) projection rule, which is a kind of generalised version of $\langle l \rangle$ (respectively, $\langle r \rangle$). Rules $\langle e_l \rangle$ and $\langle e_r \rangle$ express the capacity to exchange R-related subwords inside R-relations, either on the left (for PMOs and PMEs) or on the right (only for PMEs).

Proposition 3.6. Whether R is a PMO ($R = \sqsubseteq$) or a PME ($R = \sim$) over L, the identities $\mathscr{L}^R = \{x \in L^\star \mid x\ R\ x\}$ and $\mathsf{A}_R = \{l \in L \mid l\ R\ l\}$ hold.

Proof. We prove the properties for $R = \sqsubseteq$. The same properties will then hold with a PME $\sim$ because any PME is also a PMO. First, it is obvious that $\{x \in L^\star \mid x \sqsubseteq x\} \subseteq \mathscr{L}^{\sqsubseteq}$. For the converse, if $xm \sqsubseteq n$ (respectively, $m \sqsubseteq xn$), then $x \sqsubseteq x$ by rule $\langle p_l \rangle$ (respectively, rule $\langle p_r \rangle$). Hence, $\mathscr{L}^{\sqsubseteq} \subseteq \{x \in L^\star \mid x \sqsubseteq x\}$. As $\mathsf{A}_{\sqsubseteq} = \mathscr{L}^{\sqsubseteq} \cap L$, we get $\mathsf{A}_{\sqsubseteq} = \{l \in L \mid l \sqsubseteq l\}$. □

3.2. PMO/PME *based Kripke semantics*

In this section we introduce a Kripke interpretation of BI and BBI formulae based on PMOs and PMEs. The framework for BI and BBI is thus common and this facilitates the building of both semantic and proof-theoretic bridges between these logics.

Definition 3.7. A BI-*frame (respectively,* BBI-*frame)* is a triple $(L, R, \Vdash)$ where L is an alphabet, R is a PMO (respectively, PME) over L, and $\Vdash$ is a *forcing relation* $\Vdash \subseteq \mathscr{L}^R \times \mathsf{Var}$ that verifies the *monotonicity property*:

$$\forall X \in \mathsf{Var},\ \forall m, n \in \mathscr{L}^R,\ (m\ R\ n \wedge m \Vdash X) \Rightarrow n \Vdash X\,.$$

We extend the forcing relation to $\Vdash_R \subseteq \mathscr{L}^R \times \mathsf{BI}$ (respectively, $\mathscr{L}^R \times \mathsf{BBI}$) by induction on formulae:

$$
\begin{array}{llllll}
m \Vdash_R \mathsf{I} & \text{iff} & \epsilon\ R\ m & m \Vdash_R \neg A & \text{iff} & m \nVdash_R A \\
m \Vdash_R \bot & \text{iff} & \text{never} & m \Vdash_R A \vee B & \text{iff} & m \Vdash_R A \text{ or } m \Vdash_R B \\
m \Vdash_R \top & \text{iff} & \text{always} & m \Vdash_R A \wedge B & \text{iff} & m \Vdash_R A \text{ and } m \Vdash_R B
\end{array}
$$

$$
\begin{array}{lll}
m \Vdash_R A \to B & \text{iff} & \forall x \in \mathscr{L}^R,\ (m\ R\ x \text{ and } x \Vdash_R A) \Rightarrow x \Vdash_R B \\
m \Vdash_R A * B & \text{iff} & \exists x, y \in \mathscr{L}^R,\ xy\ R\ m \text{ and } x \Vdash_R A \text{ and } y \Vdash_R B \\
m \Vdash_R A \mathrel{-\!\!*} B & \text{iff} & \forall x, y \in \mathscr{L}^R,\ (xm\ R\ y \text{ and } x \Vdash_R A) \Rightarrow y \Vdash_R B\,.
\end{array}
$$

We may write $\Vdash$ for $\Vdash_R$ when the relation R is obvious from the context.

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Proposition 3.8. If R is a PMO (respectively, a PME), the extended relation $\Vdash_R \subseteq \mathscr{L}^{\sqsubseteq} \times \mathsf{BI}$ (respectively, $\Vdash_R \subseteq \mathscr{L}^{\sim} \times \mathsf{BBI}$) is monotonic.

Proof. Monotonicity holds when for any $F \in \mathsf{BI}$ (respectively, $F \in \mathsf{BBI}$) and any $m, n \in \mathscr{L}^R$, the condition $(m \; R \; n \wedge m \Vdash_R F) \Rightarrow n \Vdash_R F$ holds. Proving monotonicity by induction on the formula F is standard. When F is a logical variable, the monotonicity condition holds as a direct consequence of Definition 3.7. For the additive operators $\bot$, $\top$, $\vee$ and $\wedge$, the induction step is trivial. For operators I, $\rightarrow$, and $*$, the induction step involves the use of rule $\langle t \rangle$. For the operator $-\!\!*$, the induction step involves the use of rule $\langle e_l \rangle$. For the Boolean negation $\neg$, the induction step involves the use of rule $\langle s \rangle$, but as the operator $\neg$ only exists in BBI, the relation R is thus a PME, and hence it is closed under rule $\langle s \rangle$. $\square$

When R is a PME, symmetry (rule $\langle s \rangle$) ensures that the additive implication $\rightarrow$ is interpreted pointwise or classically.

Proposition 3.9. If $R = \sim$ is a PME, then for any $m \in \mathscr{L}^{\sim}$, we have $m \Vdash_{\sim} A \rightarrow B$ if and only if $m \Vdash_{\sim} A \Rightarrow m \Vdash_{\sim} B$.

Proof. Here we just write $\Vdash$ for $\Vdash_{\sim}$. As $\sim$ is a reflexive relation when restricted to $\mathscr{L}^{\sim}$, the *only if* part is trivial. For the *if part*, we use monotonicity. Let us suppose $m \Vdash A \Rightarrow m \Vdash B$. Let $m \sim n$ and $n \Vdash A$. Then by rule $\langle s \rangle$, $n \sim m$, hence by monotonicity, $m \Vdash A$, and thus $m \Vdash B$. By monotonicity again, $n \Vdash B$. So for any n such that $m \sim n$, we have $n \Vdash A \Rightarrow n \Vdash B$. Thus $m \Vdash A \rightarrow B$. $\square$

Definition 3.10. A formula $F \in \mathsf{BI}$ is *valid* in the BI-frame $(L, \sqsubseteq, \Vdash)$ if the relation $\epsilon \Vdash_{\sqsubseteq} F$ holds. A formula $F \in \mathsf{BBI}$ is *valid* in the BBI-frame $(L, \sim, \Vdash)$ if for every $m \in \mathscr{L}^{\sim}$ the relation $m \Vdash_{\sim} F$ holds.

As a complement, we will now briefly state the relation between $\epsilon \Vdash F$ and $\forall m \in \mathscr{L}^{\sim} \; m \Vdash F$ in BBI-frames.

Proposition 3.11. For any BBI-frame $(L, \sim, \Vdash)$ and any formula $F \in \mathsf{BBI}$:

1. $\epsilon \Vdash F$ if and only if $\forall m \in \mathscr{L}^{\sim} \; m \Vdash \mathsf{I} \rightarrow F$.
2. $\forall m \in \mathscr{L}^{\sim} \; m \Vdash F$ if and only if $\epsilon \Vdash \top -\!\!* F$.

Proof. For property 1, the *if part* is trivial since $\epsilon \in \mathscr{L}^{\sim}$ and $\epsilon\epsilon \sim \epsilon$. For the *only if part*, we use Proposition 3.9. Let $m \in \mathscr{L}^{\sim}$ such that $m \Vdash \mathsf{I}$. Then $\epsilon \sim m$. As $\epsilon \Vdash F$, by monotonicity we obtain $m \Vdash F$. Hence, $m \Vdash \mathsf{I} \rightarrow F$. For property 2, the *only if part* is trivial, and for the *if part*, we just use the fact that $m\epsilon \sim m$ for any $m \in \mathscr{L}^{\sim}$. $\square$

Note that property 2 also holds for BI but the *only if part* of property 1 does not. It is also worth noting that property 2 is used in Calcagno *et al.* (2005) to establish an equivalence between validity and satisfaction problems in the spatial logic for trees.

Theorem 3.12 (Completeness of PMOs with respect to BI). A formula F of BI is valid in every partially ordered partial monoid Kripke structure if and only if it is valid in every BI-frame.

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Proof. This is an obvious but tedious proof based on quotients by equivalence relations. We will just give a brief sketch of the proof and leave the details to the reader. We first prove that if a formula of BI has a Kripke counter-model in the form of a partially ordered partial monoid, then it has a counter-model in the form of a BI-frame. Then we will prove the converse result.

Let us consider a partially ordered partial monoid Kripke structure $(\mathscr{M}, \circ, \mathsf{e}, \triangleright, \Vdash_{\circ,\triangleright})$. We take the elements of $\mathscr{M}$ as letters of the language $L = \mathscr{M}$. A word of k letters $m = m_1 \ldots m_k \in L^\star$ is (partially) mapped to an element $m_\circ = m_1 \circ \cdots \circ m_k \in \mathscr{M}$. The fact that $m_\circ$ is defined, and its value if it is, do not depend on the order in which we perform the compositions of the letters of m because of the associativity and commutativity axioms of partial monoids. By definition, for the empty word, $\epsilon_\circ$ is defined and $\epsilon_\circ = \mathsf{e}$. The binary relation $\sqsubseteq$ over $L^\star$ defined by $m \sqsubseteq n$ if and only if $m_\circ\!\downarrow \wedge\, n_\circ\!\downarrow \wedge\, m_\circ \triangleright n_\circ$ is thus a PMO over L. We define the BI-frame $(L, \sqsubseteq, \Vdash_\sqsubseteq)$ by $m \Vdash_\sqsubseteq X$ if and only if $m_\circ\!\downarrow \wedge\, m_\circ \Vdash_{\circ,\triangleright} X$. Then, by straightforward induction on BI formulae, it is possible to prove that $m \Vdash_\sqsubseteq F$ if and only if $m_\circ\!\downarrow \wedge\, m_\circ \Vdash_{\circ,\triangleright} F$. As $\epsilon_\circ = \mathsf{e}$, if $\mathscr{M}$ is a Kripke counter-model of the formula F (that is, $\mathsf{e} \nVdash_{\circ,\triangleright} F$), then $(L, \sqsubseteq, \Vdash_\sqsubseteq)$ is also counter-model of the F (that is, $\epsilon \nVdash_\sqsubseteq F$).

For the converse, we consider a BI-frame $(L, \sqsubseteq, \Vdash_\sqsubseteq)$. We define the partial equivalence relation on $L^\star$ by $\sim\, = \,\sqsubseteq \cap \sqsubseteq^{-1}$, that is, $m \sim n$ if and only if $m \sqsubseteq n \wedge n \sqsubseteq m$. We should emphasise here that $\sim$ is not necessarily reflexive. Then we define $\mathscr{M}$ as the set of partial equivalence classes of $L^\star$. Let $[m] = \{x \in L^\star \mid m \sim x\}$ and $\mathscr{M} = \{[m] \mid m \in L^\star \wedge [m] \neq \emptyset\}$. The unit e is the class $[\epsilon]$, which is not empty by rule $\langle\epsilon\rangle$. The partial composition $\circ$ is defined by $[m]\circ[n] = [mn]$ and the partial order $\triangleright$ is defined by $[m] \triangleright [n]$ if and only if $m \sqsubseteq n$. Then $(\mathscr{M}, \circ, \mathsf{e}, \triangleright)$ is a partially ordered partial monoid. We define the Kripke structure $(\mathscr{M}, \circ, \mathsf{e}, \triangleright, \Vdash_{\circ,\triangleright})$ by $[m] \Vdash_{\circ,\triangleright} X$ if and only if $m \Vdash_\sqsubseteq X$. By straightforward induction on BI formulae, it is possible to show that $[m] \Vdash_{\circ,\triangleright} F$ if and only if $m \in \mathscr{L}^\sqsubseteq \wedge m \Vdash_\sqsubseteq F$. Hence, if $(L, \sqsubseteq, \Vdash_\sqsubseteq)$ is a counter-model of the formula F (that is, $\epsilon \nVdash_\sqsubseteq F$), then $\mathscr{M}$ is a counter-model of F (that is, $[\epsilon] \nVdash_{\circ,\triangleright} F$). □

Theorem 3.13 (Completeness of PMEs with respect to BBI). A formula F of BBI is valid in every partial monoid Kripke structure if and only if it is valid in every BBI-frame.

Proof. This is a straightforward adaptation to PMEs and BBI of the previous proof. □

According to these two theorems, we can define *universal validity* and *counter-models*. A BI-counter-model for $F \in$ BI is a BI-frame in which $\epsilon \nVdash_\sqsubseteq F$. A BI formula F is *universally valid* (or BI-*valid*) when it has no BI-counter-model. A BBI-counter-model for $F \in$ BBI is a BBI-frame in which there exists $m \in \mathscr{L}^\sim$ such that $m \nVdash_\sim F$. F is *universally valid* (or BBI-*valid*) when it has no BBI-counter-model.

3.3. *Sets of constraints and other properties of* PMOs/ PMEs

Being defined by closure under some deduction rules, the classes of PMOs and PMEs are thus closed under arbitrary intersection. Thus, given a binary relation R between words described by a set of constraints, there exists a least PMO (respectively, PME) containing

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R. We are especially interested in PMOs/PMEs generated by some finite or infinite set of constraints.

Definition 3.14. Let L be an alphabet and $\mathscr{C}$ be a set of constraints over the alphabet L[†]. The PMO *generated by* $\mathscr{C}$ is the least PMO, denoted $\sqsubseteq_{\mathscr{C}}$, such that the inclusion $\mathscr{C} \subseteq \sqsubseteq_{\mathscr{C}}$ holds between those two sets of constraints. We also use $\sim_{\mathscr{C}}$ to denote the PME *generated by* $\mathscr{C}$ that is the least PME such that $\mathscr{C} \subseteq \sim_{\mathscr{C}}$.

For example, the PMO $\sqsubseteq_0$ generated by the singleton constraint $\mathscr{C}_0 = \{\epsilon - a\}$ is $\sqsubseteq_0 = \{\epsilon - \epsilon, \epsilon - a, a - a\}$, whereas the PMO $\sqsubseteq_1$ generated by the singleton constraint $\mathscr{C}_1 = \{a - \epsilon\}$ is $\sqsubseteq_1 = \{a^i - a^j \mid i \geqslant j\}$. The PME generated by the singleton constraint $\mathscr{C}_0 = \{\epsilon - a\}$ is $\sim_0 = \{a^i - a^j \mid i, j \in \mathbb{N}\}$. The proofs of these statements are left to the reader. By rule $\langle s \rangle$, $\sim_0$ is also the PME generated by the singleton constraint $\mathscr{C}_1 = \{a - \epsilon\}$. Obviously, $\sqsubseteq_0 \subset \sim_0$ and $\sqsubseteq_1 \subset \sim_0$, and the inclusion is strict.

Considering two sets of constraints $\mathscr{C} \subseteq \mathscr{D}$, we have $\sqsubseteq_{\mathscr{C}} \subseteq \sqsubseteq_{\mathscr{D}}$ and $\sim_{\mathscr{C}} \subseteq \sim_{\mathscr{D}}$. Also, $\sqsubseteq_{\mathscr{C}} \subseteq \sim_{\mathscr{C}}$ because $\sim_{\mathscr{C}}$ contains $\mathscr{C}$ as a subset and, being a PME, is also a PMO.

Definition 3.15. Let $R = \sqsubseteq$ (respectively, $R = \sim$) be a PMO (respectively, PME). Let $\mathscr{C}$ be a set of constraints. We denote by $R + \mathscr{C}$ the *extension* of R by the constraints of $\mathscr{C}$ that is the least PMO (respectively, PME) containing $R \cup \mathscr{C}$.

We should emphasise the fact that the meaning of the extension $R + \mathscr{C}$ depends on whether R is viewed as a PMO or a PME, especially since PMEs are also PMOs. Let R be a PMO or a PME over the alphabet L, and $\mathscr{C}_1$ and $\mathscr{C}_2$ be two sets of constraints over L. Then $(R + \mathscr{C}_1) + \mathscr{C}_2 = (R + \mathscr{C}_2) + \mathscr{C}_1 = R + (\mathscr{C}_1 \cup \mathscr{C}_2)$. These identities hold for both PMO and PME extensions, and their proofs are trivial and left to the reader. Moreover, for any $m, n \in L^\star$, the relation $m \; R \; n$ holds if and only if the identity $R + \{m - n\} = R$ holds; in particular, $R + \{\epsilon - \epsilon\} = R$.

Proposition 3.16. If $\mathscr{C}$ is a set of constraints over L, then the inclusion $\sqsubseteq_{\mathscr{C}} \subseteq \sim_{\mathscr{C}}$ and the identity $\mathsf{A}_{\mathscr{C}} = \mathsf{A}_{\sqsubseteq_{\mathscr{C}}} = \mathsf{A}_{\sim_{\mathscr{C}}}$ hold.

Proof. For the first property, $\sim_{\mathscr{C}}$ is a PME containing $\mathscr{C}$, and hence also a PMO containing $\mathscr{C}$. For the second property, as $\mathscr{C} \subseteq \sqsubseteq_{\mathscr{C}} \subseteq \sim_{\mathscr{C}}$ as relations, we derive $\mathsf{A}_{\mathscr{C}} \subseteq \mathsf{A}_{\sqsubseteq_{\mathscr{C}}} \subseteq \mathsf{A}_{\sim_{\mathscr{C}}}$. Thus, it is sufficient to prove that $\mathsf{A}_{\sim_{\mathscr{C}}} \subseteq \mathsf{A}_{\mathscr{C}}$. Let $\sim$ be defined by $m \sim n$ if and only if $m, n \in \mathsf{A}_{\mathscr{C}}^\star$. Then, $\mathsf{A}_{\sim} = \mathsf{A}_{\mathscr{C}}$ and $\mathscr{C} \subseteq \sim$, and $\sim$ is a PME. Hence, $\sim_{\mathscr{C}} \subseteq \sim$, and thus $\mathsf{A}_{\sim_{\mathscr{C}}} \subseteq \mathsf{A}_{\sim} = \mathsf{A}_{\mathscr{C}}$. □

Proposition 3.17 (Compactness). Let $\mathscr{C}$ be a possibly infinite set of constraints over the alphabet L. Let $m, n \in L^\star$ be such that $m \sqsubseteq_{\mathscr{C}} n$ (respectively, $m \sim_{\mathscr{C}} n$) holds. There exists a finite subset $\mathscr{C}_f \subseteq \mathscr{C}$ such that $m \sqsubseteq_{\mathscr{C}_f} n$ (respectively, $m \sim_{\mathscr{C}_f} n$) holds.

Proof. We will just do the proof for PMOs; the proof for PMEs is similar. Let $\mathscr{C}$ be a set of constraints. Let the relation R_f be (the finite approximation of $\sqsubseteq_{\mathscr{C}}$) defined by

[†] As $\mathscr{C}$ is also a relation, the alphabet of $\mathscr{C}$ is $\mathsf{A}_{\mathscr{C}} = \bigcup\{\mathsf{A}_m \cup \mathsf{A}_n \mid m - n \in \mathscr{C}\}$, that is, the set of letters occurring in at least one of the constraints of $\mathscr{C}$.

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$m\ R_f\ n$ if and only if there exists $\mathscr{D} \subseteq \mathscr{C}$ such that $\mathscr{D}$ is finite and $m \sqsubseteq_{\mathscr{D}} n$. Obviously, $\mathscr{C} \subseteq R_f \subseteq \sqsubseteq_{\mathscr{C}}$. If we can show that R_f is a PMO, we get $R_f = \sqsubseteq_{\mathscr{C}}$, which proves the proposition.

So we will now prove that R_f is a PMO. For rule $\langle\epsilon\rangle$, we have $\epsilon \sqsubseteq_{\varnothing} \epsilon$ and $\varnothing$ is a finite subset of $\mathscr{C}$, hence $\epsilon\ R_f\ \epsilon$. Thus R_f is closed under rule $\langle\epsilon\rangle$. If $x\ R_f\ y$, then, for some finite subset $\mathscr{D}$ of $\mathscr{C}$, we have $x \sqsubseteq_{\mathscr{D}} y$. Then, as $\sqsubseteq_{\mathscr{D}}$ is a PMO, we have $x \sqsubseteq_{\mathscr{D}} x$ by rule $\langle l\rangle$ and $y \sqsubseteq_{\mathscr{D}} y$ by rule $\langle r\rangle$. Hence $x\ R_f\ x$ and $y\ R_f\ y$, and R_f is closed under rules $\langle l\rangle$ and $\langle r\rangle$. The same reasoning applies to the unary rule $\langle d\rangle$. We will now consider the binary rule $\langle t\rangle$. If $x\ R_f\ y$ and $y\ R_f\ z$, there exist $\mathscr{D}$ and $\mathscr{E}$ that are finite subsets of $\mathscr{C}$ such that $x \sqsubseteq_{\mathscr{D}} y$ and $y \sqsubseteq_{\mathscr{E}} z$. Let $\mathscr{F} = \mathscr{D} \cup \mathscr{E}$. Then $\mathscr{F}$ is a finite subset of $\mathscr{C}$, and $x \sqsubseteq_{\mathscr{F}} y$ and $y \sqsubseteq_{\mathscr{F}} z$ because $\sqsubseteq_{\mathscr{D}} \subseteq \sqsubseteq_{\mathscr{F}}$ and $\sqsubseteq_{\mathscr{E}} \subseteq \sqsubseteq_{\mathscr{F}}$. Thus $x \sqsubseteq_{\mathscr{F}} z$ since $\sqsubseteq_{\mathscr{F}}$ is closed under rule $\langle t\rangle$, so $x\ R_f\ z$. Hence the relation R_f is closed under rule $\langle t\rangle$. The same reasoning applies to the other binary rule $\langle c\rangle$, so the relation R_f is indeed a PMO. □

This *compactness property* is not related to the particular nature of rules defining PMOs or PMEs but due solely to the fact that these rules only have a *finite number of premises*. Apart from generating PMOs/PMEs from sets of constraints, we will provide another basic way to build them from sub-alphabets and derive an interesting property from it.

Proposition 3.18. Let L be an alphabet and $X \subseteq L$ be a subset of L. Then:

1. $\sqsubseteq^X$ is a PMO over L where $\sqsubseteq^X$ is defined by $m \sqsubseteq^X n$ iff $(n \in X^\star \Rightarrow m \in X^\star)$.
2. $\sim^X$ is a PME over L where $\sim^X$ is defined by $m \sim^X n$ iff $(n \in X^\star \Leftrightarrow m \in X^\star)$.

Proof. The relation $\sqsubseteq^X$ is reflexive because the meta-logical implication $\Rightarrow$ of its definition is reflexive. So it is obviously closed under rules $\langle\epsilon, l, r, d\rangle$. It is also transitive because the meta-logical implication is transitive, hence $\sqsubseteq^X$ is closed under rule $\langle t\rangle$.

As $mn \in X^\star$ if and only if $m, n \in X^\star$, $\sqsubseteq^X$ is closed under rule $\langle c\rangle$: indeed, let $m, n \in L^\star$ be such that $m \sqsubseteq^X n$. If $kn \in X^\star$, then $k, n \in X^\star$. As $n \in X^\star$ and $m \sqsubseteq^X n$, we deduce $m \in X^\star$. Hence, $k, m \in X^\star$ and thus $km \in X^\star$.

The inverse relation $(\sqsubseteq^X)^{-1}$ is also a PMO for identical reasons. So $\sim^X = \sqsubseteq^X \cap (\sqsubseteq^X)^{-1}$ is a PMO as it is the intersection of two PMOs. The relation $R \cap R^{-1}$ is always symmetric, and hence closed under rule $\langle s\rangle$. So $\sim^X$ is closed under rule $\langle s\rangle$, and hence a PME. □

Proposition 3.19. Let $\mathscr{C}$ be a set of constraints on the alphabet L.

1. If no constraint of $\mathscr{C}$ is of the form $m - \epsilon$ (with $m \neq \epsilon$), then for any $m \in L^\star$, $m \sqsubseteq_{\mathscr{C}} \epsilon$ only if $m = \epsilon$.
2. If no constraint of $\mathscr{C}$ is of the form $m - \epsilon$ or $\epsilon - m$ (with $m \neq \epsilon$), then for any $m \in L^\star$, $m \sim_{\mathscr{C}} \epsilon$ only if $m = \epsilon$.

Proof. We use Proposition 3.18 in the particular case where $X = \varnothing$ is the empty sub-alphabet. For property 1, we first prove that $\mathscr{C} \subseteq \sqsubseteq^{\varnothing}$. Let $m, n \in L^\star$ be such that $m - n \in \mathscr{C}$. We will prove $m \sqsubseteq^{\varnothing} n$. As $\varnothing^\star = \{\epsilon\}$, if $n \in \varnothing^\star$, then $n = \epsilon$, so $m - \epsilon \in \mathscr{C}$. By the hypothesis on $\mathscr{C}$, we must have $m = \epsilon$ and as a consequence $m \in \varnothing^\star$. So $m \sqsubseteq^{\varnothing} n$ and we have proved that $\mathscr{C} \subseteq \sqsubseteq^{\varnothing}$. As $\sqsubseteq_{\mathscr{C}}$ is the least PMO containing $\mathscr{C}$, and $\sqsubseteq^{\varnothing}$ is a PMO by property 1 of Proposition 3.18, we have $\sqsubseteq_{\mathscr{C}} \subseteq \sqsubseteq^{\varnothing}$. Now we consider $m \in L^\star$ such that

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$m \sqsubseteq_{\mathscr{C}} \epsilon$, and deduce $m \sqsubseteq^{\varnothing} \epsilon$. As $\epsilon \in \varnothing^\star$, we must have $m \in \varnothing^\star$ by definition of $\sqsubseteq^{\varnothing}$. Thus we obtain $m = \epsilon$.

For property 2, the reasoning is similar: we prove the inclusion $\mathscr{C} \subseteq \sim^{\varnothing}$, from which we deduce $\sim_{\mathscr{C}} \subseteq \sim^{\varnothing}$. Thus no $m \in L^\star$ such that $m \sim_{\mathscr{C}} \epsilon$ can exist unless $m = \epsilon$. □

For the moment, we will pause our investigation into the different methods of building PMOs and PMEs. We will come back to it later when we will describe precisely how to compute 'freely generated' PMEs in Section 6.3, but we now have enough material to present the first building block of our embedding of BI into BBI.

4. Linking PMOs/PMEs and the Kripke semantics of BI and BBI

In this section we define the relation between PMOs and PMEs lying at the core of the embedding of BI into BBI. We begin by providing an intuition into the design of this relation between PMOs and PMEs.

The basic idea can be viewed as a variant of the embedding of intuitionistic SL into classical SL. Indeed, in Ishtiaq and O'Hearn (2001), the order relation chosen to interpret implication intuitionistically is *graph inclusion* $\subseteq$ between *heaps*:

$$m \Vdash A \to B \quad \text{iff} \quad \forall h' \; (h \subseteq h' \wedge h' \Vdash A) \Rightarrow h' \Vdash B\,.$$

But, since composition is the disjoint union of graphs, we have the relation

$$h \subseteq h' \quad \text{iff} \quad \exists g,\; g \cdot h = h'\,.$$

Thus h is below h' if it is possible to compose h with something (g here) to give h'. In other words, h' is identical to h up to some unspecified part (g here). We generalise this idea by restricting the choice of the missing part g to a space of heaps, which might be disjoint from the space of observable heaps, observable meaning observable through the Kripke semantics.

4.1. *Building* PMO*s with* PME*s*

In our words and constraints based semantics, heaps are abstracted by words. To distinguish observable words from potentially unobservable words, we divide the alphabet L' into two sub-alphabets L and K, which may be disjoint, $L^\star$ representing observable words and $K^\star$ unobservable words.

Definition 4.1. Let $\sim$ be a PME over L' and L, K be two subsets of L', that is, $L \cup K \subseteq L'$. We define the relation $\sqsubseteq^{L,K}_{\sim} \subseteq L^\star \times L^\star$ by

$$m \sqsubseteq^{L,K}_{\sim} n \quad \text{iff} \quad \exists \delta \in K^\star,\; \delta m \sim n\,.$$

Thus, m is below n if m can be completed into n by some unobservable part δ. Then we prove that the relation $\exists \delta \in K^\star, \delta m \sim n$ defines a PMO over L provided the relation $\sim$ is a PME over L'. Clearly, if $\sim$ and $\sim'$ are two PMEs over L' such that $\sim \subseteq \sim'$, then $\sqsubseteq^{L,K}_{\sim} \subseteq \sqsubseteq^{L,K}_{\sim'}$.

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Lemma 4.2. If $\sim$ is a PME over L', then the relation $\sqsubseteq = \sqsubseteq_{\sim}^{L,K}$ is a PMO over L, and the identities $\mathsf{A}_{\sqsubseteq} = \mathsf{A}_{\sim} \cap L$ and $\mathscr{L}^{\sqsubseteq} = \mathscr{L}^{\sim} \cap L^{\star}$ hold.

Proof. First, note that for any $m \in L^{\star}$, $m \sqsubseteq m$ if and only if $m \sim m$: indeed, $\delta m \sim m$ implies $m \sim m$ by rule $\langle r \rangle$. Thus, by Proposition 3.6, $\mathscr{L}^{\sqsubseteq} = \mathscr{L}^{\sim} \cap L^{\star}$ and $\mathsf{A}_{\sqsubseteq} = \mathsf{A}_{\sim} \cap L$. We now prove that $\sqsubseteq$ is a PMO.

It is obvious that $\sqsubseteq$ is closed under rules $\langle \epsilon \rangle$ and $\langle d \rangle$ since $m \sqsubseteq m$ if and only if $m \sim m$ and $\sim$ is a PME. If $x \sqsubseteq y$, then $\delta x \sim y$ for some $\delta \in K^{\star}$. Then $x \sim x$ by rule $\langle p_l \rangle$, and thus $x \sqsubseteq x$. Hence $\sqsubseteq$ is closed under rule $\langle l \rangle$. Using rule $\langle p_r \rangle$ for $\sim$, we can also show that $\sqsubseteq$ is closed under rule $\langle r \rangle$. Now consider rule $\langle t \rangle$. If $x \sqsubseteq y$ and $y \sqsubseteq z$, then $\delta x \sim y$ and $\delta' y \sim z$ for some $\delta, \delta' \in K^{\star}$. Then $\delta\delta' \in K^{\star}$ and $\delta\delta' x \sim z$ by application of rule $\langle e_l \rangle$, hence $x \sqsubseteq z$. So $\sqsubseteq$ is closed under rule $\langle t \rangle$. Now consider rule $\langle c \rangle$. If $qy \sqsubseteq qy$ and $x \sqsubseteq y$, then $qy \sim qy$ and $\delta x \sim y$ for some $\delta \in K^{\star}$. By rule $\langle c \rangle$, we get $\delta qx \sim qy$. Then we have $qx \sqsubseteq qy$. So $\sqsubseteq$ is closed under rule $\langle c \rangle$. $\square$

Since we have a way to build PMOs starting from PMEs, several questions arise. For example, is this construction process general enough to represent any PMO, that is, is the map $(\sim, L, K) \mapsto \sqsubseteq_{\sim}^{L,K}$ surjective? Is it semantically compatible with some embedding of BI into BBI, that is, does it preserve Kripke semantics? We will answer the second question first.

In the next two sections, we present a map $F \mapsto F^{\circ}$ from BI-formulae to BBI-formulae and show that this map preserves the Kripke semantics provided the PMO $\sqsubseteq = \sqsubseteq_{\sim}^{L,K}$ comes from a PME $\sim$.

4.2. *An intuitive description of the map of* BI*-formulae to* BBI*-formulae*

Before we introduce the map, we single out two logical variables L and K that behave as the syntactical counterpart of the distinction between observable and unobservable words. We point out that we have intentionally chosen to give the two variables L and K the same name (that is, the same letters) as the sub-alphabets L and K that occur in the definition of $\sqsubseteq_{\sim}^{L,K}$, and use the choice of font L/L and K/K to distinguish them. The link between the (semantic) set of observable words and the (syntactic) variable L is enforced by choosing $\Vdash_{\sim}$ such that $x \in L^{\star}$ if and only if $x \Vdash_{\sim} \mathsf{L}$. The same holds for K/K, that is, $x \in K^{\star}$ if and only if $x \Vdash_{\sim} \mathsf{K}$ holds for any word x.

We will now give an informal explanation of the idea lying behind the encoding of BI into BBI and its link between observable and unobservable words. Suppose that the PMO $\sqsubseteq$ is of the form $\sqsubseteq = \sqsubseteq_{\sim}^{L,K}$ for some PME $\sim$. The Kripke interpretation of $m \Vdash_{\sqsubseteq} A * B$ is thus

$$\exists a, b \in L^{\star},\ ab \sqsubseteq m \wedge a \Vdash_{\sqsubseteq} A \wedge b \Vdash_{\sqsubseteq} B\,.$$

Using $\sqsubseteq = \sqsubseteq_{\sim}^{L,K}$, we transform this formula into

$$\exists a, b \in L^{\star}, \exists \delta \in K^{\star},\ \delta ab \sim m \wedge a \Vdash_{\sqsubseteq} A \wedge b \Vdash_{\sqsubseteq} B\,.$$

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As we have chosen to encode the set $L^\star$ with the logical variable L in the Kripke semantics $\Vdash_\sim$, we can replace $a \in L^\star$ by $a \Vdash_\sim \mathsf{L}$ (and similarly for $b/L/\mathsf{L}$ and $\delta/K/\mathsf{K}$) to give

$$\exists \delta, a, b,\ \delta ab \sim m \wedge \delta \Vdash_\sim \mathsf{K} \wedge a \Vdash_\sim \mathsf{L} \wedge a \Vdash_\sqsubseteq A \wedge b \Vdash_\sim \mathsf{L} \wedge b \Vdash_\sqsubseteq B\,.$$

Now suppose (recursively) that there are two BBI-formulae A° and B° such that for any $x \in \mathscr{L}^\sqsubseteq$, $x \Vdash_\sqsubseteq A$ if and only if $x \Vdash_\sim A^\circ$ and $x \Vdash_\sqsubseteq B$ if and only if $x \Vdash_\sim B^\circ$. Replacing $a \Vdash_\sqsubseteq A$ by $a \Vdash_\sim A^\circ$ and $b \Vdash_\sqsubseteq B$ by $b \Vdash_\sim B^\circ$, we get

$$\exists \delta, a, b,\ \delta ab \sim m \wedge \delta \Vdash_\sim \mathsf{K} \wedge a \Vdash_\sim \mathsf{L} \wedge a \Vdash_\sim A^\circ \wedge b \Vdash_\sim \mathsf{L} \wedge b \Vdash_\sim B^\circ\,,$$

or, in other words,

$$m \Vdash_\sim \mathsf{K} * (\mathsf{L} \wedge A^\circ) * (\mathsf{L} \wedge B^\circ)\,.$$

We see that we have to coerce a and b to range over observable words in $L^\star$ by stating $a \Vdash_\sim \mathsf{L}$ and $b \Vdash_\sim \mathsf{L}$ ($L^\star$ are the words on which the BI formula $A * B$ is Kripke-interpreted), whereas we coerce δ to range over unobservable words in $K^\star$ by stating $\delta \Vdash_\sim \mathsf{K}$.

4.3. *Formal definition of the embedding map of* BI*-formulae to* BBI*-formulae*

In this section we formalise the ideas of the previous section in the form of the recursively defined map $F \mapsto F^\circ$. The formulae to which the map $(\cdot)^\circ$ is applied should not contain occurrences of either L or K to enforce the distinction between observable and unobservable words.

Definition 4.3 (Embedding map). Let L and K be two different spare logical variables in Var. Given $F \in \mathsf{BI}$ containing neither L nor K, we define by induction on F the formula $F^\circ \in \mathsf{BBI}$ as follows:

$$\begin{array}{ll} X^\circ = \mathsf{K} * X \text{ for } X \in \mathsf{Var}\backslash\{\mathsf{L},\mathsf{K}\} & \mathsf{I}^\circ = \mathsf{K} * \mathsf{I} \quad \bot^\circ = \bot \quad \top^\circ = \top \\ (A \oplus B)^\circ = A^\circ \oplus B^\circ \text{ for } \oplus \in \{\wedge, \vee\} & (A * B)^\circ = \mathsf{K} * \big((\mathsf{L} \wedge A^\circ) * (\mathsf{L} \wedge B^\circ)\big) \\ (A \to B)^\circ = \mathsf{K} \mathrel{-\!\!*} \big((\mathsf{L} \wedge A^\circ) \to B^\circ\big) & (A \mathrel{-\!\!*} B)^\circ = \big(\mathsf{K} * (\mathsf{L} \wedge A^\circ)\big) \mathrel{-\!\!*} (\mathsf{L} \to B^\circ)\,. \end{array}$$

In fact, the formula F° also belongs to the language of BI, but it will be interpreted as a BBI-formula, not as a BI-formula. We now state and prove that the map $(\cdot)^\circ$ preserves the Kripke semantics provided the PMO is of the form $\sqsubseteq_\sim^{L,K}$.

Theorem 4.4. Let $\sim$ be a PME over L' and $L \cup K \subseteq L'$. Let $(L, \sqsubseteq, \Vdash_\sqsubseteq)$ be a BI-frame on $\mathsf{Var}\backslash\{\mathsf{L},\mathsf{K}\}$ such that $\sqsubseteq\, = \,\sqsubseteq_\sim^{L,K}$. We define the relation $\Vdash_\sim\, \subseteq \mathscr{L}^\sim \times \mathsf{Var}$ by

$$\begin{array}{lll} m \Vdash_\sim X & \text{iff} & \exists l \in L^\star,\ l \sim m \wedge l \Vdash_\sqsubseteq X \qquad \text{for } X \in \mathsf{Var}\backslash\{\mathsf{L},\mathsf{K}\} \\ m \Vdash_\sim \mathsf{K} & \text{iff} & \exists \delta \in K^\star,\ \delta \sim m \\ m \Vdash_\sim \mathsf{L} & \text{iff} & \exists l \in L^\star,\ l \sim m\,. \end{array}$$

Then $(L', \sim, \Vdash_\sim)$ is a BBI-frame on Var, and for any formula F of BI containing neither K nor L, and for any $m \in \mathscr{L}^\sqsubseteq$, the equivalence $m \Vdash_\sqsubseteq F$ if and only if $m \Vdash_\sim F^\circ$ holds.

Proof. Recall that $\mathscr{L}^\sqsubseteq = \mathscr{L}^\sim \cap L^\star$ (see Lemma 4.2). We first prove that $(L', \sim, \Vdash_\sim)$ is a BBI-frame, that is, that $\Vdash_\sim$ is monotonic. Let $m, n \in \mathscr{L}^\sim$ be such that $m \sim n$. If

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$X \in \mathsf{Var}\backslash\{\mathsf{L},\mathsf{K}\}$ and $m \Vdash_{\sim} X$, there exists $l \in L^\star$ such that $l \sim m$ and $l \Vdash_{\sqsubseteq} X$. By rule $\langle t \rangle$, we obtain $l \sim n$, and thus $n \Vdash_{\sim} X$. If $m \Vdash_{\sim} \mathsf{K}$, there exists $\delta \in K^\star$ such that $\delta \sim m$. Then $\delta \sim n$, and hence $n \Vdash_{\sim} \mathsf{K}$. If $m \Vdash_{\sim} \mathsf{L}$, there exists $l \in L^\star$ such that $l \sim m$. Then $l \sim n$, and hence $n \Vdash_{\sim} \mathsf{L}$. So $\Vdash_{\sim}$ is indeed monotonic.

We prove the equivalence of semantic interpretations by induction on F. The cases for F of the form $\bot$, $\top$, $A \wedge B$ or $A \vee B$ are trivial because the Kripke interpretations are obviously the same. We will just list the non-trivial cases:

— $F = X$ with $X \in \mathsf{Var}\backslash\{\mathsf{L},\mathsf{K}\}$
Let $m \in \mathscr{L}^{\sqsubseteq}$. On the one hand, if $m \Vdash_{\sqsubseteq} X$, then, as $m \in L^\star$ and $m \sim m$, we have $m \Vdash_{\sim} X$. As $\epsilon \Vdash_{\sim} \mathsf{K}$ and $\epsilon m \sim m$, we obtain $m \Vdash_{\sim} \mathsf{K} * X$, and thus $m \Vdash_{\sim} X^\circ$. On the other hand, if $m \Vdash_{\sim} \mathsf{K} * X$, there exist x, y such that $xy \sim m$, $x \Vdash_{\sim} \mathsf{K}$ and $y \Vdash_{\sim} X$. Hence there exists $\delta \in K^\star$ such that $\delta \sim x$, and there exists $l \in L^\star$ such that $l \sim y$ and $l \Vdash_{\sqsubseteq} X$. Then, by two applications of rule $\langle e_l \rangle$, $\delta l \sim m$, so $l \sqsubseteq m$. As $l \Vdash_{\sqsubseteq} X$ and $\Vdash_{\sqsubseteq}$ is monotonic, we get $m \Vdash_{\sqsubseteq} X$.

— $F = \mathsf{I}$
We have $m \Vdash_{\sim} \mathsf{K} * \mathsf{I}$ if and only if $m \Vdash_{\sim} \mathsf{K}$ if and only if $\exists \delta \in K^\star\ \delta \sim m$ if and only if $\epsilon \sqsubseteq m$ if and only if $m \Vdash_{\sqsubseteq} \mathsf{I}$.

— $F = A \to B$
We suppose $m \Vdash_{\sqsubseteq} A \to B$ with $m \in \mathscr{L}^{\sqsubseteq}$ and prove $m \Vdash_{\sim} \mathsf{K} -\!\!* ((\mathsf{L} \wedge A^\circ) \to B^\circ)$. Let k, c be such that $km \sim c$, $k \Vdash_{\sim} \mathsf{K}$ and $c \Vdash_{\sim} L \wedge A^\circ$. We will prove $c \Vdash_{\sim} B^\circ$. From $k \Vdash_{\sim} \mathsf{K}$, we obtain $\delta \in K^\star$ such that $\delta \sim k$. From $c \Vdash_{\sim} \mathsf{L}$, we get $l \in L^\star$ such that $l \sim c$. By the monotonicity of $\Vdash_{\sim}$ and rule $\langle s \rangle$, and from $c \Vdash_{\sim} A^\circ$, we deduce $l \Vdash_{\sim} A^\circ$, and hence $l \Vdash_{\sqsubseteq} A$ by the induction hypothesis. As $\delta m \sim l$, and thus $m \sqsubseteq l$, and $m \Vdash_{\sqsubseteq} A \to B$, we deduce $l \Vdash_{\sqsubseteq} B$, and hence, again by the induction hypothesis, $l \Vdash_{\sim} B^\circ$. As $l \sim c$, by the monotonicity of $\Vdash_{\sim}$, we obtain $c \Vdash_{\sim} B^\circ$. So, using $m \Vdash_{\sqsubseteq} A \to B$, we deduce $m \Vdash_{\sim} (A \to B)^\circ$.
Now we prove the converse implication. Suppose $m \Vdash_{\sim} \mathsf{K} -\!\!* ((\mathsf{L} \wedge A^\circ) \to B^\circ)$. Let $x \in \mathscr{L}^{\sqsubseteq}$ be such that $m \sqsubseteq x$ and $x \Vdash_{\sqsubseteq} A$. We will prove $x \Vdash_{\sqsubseteq} B$. From $x \Vdash_{\sqsubseteq} A$, we can deduce $x \Vdash_{\sim} A^\circ$ by the induction hypothesis. As $x \in L^\star$ and $x \sim x$ ($x \in \mathscr{L}^{\sqsubseteq} \subseteq \mathscr{L}^{\sim}$), we can deduce $x \Vdash_{\sim} \mathsf{L} \wedge A^\circ$. Since $\epsilon \Vdash_{\sim} \mathsf{K}$ and $\epsilon x \sim x$, we deduce $x \Vdash_{\sim} B^\circ$, and hence $x \Vdash_{\sqsubseteq} B$ by the induction hypothesis. So $m \Vdash_{\sqsubseteq} A \to B$.

— $F = A * B$
Suppose $m \Vdash_{\sqsubseteq} A * B$ with $m \in \mathscr{L}^{\sqsubseteq}$. There exist $x, y \in \mathscr{L}^{\sqsubseteq}$ such that $xy \sqsubseteq m$, $x \Vdash_{\sqsubseteq} A$ and $y \Vdash_{\sqsubseteq} B$. As $x, y \in L^\star$, by the induction hypothesis, we obtain $x \Vdash_{\sim} \mathsf{L} \wedge A^\circ$ and $y \Vdash_{\sim} \mathsf{L} \wedge B^\circ$. Then, as $xy \sqsubseteq m$, there exists $\delta \in K^\star$ such that $\delta xy \sim m$. Then $\delta \Vdash_{\sim} \mathsf{K}$, and hence $m \Vdash_{\sim} \mathsf{K} * ((\mathsf{L} \wedge A^\circ) * (\mathsf{L} \wedge B^\circ))$.
On the other hand, suppose $m \Vdash_{\sim} \mathsf{K} * ((\mathsf{L} \wedge A^\circ) * (\mathsf{L} \wedge B^\circ))$. Then there exists k, a, b such that $kab \sim m$, $k \Vdash_{\sim} \mathsf{K}$, $a \Vdash_{\sim} \mathsf{L} \wedge A^\circ$ and $b \Vdash_{\sim} \mathsf{L} \wedge B^\circ$. Then $a \Vdash_{\sim} \mathsf{L}$ and $b \Vdash_{\sim} \mathsf{L}$, and there exists $\delta \in K^\star$ and $x, y \in L^\star$ such that $\delta \sim k$, $x \sim a$ and $y \sim b$. By three applications of rule $\langle e_l \rangle$, we have $\delta xy \sim m$, and hence $xy \sqsubseteq m$. By the monotonicity of $\Vdash_{\sim}$ and rule $\langle s \rangle$, since $a \Vdash_{\sim} A^\circ$ and $b \Vdash_{\sim} B^\circ$, we obtain $x \Vdash_{\sim} A^\circ$ and $y \Vdash_{\sim} B^\circ$, and thus, by the induction hypothesis, $x \Vdash_{\sqsubseteq} A$ and $y \Vdash_{\sqsubseteq} B$. So $m \Vdash_{\sqsubseteq} A * B$.

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— $F = A \mathrel{-\!\!*} B$
Suppose $m \Vdash_{\sqsubseteq} A \mathrel{-\!\!*} B$ with $m \in \mathscr{L}^{\sqsubseteq}$. We will prove $m \Vdash_{\sim} (\mathsf{K} * (\mathsf{L} \wedge A^\circ)) \mathrel{-\!\!*} (\mathsf{L} \rightarrow B^\circ)$. So, let c, d be such that $cm \sim d$, $c \Vdash_{\sim} \mathsf{K} * (\mathsf{L} \wedge A^\circ)$ and $d \Vdash_{\sim} \mathsf{L}$. We will prove that $d \Vdash_{\sim} B^\circ$. There exist k, a such that $ka \sim c$, $k \Vdash_{\sim} \mathsf{K}$, $a \Vdash_{\sim} \mathsf{L}$ and $a \Vdash_{\sim} A^\circ$. Then there exists $\delta \in K^\star$ and $x \in L^\star$ such that $\delta \sim k$ and $x \sim a$. By monotonicity, $x \Vdash_{\sim} A^\circ$, and hence, by the induction hypothesis, we obtain $x \Vdash_{\sqsubseteq} A$. Moreover, $\delta xm \sim d$ by three applications of rule $\langle e_l \rangle$. As $d \Vdash_{\sim} \mathsf{L}$, let $y \in L^\star$ be such that $y \sim d$. Then $\delta xm \sim y$ by rule $\langle e_r \rangle$. Thus we have $xm \sqsubseteq y$ and $m \Vdash_{\sqsubseteq} A \mathrel{-\!\!*} B$, and then $y \Vdash_{\sqsubseteq} B$. By the induction hypothesis, we obtain $y \Vdash_{\sim} B^\circ$, and by monotonicity, $d \Vdash_{\sim} B^\circ$. We have now proved that $m \Vdash_{\sim} (\mathsf{K} * (\mathsf{L} \wedge A^\circ)) \mathrel{-\!\!*} (\mathsf{L} \rightarrow B^\circ)$.
On the other hand, suppose $m \Vdash_{\sim} (\mathsf{K} * (\mathsf{L} \wedge A^\circ)) \mathrel{-\!\!*} (\mathsf{L} \rightarrow B^\circ)$. Let $x, y \in \mathscr{L}^{\sqsubseteq}$ be such that $xm \sqsubseteq y$ and $x \Vdash_{\sqsubseteq} A$. We will prove $y \Vdash_{\sqsubseteq} B$. There exists $\delta \in K^\star$ such that $\delta xm \sim y$. As $x \in L^\star$ and $x \Vdash_{\sim} A^\circ$ (by the induction hypothesis), we then obtain $x \Vdash_{\sim} \mathsf{L} \wedge A^\circ$. As $\delta \Vdash_{\sim} \mathsf{K}$ and $\delta x \sim \delta x$ (by rule $\langle p_l \rangle$), we get $\delta x \Vdash_{\sim} \mathsf{K} * (\mathsf{L} \wedge A^\circ)$. As $\delta xm \sim y$, we derive $y \Vdash_{\sim} \mathsf{L} \rightarrow B^\circ$. But $y \in L^\star$, so $y \Vdash_{\sim} \mathsf{L}$, and hence $y \Vdash_{\sim} B^\circ$, so $y \Vdash_{\sqsubseteq} B$ by the induction hypothesis. We have now proved that $m \Vdash_{\sqsubseteq} A \mathrel{-\!\!*} B$.

We have now inductively proved for any $m \in \mathscr{L}^{\sqsubseteq}$ and any formula F of BI containing neither L nor K, that $m \Vdash_{\sqsubseteq} F$ if and only if $m \Vdash_{\sim} F^\circ$. □

With this result, we have established the first step in our embedding of BI into BBI. Indeed, provided a BI-counter-model of F can be chosen with the form $\sqsubseteq_{\sim}^{L,K}$ for some PME $\sim$, we will automatically obtain a BBI-counter-model of F°. So we are now going to study more precisely the counter-models of BI based on PMOs to show that this condition is not restrictive.

5. Tableau proof systems for BI and BBI

Tableau systems are refutation-based procedures that produce *statements* like $\mathbb{T}A$ or $\mathbb{F}A$. Sometimes statements may also be written A or $\neg A$, as in the reference textbook Fitting (1990). The statement $\mathbb{T}A$ expresses the fact that the tableau refutation process tries to build a model of the formula A, whereas the statement $\mathbb{F}A$ expresses the fact that the refutation process tries to build a counter-model of A.

Tableaux for a formula G are finite trees indexed with statements obtained by some *branch expansion* process described by expansion rules and starting from the one-node tree $\mathbb{F}G$. So a tableau for G contains the trace of a process that tries to refute G. The formula A occurring in the statements $\mathbb{T}A$ or $\mathbb{F}A$ produced by the branch expansion process are usually sub-formulae of the initial formula G, although this is not always the case for some non-classical logics.

The expansion process works as follows: to refute a branch $\gamma_0 = [\ldots, \mathbb{F}A \vee B, \ldots]$ containing a statement $\mathbb{F}A \vee B$, we expand γ_0 into one branch, $[\gamma_0, \mathbb{F}A, \mathbb{F}B]$; but to refute a branch $\gamma_1 = [\ldots, \mathbb{T}A \vee B, \ldots]$ containing a statement $\mathbb{T}A \vee B$, we expand γ_1 into two branches, $[\gamma_1, \mathbb{T}A]$ and $[\gamma_1, \mathbb{T}B]$. These two instances of the branch expansion process are represented by the following two branch expansion rules, with $\mathbb{F}\vee$ on the left-hand side

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and $\mathbb{T}\vee$ on the right-hand side:

$$\begin{array}{c} \mathbb{F}A \vee B \\ | \\ \mathbb{F}A \\ \mathbb{F}B \end{array} \qquad\qquad \begin{array}{c} \mathbb{T}A \vee B \\ \diagup \quad \diagdown \\ \mathbb{T}A \qquad \mathbb{T}B \end{array}$$

The justification for these rules lies in the following semantic arguments: for $A \vee B$ to be invalid, we require that both A and B be invalid; whereas for $A \vee B$ to be valid, it is sufficient for either A or B to be valid.

The expansion process stops either when branch expansion does not generate new statements, and then the branch is said to be *saturated*, or when a contradiction occurs, as in the branch $[\ldots, \mathbb{F}A, \ldots, \mathbb{T}A, \ldots]$, in which case the branch is said to be *closed*. It is not possible to refute such a closed branch because the formula A cannot be both valid and invalid. A tableau for G that only has closed branches is called a *closed tableau* and is generally a witness of the universal validity of G, depending, of course, on the soundness theorem for the tableau method.

When considering non-classical logics, it is sometimes useful to enrich statements with *labels* like $\mathbb{T}A : m$ or $\mathbb{F}A : m$. In $\mathbb{T}A : m$, the label m carries some semantic information about the world in which the Kripke forcing relation $m \Vdash A$ holds. For intuitionistic logic IL for example, labelled statements and the *unification of prefixes* are sufficient to provide a sound and complete proof system (Otten and Kreitz 1996), although the system is based on Wallen's matrix characterisation rather than the tableau method.

Unfortunately, labels alone do not carry enough information to provide a sound and complete tableau proof system for BI. The statements also have to be enriched with *constraints* of the form $m \sqsubseteq n$, which are relations that are supposed to hold between labels. A sound and complete labelled tableau proof system was proposed for BI in Galmiche and Méry (2003), and we briefly recall it in the next section. We also adapt the proof system to BBI and prove its soundness. In fact, we propose a common framework to describe both BI- and BBI-tableaux.

5.1. *Labelled tableaux with constraints for* BI *and* BBI

In this section we define the notion of a tableau with constraints for BI (respectively, BBI) providing a proof system called TBI (respectively, TBBI). Note that the following definition refers to *tableaux expansion rules*, which are described a bit later, so a full appreciation of the definition should be suspended until the rules have been read and understood.

Definition 5.1 (TBI- and TBBI-tableaux). Let L be an alphabet. A TBI- *(respectively,* TBBI-*) tableau with constraints for a formula* G is a finite tree with nodes labelled either by *statements of the form* $\mathbb{S}A : m$ where $\mathbb{S} \in \{\mathbb{T}, \mathbb{F}\}$, $A \in$ BI (respectively, $A \in$ BBI) and $m \in L^\star$ or by *assertions that are constraints of the form* $m - n$ where $m, n \in L^\star$, and built according to the following rules:

— the single-node tree $[\mathbb{F}G : \epsilon]$ is a TBI-tableau for G;
— the two-node tree $[a - b, \mathbb{F}G : a]$ is a TBBI-tableau for G whenever $a \neq b \in L$;

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	$\mathbb{T}A \wedge B : m$ $\mathbb{T}A : m$ $\mathbb{T}B : m$	$\mathbb{T}A \vee B : m$ $\mathbb{T}A : m \;\;\big\vert\;\; \mathbb{T}B : m$
	$\mathbb{F}A \wedge B : m$ $\mathbb{F}A : m \;\;\big\vert\;\; \mathbb{F}B : m$	$\mathbb{F}A \vee B : m$ $\mathbb{F}A : m$ $\mathbb{F}B : m$

$\mathbb{T}\mathsf{I} : m$ ass : $\epsilon \mathrel{-\!\!\cdot\!\!-} m$	$\mathbb{T}A * B : m$ ass : $ab \mathrel{-\!\!\cdot\!\!-} m$ $\mathbb{T}A : a$ $\mathbb{T}B : b$	$\mathbb{T}A \mathrel{-\!\!*} B : m$ req : $xm \; R \; y$ $\mathbb{F}A : x \;\;\big\vert\;\; \mathbb{T}B : y$
	$\mathbb{F}A * B : m$ req : $xy \; R \; m$ $\mathbb{F}A : x \;\;\big\vert\;\; \mathbb{F}B : y$	$\mathbb{F}A \mathrel{-\!\!*} B : m$ ass : $am \mathrel{-\!\!\cdot\!\!-} b$ $\mathbb{T}A : a$ $\mathbb{F}B : b$

Table 2. *Tableau expansion rules common to both* TBI *and* TBBI*: for the additives (* $\wedge$, $\vee$ *) in the upper part and for the multiplicatives (* I, $*$, $\mathrel{-\!\!*}$ *) in the lower part.*

$\mathbb{T}A \to B : m$ req : $m \sqsubseteq x$ $\mathbb{F}A : x \;\;\big\vert\;\; \mathbb{T}B : x$	$\mathbb{T}A \to B : m$ $\mathbb{F}A : m \;\;\big\vert\;\; \mathbb{T}B : m$	$\mathbb{T}\neg A : m$ $\mathbb{F}A : m$
$\mathbb{F}A \to B : m$ ass : $m \mathrel{-\!\!\cdot\!\!-} b$ $\mathbb{T}A : b$ $\mathbb{F}B : b$	$\mathbb{F}A \to B : m$ $\mathbb{T}A : m$ $\mathbb{F}B : m$	$\mathbb{F}\neg A : m$ $\mathbb{T}A : m$

Table 3. *Tableau expansion rules specific to* TBI *(on the left) and tableaux expansion rules specific to* TBBI *(on the right).*

— any (maximal) branch of a tableau for G can be expanded according to the tableau expansion rules of TBI (respectively, TBBI).

We sometimes speak of a tableau $\mathscr{T}$ without specifying the formula G, meaning simply that $\mathscr{T}$ is a tableau for some formula G (which can be recovered by looking at the root of the tableau tree).

The tableau expansion rules for both TBI and TBBI are described in Tables 2 and 3. The rules in Table 2 are common to both TBI and TBBI ($\{\mathbb{T},\mathbb{F}\}\{\wedge,\vee,*,\mathrel{-\!\!*}\}$ and $\mathbb{T}\mathsf{I}$). Table 3 contains rules specific either to TBI ($\{\mathbb{T},\mathbb{F}\}\to$ on the left) or TBBI ($\{\mathbb{T},\mathbb{F}\}\{\to,\neg\}$ on the right).

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In these expansion rules, existing statements (like $\mathbb{T}A * B : m$ in rule $\mathbb{T}*$) are decomposed into new statements (for example, $\mathbb{T}A : a$ and $\mathbb{T}B : b$) and new assertions (for example, ass : $ab \twoheadrightarrow m$). These are the products of the decomposition rule. Rules $\mathbb{T}*$ and $\mathbb{F}\mathrel{-\!\!*}$ have a side condition: the letters a and b should be chosen new in the current tableau branch. Rule $\mathbb{F}*$ (respectively, $\mathbb{T}\mathrel{-\!\!*}$) has another kind of side condition: the words x and y should be chosen such that the relation $xy\ R_\gamma\ m$ (respectively, $xm\ R_\gamma\ y$) holds with R_γ being either the PMO $\sqsubseteq_\gamma$ (for TBI) or PME $\sim_\gamma$ (for TBBI) generated by the assertions of the current tableau branch γ. For the rules specific to either TBI or TBBI, the side conditions are: in rule $\mathbb{F}\rightarrow$, the letter b should be new; in rule $\mathbb{T}\rightarrow$, the word x should verify relation $m \sqsubseteq_\gamma x$ where $\sqsubseteq_\gamma$ is the PMO generated by the current branch γ.

The process of building tableaux in tableau systems is based on the notion of branch expansion. This process is explained in full detail in Section 5.2, along with details of how tableau expansion rules are supposed to be applied. We also explain how to build the PMO $\sqsubseteq_\gamma$ (respectively, PME $\sim_\gamma$) generated by a branch γ of a tableau. The expansion process may stop when a *closure condition* is fulfilled.

Definition 5.2 (Closure conditions). A *branch γ of a* TBI- *or* TBBI-*tableau is closed* if one of the following conditions is satisfied for some propositional variable $X \in \mathsf{Var}$ and some $m, n \in L^\star$:

(1) $\mathbb{T}X : m \in \gamma, \mathbb{F}X : n \in \gamma$ and $m\ R_\gamma\ n$

(2) $\mathbb{F}\mathsf{I} : m \in \gamma$ and $\epsilon\ R_\gamma\ m$

(3) $\mathbb{T}\bot : m \in \gamma$

(4) $\mathbb{F}\top : m \in \gamma$

where R_γ is either $\sqsubseteq_\gamma$ or $\sim_\gamma$ depending on whether we are considering a TBI- or TBBI-tableau. If a branch is not closed, then it is *open*. A *tableau is closed* if all of its branches are closed.

5.2. *Explanations on tableau proof rules and the branch expansion process*

Each branch γ of a tableau tree contains a sequence of assertions and statements. Assertions are constraints, and we collect them in a sequence or set of constraints $\mathscr{C}_\gamma$ (the order or potential duplication of constraints is irrelevant). The PMO $\sqsubseteq_\gamma = \sqsubseteq_{\mathscr{C}_\gamma}$ (respectively, PME $\sim_\gamma = \sim_{\mathscr{C}_\gamma}$) is associated to the branch γ: hence $\sqsubseteq_\gamma / \sim_\gamma$ is the PMO/PME generated by the assertions of γ (see Section 6.2 for an example). Also, we use A_γ to denote the alphabet $\mathsf{A}_{\sqsubseteq_\gamma}$ (respectively, $\mathsf{A}_{\sim_\gamma}$) of the relation $\sqsubseteq_\gamma$ (respectively, $\sim_\gamma$), which is exactly the set of letters that occur in the assertions of $\mathscr{C}_\gamma$ (see Proposition 3.16).

As explained in the previous section, some of the rules are common to both systems (see Table 2), while others are not (see Table 3). The rules that differ are $\mathbb{T}\rightarrow$, $\mathbb{F}\rightarrow$[†], and, of course, $\mathbb{T}\neg$ and $\mathbb{F}\neg$ because $\neg A$ is not a BI formula. The fact that some rules have the same shape for TBI and TBBI does not imply that they can always be used

[†] The TBI-rules $\mathbb{T}\rightarrow$ and $\mathbb{F}\rightarrow$ are also valid and even complete for BBI but they are profitably replaced with simpler rules because $A \rightarrow B$ is logically equivalent to $\neg A \vee B$ in BBI (see Proposition 3.9) whereas this equivalence does not hold in BI.

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when the constraints in the branch are identical: the corresponding PMO $\sqsubseteq_\gamma$ and PME $\sim_\gamma$ may differ, implying different situations for the side conditions. Indeed, some rules have a *requirement* $x \ R \ y$ that imposes the condition $x \sqsubseteq_\gamma y$ ($R = \sqsubseteq_\gamma$) in TBI, whereas the condition is $x \sim_\gamma y$ ($R = \sim_\gamma$) in TBBI. And, of course, $\sqsubseteq_\gamma$ and $\sim_\gamma$ are generally not identical relations. Some further observations on the assertions $x \mathrel{-\!\!\!\cdot\!\!\!-} y$ and requirements $x \ R \ y$ that occur in the tableaux expansion rules are in order here:

— The assertion $x \mathrel{-\!\!\!\cdot\!\!\!-} y$ indicates that the constraint $x \mathrel{-\!\!\!\cdot\!\!\!-} y$ should also be added to the branch. The rules concerned are $\mathbb{T}\mathrm{I} : m$, $\mathbb{T}A * B : m$, $\mathbb{F}A \mathrel{-\!\!*} B : m$ (for both TBI and TBBI) and also $\mathbb{F}A \to B : m$ (but just for TBI). Consider the example of rule $\mathbb{T}A * B : m$ for TBI. It can be applied to a branch γ of a tableau provided the statement $\mathbb{T}A * B : m$ occurs in γ (but not necessarily at the leaf of γ) and, as a side condition, the letters $a \neq b \in L$ are new to γ, that is, $a, b \notin \mathsf{A}_\gamma$. Then γ is expanded into the single branch $[\gamma, ab \mathrel{-\!\!\!\cdot\!\!\!-} m, \mathbb{T}A : a, \mathbb{T}B : b]$ as shown below:

$$
\begin{array}{c}
\vdots \\
\surd_\alpha \ \mathbb{T}A * B : m \\
\vdots \\
\boxed{\gamma} \\
| \\
\mathrm{ass}_\alpha : \ ab \mathrel{-\!\!\!\cdot\!\!\!-} m \\
\mathbb{T}A : a \\
\mathbb{T}B : b
\end{array}
$$

— The requirement $x \ R \ y$ is just a side condition that should be fulfilled so that the expansion rule can be applied. For example, in the case of TBBI, to apply rule $\mathbb{T}A \mathrel{-\!\!*} B : m$ to the branch γ, the statement $\mathbb{T}A \mathrel{-\!\!*} B : m$ should occur in γ and the chosen words $x, y \in L^\star$ should verify $xm \sim_\gamma y$, R being interpreted as $R = \sim_\gamma$. Then the branch γ can be expanded into the two branches $[\gamma, \mathbb{F}A : x]$ and $[\gamma, \mathbb{T}B : y]$ as shown below:

$$
\begin{array}{c}
\vdots \\
\surd_\beta \ \mathbb{T}A \mathrel{-\!\!*} B : m \\
\vdots \\
\boxed{\gamma} \\
| \\
\mathrm{req}_\beta : \ xm \sim_\gamma y \\
\diagup \quad \diagdown \\
\mathbb{F}A : x \qquad \mathbb{T}B : y
\end{array}
$$

Note that *we generally tag the tableau trees* with requirements (like req_β, they will generally displayed in a grey box) and *history information* (like $\surd_\alpha$, $\surd_\beta$ and α, β in ass_α and req_β) so that it is easier to check which rule is applied and the reason that the conditions for its application are fulfilled. Formally, the requirements and history are not part of the tableau tree. On the other hand, assertions are not boxed but displayed with the prefix ass. Unlike requirements, assertions are critical bits of semantic information, and not just a guideline for checking that the tableau tree is well formed. Finally, statements, which also constitute critical bits of information, are not prefixed because they always start with either $\mathbb{T}$ or $\mathbb{F}$.

Proposition 5.3. For any branch γ in a TBI- (respectively, TBBI-) tableau, if the statement $\mathbb{S}F : m$ occurs in γ, then the relation $m \sqsubseteq_\gamma m$ (respectively, $m \sim_\gamma m$) holds.

Proof. We use induction on the tableau expansion process using rules $\langle p_l, p_r \rangle$. The only tableaux rules that introduce statements with new labels are:

— $\mathbb{T}*$ (respectively, $\mathbb{F}{-\!\!*}$)
In this case the labels a and b are defined by the assertion $ab \multimap m$ (respectively, $am \multimap b$), that is, if $R' = R + \{ab \multimap m\}$ (respectively, $R' = R + \{am \multimap b\}$), then $a\ R'\ a$ and $b\ R'\ b$ where $R \equiv \sqsubseteq$ is a PMO (for TBI) or $R \equiv \sim$ is a PME (for TBBI). Simply apply the rules $\langle p_l, p_r \rangle$.

— $\mathbb{F}*$ (respectively, $\mathbb{T}{-\!\!*}$)
In this case the labels x and y must already be defined because $xm\ R\ y$ (respectively, $xm\ R\ y$) implies $x\ R\ x$ and $y\ R\ y$ where R is a PMO (for TBI) or PME (for TBBI).

— $\mathbb{T}\rightarrow$ (for TBI)
In this case the label x must already be defined because $m \sqsubseteq x$ implies $x \sqsubseteq x$.

— $\mathbb{F}\rightarrow$ (for TBI)
In this case the label b is defined by the assertion $m \multimap b$, that is, if $\sqsubseteq' = \sqsubseteq + \{m \multimap b\}$, then $b \sqsubseteq' b$.

The other tableau expansion rules introduce statements with labels that must already exist in a previous statement in the branch γ, hence we must already have $m \sqsubseteq_\gamma m$ (respectively, $m \sim_\gamma m$) by the induction hypothesis. For example, for rule $\vee$, the statement $\mathbb{T}A \vee B : m$ must already occur in the branch γ before the rule is applied to produce the two branches $[\gamma, \mathbb{T}A : m]$ and $[\gamma, \mathbb{T}B : m]$. □

5.3. *Elementary and simple* PMO*s*/ BI *frames*

The TBI-tableau method provides some insights into the semantics of BI. Indeed, the TBI-tableau rules that introduce assertions, namely $\mathbb{F}\rightarrow$, $\mathbb{T}\mathsf{I}$, $\mathbb{T}*$ and $\mathbb{F}{-\!\!*}$, do not introduce constraints of arbitrary form. All the possible forms are collected together in the following definitions, and Proposition 5.7 proves that this collection is adequate.

Definition 5.4. Given a PMO $\sqsubseteq$ over L, a constraint is BI-*elementary with respect to* $\sqsubseteq$ when it has one of the following five forms:

(1) $ab \multimap m$ with $m \sqsubseteq m, m \neq \epsilon$ and $a \neq b \in L \backslash \mathsf{A}_\sqsubseteq$
(2) $am \multimap b$ with $m \sqsubseteq m$ and $a \neq b \in L \backslash \mathsf{A}_\sqsubseteq$
(3) $m \multimap b$ with $m \sqsubseteq m$ and $b \in L \backslash \mathsf{A}_\sqsubseteq$
(4) $\epsilon \multimap m$ with $m \sqsubseteq m$ and $m \neq \epsilon$
(5) $\epsilon \multimap \epsilon$.

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Let $(x_i \multimap y_i)_{i<k}$ be a sequence of constraints with $k \in \mathbb{N} \cup \{\infty\}$, and $\mathscr{C}_p$ be the set of constraints $\mathscr{C}_p = \{x_i \multimap y_i \mid i < p\}$ for $p < k$. We suppose that for any $p < k$, the constraint $x_p \multimap y_p$ is BI-elementary with respect to $\sqsubseteq_{\mathscr{C}_p}$ (respectively, $\sim_{\mathscr{C}_p}$). If $k < \infty$, *the sequence* $(x_i \multimap y_i)_{i<k}$ *is said to be* BI-*elementary*. This definition implies, in particular, that the empty sequence of constraints is BI-elementary. If $k = \infty$, *the sequence* $(x_i \multimap y_i)_{i<\infty}$ *is said to be* BI-*simple*.

Definition 5.5. A PMO is BI-elementary (respectively, BI-simple) if it is of the form $\sqsubseteq_{\mathscr{C}}$ where $\mathscr{C} = \{x_i \multimap y_i \mid i < k\}$ and $(x_i \multimap y_i)_{i<k}$ is a BI-elementary (respectively, BI-simple) sequence of constraints.

It is obvious that according to these definitions, if $\sqsubseteq$ is a BI-elementary PMO and the constraint $x \multimap y$ is BI-elementary with respect to $\sqsubseteq$, then the PMO extension $\sqsubseteq + \{x \multimap y\}$ is a BI-elementary PMO.

Using case 5 ($\epsilon \multimap \epsilon$) of Definition 5.4, any finite BI-elementary sequence can be completed into an infinite BI-simple sequence by repeated use of the constraint $\epsilon \multimap \epsilon$. Since adding this constraint does not change the corresponding PMO (because of rule $\langle\epsilon\rangle$), BI-elementary PMOs are also BI-simple. Of course, the converse is not true. Indeed, the language of a BI-elementary PMO is always finite, whereas the language of a BI-simple PMO can be infinite. So the difference between BI-elementary and BI-simple PMOs is that in the later case, the underlying sequence can be infinite, whereas it must be finite for BI-elementary PMOs.

We will now prove that the PMOs occurring in the branches of TBI-tableau are BI-elementary. But we must first establish that $m \sqsubseteq \epsilon$ never holds for BI-elementary PMOs, unless, of course, $m = \epsilon$.

Proposition 5.6. If $\sqsubseteq$ is a BI-elementary or BI-simple PMO over L, then for any $m \in L^\star$, $m \sqsubseteq \epsilon$ only if $m = \epsilon$.

Proof. According to the definition of BI-elementary constraints, they can be of the form $m \multimap \epsilon$ only if $m = \epsilon$ (case 5 of Definition 5.4). So we can apply property 1 of Proposition 3.19. □

Proposition 5.7. Statements of the form $\mathbb{T}F : \epsilon$ for some $F \in$ BI never occur in a TBI-tableau, and for every branch γ of a TBI-tableau, the sequence of assertions occurring in γ is BI-elementary.

Proof. The proof of these two properties is by mutual induction on the TBI-tableau construction process for G. Of course, the properties are valid for the single-node TBI-tableau $[\mathbb{F}G : \epsilon]$. Indeed there is only one branch $\gamma = [\mathbb{F}G : \epsilon]$ that has no assertion, the empty sequence of constraints is BI-elementary, and $\mathbb{T}F : \epsilon$ does not occur in γ.

Now we consider tableau expansion rules. The only rules that may introduce a statement of the form $\mathbb{T}F : \epsilon$ are $\mathbb{T}\wedge$, $\mathbb{T}\vee$, $\mathbb{T}\rightarrow$ and $\mathbb{T}\text{-}\!\!*$:

— For $\mathbb{T}\wedge$ (respectively, $\mathbb{T}\vee$), it would mean that $\mathbb{T}A \wedge B : \epsilon$ (respectively, $\mathbb{T}A \vee B : \epsilon$) already occurs in the branch, which is false by induction.

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— For $\mathbb{T}A \rightarrow B : m$, it would mean that $m \sqsubseteq_\gamma \epsilon$ holds. But by Proposition 5.6, since $\sqsubseteq_\gamma$ is BI-elementary by the induction hypothesis, we must have $m = \epsilon$, hence $\mathbb{T}A \rightarrow B : \epsilon$ occurs in γ, which contradicts the induction hypothesis.

— For $\mathbb{T}A \mathrel{-\!*} B : m$, it would mean that $xm \sqsubseteq_\gamma \epsilon$ holds and thus $xm = \epsilon$, which implies $m = \epsilon$, and thus that $\mathbb{T}A \mathrel{-\!*} B : \epsilon$ occurs in γ, which gives a contradiction.

So we have proved that branch expansion cannot introduce $\mathbb{T}F : \epsilon$ in a branch. We now prove that the sequences of assertions remain BI-elementary in the expanded branches. Four rules introduce new assertions: $\mathbb{T}*$, $\mathbb{F}\mathrel{-\!*}$, $\mathbb{F}\rightarrow$ and $\mathbb{T}\mathsf{I}$. The corresponding constraints are $ab \mathrel{-\!\!-} m$, $am \mathrel{-\!\!-} b$, $m \mathrel{-\!\!-} b$ and $\epsilon \mathrel{-\!\!-} m$, respectively. It is not a coincidence that these are exactly the first four constraint types in the definition of BI-elementary with respect to $\sqsubseteq_\gamma$, see Definition 5.4. In the case of $ab \mathrel{-\!\!-} m$ or $\epsilon \mathrel{-\!\!-} m$, we have $m \sqsubseteq_\gamma m$ by Proposition 5.3 and we have $m \neq \epsilon$ because otherwise $\mathbb{T}A * B : \epsilon$ or $\mathbb{T}\mathsf{I} : \epsilon$ would occur in γ, which contradicts the induction hypothesis. □

So we have singled out a sub-class of PMOs, namely BI-elementary PMOs and their limits, the BI-simple PMOs, that are those that occur in the branches of TBI-tableau trees. In the following sections we will exploit this sub-class and its properties to establish a direct link between the models of BI and the models of BBI.

5.4. *Soundness and completeness of the* TBI-*tableau system*

In this section we state the completeness of the TBI-tableau method for BI. We also state that the class of BI-simple PMOs is complete for BI, and hence BI-counter-models can always be chosen to be BI-simple. The proof of the following theorems can be found in Galmiche *et al.* (2005) and in Daniel Méry's thesis (Méry 2004).

Theorem 5.8 (Soundness and completeness of TBI). Provided the alphabet L is infinite, there exists a closed TBI-tableau for the formula G if and only if G is a BI-valid formula.

Theorem 5.9 (Completeness for simple PMOs). Every invalid formula of BI has a BI-counter-model having the form of the BI-frame $(L, \sqsubseteq, \Vdash)$ where $\sqsubseteq$ is a BI-simple PMO.

The idea is that the counter-model is extracted from an open, saturated and potentially infinite branch of a TBI-tableaux sequence. This counter-model is composed of the infinite sequence of assertions occurring in the branch, and is thus a BI-simple PMO.

5.5. *Soundness of the* TBBI-*tableau system*

Soundness and completeness also hold for the TBBI-tableau system. However, there is currently no published proof of these results. In this section, we give a soundness proof for the TBBI-tableau system with respect to BBI-frames. We do not provide the proof of completeness for two reasons:

(1) We do not need completeness since we only need soundness of the TBBI-tableau system for our embedding of BI into BBI.

(2) The completeness proof is much more complicated than the soundness proof. In particular, it involves the manipulation of infinite branches of tableaux. In this paper, we have chosen a definition of tableaux as finite trees because the definition is better suited to graphical representation, as in Section 7.1. For the formalisation of the completeness proof, it is much easier to represent tableaux as sets of branches, the branches being finite or infinite sets of statements.

As usual, the soundness proof is divided into two parts: first we prove that branch expansion preserves *realisability*; then we show that closed branches are not realisable.

We consider TBBI-tableaux over the alphabet L and BBI-frames over the alphabet K, where L and K are not necessarily identical. So in statements $\mathbb{S}F : m$ or assertions $m - n$, m, n belong to $L^\star$, whereas the relation $q \Vdash F$ or $q \nVdash F$ in the frame $(K, \sim, \Vdash)$ involves q belonging to $K^\star$. Given a total map $\rho : L \longrightarrow K^\star$, for $m = m_1 \ldots m_p \in L^\star$, we define $m_\rho = \rho(m_1) \ldots \rho(m_p)$ and obtain a morphism of (commutative) monoids $(\cdot)_\rho : L^\star \longrightarrow K^\star$.

Given $\mathscr{K} = (K, \sim, \Vdash, \rho)$, we say that *the statement* $\mathbb{T}A : m$ *(respectively,* $\mathbb{F}A : m$*) is satisfied in* $\mathscr{K}$ if $m_\rho \in \mathscr{L}^\sim$ and $m_\rho \Vdash A$ (respectively, $m_\rho \nVdash A$). We say that *the assertion* $m - n$ *is satisfied in* $\mathscr{K}$ if $m_\rho \sim n_\rho$.

Definition 5.10. We say that *a branch of a tableau is satisfied in* $\mathscr{K} = (K, \sim, \Vdash, \rho)$ if all its statements and all its assertions are satisfied in $\mathscr{K}$. We say *a tableau* $\mathscr{T}$ *for G is realisable* if there exists $\mathscr{K}$ such that at least one of the branches of $\mathscr{T}$ is satisfied in $\mathscr{K}$.

Proposition 5.11. If a branch γ of $\mathscr{T}$ is satisfied in $\mathscr{K} = (K, \sim, \Vdash, \rho)$ and $m, n \in L^\star$ verify $m \sim_\gamma n$, then $m_\rho \sim n_\rho$.

Proof. As all the assertions of γ are satisfied in $\mathscr{K}$, the binary relation $\sim' \subseteq L^\star \times L^\star$ defined by $m \sim' n$ if and only if $m_\rho \sim n_\rho$ contains all the assertions of γ. Moreover, it is straightforward to prove that the relation $\sim'$ is a PME over L from the fact that $\sim$ is a PME over K and $m \mapsto m_\rho$ is a morphism of monoids. Hence, since $\sim_\gamma$ is the least PME containing all the assertions of γ, we have $\sim_\gamma \subseteq \sim'$. □

If a branch γ is satisfied in $\mathscr{K}$, then all of its requirements, that is, all the constraints in the PME $\sim_\gamma$, are also satisfied in $\mathscr{K}$. Moreover, if $\mathbb{S}A : m \in \gamma$, then, by Proposition 5.3, we have $m \sim_\gamma m$, and hence $m_\rho \sim m_\rho$.

Proposition 5.12. Closed TBBI-tableaux are not realisable.

Proof. We prove that a closed branch γ cannot be satisfied in any $(K, \sim, \Vdash, \rho)$. Let us suppose the contrary and proceed by case analysis on the closure condition:

— If $\mathbb{T}X : m \in \gamma$, $\mathbb{F}X : n \in \gamma$ and $m \sim_\gamma n$, then, as γ is satisfied in $\mathscr{K}$, we have $m_\rho \sim n_\rho$ by Proposition 5.11. Moreover, both $\mathbb{T}X : m$ and $\mathbb{F}X : n$ are satisfied, so $m_\rho \Vdash X$ and $n_\rho \nVdash X$. As $m_\rho \sim n_\rho$, we obtain a contradiction by monotonicity of $\Vdash$.

— If $\mathbb{F}\mathsf{I} : m \in \gamma$ and $\epsilon \sim_\gamma m$, then, as γ is satisfied in $\mathscr{K}$, we have $\epsilon = \epsilon_\rho \sim m_\rho$ and $m_\rho \nVdash \mathsf{I}$. But then we should have $\epsilon \nsim m_\rho$, and we get a contradiction.

— If $\mathbb{T}\bot : m \in \gamma$, then, as γ is satisfied in $\mathscr{K}$, we have $m_\rho \sim m_\rho$ and $m_\rho \Vdash \bot$, which is impossible.

— If $\mathbb{F}\top : m \in \gamma$, then, as γ is satisfied in $\mathscr{K}$, we have $m_\rho \sim m_\rho$ and $m_\rho \nVdash \top$, which is impossible.

So we obtain a contradiction in all cases, so a closed branch cannot be satisfied, and thus closed TBBI-tableaux are not realisable. □

Lemma 5.13. TBBI-tableaux expansion rules preserve realisability.

Proof. Let $\mathscr{T}$ be a realisable BBI-tableau and let $\mathscr{K} = (K, \sim, \Vdash, \rho)$ be such that at least one branch of $\mathscr{T}$ is satisfied in $\mathscr{K}$. We consider the expansion of one of the branches of $\mathscr{T}$ by one of the rules of the TBBI-tableau system. If the expanded branch is not among the satisfied ones, the satisfied branches are unchanged by the application of the rule and hence the resulting tableau $\mathscr{T}'$ is still realisable.

So we consider the case when the branch γ we expand is among the satisfied ones. We proceed by case analysis depending on the rule applied:

— $\mathbb{T}\neg A : m$
This statement is satisfied in $\mathscr{K}$, so $m_\rho \in \mathscr{L}^\sim$ and $m_\rho \Vdash \neg A$. Thus $m_\rho \nVdash A$ and $\mathbb{F}A : m$ is satisfied in $\mathscr{K}$. So the new branch $[\gamma, \mathbb{F}A : m]$ of $\mathscr{T}'$ is satisfied in $\mathscr{K}$.

— $\mathbb{F}\neg A : m$
This case is similar to case $\mathbb{T}\neg$.

— $\mathbb{T}A \wedge B : m$
This statement is satisfied in $\mathscr{K}$, so $m_\rho \in \mathscr{L}^\sim$ and $m_\rho \Vdash A \wedge B$. Hence, $m_\rho \Vdash A$ and $m_\rho \Vdash B$, so $\mathbb{T}A : m$ and $\mathbb{T}B : m$ are satisfied in $\mathscr{K}$. So the new branch $[\gamma, \mathbb{T}A : m, \mathbb{T}B : m]$ of $\mathscr{T}'$ is satisfied in $\mathscr{K}$.

— $\mathbb{F}A \wedge B : m$
This statement is satisfied in $\mathscr{K}$, so $m_\rho \in \mathscr{L}^\sim$ and either $m_\rho \nVdash A$ or $m_\rho \nVdash B$. Hence either $\mathbb{F}A : m$ or $\mathbb{F}B : m$ is satisfied in $\mathscr{K}$. So at least one of the two new branches of $\mathscr{T}'$ (namely $[\gamma, \mathbb{F}A : m]$ or $[\gamma, \mathbb{F}B : m]$) is satisfied in $\mathscr{K}$.

— $\mathbb{T}A \vee B : m$
This case is similar to case $\mathbb{F}\wedge$.

— $\mathbb{F}A \vee B : m$
This case is similar to case $\mathbb{T}\wedge$.

— $\mathbb{T}A \to B : m$
This case is similar to case $\mathbb{F}\wedge$.

— $\mathbb{F}A \to B : m$
This case is similar to case $\mathbb{T}\wedge$.

— $\mathbb{T}\mathsf{I} : m$
This statement is satisfied in $\mathscr{K}$, so $m_\rho \in \mathscr{L}^\sim$ and $m_\rho \Vdash \mathsf{I}$. Thus $\epsilon \sim m_\rho$. As $\epsilon_\rho = \epsilon$, we obtain $\epsilon_\rho \sim m_\rho$ and thus the assertion $\epsilon - m$ is satisfied in $\mathscr{K}$. So the new branch $[\gamma, \epsilon - m]$ of $\mathscr{T}'$ is satisfied in $\mathscr{K}$.

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— $\mathbb{T}A * B : m$
This statement is satisfied in $\mathscr{K}$, so $m_\rho \in \mathscr{L}^\sim$ and $m_\rho \Vdash A * B$. So there exist $x, y \in \mathscr{L}^\sim$ such that $xy \sim m_\rho$, $x \Vdash A$ and $y \Vdash B$. We define $\rho' = \rho[a \mapsto x, b \mapsto y]$ (this is possible because $a \neq b$). Then for any $m, n \in L^\star$ such that $m \sim_\gamma n$, we have $m, n \in \mathsf{A}_\gamma^\star$, so $m_{\rho'} = m_\rho$ and $n_{\rho'} = n_\rho$ (ρ and ρ' are identical maps when restricted to A_γ because $a, b \notin \mathsf{A}_\gamma$). Thus γ is satisfied in $\mathscr{K}' = (K, \sim, \Vdash, \rho')$. Moreover, $ab - m$ is satisfied in $\mathscr{K}'$ (because $(ab)_{\rho'} = xy$, $m_{\rho'} = m_\rho$ and $xy \sim m_\rho$), $\mathbb{T}A : a$ is satisfied (because $a_{\rho'} = x$ and $x \Vdash A$), and $\mathbb{T}B : b$ is satisfied (because $b_{\rho'} = y$ and $y \Vdash B$). So the (new) branch $[\gamma, ab - m, \mathbb{T}A : a, \mathbb{T}B : b]$ of $\mathscr{T}'$ is satisfied in $\mathscr{K}'$.

— $\mathbb{F}A * B : m$
This statement is satisfied in $\mathscr{K}$, so $m_\rho \in \mathscr{L}^\sim$ and $m_\rho \nVdash A * B$. γ is expanded into two branches $[\gamma, \mathbb{F}A : x]$ and $[\gamma, \mathbb{F}B : y]$ with $xy \sim_\gamma m$. Then $x_\rho y_\rho \sim m_\rho$, so $x_\rho, y_\rho \in \mathscr{L}^\sim$. So either $x_\rho \nVdash A$ or $y_\rho \nVdash B$. Thus at least one of the two new branches of $\mathscr{T}'$ (namely, $[\gamma, \mathbb{F}A : x]$ and $[\gamma, \mathbb{F}B : y]$) is satisfied in $\mathscr{K}$.

— $\mathbb{T}A \mathrel{-\!\!*} B : m$
This case is similar to case $\mathbb{F}*$.

— $\mathbb{F}A \mathrel{-\!\!*} B : m$
This case is similar to case $\mathbb{T}*$.

So in all cases there exists a satisfiable branch in $\mathscr{T}'$, and thus $\mathscr{T}'$ is realisable. □

Theorem 5.14 (Soundness of TBBI). If there exists a closed TBBI-tableau for the formula G, then G is a valid BBI formula.

Proof. Let us suppose that G has a counter-model $(K, \sim, \Vdash)$, that is, there exists $m \in \mathscr{L}^\sim$ such that $m \nVdash G$. Then for $c_0 \neq d_0 \in L$, the unique branch of the TBBI-tableau $[c_0 - d_0, \mathbb{F}G : c_0]$ is satisfied in $(K, \sim, \Vdash, \rho)$ where $\rho = x \mapsto m$ (in particular, $\rho(c_0) = \rho(d_0) = m$). So any initial TBBI-tableau for G is realisable. Hence, as branch expansion preserves realisability, all the TBBI-tableaux for G are realisable. Therefore, G cannot have a closed TBBI-tableau. □

6. Representing simple PMOs by PMEs

In Section 4.1, we presented the map $\sim \mapsto \sqsubseteq_\sim^{L,K}$, which transforms a PME into a PMO. We also asked whether this transformation is general enough to produce any PMO. We do not know the answer to this question yet, and will not provide an answer in this paper. But we do have a positive answer in the case of BI-simple PMOs, as we will show in this section.

6.1. *Elementary and simple* PME*s*

We introduce the notion of BBI-elementary and BBI-simple PMEs in a similar way to the case of PMOs. However, there is a major difference between the two cases. BI-elementary (and BI-simple) PMOs were designed to capture those PMOs occurring in TBI-tableau proofs. BBI-elementary (and BBI-simple) PMEs are not designed to capture the PMEs occurring in TBBI-tableau proofs, and are not suitable for such a goal, as explained later

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in Section 6.2. The study of the properties of PMEs occurring in TBBI-tableau proofs is way beyond the scope of this paper.

BBI-elementary (and BBI-simple) PMEs are in fact specifically designed to represent BI-elementary (and BI-simple) PMOs through the map $\sim \mapsto \sqsubseteq_{\sim}^{L,K}$ and the notion of (L,K,M) elementary representation defined in Section 6.5.

Definition 6.1. Given a PME $\sim$ over L, a constraint is BBI-*elementary with respect to* $\sim$ when it has one of the following five forms:

(1) $ab - m$ with $m \sim m$, $m \nsim \epsilon$ and $a \neq b \in L \backslash \mathsf{A}_{\sim}$
(2) $am - b$ with $m \sim m$ and $a \neq b \in L \backslash \mathsf{A}_{\sim}$
(3) $m - b$ with $m \sim m$, $m \nsim \epsilon$ and $b \in L \backslash \mathsf{A}_{\sim}$
(4) $\epsilon - b$ with $b \in L \backslash \mathsf{A}_{\sim}$
(5) $\epsilon - \epsilon$.

Let $(x_i - y_i)_{i<k}$ be a sequence of constraints with $k \in \mathbb{N} \cup \{\infty\}$, and $\mathscr{C}_p$ be the set of constraints $\mathscr{C}_p = \{x_i - y_i \mid i < p\}$ for $p < k$. We suppose that for any $p < k$, the constraint $x_p - y_p$ is BBI-elementary with respect to $\sqsubseteq_{\mathscr{C}_p}$ (respectively, $\sim_{\mathscr{C}_p}$). If $k < \infty$, *the sequence* $(x_i - y_i)_{i<k}$ *is said to be* BBI-*elementary*. If $k = \infty$, *the sequence* $(x_i - y_i)_{i<\infty}$ *is said to be* BBI-*simple*.

Definition 6.2. A PME is BBI-elementary (respectively, BBI-simple) if it is of the form $\sim_{\mathscr{C}}$ where $\mathscr{C} = \{x_i - y_i \mid i < k\}$ and $(x_i - y_i)_{i<k}$ is a BBI-elementary (respectively, BBI-simple) sequence of constraints.

It is obvious that according to these definitions, if $\sim$ is a BI-elementary PME and the constraint $x - y$ is BBI-elementary with respect to $\sim$, then the PME extension $\sim + \{x - y\}$ is BBI-elementary. Using case 5 of Definition 6.1, any finite BBI-elementary sequence can be completed into an infinite BBI-simple sequence by repeated use of the constraint $\epsilon - \epsilon$, so BBI-elementary PMEs are also BBI-simple.

6.2. *A* PME *occurring in a* TBBI-*tableau that is not simple*

In this section we present an example of a PME that is not BBI-simple, but, nevertheless, comes from a branch of a TBBI-tableau: remember that unlike BI-elementary and BI-simple PMOs, BBI-elementary and BBI-simple PMEs are not designed to capture those PMEs generated by TBBI-tableaux.

Consider the set of constraints $\mathscr{C} = \{c_0 - d_0, \epsilon - c_0, ab - c_0\}$. In the given order, it is obvious that this sequence of constraints is not BBI-elementary: $\epsilon - c_0$ is not BBI-elementary with respect to $\sim_{\{c_0 - d_0\}} = \{\epsilon - \epsilon, c_0 - c_0, d_0 - d_0, c_0 - d_0\}$ because c_0 is not new. But this does not prove that the corresponding PME $\sim_{\mathscr{C}}$ is not BBI-elementary or BBI-simple. This sequence of constraints arises as the sequence of assertions of the unique

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branch of the following TBBI-tableau for $\neg(\mathsf{I} \wedge A * B)$:

$$
\begin{array}{c}
\text{ass}_0 : c_0 - d_0 \\
\surd_1 \ \mathbb{F}\neg\big(\mathsf{I} \wedge (A * B)\big) : c_0 \\
| \\
\surd_2 \ \mathbb{T}\mathsf{I} \wedge (A * B) : c_0 \\
| \\
\surd_3 \ \mathbb{T}\mathsf{I} : c_0 \\
\surd_4 \ \mathbb{T}A * B : c_0 \\
| \\
\text{ass}_3 : \epsilon - c_0 \\
| \\
\text{ass}_4 : ab - c_0 \\
\mathbb{T}A : a \\
\mathbb{T}B : b
\end{array}
$$

It is possible to compute the form of the PME $\sim_{\mathscr{C}}$ explicitly. Indeed, one can check the following identity by double inclusion (arguably, after a certain amount of work):

$$\sim_{\mathscr{C}} = \{a^{i_0}b^{j_0}x - a^{i_1}b^{j_1}y \mid i_0 + j_1 = i_1 + j_0 \text{ and } x, y \in \{c_0, d_0\}^\star\}.$$

Then it is straightforward to check that $a^2 \sim_{\mathscr{C}} a^2$ and $a \nsim_{\mathscr{C}} \epsilon$. The one letter word a is squarable in $\sim_{\mathscr{C}}$ but, nevertheless, not equivalent to ϵ. Then, according to Corollary 6.10 (see later), the PME $\sim_{\mathscr{C}}$ cannot be BBI-simple. It is, nevertheless, associated to some branch of a TBBI-tableau.

This example shows the conceptual differences between BI-elementary PMOs, which capture those PMOs occurring in TBI-tableaux, and BBI-elementary PMEs, which do not capture those PMEs occurring in TBBI-tableaux. The role played by BBI-elementary PMEs and the justification of the introduction of this concept will become clearer when elementary representations are introduced in Section 6.5.

6.3. *Free* PME *extensions*

In order to prove further properties of BBI-elementary and BBI-simple PMEs, we will now introduce some general results that explicitly compute 'free' PME extensions like $\sim + \{ab - m\}$ or $\sim + \{am - b\}$ where m is already defined in $\sim$ (that is, $m \sim m$) and $a \neq b$ are two letters new to $\sim$ (that is, $a \nsim a$ and $b \nsim b$).

The three following results are essential as a basis for reasoning about these 'free' PME extensions. The case $\sim + \{\alpha - m\}$ covers both the extension $\sim + \{ab - m\}$ (where a and b are new) and $\sim + \{m - b\}$ (where b is new). Apart from the first result, $\sim + \{\epsilon - b\}$, neither the shape of $\sim + \{\alpha - m\}$ or $\sim + \{\alpha m - b\}$ nor the hypotheses on m, α and b are obvious.

The tedious proofs of these results are carried out using basic arguments. They are provided for completeness, but are postponed to Appendix A because they are quite long and we feel that they would be a distraction at this point.

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Proposition 6.3 ($\sim + \{\epsilon - b\}$). Let $\sim$ be a PME over L, and b be new to $\sim$, that is, $b \in L \setminus \mathsf{A}_\sim$. Then $\sim + \{\epsilon - b\} = \sim'$ with $\sim' = \{b^p x - b^q y \mid x \sim y \wedge p, q \geqslant 0\}$ and $\mathsf{A}_{\sim'} = \mathsf{A}_\sim \cup \{b\}$.

Lemma 6.4 ($\sim + \{\alpha - m\}$). Let $\sim$ be a PME over L. Let $m \in L^\star$ and $\alpha \in L^\star$ be such that $m \sim m$, $mm \nsim mm$, $\alpha \neq \epsilon$ and $\mathsf{A}_\alpha \cap \mathsf{A}_\sim = \emptyset$. Then $\sim + \{\alpha - m\} = \sim'$ with

$$\begin{aligned}\sim' = \sim &\cup \{\delta x - \delta y \mid x \sim y \wedge mx \sim my \wedge \delta \prec \alpha \wedge \delta \notin \{\epsilon, \alpha\}\}\\ &\cup \{\alpha x - \alpha y \mid mx \sim my\}\\ &\cup \{\alpha x - y \mid mx \sim y\}\\ &\cup \{x - \alpha y \mid x \sim my\}\end{aligned}$$

and $\mathsf{A}_{\sim'} = \mathsf{A}_\sim \cup \mathsf{A}_\alpha$.

Lemma 6.5 ($\sim + \{\alpha m - b\}$). Let $\sim$ be a PME over L. Let $m \in L^\star$, $\alpha \in L^\star$, $b \in L$ be such that $m \sim m$, $\alpha \neq \epsilon$, $\mathsf{A}_\alpha \cap \mathsf{A}_\sim = \emptyset$ and $b \notin \mathsf{A}_\sim \cup \mathsf{A}_\alpha$. Then $\sim + \{\alpha m - b\} = \sim'$ with

$$\begin{aligned}\sim' = \sim &\cup \{\delta x - \delta y \mid x \sim y \wedge \epsilon \neq \delta \prec \alpha \wedge \exists k\ xk \sim m\}\\ &\cup \{\alpha x - jb \mid x \sim jm \wedge \exists k\ jkm \sim m\}\\ &\cup \{ib - \alpha y \mid y \sim im \wedge \exists k\ ikm \sim m\}\\ &\cup \{ib - jb \mid \exists k\ (ikm \sim m \wedge jkm \sim m)\}\end{aligned}$$

and $\mathsf{A}_{\sim'} = \mathsf{A}_\sim \cup \mathsf{A}_\alpha \cup \{b\}$.

6.4. *No square in simple* PMEs

Note that one of the hypotheses on m in the free PME extension $\sim + \{ab - m\}$ is that the square of m is not defined in $\sim$, that is, $mm \nsim mm$, see Lemma 6.4 with $\alpha = ab$. In order to use the equation of this lemma to compute BBI-elementary PMEs, we first establish that they do not contain squares, in a kind of relaxed way.

Definition 6.6. Let $\sim$ be a PME over L. We define $\mathsf{I}_\sim = \{i \in L \mid i \sim \epsilon\}$. We say that $\sim$ *has no square* if for any letter $c \in L$, we have $cc \sim cc$ only if $c \sim \epsilon$.

This is not exactly the same as stating that no word can be squared unless it is ϵ, but amounts to saying that no word m can be squared unless it is equivalent to ϵ (that is, $m \sim \epsilon$). The set $\mathsf{I}_\sim$ is the set of letters that are equivalent to ϵ as one letter words.

Proposition 6.7. If the PME $\sim$ over L has no square, then the following properties hold:

1. For any $m \in L^\star$, we have $mm \sim mm$ if and only if $m \in \mathsf{I}_\sim^\star$ if and only if $m \sim \epsilon$.
2. For any $i, k, m \in L^\star$, if $m \sim m$, then $ikm \sim m$ if and only if $i \sim \epsilon \wedge k \sim \epsilon$.
3. For any $i, j \in \mathsf{I}_\sim^\star$, $m, n \in L^\star$, we have $im \sim jn$ if and only if $m \sim n$.

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Proof. We define $I = I_\sim$. For property 1, let $m \in L^\star$. If $mm \sim mm$, we consider two cases: $m = \epsilon$ and $m \neq \epsilon$. If $m = \epsilon$, then $m \in I^\star$. If $m \neq \epsilon$, let c be a letter of m. Then $cm' = m$ for some $m' \in L^\star$. We get $cm'cm' \sim cm'cm'$, hence $cc \sim cc$ by rule $\langle d \rangle$. So $c \in I$. As $c \in I$ for any letter of m, we deduce $m \in I^\star$. Hence, $mm \sim mm$ only if $m \in I^\star$.

We now suppose $m \in I^\star$ and prove that $m \sim \epsilon$ by induction on the length of m. If $m = \epsilon$, then $m \sim \epsilon$ by rule $\langle \epsilon \rangle$. Otherwise, $m = cm'$ with $c \in I$ and $m' \in I^\star$. By induction, we have $m' \sim \epsilon$. As $c \in I$, we have $c \sim \epsilon$. By rule $\langle e_l \rangle$, from $\epsilon m' \sim \epsilon$ we deduce $cm' \sim \epsilon$, and hence $m \sim \epsilon$. So $m \in I^\star$ only if $m \sim \epsilon$.

We now suppose $m \sim \epsilon$. Then from $\epsilon m \sim \epsilon$ we deduce $mm \sim \epsilon$ by rule $\langle e_l \rangle$. Hence $mm \sim mm$ by rule $\langle l \rangle$. So we have proved property 1.

For property 2, let $i, k, m \in L^\star$ be such that $m \sim m$. On the one hand, if $ikm \sim m$, then $ik(ikm) \sim m$ by rule $\langle e_l \rangle$. Hence $(ik)(ik) \sim (ik)(ik)$ by rule $\langle p_l \rangle$. Thus $ik \in I^\star$ by property 1. Thus $i, k \in I^\star$, so $i \sim \epsilon$ and $k \sim \epsilon$. On the other hand, if $i \sim \epsilon$ and $k \sim \epsilon$, then from $\epsilon\epsilon m \sim m$, we get $ikm \sim m$ by two applications of rule $\langle e_l \rangle$.

For property 3, we have both $i \sim \epsilon$, $j \sim \epsilon$, $\epsilon \sim i$ and $\epsilon \sim j$ by property 1 and rule $\langle s \rangle$. The equivalence is obtained by application of rules $\langle e_l \rangle$ and $\langle e_r \rangle$. □

When $\sim$ has no square, the explicit form of the PME extension $\sim + \{am - b\}$ can be simplified a bit.

Proposition 6.8. Let $\sim$ be a PME over L, $m \sim m$ and $a \neq b \in L \backslash A_\sim$. If $\sim$ has no square, the following identity holds:

$$\begin{aligned} \sim + \{am - b\} = \sim &\cup \{ax - ay \mid x \sim y \wedge \exists k\ xk \sim m\} \\ &\cup \{ax - jb, jb - ax \mid x \sim m \wedge j \sim \epsilon\} \\ &\cup \{ib - jb \mid i \sim \epsilon \wedge j \sim \epsilon\}. \end{aligned}$$

Proof. Starting from the identity of Lemma 6.5, as $\sim$ has no square, with property 2 of Proposition 6.7, we can simplify the condition $x \sim jm \wedge \exists k\ jkm \sim m$ into the equivalent $x \sim m \wedge j \sim \epsilon$. We can also simplify $\exists k\,(ikm \sim m \wedge jkm \sim m)$ into the equivalent $i \sim \epsilon \wedge j \sim \epsilon$. □

Now we prove that BBI-elementary extensions preserve the property of 'having no square' and compute the sets $I_\sim$ accordingly.

Proposition 6.9. If the PME $\sim$ has no square and $x - y$ is BBI-elementary with respect to $\sim$, then $\sim' = \sim + \{x - y\}$ has no square. Moreover, $I_{\sim'} = I_\sim$ in all cases except case 4 ($\sim' = \sim + \{\epsilon - b\}$) where $I_{\sim'} = I_\sim \cup \{b\}$.

Proof. Let $A = A_\sim$, $I = I_\sim$, $\sim' = \sim + \{x - y\}$, $A' = A_{\sim'}$, $I' = I_{\sim'}$. Then, as $\sim \subseteq \sim'$, we deduce $A \subseteq A'$ and $I \subseteq I'$. We consider each case for $x - y$ according to Definition 6.1,

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using Proposition 6.3, Lemma 6.4 and Proposition 6.8:

— $ab \multimap m$ where $m \sim m$, $m \nsim \epsilon$, $a \neq b \in L \backslash A$.
As $\sim$ has no square, from $m \nsim \epsilon$ we deduce $mm \nsim mm$ (see property 1 of Proposition 6.7) and thus, by Lemma 6.4,

$$\begin{aligned} \sim' = \sim + \{ab \multimap m\} = \sim &\cup \{ax \multimap ay, bx \multimap by \mid x \sim y \wedge mx \sim my\} \\ &\cup \{abx \multimap aby \mid mx \sim my\} \\ &\cup \{abx \multimap y, y \multimap abx \mid mx \sim y\}. \end{aligned}$$

Let c be a letter such that $cc \sim' cc$. As a and b are two different letters not occurring in $A = A_\sim$, it is obvious from the form of $\sim'$ that the only option is $cc \sim cc$, hence $c \sim \epsilon$ because $\sim$ has no square. Thus $c \sim' \epsilon$. We have proved that $\sim'$ has no square and $I' \subseteq I$, so $I' = I$.

— $am \multimap b$ where $m \sim m$ and $a \neq b \in L \backslash A$.
As $\sim$ has no square, by Proposition 6.8, we obtain the identity

$$\begin{aligned} \sim' = \sim + \{am \multimap b\} = \sim &\cup \{ax \multimap ay \mid x \sim y \wedge \exists k\ xk \sim m\} \\ &\cup \{ax \multimap jb, jb \multimap ax \mid x \sim m \wedge j \sim \epsilon\} \\ &\cup \{ib \multimap jb \mid i \sim \epsilon \wedge j \sim \epsilon\}. \end{aligned}$$

Let c be a letter such that $cc \sim' cc$. It is obvious from the form of $\sim'$ that the only option is $cc \sim cc$, so $c \sim \epsilon$ because $\sim$ has no square. Thus $c \sim' \epsilon$. Hence $\sim'$ has no square and $I' = I$.

— $m \multimap b$ where $m \sim m$, $m \nsim \epsilon$, $b \in L \backslash A$.
As $\sim$ has no square, we deduce $mm \nsim mm$ from $m \nsim \epsilon$, and thus, by Lemma 6.4,

$$\begin{aligned} \sim' = \sim + \{b \multimap m\} = \sim &\cup \{bx \multimap by \mid mx \sim my\} \\ &\cup \{bx \multimap y, y \multimap bx \mid mx \sim y\}. \end{aligned}$$

But $\sim' = \sim + \{m \multimap b\} = \sim + \{b \multimap m\}$ by rule $\langle s \rangle$. Let c be a letter such that $cc \sim' cc$. It is obvious from the form of $\sim' = \sim + \{b \multimap m\}$ that the only option is $cc \sim cc$, hence $c \sim \epsilon$ because $\sim$ has no square. Thus $c \sim' \epsilon$. So $\sim'$ has no square and $I' = I$.

— $\epsilon \multimap b$ with $b \in L \backslash A$.
By Proposition 6.3:

$$\sim' = \sim + \{\epsilon \multimap b\} = \{b^p x \multimap b^q y \mid x \sim y \wedge p, q \geqslant 0\}.$$

Let $c \in L$. There are two options for $cc \sim' cc$: either $cc \sim cc$ or $c = b$. When $cc \sim cc$, we obtain $c \sim \epsilon$ because $\sim$ has no square, hence $c \sim' \epsilon$. When $c = b$, we get $c = b \sim' \epsilon$ by rule $\langle s \rangle$. Hence $\sim'$ has no square and $I' = I \cup \{b\}$;

— $\epsilon \multimap \epsilon$
In this case $\sim' = \sim + \{\epsilon \multimap \epsilon\} = \sim$ has no square and $I' = I$.

We have now proved in all cases that $\sim'$ has no square, and have computed $I_{\sim'}$ accordingly. □

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Corollary 6.10. BBI-elementary and BBI-simple PMEs have no square.

Proof. We prove the result for BBI-simple PMEs as BBI-elementary PMEs are also BBI-simple PMEs. Let $\sim$ be a BBI-simple PME. Then there exists $(x_i - y_i)_{i<\infty}$ that is a BBI-simple sequence of constraints such that $\sim = \sim_{\mathscr{C}}$ with $\mathscr{C} = \{x_i - y_i \mid i < \infty\}$. Let $\sim_p = \sim_{\mathscr{C}_p}$ with $\mathscr{C}_p = \{x_i - y_i \mid i < p\}$ for $p \in \mathbb{N} \cup \{\infty\}$. Then $\sim = \sim_\infty$.

For $p = 0$, we have $\sim_0 = \sim_\varnothing = \{\epsilon - \epsilon\}$, which of course has no square. With Proposition 6.9, it is trivial to prove the induction step, that is, $\sim_p$ has no square implies $\sim_{p+1} = \sim_p + \{x_p - y_p\}$ has no square, because $x_p - y_p$ is BBI-elementary with respect to $\sim_p$. By induction, for any $p < \infty$, the PME $\sim_p$ has no square.

Let $c \in L$ be such that $cc \sim_\infty cc$. By compactness (see Proposition 3.17), there exists $p < \infty$ such that $cc \sim_p cc$. Hence, as $\sim_p$ has no square, we deduce $c \sim_p \epsilon$, and, as $\sim_p \subseteq \sim_\infty$, we have $c \sim_\infty \epsilon$. Thus $\sim = \sim_\infty$ has no square. □

We will now briefly discuss the incremental computation of BBI-elementary PMEs. This problem consists of the computation of BBI-elementary extensions of $\sim$ where $\sim$ is itself a BBI-elementary PME. Hence, $\sim$ has no square and thus, since cases 1 and 3 of Definition 6.1 contain the condition $m \nsim \epsilon$, by Proposition 6.7, we deduce $mm \nsim mm$. Hence, it is legitimate to use the equation of Lemma 6.4 to compute the BBI-elementary extensions $\sim + \{ab - m\}$ and $\sim + \{m - b\}$ as already done in the proof of Proposition 6.9. To complete the description, for case 2, we use Proposition 6.8 and for case 4, we use Proposition 6.3.

6.5. *Elementary representations*

Having defined BI-elementary PMOs and BBI-elementary PMEs, and having described how to compute BBI-elementary PMEs, we are now in position to state and prove the fundamental lemma of this paper. It describes how BI-elementary extensions of BI-elementary PMOs are related to BBI-elementary extensions of BBI-elementary PMEs. From this, we deduce a procedure that, given a BI-simple PMO $\sqsubseteq$ over L, computes a BBI-simple PME $\sim$ over $L \cup K \cup M$ such that $\sim$ represents $\sqsubseteq$, that is, $\sqsubseteq = \sqsubseteq^{L,K}_{\sim}$.

Lemma 6.13 is the foundation on which the adequacy and the faithfulness of our embedding of BI into BBI relies.

Definition 6.11. Let L, K and M be three mutually disjoint alphabets. We say that the pair $(\sqsubseteq, \sim)$ is an (L, K, M) *elementary representation* if the following properties hold:

(1) $\sqsubseteq$ is a BI-elementary PMO over L.
(2) $\sim$ is a BBI-elementary PME over $L \cup K \cup M$.
(3) The inclusion $I_\sim \subseteq M$ holds.
(4) For any $d \in M$, if $d \sim d$ then $x\alpha \sim d$ for some $x \in L^\star$ and $\alpha \in K^\star$.
(5) The identity $\sqsubseteq = \sqsubseteq^{L,K}_{\sim}$ holds.

Thus, if $(\sqsubseteq, \sim)$ is an (L, K, M) elementary representation and $m \in L^\star$, then $m \sqsubseteq m$ if and only if $m \sim m$ (see Lemma 4.2).

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Proposition 6.12. If the pair $(\sqsubseteq, \sim)$ is an (L, K, M) elementary representation, then $A_{\sqsubseteq} = A_{\sim} \cap L$, $\mathscr{L}^{\sqsubseteq} = L^{\star} \cap \mathscr{L}^{\sim}$, and for any $k \in (L \cup K \cup M)^{\star}$ such that $k \sim k$, there exists $x \in L^{\star}$ and $\alpha \in K^{\star}$ such that $x\alpha \sim k$.

Proof. The identity $A_{\sqsubseteq} = A_{\sim} \cap L$ and $\mathscr{L}^{\sqsubseteq} = L^{\star} \cap \mathscr{L}^{\sim}$ are a direct consequence of Lemma 4.2. Let $k \in (L \cup K \cup M)^{\star}$. There exist $l \in L^{\star}$, $\delta \in K^{\star}$ and $m \in M^{\star}$ such that $l\delta m = k$. Let $m = d_1 \ldots d_p$ where $d_1, \ldots, d_p \in M$ are the letters of m.

As $k \sim k$, we get $l\delta m \sim l\delta m$, so $m \sim m$ by rule $\langle d \rangle$. Thus $d_1 \ldots d_p \sim d_1 \ldots d_p$, and for any $i \in [1, p]$, we have $d_i \sim d_i$. As $(\sqsubseteq, \sim)$ is an (L, K, M) elementary representation, for any $i \in [1, p]$ there exist $x_i \in L^{\star}$ and $\alpha_i \in K^{\star}$ such that $x_i\alpha_i \sim d_i$.

From $l\delta d_1 \ldots d_p \sim k$, we get $l\delta(x_1\alpha_1)\ldots(x_p\alpha_p) \sim k$ by p applications of rule $\langle e_l \rangle$. Hence we have $x\alpha \sim k$ with $x = lx_1 \ldots x_p \in L^{\star}$ and $\alpha = \delta\alpha_1 \ldots \alpha_p \in K^{\star}$. □

Lemma 6.13. Let L, K and M be three mutually disjoint alphabets. Let $(\sqsubseteq, \sim)$ be an (L, K, M) elementary representation. Let m, a, b, δ and c be such that $m \sqsubseteq m$, $a \neq b \in L \backslash A_{\sqsubseteq}$, $\delta \in K \backslash A_{\sim}$ and $c \in M \backslash A_{\sim}$. Then in each of the following cases, $(\sqsubseteq', \sim')$ is an (L, K, M) elementary representation:

(1)	$\sqsubseteq' = \sqsubseteq + \{ab - m\}$	and	$\sim' = \sim + \{\delta c - m, ab - c\}$	when $m \neq \epsilon$
(2)	$\sqsubseteq' = \sqsubseteq + \{am - b\}$	and	$\sim' = \sim + \{cm - b, \delta a - c\}$	
(3)	$\sqsubseteq' = \sqsubseteq + \{m - b\}$	and	$\sim' = \sim + \{\delta m - b\}$	
(3′)	$\sqsubseteq' = \sqsubseteq + \{m - b\}$	and	$\sim' = \sim + \{\delta m - b, \epsilon - \epsilon\}$	
(4)	$\sqsubseteq' = \sqsubseteq + \{\epsilon - m\}$	and	$\sim' = \sim + \{\delta c - m, \epsilon - c\}$	when $m \neq \epsilon$
(4′)	$\sqsubseteq' = \sqsubseteq + \{\epsilon - m\}$	and	$\sim' = \sim + \{\epsilon - c, m - \delta\}$	when $m \neq \epsilon$.

Proof. Here we only provide the proof of case 2 as an illustration of the type of arguments involved. The rest of the proof (cases 1, 3, 3′, 4 and 4′) is postponed to Appendix B because of the overall length of the argument.

For case 2 we have $\sqsubseteq' = \sqsubseteq + \{am - b\}$ and $\sim' = \sim + \{cm - b, \delta a - c\}$. First, $\sqsubseteq'$ is clearly BI-elementary. The constraint $cm - b$ is obviously BBI-elementary with respect to $\sim$. Then, by Proposition 6.9, $\sim'' = \sim + \{cm - b\}$ has no square and $I_{\sim''} = I_{\sim}$. As $c \notin I_{\sim}$ (because $c \notin A_{\sim}$), we have $c \notin I_{\sim''}$, hence $c \nsim'' \epsilon$. Thus $\delta a - c$ is BBI-elementary with respect to $\sim''$. So $\sim' = \sim'' + \{\delta a - c\}$ is BBI-elementary and has no square, and $I_{\sim'} = I_{\sim''} = I_{\sim} \subseteq M$.

We have $A_{\sim'} = A_{\sim} \cup \{a, b, \delta, c\}$. Let $d \in M$ be such that $d \sim' d$. Then either $d \in A_{\sim}$ or $d = c$. On the one hand, if $d \in A_{\sim}$, then $d \sim d$, and we let $x \in L^{\star}$ and $\alpha \in K^{\star}$ be such that $x\alpha \sim d$. Hence $x\alpha \sim' d$ because $\sim \subseteq \sim'$. On the other hand, if $d = c$, then $a\delta \sim' d$ with $a \in L^{\star}$ and $\delta \in K^{\star}$.

As $cm \sim' b$ and $\delta a \sim' c$, by rule $\langle e_l \rangle$, we obtain $\delta am \sim' b$, hence $am \sqsubseteq_{\sim'}^{L,K} b$. As $\sqsubseteq = \sqsubseteq_{\sim}^{L,K} \subseteq \sqsubseteq_{\sim'}^{L,K}$, we get $\sqsubseteq \cup \{am - b\} \subseteq \sqsubseteq_{\sim'}^{L,K}$ and obtain $\sqsubseteq' \subseteq \sqsubseteq_{\sim'}^{L,K}$.

We now consider the converse inclusion $\sqsubseteq_{\sim'}^{L,K} \subseteq \sqsubseteq'$, which is the tricky part of the proof. We have the following identities according to Proposition 6.8 and Lemma 6.4 ($c \nsim'' \epsilon$ and thus $cc \not\sim'' cc$):

$$\begin{aligned}
\sim'' = \sim + \{cm - b\} = \sim &\cup \{cx - cy \mid x \sim y \wedge \exists k\ xk \sim m\} \\
&\cup \{cx - jb, jb - cx \mid x \sim m \wedge j \sim \epsilon\} \\
&\cup \{ib - jb \mid i \sim \epsilon \wedge j \sim \epsilon\} \\
\sim' = \sim'' + \{\delta a - c\} = \sim'' &\cup \{\delta x - \delta y, ax - ay \mid x \sim'' y \wedge cx \sim'' cy\} \\
&\cup \{\delta ax - \delta ay \mid cx \sim'' cy\} \\
&\cup \{\delta ax - y, y - \delta ax \mid cx \sim'' y\}.
\end{aligned}$$

Let $\gamma \in K^\star$ and $x, y \in L^\star$ be such that $\gamma x \sim' y$. We now prove that $x \sqsubseteq' y$ by considering each of the possible forms taken by $(\gamma x, y)$:

— $\gamma x \sim'' y$.
According to the equations for $\sim''$, the only two possibilities for this case are when $\gamma x \sim y$ or $(\gamma x, y) = (ib, jb)$ with $i \sim \epsilon$ and $j \sim \epsilon$ (otherwise the letter $c \notin L \cup K$ occurs either on the left or on the right). Clearly, if $\gamma x \sim y$, then $x \sqsubseteq y$, so $x \sqsubseteq' y$. Consider the case where $(\gamma x, y) = (ib, jb)$ with $i \sim \epsilon$ and $j \sim \epsilon$. Then $i, j \in I_\sim^\star$. As $I_\sim \subseteq M$ and $i \prec \gamma x \in (L \cup K)^\star$, we must have $i = \epsilon$. As $j \prec y \in L^\star$, we must have $j = \epsilon$. Hence $(\gamma x, y) = (b, b)$. Thus $\gamma = \epsilon$ and $(x, y) = (b, b)$. As $am \sqsubseteq' b$, by rule $\langle r \rangle$, we get $b \sqsubseteq' b$, so $x \sqsubseteq' y$.

— $(\gamma x, y) = (\delta x', \delta y')$.
This case is impossible because $\delta \not\prec y$ ($\delta \notin L$).

— $(\gamma x, y) = (ax', ay')$ with $x' \sim'' y'$ and $cx' \sim'' cy'$.
The only possibility for $cx' \sim'' cy'$ is when $x' \sim y'$ and $x'k \sim m$ for some k. Thus $y'k \sim m$ by rules $\langle s \rangle$ and $\langle e_l \rangle$ and $k \sim k$ by rule $\langle p_l \rangle$. By Proposition 6.12, there exists $z \in L^\star$ and $\alpha \in K^\star$ such that $z\alpha \sim k$. So $y'z\alpha \sim m$ by rule $\langle e_l \rangle$. As $y' \prec y \in L^\star$, we get $y'z \sqsubseteq m$. As $\gamma x = ax'$ and $a \notin A_\gamma \subseteq K$, we have $a \prec x$ and let $ax'' = x$. Hence $x' = \gamma x''$, so $\gamma x'' \sim y'$ and thus $x'' \sqsubseteq y'$. As $\sqsubseteq \subseteq \sqsubseteq'$, we get $y'z \sqsubseteq' m$ and $x'' \sqsubseteq' y'$. Consider the following deduction tree:

$$\dfrac{\dfrac{\dfrac{y'z \sqsubseteq' m \quad am \sqsubseteq' b}{ay'z \sqsubseteq' b}\langle e_l \rangle}{ay' \sqsubseteq' ay'}\langle p_l \rangle \qquad x'' \sqsubseteq' y'}{ax'' \sqsubseteq' ay'}\langle c \rangle$$

Hence $x = ax'' \sqsubseteq' ay' = y$.

— $(\gamma x, y) = (\delta ax', \delta ay')$.
This case is impossible because $\delta \not\prec y$.

— $(\gamma x, y) = (\delta ax', y)$ with $cx' \sim'' y$.
The only possibility according to the equations for $\sim''$ is $(cx', y) = (cx', jb)$ with $x' \sim m$ and $j \sim \epsilon$. Then $j \in I_\sim^\star$, so $j \in M^\star$. But $j \prec y \in L^\star$. So $j = \epsilon$ and $y = b$. As $\gamma x = \delta ax'$, we have $\delta \prec \gamma$ and $a \prec x$. Let $ax'' = x$ and $\delta\gamma' = \gamma$. Then $\delta\gamma' ax'' = \delta ax'$, so $x' = \gamma' x''$.

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Hence $\gamma' x'' \sim m$. As $\gamma' \prec \gamma \in K^\star$, $x'' \prec x \in L^\star$ and $m \in A^\star_\sqsubseteq \subseteq L^\star$, we deduce $x'' \sqsubseteq m$. Hence $x'' \sqsubseteq' m$. As $am \sqsubseteq' b$, we obtain $ax'' \sqsubseteq' b$ by rule $\langle e_l \rangle$. But $ax'' = x$ and $b = y$, so $x \sqsubseteq' y$.

— $(\gamma x, y) = (\gamma x, \delta a y')$.
This case is impossible because $\delta \not\prec y$.

We have proved that for any $\gamma \in K^\star$ and $x, y \in L^\star$, if $\gamma x \sim' y$, then $x \sqsubseteq' y$. Thus $\sqsubseteq^{L,K}_{\sim'} \subseteq \sqsubseteq'$. So we have indeed proved that $\sqsubseteq' = \sqsubseteq^{L,K}_{\sim'}$ for case 2. □

Note that even if in case 4, $\sim'$ does not 'look' BBI-elementary (because c is not new to $\sim + \{\delta c - m\}$), it is in fact BBI-elementary when viewed in the form of case 4′ (see the proof in Appendix B).

6.6. *From simple* PMOs *to simple* PMEs

The notion of an elementary representation is thus a useful tool for maintaining the relation $\sqsubseteq = \sqsubseteq^{L,K}_\sim$ between the BI-elementary PMO $\sqsubseteq$ and the BBI-elementary $\sim$ when $\sqsubseteq$ is enriched with new BI-elementary constraints. What happens with the limit of increasing sequences of BI-elementary PMOs, that is, BI-simple PMOs? The following theorem provides an answer to this question.

Theorem 6.14. Let L be an alphabet and $\sqsubseteq$ be a BI-simple PMO over L. There exist two alphabets K and L' and a BBI-simple PME $\sim$ over L' such that $L \cup K \subseteq L'$ and $\sqsubseteq = \sqsubseteq^{L,K}_\sim$.

Proof. Let $\sqsubseteq$ be described by the BI-simple sequence of constraints $(x_i - y_i)_{i<\infty}$ over the alphabet L: we have $\sqsubseteq = \sqsubseteq_{\mathscr{C}}$ with $\mathscr{C} = \{x_i - y_i \mid i < \infty\}$. Let $\sqsubseteq_p = \sqsubseteq_{\mathscr{C}_p}$ with $\mathscr{C}_p = \{x_i - y_i \mid i < p\}$ for $p \in \mathbb{N} \cup \{\infty\}$. Then $\sqsubseteq = \sqsubseteq_\infty$.

Let $K = \{\delta_0, \delta_1, \ldots\}$ and $M = \{c_0, c_1, \ldots\}$ be two infinite, countable and disjoint sets such that $(K \cup M) \cap L = \varnothing$. Let $L' = L \cup K \cup M$. We build the sequence $(x'_i - y'_i)_{i<\infty}$ of constraints over L' according to the following table (the case column refers to the terminology of Lemma 6.13):

case	$x_i - y_i$	$x'_{2i} - y'_{2i}$	$x'_{2i+1} - y'_{2i+1}$
1	$ab - m$	$\delta_i c_i - m$	$ab - c_i$
2	$am - b$	$c_i m - b$	$\delta_i a - c_i$
3′	$m - b$	$\delta_i m - b$	$\epsilon - \epsilon$
4′	$\epsilon - m$	$\epsilon - c_i$	$m - \delta_i$
	$\epsilon - \epsilon$	$\epsilon - \epsilon$	$\epsilon - \epsilon$

Let $\mathscr{D}_p = \{x'_i - y'_i \mid i < p\}$ with $p \in \mathbb{N} \cup \{\infty\}$ and let $\sim_p = \sim_{\mathscr{D}_p}$. By Lemma 6.13, one can check by induction on p that for any $p < \infty$, $(\sqsubseteq_p, \sim_{2p})$ is an (L, K, M) elementary representation. Hence $\sim_{2p}$ is a BBI-elementary PME, $I_{\sim_{2p}} \subseteq M$ and $\sqsubseteq_p = \sqsubseteq^{L,K}_{\sim_{2p}}$.

We will now prove that $\sqsubseteq_\infty = \sqsubseteq^{L,K}_{\sim_\infty}$. If $x \sqsubseteq_\infty y$, then, by compactness (see Proposition 3.17), there exists $p < \infty$ such that $x \sqsubseteq_p y$. Then $x \sqsubseteq^{L,K}_{\sim_{2p}} y$. So there exists $\delta \in K^\star$ such that $\delta x \sim_{2p} y$. Then $\delta x \sim_\infty y$ as $\sim_{2p} \subseteq \sim_\infty$. Hence $x \sqsubseteq^{L,K}_{\sim_\infty} y$. Conversely, if $x \sqsubseteq^{L,K}_{\sim_\infty} y$, there exists $\delta \in K^\star$ such that $\delta x \sim_\infty y$. By compactness again, there exists $q < \infty$ such that $\delta x \sim_q y$. Then, as $\sim_q \subseteq \sim_{2q}$ (because $\mathscr{D}_q \subseteq \mathscr{D}_{2q}$), we have $\delta x \sim_{2q} y$, so $x \sqsubseteq_q y$. Thus $x \sqsubseteq_\infty y$ as $\sqsubseteq_q \subseteq \sqsubseteq_\infty$.

So we have proved that $\sqsubseteq = \sqsubseteq_\infty = \sqsubseteq^{L,K}_{\sim_\infty}$ where $\sim_\infty = \sim_{\mathscr{D}_\infty}$ and $\mathscr{D}_\infty = \{x'_i - y'_i \mid i < \infty\}$. The sequence $(x'_i - y'_i)_{i<\infty}$ is a BBI-simple sequence of constraints because, for each $p < \infty$, $x_p - y_p$ is BBI-elementary with respect to $\sim_p$ (indeed, $p < 2p + 2$ and $\sim_{2p+2}$ is BBI-elementary). Thus $\sim_\infty$ is a BBI-simple PME. □

7. Soundness of the embedding of BI into BBI

The map $F \mapsto F^\circ$ looks like a good candidate for embedding BI into BBI. Indeed, given an invalid formula F of BI, by Theorem 5.9, it is possible to obtain a counter-model of F of the form of a BI-simple PMO $\sqsubseteq$ over some language L, that is, $\epsilon \nVdash_\sqsubseteq F$. Then by Theorem 6.14, there exists a (BBI-simple) PME $\sim$ such that $\sqsubseteq = \sqsubseteq^{L,K}_\sim$. So, by Theorem 4.4, we obtain $\epsilon \nVdash_\sim F^\circ$, and thus a counter-model of F°.

It may seem that we have our embedding, but, unfortunately, F° is not necessarily BBI-valid when F is BI-valid. The mapping $F \mapsto F^\circ$ is not exactly the embedding we are looking for. It preserves counter-models but does not preserve provability.

Indeed, nothing in F° captures the special roles played by the two spare variables L and K. We have to incorporate some information about L and K that logically encodes the way they are interpreted in the particular model of Theorem 4.4, where they are forced by words belonging to sub-languages generated by sub-alphabets $L^\star$ and $K^\star$, respectively. So let H be the formula

$$\mathsf{H} \equiv (\mathsf{L} \wedge \mathsf{K}) \wedge \big((\top -\!\!* (\mathsf{L} * \mathsf{L} \to \mathsf{L})) \wedge (\top -\!\!* (\mathsf{K} * \mathsf{K} \to \mathsf{K}))\big).$$

For example, the sub-formula $\top -\!\!* (\mathsf{L} * \mathsf{L} \to \mathsf{L})$ encodes the property that the decomposition of words forcing L yields words forcing L, a subword property typical of sub-languages generated by sub-alphabets.

We are going to state and prove that $(\mathsf{I} \wedge \mathsf{H}) \to G^\circ$ is BBI-valid whenever G is BI-valid. And then we will prove that $(\mathsf{I} \wedge \mathsf{H}) \to G^\circ$ is BBI-invalid whenever G is BI-invalid.

7.1. *From* TBI-*tableaux to* TBBI-*tableaux*

In this section we describe how to process a TBI-tableau for G and obtain a corresponding TBBI-tableau for $(\mathsf{I} -\!\!* (\mathsf{I} \wedge \mathsf{H})) \to (\mathsf{I} * G^\circ)$. We have chosen this translation instead of the simpler $(\mathsf{I} \wedge \mathsf{H}) \to G^\circ$ (these two formulae are logically equivalent in BBI, see Proposition 7.5) because we can provide a direct tableau translation procedure for it, as described in the following results.

Lemma 7.1. Let $L' = L \cup K \cup M$ be a partition of L' where K and M are two disjoint infinite sets of spare letters. Let $\mathscr{T}$ be a TBI-tableau for $G \in \mathrm{BI}$ over L. There exists a

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TBBI-tableau $\mathscr{T}'$ for $(\mathsf{I} \twoheadrightarrow (\mathsf{I} \wedge \mathsf{H})) \to (\mathsf{I} * G^\circ)$ over L' and an injective map φ from the branches of $\mathscr{T}$ to the branches of $\mathscr{T}'$ such that:

(1) Each branch γ' of $\mathscr{T}'$ contains either $\mathbb{F}\mathsf{I} : \epsilon$ or the following set of statements:

$$\{\mathbb{T}\mathsf{L} : \epsilon, \mathbb{T}\top \twoheadrightarrow (\mathsf{L} * \mathsf{L} \to \mathsf{L}) : \epsilon\} \cup \{\mathbb{T}\mathsf{K} : \epsilon, \mathbb{T}\top \twoheadrightarrow (\mathsf{K} * \mathsf{K} \to \mathsf{K}) : \epsilon\}$$
$$\cup \{\mathbb{T}\mathsf{L} : a \mid a \in L \cap \mathsf{A}_{\gamma'}\} \qquad \cup \{\mathbb{T}\mathsf{K} : \delta_0 \mid \delta_0 \in K \cap \mathsf{A}_{\gamma'}\}.$$

(2) For each branch γ of $\mathscr{T}$, with $\gamma' = \varphi(\gamma)$, the following two conditions hold:

(2.1) $(\sqsubseteq_\gamma, \sim_{\gamma'})$ is an (L, K, M) elementary representation.

(2.2) For every statement $\mathbb{S}A : m$ of γ, the statement $\mathbb{S}A^\circ : m$ occurs in γ'.

(3) For each other branch γ' of $\mathscr{T}'$ (that is, ones that are not the image $\varphi(\gamma)$ of some branch γ of $\mathscr{T}$), at least one of the following conditions hold:

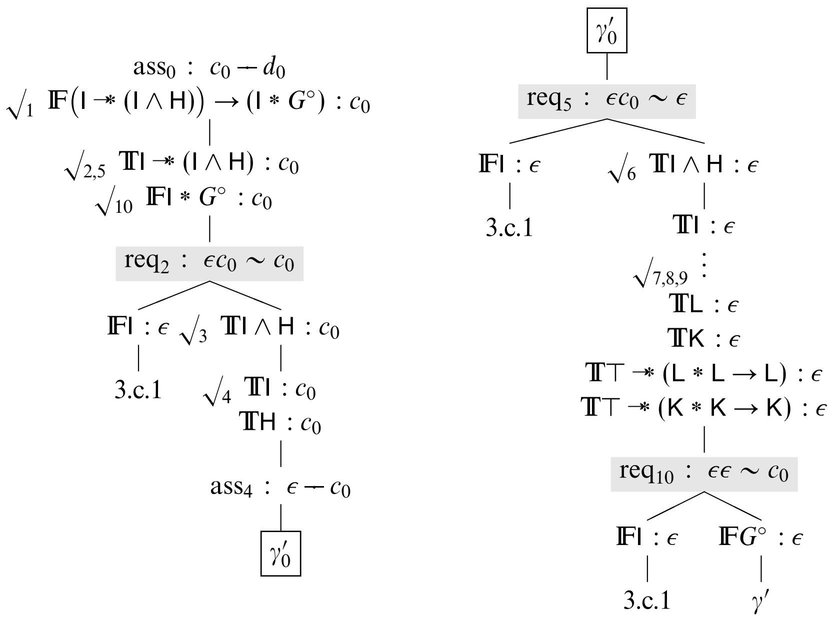
(3.c.1) $\mathbb{F}\mathsf{I} : \epsilon \in \gamma'$.

(3.c.2) $\mathbb{F}\mathsf{L} : l \in \gamma'$ for some $l \in L^\star$.

(3.c.3) $\mathbb{F}\mathsf{K} : \delta \in \gamma'$ for some $\delta \in K^\star$.

Proof. We build $\mathscr{T}'$ and φ by following the construction process of $\mathscr{T}$, the TBI-tableau of G.

— First consider the initial TBI-tableau $[\mathbb{F}G : \epsilon]$. We choose two letters $c_0 \neq d_0 \in M$. The following diagram shows a TBBI-tableau for $(\mathsf{I} \twoheadrightarrow (\mathsf{I} \wedge \mathsf{H})) \to (\mathsf{I} * G^\circ)$ that fulfills conditions 1, 2 and 3. It has been split into two parts to save space on the page: the two parts should be glued together at point γ'_0:



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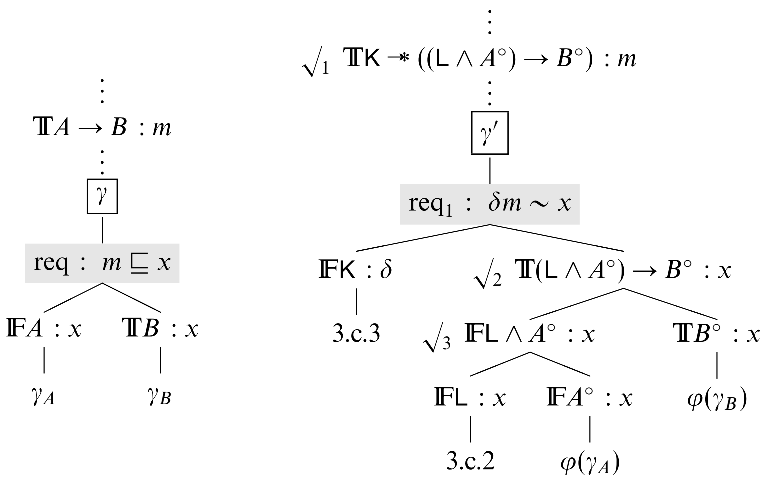
The reader is invited to check that this is indeed a valid TBBI-tableau. Note that the formula $\mathsf{H} \equiv (\mathsf{L} \wedge \mathsf{K}) \wedge ((\top \twoheadrightarrow (\mathsf{L} * \mathsf{L} \rightarrow \mathsf{L})) \wedge (\top \twoheadrightarrow (\mathsf{K} * \mathsf{K} \rightarrow \mathsf{K}))$ is developed at history checkpoints 7, 8 and 9 by three applications of expansion rule $\mathbb{T}\wedge$ (in fact, these have been omitted from the TBBI-tableau in the diagram to shorten its length). This TBBI-tableau has four branches, three of which stop at $\mathbb{F}\mathsf{I} : \epsilon$ (hence fulfilling conditions 1 and 3.c.1), and the last one containing $\mathbb{F}G^\circ : \epsilon$ as required by condition 2.2. This is the branch $\gamma' = \varphi(\gamma)$ associated through φ to the unique branch γ of the one node TBI-tableau $[\mathbb{F}G : \epsilon]$. It is obvious that γ' fulfills condition 1 because $\mathsf{A}_{\gamma'} = \{c_0, d_0\} \subseteq M$, so $\mathsf{A}_{\gamma'} \cap (L \cup K) = \varnothing$. It fulfills condition 2.1 for the following reasons. For $\mathscr{C} = \{\epsilon - c_0, \epsilon - d_0\}$ and $\mathscr{C}_{\gamma'} = \{c_0 - d_0, \epsilon - c_0\}$, we have $\sim_{\mathscr{C}} = \sim_{\mathscr{C}_{\gamma'}}$, so $\sqsubseteq_\gamma = \sqsubseteq_\varnothing = \{\epsilon - \epsilon\}$ and $\sim_{\gamma'} = \sim_{\mathscr{C}}$ where $\mathscr{C} = \{\epsilon - c_0, \epsilon - d_0\}$ and $c_0, d_0 \in M$. Thus $\sim_{\gamma'} = \{x - y \mid x, y \in \{c_0, d_0\}^\star\}$, and we can check that $\sqsubseteq_\gamma = \sqsubseteq^{L,K}_{\sim_{\gamma'}}$. So $(\sqsubseteq_\gamma, \sim_{\gamma'})$ forms an (L, K, M) elementary representation.

— If $\mathscr{T}$ is a TBI-tableau of G obtained by expansion using rule $\mathbb{T}\wedge$, $\mathbb{F}\wedge$, $\mathbb{T}\vee$ or $\mathbb{F}\vee$, we trivially use the same expansion rule for $\mathscr{T}'$ and fix the mapping φ accordingly.

— If $\mathscr{T}$ is obtained by expansion of $\mathbb{T}A \rightarrow B : m$ in branch γ, the requirement $m \sqsubseteq_\gamma x$ must hold, so there exists $\delta \in K^\star$ such that $\delta m \sim_{\gamma'} x$ holds. We extend γ' as follows:



And we extend φ so that the branch containing $\mathbb{F}A : x$ (respectively, $\mathbb{T}B : x$) in $\mathscr{T}$ corresponds to the branch of $\mathbb{F}A^\circ : x$ (respectively, $\mathbb{T}B^\circ : x$) in $\mathscr{T}'$. No assertion is generated, so $(\sqsubseteq_\gamma, \sim_{\gamma'})$ does not change on either branches $(\gamma_A, \varphi(\gamma_A))$ or branches $(\gamma_B, \varphi(\gamma_B))$, and is thus still an elementary representation. We also see that the two remaining branches contain $\mathbb{F}\mathsf{K} : \delta$ with $\delta \in K^\star$ and $\mathbb{F}\mathsf{L} : x$ with $x \in L^\star$, respectively, thereby fulfilling condition 3.c.3 and 3.c.2, respectively.

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— If $\mathscr{T}$ is obtained by expansion of $\mathbb{F}A \to B : m$ in branch γ, then $b \in L \backslash \mathsf{A}_\gamma$. As $\mathsf{A}_\gamma = \mathsf{A}_{\gamma'} \cap L$, we deduce $b \in L \backslash \mathsf{A}_{\gamma'}$. We choose $\delta_1 \in K \backslash \mathsf{A}_{\gamma'}$ (this is possible because K is infinite and $\mathsf{A}_{\gamma'}$ is finite as $\sim_{\gamma'}$ is BBI-elementary) and apply the following expansion rules to the branch γ':

$$
\begin{array}{c}
\vdots \\
\mathbb{F}A \to B : m \\
\vdots \\
\boxed{\gamma} \\
| \\
\mathrm{ass} : \ m - b \\
\mathbb{T}A : b \\
\mathbb{F}B : b \\
| \\
\gamma_{\to}
\end{array}
\qquad\qquad
\begin{array}{c}
\vdots \\
\surd_1 \ \mathbb{F}\mathsf{K} \twoheadrightarrow ((\mathsf{L} \wedge A^\circ) \to B^\circ) : m \\
\vdots \\
\boxed{\gamma'} \\
| \\
\mathrm{ass}_1 : \ \delta_1 m - b \\
\mathbb{T}\mathsf{K} : \delta_1 \\
\surd_2 \ \mathbb{F}(\mathsf{L} \wedge A^\circ) \to B^\circ : b \\
| \\
\surd_3 \ \mathbb{T}\mathsf{L} \wedge A^\circ : b \\
\mathbb{F}B^\circ : b \\
| \\
\mathbb{T}\mathsf{L} : b \\
\mathbb{T}A^\circ : b \\
| \\
\varphi(\gamma_{\to})
\end{array}
$$

Then $(\sqsubseteq_\gamma + \{m - b\}, \sim_{\gamma'} + \{\delta_1 m - b\})$ is an elementary representation according to Lemma 6.13 case 3 and φ is extended in the obvious way. We also note that $\mathbb{T}\mathsf{K} : \delta_1$ and $\mathbb{T}\mathsf{L} : b$ are introduced, thereby fulfilling condition 1.

— If $\mathscr{T}$ is a TBI-tableau of G obtained by expanding $\mathbb{T}\mathsf{I} : m$ on branch γ, then let $\gamma' = \varphi(\gamma)$ and choose $\delta_1 \in K \backslash \mathsf{A}_{\gamma'}$ and $c_1 \in M \backslash \mathsf{A}_{\gamma'}$. We then apply the following expansion rules to the branch γ':

$$
\begin{array}{c}
\vdots \\
\mathbb{T}\mathsf{I} : m \\
\vdots \\
\boxed{\gamma} \\
| \\
\mathrm{ass} : \ \epsilon - m \\
| \\
\gamma_{\mathsf{I}}
\end{array}
\qquad\qquad
\begin{array}{c}
\vdots \\
\surd_1 \ \mathbb{T}\mathsf{K} * \mathsf{I} : m \\
\vdots \\
\boxed{\gamma'} \\
| \\
\mathrm{ass}_1 : \ \delta_1 c_1 - m \\
\mathbb{T}\mathsf{K} : \delta_1 \\
\surd_2 \ \mathbb{T}\mathsf{I} : c_1 \\
| \\
\mathrm{ass}_2 : \ \epsilon - c_1 \\
| \\
\varphi(\gamma_{\mathsf{I}})
\end{array}
$$

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Then $(\sqsubseteq_\gamma + \{\epsilon - m\}, \sim_{\gamma'} + \{\delta_1 c_1 - m, \epsilon - c_1\})$ forms an elementary representation according to Lemma 6.13 case 4 because $m \neq \epsilon$ (Proposition 5.7 applied to $\mathscr{T}$). We also note that the statement $\mathbb{T}\mathsf{K} : \delta_1$ is introduced, which fulfills condition 1.

— If $\mathscr{T}$ is obtained by expansion of $\mathbb{T}A * B : m$ in branch γ, then a, b must have been chosen such that $a \neq b \in L \backslash \mathsf{A}_\gamma$. As $(\sqsubseteq_\gamma, \sim_{\gamma'})$ is an elementary representation, we have $\mathsf{A}_\gamma = \mathsf{A}_{\gamma'} \cap L$ (see Proposition 6.12), and hence $a \neq b \in L \backslash \mathsf{A}_{\gamma'}$. We choose $\delta_1 \in K \backslash \mathsf{A}_{\gamma'}$ and $c_1 \in M \backslash \mathsf{A}_{\gamma'}$, and apply the following expansion rules (on the right-hand side) to the branch γ':

$$
\begin{array}{c}
\vdots \\
\mathbb{T}A * B : m \\
\vdots \\
\boxed{\gamma} \\
| \\
\text{ass} : \ ab - m \\
\mathbb{T}A : a \\
\mathbb{T}B : b \\
| \\
\gamma_*
\end{array}
\qquad\qquad
\begin{array}{c}
\vdots \\
\surd_1 \ \mathbb{T}\mathsf{K} * \big((\mathsf{L} \wedge A^\circ) * (\mathsf{L} \wedge B^\circ)\big) : m \\
\vdots \\
\boxed{\gamma'} \\
| \\
\text{ass}_1 : \ \delta_1 c_1 - m \\
\mathbb{T}\mathsf{K} : \delta_1 \\
\surd_2 \ \mathbb{T}(\mathsf{L} \wedge A^\circ) * (\mathsf{L} \wedge B^\circ) : c_1 \\
| \\
\text{ass}_2 : \ ab - c_1 \\
\surd_3 \ \mathbb{T}\mathsf{L} \wedge A^\circ : a \\
\surd_4 \ \mathbb{T}\mathsf{L} \wedge B^\circ : b \\
| \\
\mathbb{T}\mathsf{L} : a \\
\mathbb{T}A^\circ : a \\
| \\
\mathbb{T}\mathsf{L} : b \\
\mathbb{T}B^\circ : b \\
| \\
\varphi(\gamma_*)
\end{array}
$$

Then $(\sqsubseteq_\gamma + \{ab - m\}, \sim_{\gamma'} + \{\delta_1 c_1 - m, ab - c_1\})$ is an (L, K, M) elementary representation according to Lemma 6.13 case 1, having checked that $m \neq \epsilon$ (Proposition 5.7 applied to $\mathscr{T}$). The injective map φ is extended in the obvious way since there is only one new branch and this branch fulfills condition 2. We also observe that $\mathbb{T}\mathsf{K} : \delta_1$, $\mathbb{T}\mathsf{L} : a$ and $\mathbb{T}\mathsf{L} : b$ are introduced, thereby fulfilling condition 1.

— If $\mathscr{T}$ is obtained by expansion of $\mathbb{F}A * B : m$ in branch γ, the requirement $xy \sqsubseteq_\gamma m$ must hold. As $\sqsubseteq_\gamma = \sqsubseteq^{L,K}_{\sim_{\gamma'}}$, there exists $\delta \in K^\star$ such that $\delta xy \sim_{\gamma'} m$. We extend γ' as

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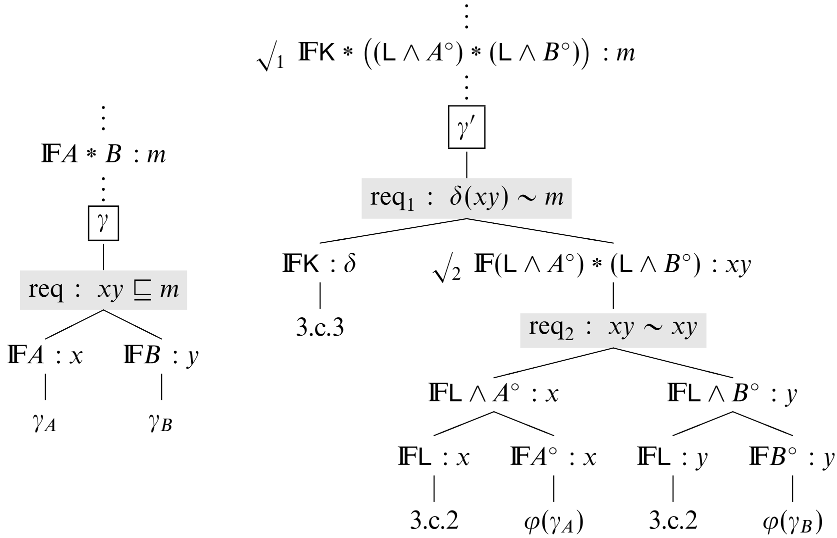
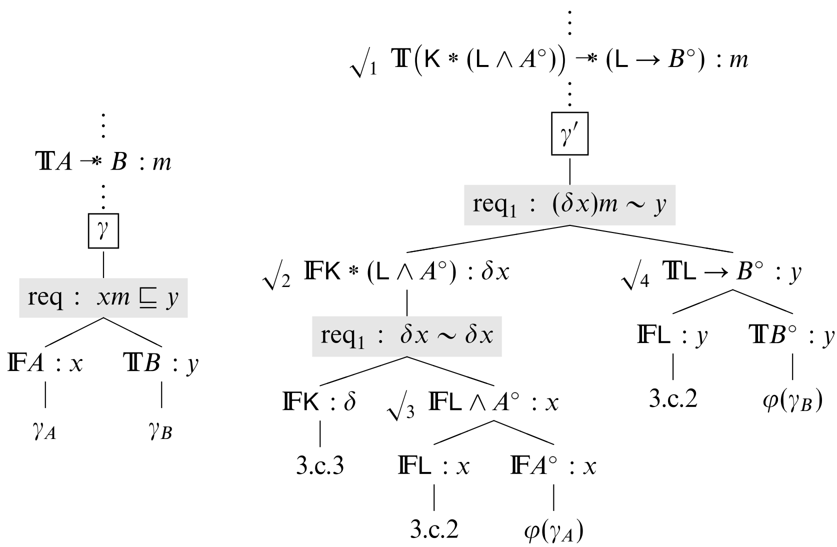
follows (on the right-hand side):



Then, whichever new branch is considered, no new assertion is introduced, so condition 1 is fulfilled and $(\sqsubseteq_\gamma, \sim_{\gamma'})$ is unchanged, so it is still an elementary representation. We extend φ so that the branch containing $\mathbb{F}A : x$ (respectively, $\mathbb{F}B : y$) in $\mathscr{T}$ corresponds to the branch of $\mathbb{F}A^\circ : x$ (respectively, $\mathbb{F}B^\circ : y$) in $\mathscr{T}'$. Hence condition 2 is fulfilled. We also observe that the three remaining branches contain $\mathbb{F}\mathsf{K} : \delta$ with $\delta \in K^\star$, $\mathbb{F}\mathsf{L} : x$ with $x \in L^\star$ and $\mathbb{F}\mathsf{L} : y$ with $y \in L^\star$, respectively, thereby fulfilling condition 3.

— If $\mathscr{T}$ is obtained by expansion of $\mathbb{T}A \twoheadrightarrow B : m$ in branch γ, the requirement $xm \sqsubseteq_\gamma y$ must hold. Then there exists $\delta \in K^\star$ such that $\delta x m \sim_{\gamma'} y$. We extend γ' as follows:



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Then $(\sqsubseteq_\gamma, \sim_{\gamma'})$ is unchanged and is still an elementary representation. We extend φ so that the branch containing $\mathbb{F}A : x$ (respectively, $\mathbb{T}B : y$) in $\mathscr{T}$ corresponds to the branch of $\mathbb{F}A^\circ : x$ (respectively, $\mathbb{T}B^\circ : y$) in $\mathscr{T}'$. We also see that the three remaining branches contain $\mathbb{F}\mathsf{K} : \delta$ with $\delta \in K^\star$, $\mathbb{F}\mathsf{L} : x$ with $x \in L^\star$ and $\mathbb{F}\mathsf{L} : y$ with $y \in L^\star$, respectively, thereby fulfilling condition 3.c.3, 3.c.2 and 3.c.2, respectively.

— If $\mathscr{T}$ is obtained by expansion of $\mathbb{F}A \mathrel{-\!\!*} B : m$ in branch γ, then $a \neq b \in L \backslash \mathsf{A}_\gamma$. As $(\sqsubseteq_\gamma, \sim_{\gamma'})$ is an elementary representation, we deduce $a \neq b \in L \backslash \mathsf{A}_{\gamma'}$. We choose $\delta_1 \in K \backslash \mathsf{A}_{\gamma'}$ and $c_1 \in M \backslash \mathsf{A}_{\gamma'}$ and apply the following expansion rules to γ':

$$
\begin{array}{c}
\vdots \\
\mathbb{F}A \mathrel{-\!\!*} B : m \\
\vdots \\
\boxed{\gamma} \\
| \\
\mathrm{ass} : \ am \mathbin{-} b \\
\mathbb{T}A : a \\
\mathbb{F}B : b \\
| \\
\gamma_{-\!*}
\end{array}
\qquad\qquad
\begin{array}{c}
\vdots \\
\surd_1\ \mathbb{F}\big(\mathsf{K} * (\mathsf{L} \wedge A^\circ)\big) \mathrel{-\!\!*} (\mathsf{L} \to B^\circ) : m \\
\vdots \\
\boxed{\gamma'} \\
| \\
\mathrm{ass}_1 : \ c_1 m \mathbin{-} b \\
\surd_2\ \mathbb{T}\mathsf{K} * (\mathsf{L} \wedge A^\circ) : c_1 \\
\surd_4\ \mathbb{F}\mathsf{L} \to B^\circ : b \\
| \\
\mathrm{ass}_2 : \ \delta_2 a \mathbin{-} c_1 \\
\mathbb{T}\mathsf{K} : \delta_2 \\
\surd_3\ \mathbb{T}\mathsf{L} \wedge A^\circ : a \\
| \\
\mathbb{T}\mathsf{L} : a \\
\mathbb{T}A^\circ : a \\
| \\
\mathbb{T}\mathsf{L} : b \\
\mathbb{F}B^\circ : b \\
| \\
\varphi(\gamma_{-\!*})
\end{array}
$$

Then $(\sqsubseteq_\gamma + \{am \mathbin{-} b\}, \sim_{\gamma'} + \{c_1 m \mathbin{-} b, \delta_1 a \mathbin{-} c_1\})$ is an elementary representation according to Lemma 6.13 case 2, having checked that $m \neq \epsilon$ (Proposition 5.7 applied to $\mathscr{T}$). We extend φ in the obvious way. We also observe that $\mathbb{T}\mathsf{K} : \delta_1$, $\mathbb{T}\mathsf{L} : a$ and $\mathbb{T}\mathsf{L} : b$ are introduced, thereby fulfilling condition 1. □

7.2. *From* BI-*proofs to* BBI-*proofs*

We have shown how a TBI-tableau can be transformed into a TBBI-tableau. We now show that a closed TBI-tableau (that is, a proof) can be transformed into a closed TBBI-tableau, thus obtaining the soundness part of our embedding. This is done in two steps: we first obtain a pseudo-closed TBBI-tableau and then close the pseudo-closed TBBI-tableau.

Definition 7.2. A (L, K) *pseudo-closed* TBBI-*tableau* is a TBBI-tableau $\mathscr{T}$ in which every open branch γ verifies the following two conditions:

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(1) γ contains the following set of statements:

$$\begin{array}{ll} \{\mathbb{T}\mathsf{L} : \epsilon, \mathbb{T}\top \mathrel{-\!\!*} (\mathsf{L} * \mathsf{L} \rightarrow \mathsf{L}) : \epsilon\} & \cup\ \{\mathbb{T}\mathsf{K} : \epsilon, \mathbb{T}\top \mathrel{-\!\!*} (\mathsf{K} * \mathsf{K} \rightarrow \mathsf{K}) : \epsilon\} \\ \cup\ \{\mathbb{T}\mathsf{L} : a \mid a \in L \cap \mathsf{A}_\gamma\} & \cup\ \{\mathbb{T}\mathsf{K} : \delta_0 \mid \delta_0 \in K \cap \mathsf{A}_\gamma\}. \end{array}$$

(3) γ verifies at least one of the following two conditions:

(3.c.2) $\mathbb{F}\mathsf{L} : l \in \gamma$ for some $l \in L^\star$.

(3.c.3) $\mathbb{F}\mathsf{K} : \delta \in \gamma$ for some $\delta \in K^\star$.

Note that we have maintained the same numbering as in Lemma 7.1 for the conditions that remain, which explains the gap in the numbering of the conditions this time.

Proposition 7.3. If the formula $G \in \mathsf{BI}$ has a closed TBI-tableau over the alphabet L, then the formula $(\mathsf{I} \mathrel{-\!\!*} (\mathsf{I} \wedge \mathsf{H})) \rightarrow (\mathsf{I} * G^\circ) \in \mathsf{BBI}$ has an (L, K) pseudo-closed TBBI-tableau for some alphabet K.

Proof. Let $\mathscr{T}$ be a closed TBI-tableau for G. According to Lemma 7.1, we build a corresponding TBBI-tableau $\mathscr{T}'$ for $G' = (\mathsf{I} \mathrel{-\!\!*} (\mathsf{I} \wedge \mathsf{H})) \rightarrow (\mathsf{I} * G^\circ)$ over $L \cup K \cup M$ and the injective map φ from (maximal) branches of $\mathscr{T}$ to (maximal) branches of $\mathscr{T}'$. Since $\mathscr{T}'$ verifies condition 1 of Lemma 7.1, each branch γ' of $\mathscr{T}'$ verifies condition 1 of Definition 7.2 because $\mathbb{F}\mathsf{I} : \epsilon \in \gamma'$ is a closure condition for TBBI-tableaux branches.

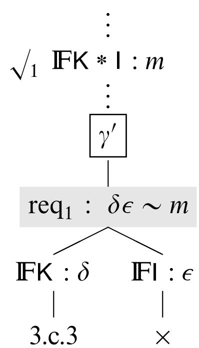
There are two kinds of branches in $\mathscr{T}'$: those that are images $\gamma' = \varphi(\gamma)$ of branches of $\mathscr{T}$ and those that are not. We consider the latter case first. According to condition 3 of Lemma 7.1, such a branch is either closed by $\mathbb{F}\mathsf{I} : \epsilon \in \gamma'$ or satisfies conditions 3.c.2 or 3.c.3. In all cases, the open branches that are not of the form $\gamma' = \varphi(\gamma)$ verify condition 3.c.2 or 3.c.3 of Definition 7.2.

Now consider the former case, in which branches are of the form $\gamma' = \varphi(\gamma)$. Since $\mathscr{T}$ is closed, each of its branches are closed, including γ in particular. The branch γ is closed by one of the following conditions:

— If $\mathbb{T}\bot : m \in \gamma$, then $\mathbb{T}\bot^\circ : m \in \gamma'$, and as $\bot^\circ = \bot$, the branch γ' is closed.

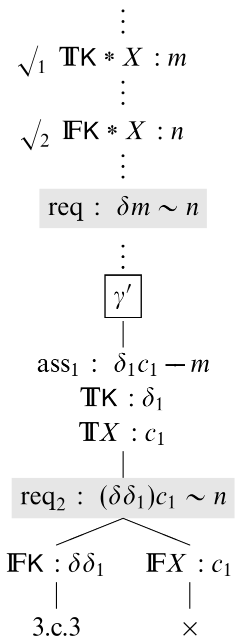
— For the same reason, if $\mathbb{F}\top : m \in \gamma$, then $\mathbb{F}\top : m \in \gamma'$ and γ' is closed.

— If $\mathbb{F}\mathsf{I} : m \in \gamma$ and $\epsilon \sqsubseteq_\gamma m$, then $\mathbb{F}\mathsf{K} * \mathsf{I} : m \in \gamma'$ (because $\mathsf{I}^\circ = \mathsf{K} * \mathsf{I}$) and there exists $\delta \in K^\star$ such that $\delta\epsilon \sim_{\gamma'} m$. Then we apply the following branch expansion rules:



We then replace γ' with two branches, one of which is closed by $\mathbb{F}\mathsf{I} : \epsilon$ and the other satisfying condition 3.c.3 of Definition 7.2. We also observe that condition 1 is still fulfilled by the left branch because, should it remain open, no new assertion is introduced, and thus the alphabet of the branch is left unchanged.

— If $\mathbb{T}X : m \in \gamma$, $\mathbb{F}X : n \in \gamma$ and $m \sqsubseteq_\gamma n$, then $\mathbb{T}\mathsf{K} * X : m \in \gamma$, $\mathbb{F}\mathsf{K} * X : n \in \gamma$ and there exists $\delta \in K^\star$ such that $\delta m \sim_{\gamma'} n$. We choose $\delta_1 \in K \backslash \mathsf{A}_{\gamma'}$ and $c_1 \in M \backslash \mathsf{A}_{\gamma'}$, and then apply the following branch expansion rules:



We check that $(\delta\delta_1)c_1 \sim_{\gamma''} n$ by rule $\langle e_l \rangle$ with $\gamma'' = [\gamma', \delta_1 c_1 - m, \mathbb{T}\mathsf{K} : \delta_1, \mathbb{T}X : c_1]$. γ' is replaced by two branches $\gamma'_l = [\gamma'', \mathbb{F}\mathsf{K} : \delta\delta_1]$ and $\gamma'_r = [\gamma'', \mathbb{F}X : c_1]$, the latter γ'_r being closed by $\mathbb{T}X : c_1$ and $\mathbb{F}X : c_1$, the former γ'_l satisfying condition 3.c.3 of Definition 7.2, should it be open. We also note that the statement $\mathbb{T}\mathsf{K} : \delta_1$ is introduced in γ''. So condition 1 is fulfilled because the alphabet $\mathsf{A}_{\gamma'_l}$ of the left branch verifies the equation $\mathsf{A}_{\gamma'_l} = \mathsf{A}_{\gamma'} \cup \{\delta_1, c_1\}$.

Applying these transformations for every branch $\gamma' = \varphi(\gamma)$ of $\mathscr{T}'$, we obtain a TBBI-tableau $\mathscr{T}''$ in which every open branch satisfies condition 1 and either condition 3.c.2 or 3.c.3 of Definition 7.2. Hence $\mathscr{T}''$ is a (L, K) pseudo-closed TBBI-tableau. □

Proposition 7.4. If a formula of BBI has an (L, K) pseudo-closed TBBI-tableau, it has a closed TBBI-tableau.

Proof. We define the *weight of a branch* γ' by 0 for a closed branch and otherwise by the length of the shortest word $x \in \mathsf{A}^\star_{\gamma'}$ such that either ($x \in L^\star$ and $\mathbb{F}\mathsf{L} : x \in \gamma'$) or ($x \in K^\star$ and $\mathbb{F}\mathsf{K} : x \in \gamma'$). The weight exists because condition 3.c.2 or 3.c.3 is fulfilled for

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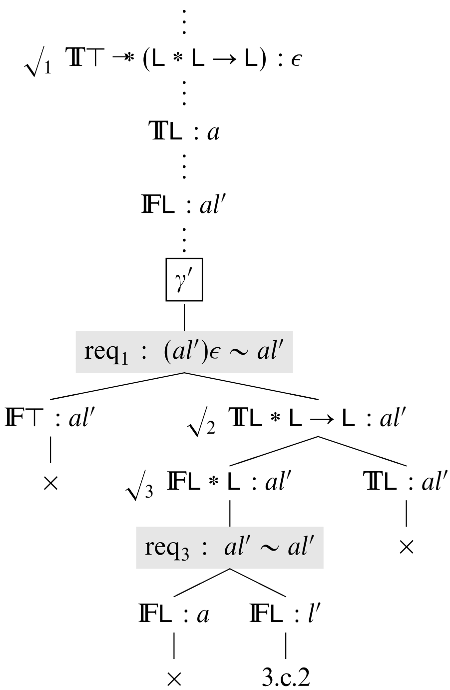
any open branch of a (L, K) pseudo-closed TBBI-tableau. The *weight of a whole* (L, K) *pseudo closed* TBBI-*tableau* is the sum of the weights of all its branches.

We will now prove that any (L, K) pseudo-closed TBBI-tableau can be expanded into a closed TBBI-tableau by induction on the weight of the tableau:

— The weight of the tableau is 0.
Consider any branch γ'. It must have weight 0 (because weights are positive numbers), so either γ' is closed or $\mathbb{F}\mathsf{L} : \epsilon \in \gamma'$ or $\mathbb{F}\mathsf{K} : \epsilon \in \gamma'$, ϵ being the only word of length 0. Reasoning by contradiction, if γ' is not closed, then by condition 1, $\mathbb{T}\mathsf{L} : \epsilon$ and $\mathbb{T}\mathsf{K} : \epsilon$ occur in γ'. Hence γ' is closed because either $\mathbb{F}\mathsf{L} : \epsilon$ or $\mathbb{F}\mathsf{K} : \epsilon$ occur in γ'. So if the weight of the tableau is 0, the tableau is closed because all its branches are closed, so there is no need to expand it.

— The weight of the tableau is not 0.
We choose a branch γ' of strictly positive weight $p > 0$. Then γ' is an open branch and we let $\mathbb{F}\mathsf{L} : l$ occur in γ' with $l \in L^\star$ of length p (the case $\mathbb{F}\mathsf{K} : \delta$ with $\delta \in K^\star$ of length p is treated similarly). As $p > 0$, we write $l = al'$ with $a \in L$ and $l' \in L^\star$. As $\mathbb{F}\mathsf{L} : l$ occurs in γ', we must have $l \sim_{\gamma'} l$ by Proposition 5.3, hence $al' \sim_{\gamma'} al'$. Then $a \in L \cap \mathsf{A}_{\gamma'}$, and by condition 1, the statements $\mathbb{T}\mathsf{L} : a$ and $\mathbb{T}\top \twoheadrightarrow (\mathsf{L} * \mathsf{L} \rightarrow \mathsf{L}) : \epsilon$ both occur in γ'. We apply to the branch γ' the expansion rules described by



The order in which the statements $\mathbb{T}\top \twoheadrightarrow (\mathsf{L} * \mathsf{L} \rightarrow \mathsf{L}) : \epsilon$, $\mathbb{T}\mathsf{L} : a$ and $\mathbb{F}\mathsf{L} : al'$ occur in the branch γ' is of no importance. The branch γ' is expanded into four branches, three of which are closed, and thus of weight 0, and the last one containing $\mathbb{F}\mathsf{L} : l'$, and thus of weight lower than $p-1$, which is the length of l'. So the TBBI-tableau obtained after such branch expansion has a strictly lower weight. It is also an (L, K) pseudo-closed TBBI-tableau because no assertion is inserted, hence condition 1 is still fulfilled, and the only new branch that is potentially open verifies condition 3.c.2. By the induction hypothesis, this TBBI-tableau can itself be expanded into a closed TBBI-tableau.

We have now proved by induction on the weight that any (L, K) pseudo-closed TBBI-tableau can be expanded into a closed TBBI-tableau. □

Proposition 7.5. For any two BBI-formulae A and B, the formulae $A \rightarrow B$ and $(\mathsf{I} -\!\!* A) \rightarrow (\mathsf{I} * B)$ are logically equivalent in BBI.

Proof. Let $(L, \sim, \Vdash)$ be a BBI-frame. For any $m \in \mathscr{L}^{\sim}$, by monotonicity of $\Vdash$, we can show that $m \Vdash A$ holds if and only if $m \Vdash \mathsf{I} -\!\!* A$ holds, and that $m \Vdash B$ holds if and only if $m \Vdash \mathsf{I} * B$. Thus $m \Vdash A \rightarrow B$ holds if and only if $m \Vdash (\mathsf{I} -\!\!* A) \rightarrow (\mathsf{I} * B)$ holds. □

Theorem 7.6 (Soundness of the embedding). Let G be a BI-formula not containing the spare logical variables L and K. If G is BI-valid, then $(\mathsf{I} \wedge \mathsf{H}) \rightarrow G^{\circ}$ is BBI-valid.

Proof. Let $G' = (\mathsf{I} -\!\!* (\mathsf{I} \wedge \mathsf{H})) \rightarrow (\mathsf{I} * G^{\circ})$. If G is BI-valid, then, by completeness of the TBI-tableau system (see Theorem 5.8), G has a closed TBI-tableau. Then, according to Proposition 7.3, G' has a (L, K) pseudo-closed TBBI-tableau. So, by Proposition 7.4, G' has a closed TBBI-tableau, and hence is BBI-valid by soundness of the TBBI-tableau system (see Theorem 5.14). Hence $(\mathsf{I} \wedge \mathsf{H}) \rightarrow G^{\circ}$ is BBI-valid because it is BBI-equivalent to G' by Proposition 7.5 (with $A = \mathsf{I} \wedge \mathsf{H}$ and $B = G^{\circ}$). □

8. Faithfulness by counter-model transformation

We have proved that if the BI-formula G is BI-valid, then the formula $(\mathsf{I} \wedge \mathsf{H}) \rightarrow G^{\circ}$ is BBI-valid. In this section we will show that if G has a BI-counter-model, then $(\mathsf{I} \wedge \mathsf{H}) \rightarrow G^{\circ}$ has a BBI-counter-model.

8.1. *From* BI-*counter-models to* BBI-*counter-models*

Theorem 8.1 (Faithfulness). Let G be a formula of BI not containing the variables L or K. If G has a BI-counter-model, the BBI-formula $(\mathsf{I} \wedge \mathsf{H}) \rightarrow G^{\circ}$ has a BBI-counter-model.

Proof. Suppose G has a BI-counter-model. Then it is BI-invalid and, by Theorem 5.9, it is possible to obtain a counter-model of G in the form of a BI-simple PMO $\sqsubseteq$ over some language L. So let $(L, \sqsubseteq, \Vdash_{\sqsubseteq})$ be a BI-frame where $\sqsubseteq$ is a BI-simple PMO and such that $\epsilon \nVdash_{\sqsubseteq} G$. By Theorem 6.14, there exists a BBI-simple PME $\sim$ over L' such that $L \cup K \subseteq L'$ and $\sqsubseteq = \sqsubseteq_{\sim}^{L,K}$. Thus, by Theorem 4.4, in the BBI-frame $(L', \sim, \Vdash_{\sim})$, we have $\epsilon \nVdash_{\sim} G^{\circ}$.

In the frame $(L', \sim, \Vdash_{\sim})$, we observe that $\epsilon \Vdash_{\sim} \mathsf{L}$ and $\epsilon \Vdash_{\sim} \mathsf{K}$. We now prove that $\epsilon \Vdash_{\sim} \top -\!\!* (\mathsf{L} * \mathsf{L} \rightarrow \mathsf{L})$. Let $x\epsilon \sim y$, $x \Vdash_{\sim} \top$ and $y \Vdash_{\sim} \mathsf{L} * \mathsf{L}$. We now prove $y \Vdash_{\sim} \mathsf{L}$. There exists y_1, y_2 such that $y_1 y_2 \sim y$, $y_1 \Vdash_{\sim} \mathsf{L}$ and $y_2 \Vdash_{\sim} \mathsf{L}$. Thus, by the definition of $(\cdot) \Vdash_{\sim} \mathsf{L}$, there exists $l_1, l_2 \in L^{\star}$ such that $l_1 \sim y_1$ and $l_2 \sim y_2$. Hence, by two applications of rule $\langle e_l \rangle$, we obtain $l_1 l_2 \sim y$. As $l_1 l_2 \in L^{\star}$, we conclude $y \Vdash_{\sim} \mathsf{L}$. We have proved that $\epsilon \Vdash_{\sim} \top -\!\!* (\mathsf{L} * \mathsf{L} \rightarrow \mathsf{L})$. By an identical argument, we show that $\epsilon \Vdash_{\sim} \top -\!\!* (\mathsf{K} * \mathsf{K} \rightarrow \mathsf{K})$.

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So we have $\epsilon \Vdash_\sim \mathsf{H}$. To finish, we get $\epsilon \nVdash_\sim (\mathsf{I} \wedge \mathsf{H}) \to G^\circ$, so $(L', \sim, \Vdash_\sim)$ is the desired BBI-counter-model. □

8.2. *Example of counter-model transformation*

In this section we explain how a BI-counter-model of $X \vee (X \to \bot)$ is transformed into a BBI-counter-model of $F = (\mathsf{I} \wedge \mathsf{H}) \to (\mathsf{K} * X \vee \mathsf{K} \mathrel{-\!\!*} ((\mathsf{L} \wedge \mathsf{K} * X) \to \bot))$, which is the BBI-formula associated to $X \vee (X \to \bot)$. We recall that $\mathsf{H} = (\mathsf{L} \wedge \mathsf{K}) \wedge \big((\top \mathrel{-\!\!*} (\mathsf{L} * \mathsf{L} \to \mathsf{L})) \wedge (\top \mathrel{-\!\!*} (\mathsf{K} * \mathsf{K} \to \mathsf{K}))\big)$.

One possible BI-counter-model for $X \vee (X \to \bot)$ is based on the PMO $\sqsubseteq = \sqsubseteq_{\mathscr{C}}$ over the language $L = \{b\}$ with $\mathscr{C} = \{\epsilon - b\}$. We can check that

$$\sqsubseteq = \sqsubseteq_{\mathscr{C}} = \{\epsilon - \epsilon, b - b, \epsilon - b\}.$$

We complete the BI-frame $(L = \{b\}, \sqsubseteq, \Vdash_\sqsubseteq)$ with $\epsilon \nVdash_\sqsubseteq X$ and $b \Vdash_\sqsubseteq X$ and check the monotonicity of the relation $\Vdash_\sqsubseteq$. Then we verify that we have a BI-counter-model of $X \vee (X \to \bot)$. Indeed, $b \nVdash_\sqsubseteq \bot$, and thus, as $\epsilon \sqsubseteq b$, we get $\epsilon \nVdash_\sqsubseteq X \to \bot$. Hence, $\epsilon \nVdash_\sqsubseteq X \vee (X \to \bot)$. This is the usual Kripke counter-model of the intuitionistic propositional formula $X \vee (X \to \bot)$.

The PMO $\sqsubseteq$ is clearly BI-elementary (case 3 of Definition 5.4 with $m = \epsilon$). According to Lemma 6.13 case 3, we compute $\mathscr{D} = \{\delta - b\}$ with $\delta \in K$: $(\sqsubseteq_\varnothing, \sim_\varnothing)$ is an (L, K, M) elementary representation, so $(\sqsubseteq_{\mathscr{C}}, \sim_{\mathscr{D}})$ is an (L, K, M) elementary representation.

Let $\sim = \sim_{\mathscr{D}}$. Then we can check the following identity:

$$\sim = \sim_{\mathscr{D}} = \{\epsilon - \epsilon, b - b, \delta - \delta, \delta - b, b - \delta\}.$$

As $\mathsf{A}_\sim \cap L = \{b\}$, $\mathsf{A}_\sim \cap K = \{\delta\}$ and $\mathsf{A}_\sim \cap M = \varnothing$, we can verify $\sqsubseteq = \sqsubseteq_\sim^{L,K}$.

As $(\{b\}, \sqsubseteq, \Vdash_\sqsubseteq)$ is a BI-counter-model for $X \vee (X \to \bot)$, let us complete $(\{b, \delta\}, \sim, \Vdash_\sim)$ into a BBI-counter-model for the translation F of $X \vee (X \to \bot)$. The forcing relation $\Vdash_\sim$ is given by Theorem 4.4: we have $\epsilon \Vdash_\sim \mathsf{L}, \mathsf{K}$, $\epsilon \nVdash_\sim X$, $b \Vdash_\sim X, \mathsf{L}, \mathsf{K}$ and $\delta \Vdash_\sim X, \mathsf{L}, \mathsf{K}$.

We now check that we do indeed have a BBI-counter-model of F. We have $\epsilon, b, \delta \Vdash_\sim \mathsf{L} * \mathsf{L} \to \mathsf{L}$ and $\epsilon, b, \delta \Vdash_\sim \mathsf{K} * \mathsf{K} \to \mathsf{K}$, so $\epsilon \Vdash_\sim \top \mathrel{-\!\!*} (\mathsf{L} * \mathsf{L} \to \mathsf{L})$ and $\epsilon \Vdash_\sim \top \mathrel{-\!\!*} (\mathsf{K} * \mathsf{K} \to \mathsf{K})$. We obtain $\epsilon \Vdash_\sim \mathsf{H}$ and thus $\epsilon \Vdash_\sim \mathsf{I} \wedge \mathsf{H}$. We also have $b, \delta \Vdash_\sim \mathsf{K} * X$ and $\epsilon \nVdash_\sim \mathsf{K} * X$, so $b, \delta \Vdash_\sim \mathsf{L} \wedge \mathsf{K} * X$ and $\epsilon \nVdash_\sim \mathsf{L} \wedge \mathsf{K} * X$. So $b, \delta \nVdash_\sim (\mathsf{L} \wedge \mathsf{K} * X) \to \bot$ and $\epsilon \Vdash_\sim (\mathsf{L} \wedge \mathsf{K} * X) \to \bot$. Hence, $\epsilon \nVdash_\sim \mathsf{K} \mathrel{-\!\!*} ((\mathsf{L} \wedge \mathsf{K} * X) \to \bot)$, because $\delta \Vdash_\sim \mathsf{K}$ and $\delta\epsilon \nVdash_\sim (\mathsf{L} \wedge \mathsf{K} * X) \to \bot$, so $\epsilon \nVdash_\sim \mathsf{K} * X \vee \mathsf{K} \mathrel{-\!\!*} ((\mathsf{L} \wedge \mathsf{K} * X) \to \bot)$, and then $\epsilon \nVdash_\sim F$, so we do indeed have a BBI-counter-model of the formula F, which is the translation of $X \vee (X \to \bot)$.

8.3. *Faithfully embedding* BI *into* BBI

We conclude with the central result of this paper. If we add two spare logical variables L and K to the language of BI, we can provide a map from BI-formulae (without L and K) to BBI-formulae that preserves both validity and invalidity.

Theorem 8.2. Let L and K be two different spare logical variables. The map $G \mapsto (\mathsf{I} \wedge \mathsf{H}) \to G^\circ$, where $\mathsf{H} \equiv (\mathsf{L} \wedge \mathsf{K}) \wedge ((\top \mathrel{-\!\!*} (\mathsf{L} * \mathsf{L} \to \mathsf{L})) \wedge (\top \mathrel{-\!\!*} (\mathsf{K} * \mathsf{K} \to \mathsf{K})))$, is a sound and faithful embedding of BI into BBI.

Proof. The proof is a direct combination of Theorems 7.6 and 8.1. □

9. Application to the expressive power of BI

In this section we give a brief presentation of one possible application of some of the semantical results of this paper related to the expressivity of BI on BI-frames. These results can be transferred trivially to partially ordered partial monoids.

The property of 'being squarable' in a BI-frame can be expressed by the first-order logic atomic formula $H'(x) = xx \sqsubseteq xx$. We show that it cannot be represented by any BI-formula: there is no formula F of BI such that for any BI-frame $(L, \sqsubseteq, \Vdash_{\sqsubseteq})$ and for any $m \in \mathscr{L}^{\sqsubseteq}$, we have $H'(m)$ holds if and only if $m \Vdash_{\sqsubseteq} F$.

Proposition 9.1. BI-simple PMOs do not have square words, except of course ϵ.

Proof. Let $\sqsubseteq$ be a BI-simple PMO over L. We want to show that for any $m \in L^\star$, if $mm \sqsubseteq mm$, then $m = \epsilon$. According to Theorem 6.14, let L', K and $\sim$, where $\sim$ is a BBI-simple PME over L' be such that $L \cup K \subseteq L'$ and $\sqsubseteq = \sqsubseteq_{\sim}^{L,K}$. Let $m \in L^\star$ be such that $mm \sqsubseteq mm$. Then we have $\delta mm \sim mm$ for some $\delta \in K^\star$. So we have $mm \sim mm$ by rule $\langle r \rangle$. By Corollary 6.10 and property 1 of Proposition 6.7, we obtain $m \sim \epsilon$. Hence $\epsilon m \sim \epsilon$ with $\epsilon \in K^\star$, so, as $\sqsubseteq = \sqsubseteq_{\sim}^{L,K}$, we get $m \sqsubseteq \epsilon$, and by Proposition 5.6, we have $m = \epsilon$. □

Proposition 9.2. Let $H(\sqsubseteq)$ be a property ranging over PMOs verifying:

(1) $H(\sqsubseteq)$ is true for every BI-simple PMO $\sqsubseteq$.
(2) $H(\sqsubseteq_0)$ is false for some (other) PMO $\sqsubseteq_0$.

Then the property $H(\sqsubseteq)$ cannot be represented by a BI-formula, that is, there is no formula F of BI such that for any BI-frame $(L, \sqsubseteq, \Vdash)$, $H(\sqsubseteq)$ holds if and only if $\epsilon \Vdash_{\sqsubseteq} F$.

Proof. We suppose that such a formula F exists and show a contradiction. So we assume $H(\sqsubseteq)$ holds if and only if $\epsilon \Vdash_{\sqsubseteq} F$. Thus $\epsilon \Vdash_{\sqsubseteq} F$ holds in every BI-simple PMO. Hence, F cannot have a BI-simple counter-model and is thus BI-valid by Theorem 5.9.

On the other hand, let $(L, \sqsubseteq, \Vdash)$ be a (non-simple) PMO such that $H(\sqsubseteq)$ does not hold. Then in this frame we have $\epsilon \nVdash_{\sqsubseteq} F$. Hence F has a BI-counter-model, which contradicts its validity. □

Corollary 9.3. The property $mm \sqsubseteq mm$ ('being squarable') is not BI-definable.

Proof. Let $H'(x) = xx \sqsubseteq xx$, which expresses the property of being squarable. Suppose the BI-formula F represents H', that is, for any $m \in \mathscr{L}^{\sqsubseteq}$, $H'(m)$ holds if and only if $m \Vdash F$. We now prove that the existence of F leads to a contradiction. Consider the property $H(\sqsubseteq) = \forall m \in \mathscr{L}^{\sqsubseteq} (H'(m) \Rightarrow \epsilon \sqsubseteq m)$, which is represented by the formula $F \mathrel{-\!\!*} \mathsf{I}$, that is, $H(\sqsubseteq)$ holds if and only if $\epsilon \Vdash_{\sqsubseteq} F \mathrel{-\!\!*} \mathsf{I}$.

By Proposition 9.1, $H(\sqsubseteq)$ is true for every BI-simple PMO $\sqsubseteq$. On the other hand, let $\sqsubseteq_0 = \{\epsilon - \epsilon, a - a, aa - aa\}$ where a is an arbitrary letter. It is easy to check that $\sqsubseteq_0$ is a PMO but $H(\sqsubseteq_0)$ does not hold because $a \in \mathscr{L}^{\sqsubseteq_0}$, $aa \sqsubseteq_0 aa$ and $\epsilon \not\sqsubseteq_0 a$. Thus, by Proposition 9.2, $H(\sqsubseteq)$ cannot be represented by $F \mathrel{-\!\!*} \mathsf{I}$, which gives a contradiction. □

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Corollary 9.4. The property $m \sqsubseteq \epsilon$ is not BI-definable.

Proof. Suppose $H'(x) = x \sqsubseteq \epsilon$ is expressed by the BI-formula F, that is, $m \sqsubseteq \epsilon$ holds if and only if $m \Vdash F$, and consider $H(\sqsubseteq) = \forall m \in \mathscr{L}^{\sqsubseteq}(m \sqsubseteq \epsilon \Rightarrow \epsilon \sqsubseteq m)$. We have $H(\sqsubseteq)$ holds if and only if $\epsilon \Vdash_{\sqsubseteq} F \mathrel{-\!\!*} \mathsf{I}$. If $\sqsubseteq$ is a BI-simple PMO, then $m \sqsubseteq \epsilon$ holds for no word m other than ϵ, see Proposition 5.6. Thus $H(\sqsubseteq)$ holds whenever $\sqsubseteq$ is a BI-simple PMO. On the other hand, let $\sqsubseteq_0 = \{a^i \mathrel{-} a^j \mid i \geqslant j\}$ be the PMO generated by the singleton constraint $\{a \mathrel{-} \epsilon\}$, which, of course, is not BI-elementary. $H(\sqsubseteq_0)$ does not hold because $a \sqsubseteq_0 \epsilon$ but $\epsilon \not\sqsubseteq_0 a$, which gives a contradiction. □

10. Conclusion and perspectives

In this paper, we have proved that there exists a sound and faithful embedding of intuitionistic BI logic into Boolean BI. The result is based on the study of the relations between constraint-based models of BI and BBI, namely PMOs and PMEs, the completeness of the class of simple PMOs with respect to intuitionistic BI, and the soundness of the TBBI-tableau method for BBI. We have also pointed out some immediate consequences of our intermediary results on the expressivity of BI.

Another quite direct application of our results would be a new proof and counter-model search method for BI derived from our embedding and based on partial monoidal constraints (PMEs) instead of the existing resource graphs (Galmiche *et al.* 2005). Resource graphs are mainly a graphical representation for PMOs. The embedding we have obtained was quite unexpected and is based on the intuition to represent the order relation $m \sqsubseteq n$ by composition with some unobservable word δ such that $\delta m \sim n$. There may be some practical applications of this idea to distinguish observable and unobservable words. In particular, we aim to describe a concrete and complete class of separation logic style models for (intuitionistic) BI.

Of course, the proof of the completeness theorem of the TBBI-tableau method (completeness only, not the already achieved soundness) is one of our immediate goals. The complete study and characterisation of constraints based models of BBI is a natural evolution of our work. In particular, the TBBI-tableau method may introduce constraints like $\epsilon \mathrel{-} ab$, in which case a and b become invertible words. Our aim is a generalisation of the notion of resource graph to take invertible elements into account. Potentially, this would constitute a major evolution because invertible elements do not occur in the TBI-tableau method.

The characterisation of TBBI-constraints can lead to an effective decision procedure for these partial monoidal constraints. We wish to compute the explicit form of the extension $\sim + \{\epsilon \mathrel{-} m\}$, not only when m is a new letter (as in the present paper), but also when m is any defined word. Combined with the other explicit forms ($\sim + \{ab \mathrel{-} m\}$ and $\sim + \{am \mathrel{-} b\}$) described in the present paper, this can lead to specific properties of TBBI-generated constraints and then to expressivity results for BBI. For example, we think that in a TBBI-generated PME $\sim$, no word m is squarable ($mm \sim mm$) unless it is invertible ($\exists a\, am \sim \epsilon$). As invertibility is BBI-defined by the

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formula $\neg(\top \mathrel{-\!\!*} \neg \mathsf{I})$, the consequence would be that, as with BI, 'being squarable' is not BBI-definable.

Appendix A. Proofs of free PME extensions identities

Proposition 6.3. Let $\sim$ be a PME over L, and b be new to $\sim$, that is, $b \in L \backslash A_\sim$. Then $\sim + \{\epsilon - b\} = \sim'$ with $\sim' = \{b^p x - b^q y \mid x \sim y \wedge p, q \geqslant 0\}$ and $A_{\sim'} = A_\sim \cup \{b\}$.

Proof. First, it is obvious that $A_{\sim'} = A_\sim \cup \{b\}$ (see Proposition 3.16).

Let $\sim'' = \sim + \{\epsilon - b\}$. We will prove that $\sim' \subseteq \sim''$ and that $\sim'$ is a PME, which is sufficient to establish $\sim'' = \sim'$ because it is obvious that $\sim \cup \{\epsilon - b\} \subseteq \sim'$ (we have $b^0\epsilon \sim' b^1\epsilon$).

As $b \sim'' \epsilon$ (by rule $\langle s \rangle$), for any $p \geqslant 1$, by p applications of rule $\langle e_l \rangle$, we can show that $b^p \sim'' \epsilon$ (since $b \sim'' \epsilon$ and $\epsilon^p \sim'' \epsilon$). Also, $b^0 = \epsilon \sim'' \epsilon$. Hence, $b^p \sim'' \epsilon$ and $b^q \sim'' \epsilon$ for any $p, q \geqslant 0$. Let x, y be such that $x \sim y$. Then $x \sim'' y$ as $\sim \subseteq \sim''$. Thus $\epsilon x \sim'' \epsilon y$, and by applications of rule $\langle e_l \rangle$ and $\langle e_r \rangle$, we obtain $b^p x \sim'' b^q y$. We have now proved that $\sim' \subseteq \sim''$.

The relation $\sim'$ is obviously closed under rules $\langle \epsilon, s \rangle$. For rule $\langle t \rangle$, we consider x, y, z such that $x \sim' y$ and $y \sim' z$. Then $(x, y) = (b^p x', b^q y')$ for some $p, q \geqslant 0$ and some x', y' such that $x' \sim y'$. Also, $(y, z) = (b^r y'', b^s z')$ with $r, s \geqslant 0$ and $y'' \sim z'$. As $y', y'' \in A_\sim^\star$ and $b \notin A_\sim$, from $y = b^q y' = b^r y''$, we deduce $q = r$ and $y' = y''$. Hence $y' \sim z'$, so $x' \sim z'$ since $\sim$ is closed under rule $\langle t \rangle$. So $x = b^p x' \sim' b^s z' = z$. The relation $\sim'$ is thus closed under rule $\langle t \rangle$.

For rules $\langle d, c \rangle$, the core argument is the same: $b \notin A_\sim$, so the decomposition $x = b^p x'$ with $x' \in A_\sim^\star$ is unique. Hence, having proved that it is closed under rules $\langle d \rangle$ and $\langle c \rangle$, the relation $\sim'$ is a PME, which completes the proof. □

Lemma 6.4. Let $\sim$ be a PME over L. Let $m \in L^\star$ and $\alpha \in L^\star$ be such that $m \sim m$, $mm \nsim mm$, $\alpha \neq \epsilon$ and $A_\alpha \cap A_\sim = \varnothing$. Then $\sim + \{\alpha - m\} = \sim'$ with

$$
\begin{aligned}
\sim' = \sim &\cup \{\delta x - \delta y \mid x \sim y \wedge mx \sim my \wedge \delta \prec \alpha \wedge \delta \notin \{\epsilon, \alpha\}\} \\
&\cup \{\alpha x - \alpha y \mid mx \sim my\} \\
&\cup \{\alpha x - y \mid mx \sim y\} \\
&\cup \{x - \alpha y \mid x \sim my\}
\end{aligned}
$$

and $A_{\sim'} = A_\sim \cup A_\alpha$.

Proof. First, it is obvious that $A_{\sim'} \subseteq A_\sim \cup A_\alpha$.

We also have $A_\sim \subseteq A_{\sim'}$ and $\alpha \sim' \alpha$ (because $m\epsilon \sim m\epsilon$), so $A_\alpha \subseteq A_{\sim'}$. Thus, $A_{\sim'} = A_\sim \cup A_\alpha$. As $A_\alpha \cap A_\sim = \varnothing$, we have the following property: whenever $x \sim' y$ holds, x and y can be uniquely decomposed into $x = x_1 x_2$ and $y = y_1 y_2$ such that $x_1, y_1 \in A_\alpha^\star$ and $x_2, y_2 \in A_\sim^\star$. Let $\sim'' = \sim + \{\alpha - m\}$. We will prove that $\sim' \subseteq \sim''$, and that $\sim'$ is a PME, which is sufficient to establish $\sim'' = \sim'$ because it is obvious that $\sim \cup \{\alpha - m\} \subseteq \sim'$ (we have $\alpha\epsilon \sim' m$).

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For $\sim' \subseteq \sim''$, we already have $\sim \subseteq \sim''$. We also have the following deduction trees:

$$\dfrac{\dfrac{\dfrac{\alpha \sim'' m \qquad \dfrac{\dfrac{mx \sim my}{my \sim my}\,\langle r\rangle}{my \sim'' my}\,\sim \subseteq \sim''}{\alpha y \sim'' my}\,\langle e_l\rangle}{\delta y \sim'' \delta y}\,\langle p_l\rangle,\ \delta \prec \alpha \qquad \dfrac{x \sim y}{x \sim'' y}\,\sim \subseteq \sim''}{\delta x \sim'' \delta y}\,\langle c\rangle \qquad\qquad \dfrac{\alpha \sim'' m \qquad \dfrac{mx \sim y}{mx \sim'' y}\,\sim \subseteq \sim''}{\alpha x \sim'' y}\,\langle e_l\rangle$$

$$\dfrac{\alpha \sim'' m \qquad \dfrac{\alpha \sim'' m \qquad \dfrac{mx \sim my}{mx \sim'' my}\,\sim \subseteq \sim''}{mx \sim'' \alpha y}\,\langle e_r\rangle}{\alpha x \sim'' \alpha y}\,\langle e_l\rangle \qquad\qquad \dfrac{\alpha \sim'' m \qquad \dfrac{x \sim my}{x \sim'' my}\,\sim \subseteq \sim''}{x \sim'' \alpha y}\,\langle e_r\rangle$$

We will now prove that $\sim'$ is a PME. It is obvious that $\sim'$ is closed under rules $\langle \epsilon \rangle$ and $\langle s \rangle$ (observe the symmetry between x and y in the definition of $\sim'$).

We now consider the other rules in turn:

Rule $\langle d \rangle$: Let x, y be such that $xy \sim' xy$. If $xy \sim xy$, then $x \sim x$, so $x \sim' x$.

We now consider the other cases:

- As $\alpha \neq \epsilon$ and $\mathsf{A}_\alpha \cap \mathsf{A}_\sim = \emptyset$, the cases $(xy, xy) = (\alpha x', y')$ and $(xy, xy) = (x', \alpha y')$ are impossible.
- So $xy = \delta z'$ with $mz' \sim mz'$ and $\epsilon \neq \delta \prec \alpha$ (this covers the cases $\delta \neq \alpha$ and $\delta = \alpha$). As $\mathsf{A}_\alpha \cap \mathsf{A}_\sim = \emptyset$, we let $x = x_1 x_2$ and $y = y_1 y_2$ with $x_1, y_1 \in \mathsf{A}_\alpha^\star$ and $x_2, y_2 \in \mathsf{A}_\sim^\star$. From $xy = x_1 x_2 y_1 y_2 = \delta z'$, we obtain $\delta = x_1 y_1$ and $z' = x_2 y_2$. We have $mx_2 y_2 \sim mx_2 y_2$, so $mx_2 \sim mx_2$ and $x_2 \sim x_2$ by rule $\langle d \rangle$. If $x_1 = \epsilon$, then $x = x_2 \sim x_2 = x$, so $x \sim' x$. If $x_1 \neq \epsilon$, then $\epsilon \neq x_1 \prec x_1 y_1 = \delta \prec \alpha$, so, as $mx_2 \sim mx_2$ and $x_2 \sim x_2$, we get $x = x_1 x_2 \sim' x_1 x_2 = x$.

Rule $\langle t \rangle$: Let x, y, z be such that $x \sim' y$ and $y \sim' z$. We want to prove $x \sim' z$. In theory, there are $5 \times 5 = 25$ cases to study for (x, y) and (y, z). But as we have already proved that $\sim'$ is symmetric, we only need to consider 'half' of the matrix of cases, that is, $5 + \cdots + 1 = 15$ cases:

- $x \sim y$ and $y \sim z$.
  So $x \sim z$, and thus $x \sim' z$.
- $x \sim y$ and $(y, z) = (\delta y', \delta z')$ with $\delta \prec \alpha$ and $\delta \neq \epsilon$.
  This case is impossible because $\delta \in \mathsf{A}_\alpha^\star$, $\delta \prec y \in \mathsf{A}_\sim^\star$ and $\mathsf{A}_\alpha \cap \mathsf{A}_\sim = \emptyset$.

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— $x \sim y$ and $(y, z) = (\alpha y', \alpha z')$.
This case is impossible by the same argument, $\alpha \neq \epsilon$.

— $x \sim y$ and $(y, z) = (\alpha y', z)$.
This case is impossible because $\alpha \neq \epsilon$.

— $x \sim y$ and $(y, z) = (y, \alpha z')$ with $y \sim mz'$.
So $x \sim mz'$ and thus $x \sim' \alpha z' = z$.

— $(x, y) = (\delta x', \delta y')$ and $(y, z) = (\delta' y'', \delta' z')$ with $x' \sim y'$, $mx' \sim my'$, $y'' \sim z'$, $my'' \sim mz'$, $\delta \prec \alpha$, $\delta' \prec \alpha$ and $\delta, \delta' \notin \{\epsilon, \alpha\}$.
From $y = \delta y' = \delta' y''$, we deduce $\delta = \delta'$ and $y' = y''$. We have $y' \sim z'$ and $my' \sim mz'$, so by rule $\langle t \rangle$, we have $x' \sim z'$ and $mx' \sim mz'$. So $\delta x' \sim' \delta z'$, and thus $x \sim' z$.

— $(x, y) = (\delta x', \delta y')$ and $(y, z) = (\alpha y'', \alpha z')$.
This case is impossible because $\delta \neq \alpha$.

— $(x, y) = (\delta x', \delta y')$ and $(y, z) = (\alpha y'', z)$.
This case is impossible because $\delta \neq \alpha$.

— $(x, y) = (\delta x', \delta y')$ and $(y, z) = (y, \alpha z')$.
This case is impossible because $\delta \neq \epsilon$.

— $(x, y) = (\alpha x', \alpha y')$ and $(y, z) = (\alpha y'', \alpha z')$ with $mx' \sim my'$ and $my'' \sim mz'$.
We have $y = \alpha y' = \alpha y''$, so $y' = y''$ and $my' \sim mz'$. So $mx' \sim mz'$, and we get $\alpha x' \sim' \alpha z'$, and thus $x \sim' z$.

— $(x, y) = (\alpha x', \alpha y')$ and $(y, z) = (\alpha y'', z)$ with $mx' \sim my'$ and $my'' \sim z$.
We have $y' = y''$ and thus $mx' \sim z$, so $x = \alpha x' \sim' z$.

— $(x, y) = (\alpha x', \alpha y')$ and $(y, z) = (y, \alpha z')$.
This case is impossible because $\alpha \neq \epsilon$.

— $(x, y) = (\alpha x', y)$ and $(y, z) = (\alpha y', z)$.
This case is impossible because $\alpha \neq \epsilon$.

— $(x, y) = (\alpha x', y)$ and $(y, z) = (y, \alpha z')$ with $mx' \sim y$ and $y \sim mz'$.
We have $mx' \sim mz'$ and thus $\alpha x' \sim' \alpha z'$, so $x \sim' z$.

— $(x, y) = (x, \alpha y')$ and $(y, z) = (y, \alpha z')$.
This case is impossible because $\alpha \neq \epsilon$.

Rule $\langle c \rangle$**:** We consider q, x, y such that $qy \sim' qy$ and $x \sim' y$. We want to prove $qx \sim' qy$. We consider $2 \times 5 = 10$ cases:

— $qy \sim qy$ and $x \sim y$.
We have $qx \sim qy$, so $qx \sim' qy$.

— $qy \sim qy$ and $(x, y) = (\delta x', \delta y')$.
This case is impossible because we would have $\delta \prec qy$ with $\epsilon \neq \delta \in A_\alpha^\star$, $qy \in A_\sim^\star$ and $A_\alpha \cap A_\sim = \emptyset$.

— $qy \sim qy$ and $(x, y) = (\alpha x', \alpha y')$.
This case is impossible because $\alpha \not\prec qy$.

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— $qy \sim qy$ and $(x,y) = (\alpha x', y)$ with $mx' \sim y$.
We have $qmx' \sim qy$ by rule $\langle c \rangle$, so $qx = \alpha qx' \sim' qy$.

— $qy \sim qy$ and $(x,y) = (x, \alpha y')$.
This case is impossible because $\alpha \not\prec qy$.

— $qy = \delta z'$ and $x \sim y$ with $mz' \sim mz'$, $\epsilon \neq \delta \prec \alpha$.
Let $q = q_1 q_2$ with $q_1 \in A_\alpha^\star$ and $q_2 \in A_\sim^\star$. As $y, z' \in A_\sim^\star$, from $qy = q_1 q_2 y = \delta z'$, we get $q_1 = \delta$ and $q_2 y = z'$. Hence $mq_2 y \sim mq_2 y$. So $q_2 y \sim q_2 y$, and we get $mq_2 x \sim mq_2 y$ and $q_2 x \sim q_2 y$ by rule $\langle c \rangle$. In either case ($\delta = \alpha$ or $\delta \neq \alpha$), we deduce $qx = \delta q_2 x \sim' \delta q_2 y = qy$.

— $qy = \delta z'$ and $(x,y) = (\delta' x', \delta' y')$ with $mz' \sim mz'$, $\epsilon \neq \delta \prec \alpha$, $x' \sim y'$, $mx' \sim my'$, $\delta' \prec \alpha$ and $\delta' \notin \{\epsilon, \alpha\}$.
Let $q = q_1 q_2$ with $q_1 \in A_\alpha^\star$ and $q_2 \in A_\sim^\star$. From $qy = q_1 q_2 \delta' y' = \delta z'$, we get $q_1 \delta' = \delta$ and $q_2 y' = z'$. As in the previous case, we derive $mq_2 x' \sim mq_2 y'$ and $q_2 x' \sim q_2 y'$. So $qx = q_1 q_2 \delta' x' = \delta q_2 x' \sim' \delta q_2 y' = q_1 q_2 \delta' y' = qy$.

— $qy = \delta z'$ and $(x,y) = (\alpha x', \alpha y')$ with $mz' \sim mz'$, $\epsilon \neq \delta \prec \alpha$ and $mx' \sim my'$.
Let $q = q_1 q_2$ with $q_1 \in A_\alpha^\star$ and $q_2 \in A_\sim^\star$. From $qy = q_1 q_2 \alpha y' = \delta z'$, we get $q_1 \alpha = \delta$ and $q_2 y' = z'$. Hence $q_1 = \epsilon$ and $\alpha = \delta$. Thus $q = q_2$, and from $mz' \sim mz'$, we get $mqy' \sim mqy'$. Combining this with $mx' \sim my'$ by rule $\langle c \rangle$, we get $mqx' \sim mqy'$. So $qx = \alpha qx' \sim' \alpha qy' = qy$.

— $qy = \delta z'$ and $(x,y) = (\alpha x', y)$ with $mz' \sim mz'$, $\epsilon \neq \delta \prec \alpha$ and $mx' \sim y$.
As $y \in A_\sim^\star$, we deduce $\delta \prec q$ from $qy = \delta z'$. Let $\delta q' = q$. We have $q' y = z'$, so $mq' y \sim mq' y$. Combining this with $mx' \sim y$ by rule $\langle c \rangle$, we derive $mq' mx' \sim mq' y$, and thus $mm \sim mm$ by rule $\langle p_l \rangle$, which contradicts the overall hypothesis $mm \nsim mm$.

— $qy = \delta z'$ and $(x,y) = (x, \alpha y')$ with $mz' \sim mz'$, $\epsilon \neq \delta \prec \alpha$ and $x \sim my'$.
Let $q = q_1 q_2$ with $q_1 \in A_\alpha^\star$ and $q_2 \in A_\sim^\star$. From $qy = q_1 q_2 \alpha y' = \delta z'$, we get $q_1 \alpha = \delta$ and $q_2 y' = z'$. Hence $q_1 = \epsilon$ and $\alpha = \delta$. So $q = q_2$ and from $mz' \sim mz'$ we get $mqy' \sim mqy'$. Combining this with $x \sim my'$ by rule $\langle c \rangle$, we obtain $qx \sim qmy'$. Hence $qx \sim' \alpha qy' = qy$.

This concludes the proof that the relation $\sim'$ is closed under all the rules defining PMEs. □

Lemma 6.5. Let $\sim$ be a PME over L. Let $m \in L^\star$, $\alpha \in L^\star$, $b \in L$ be such that $m \sim m$, $\alpha \neq \epsilon$, $A_\alpha \cap A_\sim = \emptyset$ and $b \notin A_\sim \cup A_\alpha$. Then $\sim + \{\alpha m - b\} = \sim'$ with

$$\begin{aligned} \sim' = \sim &\cup \{\delta x - \delta y \mid x \sim y \wedge \epsilon \neq \delta \prec \alpha \wedge \exists k\ xk \sim m\} \\ &\cup \{\alpha x - jb \mid x \sim jm \wedge \exists k\ jkm \sim m\} \\ &\cup \{ib - \alpha y \mid y \sim im \wedge \exists k\ ikm \sim m\} \\ &\cup \{ib - jb \mid \exists k\ (ikm \sim m \wedge jkm \sim m)\} \end{aligned}$$

and $A_{\sim'} = A_\sim \cup A_\alpha \cup \{b\}$.

Proof. First, it is obvious that $A_{\sim'} \subseteq A_\sim \cup A_\alpha \cup \{b\}$.

We have $A_\sim \subseteq A_{\sim'}$. As $m \sim m$, we get $b \sim' b$ and $\alpha m \sim' b$, and thus $A_\alpha \subseteq A_{\sim'}$ and $b \in A_{\sim'}$. So we have $A_{\sim'} = A_\sim \cup A_\alpha \cup \{b\}$. We let $\sim'' = \sim + \{\alpha m - b\}$ and prove that

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$\sim' \subseteq \sim''$, and that $\sim'$ is a PME, which is sufficient to establish $\sim'' = \sim'$ since it is obvious that $\sim \cup \{\alpha m - b\} \subseteq \sim'$ (we have $\alpha m \sim' \epsilon b$ because $m \sim \epsilon m$ and $\epsilon\epsilon m \sim m$).

For $\sim' \subseteq \sim''$, we already have $\sim \subseteq \sim''$. Let x, y, δ, k be such that $x \sim y$, $\epsilon \neq \delta \prec \alpha$ and $xk \sim m$. We have the following deduction tree, which is split into two parts:

$$\dfrac{\dfrac{\dfrac{xk \sim m}{xk \sim'' m}\sim\subseteq\sim'' \qquad \dfrac{\alpha m \sim'' b}{\alpha m \sim'' \alpha m}\langle l\rangle}{\alpha k x \sim'' \alpha m}\langle e_l\rangle}{\cdots} \qquad \dfrac{\dfrac{\dfrac{x \sim y}{y \sim x}\langle s\rangle}{y \sim'' x}\sim\subseteq\sim'' \qquad \dfrac{\dfrac{\cdots}{\alpha k x \sim'' \alpha m}\langle e_l\rangle}{\delta x \sim'' \delta x}\langle p_l\rangle,\ \delta \prec \alpha}{\delta x \sim'' \delta y}\langle e_r\rangle$$

So we have $\delta x \sim'' \delta y$.

Now let x, i, k be such that $x \sim im$ and $ikm \sim m$. We have the following two deduction trees:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{ikm \sim m}{ikm \sim'' m}\sim\subseteq\sim'' \qquad \alpha m \sim'' b}{\alpha i k m \sim'' b}\langle e_l\rangle}{i\alpha m \sim'' i\alpha m}\langle p_l\rangle \qquad \dfrac{\alpha m \sim'' b}{b \sim'' \alpha m}\langle s\rangle}{ib \sim'' i\alpha m}\langle c\rangle}{\cdots} \qquad \dfrac{\dfrac{x \sim im}{x \sim'' im}\sim\subseteq\sim'' \qquad \dfrac{\cdots}{ib \sim'' i\alpha m}\langle c\rangle}{ib \sim'' \alpha x}\langle e_r\rangle$$

So we have $ib \sim'' \alpha x$.

By rule $\langle s\rangle$, we also get $\alpha x \sim'' ib$. Consider the last line of the definition of $\sim'$: let i, j, k be such that $ikm \sim m$ and $jkm \sim m$. Then we can deduce $im \sim jm$:

$$\dfrac{\dfrac{jkm \sim m \qquad \dfrac{ikm \sim m}{im \sim im}\langle p_l\rangle}{im \sim ijkm}\langle e_r\rangle \qquad \dfrac{\dfrac{jkm \sim m}{jm \sim jm}\langle p_l\rangle \qquad ikm \sim m}{jikm \sim jm}\langle c\rangle}{im \sim jm}\langle t\rangle$$

But we have already proved that we can deduce $ib \sim'' i\alpha m$ from $ikm \sim m$ (see first part, on the left, of a previous deduction tree). Replacing i by j we can deduce $jb \sim'' j\alpha m$ from $jkm \sim m$. So we have the following deduction tree:

$$\dfrac{\dfrac{\cdots}{ib \sim'' \alpha im} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\cdots}{im \sim jm}}{im \sim'' jm}\sim\subseteq\sim'' \qquad \dfrac{\cdots}{jb \sim'' \alpha jm}}{jb \sim'' \alpha im}\langle e_r\rangle}{\alpha im \sim'' jb}\langle s\rangle}{ib \sim'' jb}\langle t\rangle$$

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So $ib \sim'' ib$, and for any x, y such that $x \sim' y$, we have proved that $x \sim'' y$. So $\sim' \subseteq \sim''$.

We will now prove that $\sim'$ is a PME. We first make the following observation: if $x \sim' x$, then either $x \sim x$ or ($x = \delta x'$ with $x' \sim x'$, $\epsilon \neq \delta \prec \alpha$ and $x'k \sim m$ for some k) or ($x = ib$ with $ikm \sim m$ for some k).

We consider each of the five rules defining PMEs in turn.

Rule $\langle\epsilon\rangle$: $\epsilon \sim' \epsilon$ since $\epsilon \sim \epsilon$, so $\sim'$ is closed under rule $\langle\epsilon\rangle$.

Rule $\langle s\rangle$: Let x, y be such that $x \sim' y$. We have five cases:

— $x \sim y$.
So we have $y \sim x$, and thus $y \sim' x$.

— $(x, y) = (\delta x', \delta y')$ with $x' \sim y'$, $\epsilon \neq \delta \prec \alpha$ and $x'k \sim m$.
We have $y' \sim x'$ and thus $y'k \sim m$ by rule $\langle e_l\rangle$. Hence $y = \delta y' \sim' \delta x' = x$.

— $(x, y) = (\alpha x', jb)$ with $x' \sim jm$ and $jkm \sim m$.
We have $y = jb \sim' \alpha x' = x$.

— $(x, y) = (ib, \alpha y')$.
The same argument applies in this case.

— $(x, y) = (ib, jb)$ with $ikm \sim m$ and $jkm \sim m$.
We have $y = jb \sim' ib = x$.

So in all cases we get $y \sim' x$, so $\sim'$ is closed under rule $\langle s\rangle$.

Rule $\langle d\rangle$: Let x, y be such that $xy \sim' xy$. We have three cases, because $\alpha z' \neq ib$ ($b \notin A_\alpha \cup A_\sim$):

— $xy \sim xy$.
So $x \sim x$ and thus $x \sim' x$.

— $xy = \delta z'$ with $z' \sim z'$, $\epsilon \neq \delta \prec \alpha$ and $z'k \sim m$ for some k.
We have $x, y \in (A_\alpha \cup A_\sim)^\star$ and let $x = x_1x_2$, $y = y_1y_2$ with $x_1, y_1 \in A_\alpha^\star$ and $x_2, y_2 \in A_\sim^\star$. So we have $\delta = x_1y_1$ and $z' = x_2y_2$. From $x_2y_2 = z' \sim z' = x_2y_2$, we deduce $x_2 \sim x_2$ by rule $\langle d\rangle$. On the one hand, if $x_1 = \epsilon$, then $x = x_2 \sim x_2 = x$, so $x \sim' x$. On the other hand, if $x_1 \neq \epsilon$, as we have $x_2(y_2k) = z'k \sim m$ and $x_1 \prec x_1y_1 = \delta \prec \alpha$, we get $x_1x_2 \sim' x_1x_2$, and thus $x \sim' x$.

— $xy = ib$ with $ikm \sim m$.
We have either $b \prec x$ or $b \not\prec x$. On the one hand, if $b \not\prec x$, then, as b is a one letter word, $b \prec y$. Hence $x(y/b)b = ib$, so $x(y/b) = i$ and thus $x(y/b)km \sim m$. By application of rule $\langle p_l\rangle$, we have $x \sim x$ and thus $x \sim' x$. On the other hand, if $b \prec x$, then $(x/b)y = i$, so $(x/b)(yk)m \sim m$. Thus, $x = (x/b)b \sim' (x/b)b = x$.

So in all cases we get $x \sim' x$, so $\sim'$ is closed under rule $\langle d\rangle$.

Rule $\langle t\rangle$: Let x, y, z be such that $x \sim' y$ and $y \sim' z$. As we have already proved that $\sim'$ is closed under rule $\langle s\rangle$, we only need to consider $5 + \ldots + 1 = 15$ cases:

— $x \sim y$ and $y \sim z$.
We have $x \sim z$, so $x \sim' z$.

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— $x \sim y$ and $(y,z) = (\delta y', \delta z')$.
This would imply $\delta \prec \delta y' = y$, which is impossible because $\epsilon \neq \delta \in A_\alpha^\star$ and $A_\sim \cap A_\alpha = \emptyset$.

— $x \sim y$ and $(y,z) = (\alpha y', jb)$.
This case is impossible because $\alpha \neq \epsilon$ and $A_\sim \cap A_\alpha = \emptyset$.

— $x \sim y$ and $(y,z) = (ib, \alpha z')$.
This case is impossible because $b \notin A_\sim$.

— $x \sim y$ and $(y,z) = (ib, jb)$.
This case is impossible because $b \notin A_\sim$.

— $(x,y) = (\delta x', \delta y')$ and $(y,z) = (\delta' y'', \delta' z')$ with $x' \sim y'$, $y'' \sim z'$, $\epsilon \neq \delta \prec \alpha$, $\epsilon \neq \delta' \prec \alpha$, $x'k \sim m$ and $y''k' \sim m$.
As $y = \delta y' = \delta' y''$ and $A_\sim \cap A_\alpha = \emptyset$, we have $\delta = \delta'$ and $y' = y''$. As $x' \sim y'$ and $y' = y'' \sim z'$, we get $x' \sim z'$. Hence, as $x'k \sim m$, we get $x = \delta x' \sim' \delta z' = z$.

— $(x,y) = (\delta x', \delta y')$ and $(y,z) = (\alpha y'', jb)$ with $x' \sim y'$, $\epsilon \neq \delta \prec \alpha$, $x'k \sim m$, $y'' \sim jm$ and $jk'm \sim m$.
We have $y = \delta y' = \alpha y''$, which implies $\delta = \alpha$ and $y' = y''$ ($A_\sim \cap A_\alpha = \emptyset$). Thus $y' \sim jm$ and hence $x' \sim jm$ by rule $\langle t \rangle$. So $jm \sim x'$ by rule $\langle s \rangle$. As $x'k \sim m$, we get $jmk \sim m$ by rule $\langle e_l \rangle$. Hence we have $x' \sim jm$ and $jkm \sim m$. Thus $x = \delta x' = \alpha x' \sim' jb = z$.

— $(x,y) = (\delta x', \delta y')$ and $(y,z) = (ib, \alpha z')$.
This case is impossible because $b \notin A_\sim \cup A_\alpha$.

— $(x,y) = (\delta x', \delta y')$ and $(y,z) = (ib, jb)$.
This case is impossible because $b \notin A_\sim \cup A_\alpha$.

— $(x,y) = (\alpha x', jb)$ and $(y,z) = (\alpha y', j'b)$.
This case is impossible because $b \notin A_\sim \cup A_\alpha$.

— $(x,y) = (\alpha x', jb)$ and $(y,z) = (i'b, \alpha z')$ with $x' \sim jm$, $jkm \sim m$, $z' \sim i'm$ and $i'k'm \sim m$.
As $y = jb = i'b$, we get $j = i'$. As $x' \sim jm$ and $jkm \sim m$, we get $x'k \sim m$ by rule $\langle e_l \rangle$. As $z' \sim i'm$, we get $jm = i'm \sim z'$ by rule $\langle s \rangle$ and thus $x' \sim z'$ by rule $\langle t \rangle$. Hence, as $\epsilon \neq \alpha \prec \alpha$, we get $x = \alpha x' \sim' \alpha z' = z$.

— $(x,y) = (\alpha x', jb)$ and $(y,z) = (i'b, j'b)$ with $x' \sim jm$, $jkm \sim m$, $i'k'm \sim m$ and $j'k'm \sim m$.
As noted previously, we necessarily have $i'm \sim j'm$. Thus $j'm \sim i'm$ by rule $\langle s \rangle$. As $y = jb = i'b$, we get $j = i'$ and thus $i'km \sim m$. As $j'm \sim i'm$ and $(i'm)k \sim m$, we get $(j'm)k \sim m$ by rule $\langle e_l \rangle$. Hence $j'km \sim m$. As $x' \sim jm = i'm$ and $i'm \sim j'm$, we get $x' \sim j'm$ by rule $\langle t \rangle$, so $x = \alpha x' \sim' j'b = z$.

— $(x,y) = (ib, \alpha y')$ and $(y,z) = (i'b, \alpha z')$.
This case is impossible because $b \notin A_\sim \cup A_\alpha$.

— $(x,y) = (ib, \alpha y')$ and $(y,z) = (i'b, j'b)$.
This case is impossible because $b \notin A_\sim \cup A_\alpha$.

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— $(x, y) = (ib, jb)$ and $(y, z) = (i'b, j'b)$ with $ikm \sim m$, $jkm \sim m$, $i'k'm \sim m$ and $j'k'm \sim m$.
We have $y = jb = i'b$, so $j = i'$, and, as noted previously, we necessarily have $im \sim jm$ and $i'm \sim j'm$. As $jm = i'm$, we get $im \sim j'm$ by rule $\langle t \rangle$. As $(j'm)k' \sim m$, we get $(im)k' \sim m$ by rule $\langle e_l \rangle$. Hence $ik'm \sim m$ and $j'k'm \sim m$, so $x = ib \sim' j'b = z$.

In all cases we get $x \sim' z$, so we have proved that $\sim'$ is closed under rule $\langle t \rangle$.

Rule $\langle c \rangle$**:** Let q, x, y be such that $qy \sim' qy$ and $x \sim' y$. We consider $3 \times 5 = 15$ cases:

— $qy \sim qy$ and $x \sim y$.
We have $qx \sim qy$, so $qx \sim' qy$.

— $qy \sim qy$ and $(x, y) = (\delta x', \delta y')$.
This would imply $\delta \prec qy$, which is impossible because $\epsilon \neq \delta \in \mathsf{A}_\alpha^\star$, $qy \in \mathsf{A}_\sim^\star$ and $\mathsf{A}_\sim \cap \mathsf{A}_\alpha = \emptyset$.

— $qy \sim qy$ and $(x, y) = (\alpha x', jb)$.
This case is impossible because $b \notin \mathsf{A}_\sim$.

— $qy \sim qy$ and $(x, y) = (ib, \alpha y')$.
This case is impossible because $\alpha \neq \epsilon$ and $\mathsf{A}_\sim \cap \mathsf{A}_\alpha = \emptyset$.

— $qy \sim qy$ and $(x, y) = (ib, jb)$.
This case is impossible because $b \notin \mathsf{A}_\sim$.

— $qy = \delta z'$ and $x \sim y$ with $z' \sim z'$, $\epsilon \neq \delta \prec \alpha$ and $z'k \sim m$.
As $\delta \prec \delta z' = qy$, $y \in \mathsf{A}_\sim^\star$, $\delta \prec \alpha$ and $\mathsf{A}_\sim \cap \mathsf{A}_\alpha = \emptyset$, we get $\delta \prec q$. So let $q = \delta q'$ hence $q'y = z'$. Then $q'y \sim q'y$, hence $q'x \sim q'y$ by rule $\langle c \rangle$. As $q'yk = z'k \sim m$ we get $(q'x)k \sim m$ by rule $\langle e_l \rangle$. Hence $qx = \delta q'x \sim' \delta q'y = qy$.

— $qy = \delta z'$ and $(x, y) = (\delta' x', \delta' y')$ with $z' \sim z'$, $\epsilon \neq \delta \prec \alpha$, $z'k \sim m$, $x' \sim y'$, $\epsilon \neq \delta' \prec \alpha$ and $x'k' \sim m$.
As $q \prec qy = \delta z'$, we have $q \in (\mathsf{A}_\alpha \cup \mathsf{A}_\sim)^\star$. So let $q = q_1 q_2$ with $q_1 \in \mathsf{A}_\alpha^\star$ and $q_2 \in \mathsf{A}_\sim^\star$. From $qy = q_1 q_2 \delta' y' = \delta z'$, we get $q_1 \delta' = \delta$ and $q_2 y' = z'$. As $z' \sim z'$, we deduce $q_2 y' \sim q_2 y'$. As $x' \sim y'$, we deduce $q_2 x' \sim q_2 y'$ by rule $\langle c \rangle$. Also, $z'k \sim m$, that is, $q_2 y'k \sim m$, so we have $q_2 x'k \sim m$ by rule $\langle e_l \rangle$, and thus $qx = q_1 q_2 \delta' x' = \delta q_2 x' \sim' \delta q_2 y' = q_1 \delta' q_2 y = qy$.

— $qy = \delta z'$ and $(x, y) = (\alpha x', jb)$.
This case is impossible because $b \notin \mathsf{A}_\sim \cup \mathsf{A}_\alpha$.

— $qy = \delta z'$ and $(x, y) = (ib, \alpha y')$ with $z' \sim z'$, $\epsilon \neq \delta \prec \alpha$, $z'k \sim m$, $y' \sim im$ and $ik'm \sim m$.
We have $qy = q\alpha y' = \delta z'$, so $\alpha \prec \delta z'$. As $\mathsf{A}_\sim \cap \mathsf{A}_\alpha = \emptyset$ and $z' \in \mathsf{A}_\sim^\star$, we get $\alpha \prec \delta$. Hence $\alpha = \delta$, and thus $qy' = z'$. We deduce $qy'k \sim m$, and thus $qy' \sim qy'$ by rule $\langle p_l \rangle$. From $y' \sim im$, we derive $im \sim y'$ by rule $\langle s \rangle$, and then $qy' \sim q(im)$ by rule $\langle e_r \rangle$. Hence $qim \sim qy'$ by rule $\langle s \rangle$. From $qy'k \sim m$, we get $qimk \sim m$ by rule $\langle e_l \rangle$. So we have $(qy') \sim (qi)m$ and $(qi)km \sim m$, and thus $qx = (qi)b \sim' \alpha(qy') = qy$.

— $qy = \delta z'$ and $(x, y) = (i'b, j'b)$.
This case is impossible because $b \notin \mathsf{A}_\sim \cup \mathsf{A}_\alpha$.

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— $qy = ib$ and $x \sim y$ with $ikm \sim m$.
We have $b \not\prec y$ because $y \in A_{\sim}^{\star}$. As b is a single letter word, we deduce $b \prec q$, and hence $(q/b)y = i$. So $((q/b)y)km \sim m$, and thus, as $x \sim y$, we get $((q/b)x)km \sim m$ by rule $\langle e_l \rangle$. So we have $qx = (q/b)xb \sim' (q/b)yb = qy$.

— $qy = ib$ and $(x, y) = (\delta x', \delta y')$.
This would imply $\delta \prec ib$, which is impossible because $\delta \neq \epsilon$ and $A_\alpha \cap (A_\sim \cup \{b\}) = \emptyset$.

— $qy = ib$ and $(x, y) = (\alpha x', j'b)$ with $ikm \sim m$, $x' \sim j'm$ and $j'k'm \sim m$.
As $qy = qj'b = ib$, we get $qj' = i$. Hence $qj'km \sim m$. So $qj'm \sim qj'm$ by rule $\langle p_l \rangle$. As $x' \sim j'm$, we get $qx' \sim (qj')m$ by rule $\langle e_l \rangle$. Since we also have $(qj')km \sim m$, we can conclude that $qx = \alpha(qx') \sim' (qj')b = qy$.

— $qy = ib$ and $(x, y) = (ib, \alpha y')$.
This would imply $\alpha \prec ib$, which is impossible because $\alpha \neq \epsilon$ and $A_\alpha \cap (A_\sim \cup \{b\}) = \emptyset$.

— $qy = ib$ and $(x, y) = (i'b, j'b)$ with $ikm \sim m$, $i'k'm \sim m$ and $j'k'm \sim m$.
As $qy = qj'b = ib$, we get $qj' = i$. So $qj'km \sim m$. From $i'k'm \sim m$ and $j'k'm \sim m$, we can deduce $i'm \sim j'm$ as noted previously. So we can derive $qk(i'm) \sim m$ from $qk(j'm) \sim m$ by rule $\langle e_l \rangle$. So $(qi')km \sim m$ and $(qj')km \sim m$, and thus $qx = (qi')b \sim' (qj')b = qy$.

In all cases we get $qx \sim' qy$, so we have proved that $\sim'$ is closed under rule $\langle c \rangle$. □

Appendix B. Complete proof of the fundamental lemma

Lemma 6.13. Let L, K and M be three mutually disjoint alphabets. Let $(\sqsubseteq, \sim)$ be an (L, K, M) elementary representation. Let m, a, b, δ and c be such that $m \sqsubseteq m$, $a \neq b \in L \backslash A_{\sqsubseteq}$, $\delta \in K \backslash A_\sim$ and $c \in M \backslash A_\sim$. Then in each of the following cases, $(\sqsubseteq', \sim')$ is an (L, K, M) elementary representation:

(1)	$\sqsubseteq' = \sqsubseteq + \{ab - m\}$	and	$\sim' = \sim + \{\delta c - m, ab - c\}$	when $m \neq \epsilon$
(2)	$\sqsubseteq' = \sqsubseteq + \{am - b\}$	and	$\sim' = \sim + \{cm - b, \delta a - c\}$	
(3)	$\sqsubseteq' = \sqsubseteq + \{m - b\}$	and	$\sim' = \sim + \{\delta m - b\}$	
(3′)	$\sqsubseteq' = \sqsubseteq + \{m - b\}$	and	$\sim' = \sim + \{\delta m - b, \epsilon - \epsilon\}$	
(4)	$\sqsubseteq' = \sqsubseteq + \{\epsilon - m\}$	and	$\sim' = \sim + \{\delta c - m, \epsilon - c\}$	when $m \neq \epsilon$
(4′)	$\sqsubseteq' = \sqsubseteq + \{\epsilon - m\}$	and	$\sim' = \sim + \{\epsilon - c, m - \delta\}$	when $m \neq \epsilon$.

Proof. Case 2 was treated as an illustration in the main body of the paper; we consider the remaining cases here.

Case 1: $\sqsubseteq' = \sqsubseteq + \{ab - m\}$ and $\sim' = \sim + \{\delta c - m, ab - c\}$ and $m \neq \epsilon$.

As $m \neq \epsilon$, $\sqsubseteq'$ is clearly BI-elementary. By Proposition 6.12, we have $m \sqsubseteq m \Rightarrow m \in \mathscr{L}^{\sqsubseteq} \Rightarrow m \in \mathscr{L}^{\sim} \Rightarrow m \sim m$. As $\sim$ has no square (see Corollary 6.10), by Proposition 6.7, we have $m \sim \epsilon \Rightarrow m \in I_\sim^\star \Rightarrow m \in L^\star \cap M^\star \Rightarrow m = \epsilon$. Hence $m \nsim \epsilon$ and thus $\delta c - m$ is BBI-elementary with respect to $\sim$ ($\delta \neq c$ and $\delta, c \notin A_\sim$). Thus $\sim'' = \sim + \{\delta c - m\}$ is BBI-elementary, and according to Proposition 6.9, $I_{\sim''} = I_\sim$. Then we have $c \sim'' \epsilon \Rightarrow c \in I_{\sim''} \Rightarrow c \in I_\sim \Rightarrow c \in A_\sim$. Thus from $c \notin A_\sim$ we deduce $c \nsim'' \epsilon$. As $A_{\sim''} = A_\sim \cup \{\delta, c\}$, we have $A_{\sim''} \cap L = A_\sim \cap L = A_{\sqsubseteq}$ and thus $L \backslash A_{\sim''} = L \backslash A_{\sqsubseteq}$. So we have $a \neq b \in L \backslash A_{\sim''}$. Thus $ab - c$ is BBI-elementary with respect to $\sim''$. Hence,

$\sim' = \sim'' + \{ab - c\}$ is BBI-elementary and has no square, and $\mathsf{I}_{\sim'} = \mathsf{I}_{\sim''}$. So we deduce $\mathsf{I}_{\sim'} = \mathsf{I}_{\sim} \subseteq M$.

It is obvious that $\mathsf{A}_{\sim'} = \mathsf{A}_{\sim} \cup \{a, b, \delta, c\}$. Let $d \in M$ be such that $d \sim' d$. Then, either $d \in \mathsf{A}_{\sim}$ or $d = c$. On the one hand, if $d \in \mathsf{A}_{\sim}$, then $d \sim d$, and we let $x \in L^\star$ and $\alpha \in K^\star$ be such that $x\alpha \sim d$. Thus $x\alpha \sim' d$ because $\sim \subseteq \sim'$. On the other hand, if $d = c$, then $(ab)\epsilon \sim' c$ with $ab \in L^\star$ and $\epsilon \in K^\star$.

We now prove $\sqsubseteq' \subseteq \sqsubseteq^{L,K}_{\sim'}$. As $\delta c \sim' m$ and $ab \sim' c$, by rule $\langle e_l \rangle$, we get $\delta ab \sim' m$, and hence $ab \sqsubseteq^{L,K}_{\sim'} m$. As $\sim \subseteq \sim'$, we obviously have $\sqsubseteq = \sqsubseteq^{L,K}_{\sim} \subseteq \sqsubseteq^{L,K}_{\sim'}$, so $\sqsubseteq \cup \{ab - m\} \subseteq \sqsubseteq^{L,K}_{\sim'}$. We get $\sqsubseteq' = \sqsubseteq + \{ab - m\} \subseteq \sqsubseteq^{L,K}_{\sim'}$.

We now consider the converse inclusion $\sqsubseteq^{L,K}_{\sim'} \subseteq \sqsubseteq'$. As $m \not\sim \epsilon$ and $c \not\sim'' \epsilon$, and thus $mm \not\sim mm$ and $cc \not\sim'' cc$, we have the following identities according to Lemma 6.4:

$$\begin{aligned}
\sim'' = \sim + \{\delta c - m\} = \sim &\cup \{\delta x - \delta y, cx - cy \mid x \sim y \wedge mx \sim my\} \\
&\cup \{\delta cx - \delta cy \mid mx \sim my\} \\
&\cup \{\delta cx - y, y - \delta cx \mid mx \sim y\} \\
\sim' = \sim'' + \{ab - c\} = \sim'' &\cup \{ax - ay, bx - by \mid x \sim'' y \wedge cx \sim'' cy\} \\
&\cup \{abx - aby \mid cx \sim'' cy\} \\
&\cup \{abx - y, y - abx \mid cx \sim'' y\}\,.
\end{aligned}$$

Let $\gamma \in K^\star$ and $x, y \in L^\star$ be such that $\gamma x \sim' y$. We prove that $x \sqsubseteq' y$ by considering each of the possible forms taken by $(\gamma x, y)$:

— $\gamma x \sim'' y$.
According to the equations for $\sim''$, the only possibility for $\gamma x \sim'' y$ is when $\gamma x \sim y$: indeed, in the other cases, either δ occurs on the right (which is impossible because we would have $\delta \prec y$ with $\delta \notin L$) or c occurs on the left or on the right (which is impossible because $c \notin L \cup K$). Since $\gamma x \sim y$, we deduce $x \sqsubseteq y$ and thus $x \sqsubseteq' y$.

— $(\gamma x, y) = (ax', ay')$ with $x' \sim'' y'$ and $cx' \sim'' cy'$.
As $\gamma x = ax'$ and $a \not\prec \gamma$, we have $a \prec x$ and let $x'' = x/a$. Then $x' = \gamma x''$, $\gamma x'' \sim'' y'$ and $c\gamma x'' \sim'' cy'$. According to the equations for $\sim''$, as $cx' \sim'' cy'$ and $\delta \not\prec y'$, we must have $x' \sim y'$ and $mx' \sim my'$. Hence, $\gamma x'' \sim y'$ and $m\gamma x'' \sim my'$. Thus, $x'' \sqsubseteq y'$ and $mx'' \sqsubseteq my'$, and, as $\sqsubseteq \subseteq \sqsubseteq'$, we get $x'' \sqsubseteq' y'$ and $mx'' \sqsubseteq' my'$. We have the following deduction tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{mx'' \sqsubseteq' my'}{my' \sqsubseteq' my'}\langle r \rangle \qquad ab \sqsubseteq' m}{aby' \sqsubseteq' my'}\langle c \rangle}{ay' \sqsubseteq' ay'}\langle p_l \rangle \qquad x'' \sqsubseteq' y'}{ax'' \sqsubseteq' ay'}\langle c \rangle$$

So $x = ax'' \sqsubseteq' ay' = y$.

— $(\gamma x, y) = (bx', by')$ with $x' \sim'' y'$ and $cx' \sim'' cy'$.
The same argument applies using b instead of a in the last two steps of the left branch of the preceding deduction tree, so $x \sqsubseteq' y$.

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— $(\gamma x, y) = (abx', aby')$ with $cx' \sim'' cy'$.
As $\gamma x = abx'$, we define $x'' = x/ab$ and have $x = abx''$ and $x' = \gamma x''$. Hence, $\gamma cx'' \sim'' cy'$. As $\delta \not\prec y'$, the only possibility is when $\gamma x'' \sim y'$ and $m\gamma x'' \sim my'$. Thus, $x'' \sqsubseteq y'$ and $mx'' \sqsubseteq my'$ and we can repeat the preceding deduction tree using ab instead of a in the last two steps of the left branch. Thus $x = abx'' \sqsubseteq' aby' = y$.

— $(\gamma x, y) = (abx', y)$ with $cx' \sim'' y$.
Let $x'' = x/ab$. We have $x = abx''$ and $x' = \gamma x''$. Thus $c\gamma x'' \sim'' y$. As $c \not\prec y$, we must have $\delta \prec \gamma x''$ and $m(\gamma x'')/\delta \sim y$. As $\delta \not\prec x''$ (because $\delta \in K$ and $x'' \in L^\star$), we have $\delta \prec \gamma$. So we have $m(\gamma/\delta)x'' \sim y$. Hence $mx'' \sqsubseteq y$. So $mx'' \sqsubseteq' y$. As $ab \sqsubseteq' m$, with rule $\langle e_l \rangle$, we get $x = abx'' \sqsubseteq' y$.

— $(\gamma x, y) = (\gamma x, aby')$ with $\gamma x \sim'' cy'$.
As $c \not\prec \gamma x$, we must have $(\gamma x, cy') = (x'', \delta cy'')$ for some x'', y'' such that $my'' \sim x''$. But then $y' = \delta y''$, so $\delta \prec y' \prec aby' = y$. But this is impossible since $y \in L^\star$ and $\delta \in K$.

We have now proved that for any $\gamma \in K^\star$ and $x, y \in L^\star$, if $\gamma x \sim' y$, then $x \sqsubseteq' y$. Thus $\sqsubseteq^{L,K}_{\sim'} \subseteq \sqsubseteq'$. So we have indeed proved that in case 1 $(\sqsubseteq', \sim')$ is an (L, K, M) elementary representation.

Case 3: $\sqsubseteq' = \sqsubseteq + \{m - b\}$ and $\sim' = \sim + \{\delta m - b\}$.

First, $\sqsubseteq'$ is clearly BI-elementary. The constraint $\delta m - b$ is obviously BBI-elementary with respect to $\sim$ because $\delta \neq b$ and $\delta, b \notin A_\sim$, so $\sim'$ is BBI-elementary. According to Proposition 6.9, $\sim' = \sim + \{\delta m - b\}$ has no square and $I_{\sim'} = I_\sim \subseteq M$.
We have $A_{\sim'} = A_\sim \cup \{b, \delta\}$. Let $d \in M$ be such that $d \sim' d$. So we must have $d \in A_\sim$. Let $x \in L^\star$ and $\alpha \in K^\star$ be such that $x\alpha \sim d$. Hence $x\alpha \sim' d$.
As $\delta m \sim' b$, we deduce $m \sqsubseteq^{L,K}_{\sim'} b$ and thus $\sqsubseteq' = \sqsubseteq + \{m - b\} \subseteq \sqsubseteq^{L,K}_{\sim'}$. We now consider the converse inclusion $\sqsubseteq^{L,K}_{\sim'} \subseteq \sqsubseteq'$. We have the following identity according to Proposition 6.8:

$$\begin{aligned} \sim'' = \sim + \{\delta m - b\} = \sim &\cup \{\delta x - \delta y \mid x \sim y \wedge \exists k\ xk \sim m\} \\ &\cup \{\delta x - jb, jb - \delta x \mid x \sim m \wedge j \sim \epsilon\} \\ &\cup \{ib - jb \mid i \sim \epsilon \wedge j \sim \epsilon\}. \end{aligned}$$

Let $\gamma \in K^\star$ and $x, y \in L^\star$ be such that $\gamma x \sim' y$. We will prove that $x \sqsubseteq' y$ by considering each of the possible forms taken by $(\gamma x, y)$:

— $\gamma x \sim y$.
We have $x \sqsubseteq y$, so $x \sqsubseteq' y$.

— $(\gamma x, y) = (\delta x', \delta y')$.
This case is impossible because $\delta \not\prec y$.

— $(\gamma x, y) = (\delta x', jb)$ with $x' \sim m$ and $j \sim \epsilon$.
We have $j \in I^\star_\sim$, so $j \in M^\star$. As $j \prec jb = y$, we have $j \in L^\star$. Thus $j \in L^\star \cap M^\star$. So $j = \epsilon$ and $y = b$. As $\delta \not\prec x$ and $\delta \prec \delta x' = \gamma x$, we have $\delta \prec \gamma$. So let γ' be such that $\delta\gamma' = \gamma$. Then $\gamma' x = x'$, and we get $\gamma' x \sim m$. So $x \sqsubseteq m$ and thus $x \sqsubseteq' m$. As $m \sqsubseteq' b$, we get $x \sqsubseteq' b = y$ by rule $\langle t \rangle$.

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— $(\gamma x, y) = (ib, \delta y')$.
This case is impossible because $\delta \not\prec y$.

— $(\gamma x, y) = (ib, jb)$ with $i \sim \epsilon$ and $j \sim \epsilon$.
We have $i, j \in I^\star_\sim$, so $i, j \in M^\star$. As $i \prec \gamma x$ and $j \prec y$, we must have $i = j = \epsilon$. Hence $x = y = b$ and $\gamma = \epsilon$. As $m \sqsubseteq' b$, we have $b \sqsubseteq' b$ by rule $\langle r \rangle$, so $x \sqsubseteq' y$.

We have proved that for any $\gamma \in K^\star$ and $x, y \in L^\star$, if $\gamma x \sim' y$, then $x \sqsubseteq' y$. Thus $\sqsubseteq^{L,K}_{\sim'} \subseteq \sqsubseteq'$. So we have indeed proved that $\sqsubseteq' = \sqsubseteq^{L,K}_{\sim'}$ in case 3.

Case 3′: $\sqsubseteq' = \sqsubseteq + \{m - b\}$ and $\sim' = \sim + \{\delta m - b, \epsilon - \epsilon\}$.

This case is trivial because $\sim + \{\delta m - b, \epsilon - \epsilon\} = (\sim + \{\delta m - b\}) + \{\epsilon - \epsilon\} = \sim + \{\delta m - b\}$ and we are thus back to case 3.

(We will skip case 4 for the moment.)

Case 4′: $\sqsubseteq' = \sqsubseteq + \{\epsilon - m\}$ and $\sim' = \sim + \{\epsilon - c, m - \delta\}$ and $m \neq \epsilon$.

First, $\sqsubseteq'$ is clearly BI-elementary. As $c \notin A_\sim$, $\epsilon - c$ is BBI-elementary with respect to $\sim$, we have, according to Proposition 6.9, that $\sim'' = \sim + \{\epsilon - c\}$ has no square and $I_{\sim''} = I_\sim \cup \{c\} \subseteq M$. We have $m \not\sim'' \epsilon$, since otherwise $m \sim'' \epsilon \Rightarrow m \in I^\star_{\sim''} \Rightarrow m \in L^\star \cap M^\star \Rightarrow m = \epsilon$. Thus $m - \delta$ is BBI-elementary with respect to $\sim''$, so $\sim' = \sim'' + \{m - \delta\}$ is BBI-elementary and, according to Proposition 6.9, we have $I_{\sim'} = I_{\sim''} = I_\sim \cup \{c\} \subseteq M$. By rule $\langle s \rangle$, we have $\delta\epsilon \sim' m$, so we deduce $\epsilon \sqsubseteq^{L,K}_{\sim'} m$, and thus $\sqsubseteq' = \sqsubseteq + \{\epsilon - m\} \subseteq \sqsubseteq^{L,K}_{\sim'}$. We now consider the converse inclusion $\sqsubseteq^{L,K}_{\sim'} \subseteq \sqsubseteq'$. As $\sim'' + \{m - \delta\} = \sim'' + \{\delta - m\}$ by rule $\langle s \rangle$, we have the following identities according to Proposition 6.3 and Lemma 6.4 (as $m \not\sim'' \epsilon$ we have $mm \not\sim'' mm$):

$$\begin{aligned} \sim'' &= \sim + \{\epsilon - c\} = \{c^p x - c^q y \mid x \sim y \wedge p, q \geqslant 0\} \\ \sim' &= \sim'' + \{\delta - m\} = \sim'' \cup \{\delta x - \delta y \mid mx \sim'' my\} \\ &\qquad \cup \{\delta x - y, y - \delta x \mid mx \sim'' y\}. \end{aligned}$$

Let $\gamma \in K^\star$ and $x, y \in L^\star$ be such that $\gamma x \sim' y$. We will prove that $x \sqsubseteq' y$ by considering each case according to the form of $(\gamma x, y)$:

— $\gamma x \sim'' y$
We have $(\gamma x, y) = (c^p x', c^q y')$ with $x' \sim y'$ and $p, q \geqslant 0$. As $c \notin L \cup K$, we must have $p = q = 0$ and thus $x' = \gamma x$, $y' = y$ and $\gamma x \sim y$. So $x \sqsubseteq y$ and we get $x \sqsubseteq' y$.

— $(\gamma x, y) = (\delta x', \delta y')$.
This case is impossible because $\delta \not\prec y$.

— $(\gamma x, y) = (\delta x', y)$ with $mx' \sim'' y$.
As $mx' \in L^\star$, $y \in L^\star$ and $c \notin L$, we must have $mx' \sim y$. Then $\delta \prec \gamma$ and we let $\gamma'\delta = \gamma$. Hence $\gamma' x = x'$, so $m\gamma' x \sim y$. So we get $mx \sqsubseteq y$, and thus $mx \sqsubseteq' y$. But $\epsilon \sqsubseteq' m$, and we can then derive $x \sqsubseteq' y$ by rule $\langle e_l \rangle$.

— $(\gamma x, y) = (\gamma x, \delta y')$.
This case is impossible because $\delta \not\prec y$.

We have now proved that for any $\gamma \in K^\star$ and $x, y \in L^\star$, if $\gamma x \sim' y$, then $x \sqsubseteq' y$. Thus $\sqsubseteq^{L,K}_{\sim'} \subseteq \sqsubseteq'$. So we have indeed proved that $\sqsubseteq' = \sqsubseteq^{L,K}_{\sim'}$ in case 4′.

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Case 4: $\sqsubseteq' = \sqsubseteq + \{\epsilon - m\}$ and $\sim' = \sim + \{\delta c - m, \epsilon - c\}$ and $m \neq \epsilon$.

Case 4 is obtained from case 4′. Let $\sim'' = \sim + \{\epsilon - c\}$. As $\epsilon \sim'' c$, we can easily prove that $\sim'' + \{\delta c - m\} = \sim'' + \{\delta - m\}$. Thus

$$\begin{aligned}
\sim + \{\delta c - m, \epsilon - c\} &= (\sim + \{\epsilon - c\}) + \{\delta c - m\} \\
&= \sim'' + \{\delta c - m\} \\
&= \sim'' + \{\delta - m\} \\
&= \sim'' + \{m - \delta\} \\
&= (\sim + \{\epsilon - c\}) + \{m - \delta\} \\
&= \sim + \{\epsilon - c, m - \delta\}
\end{aligned}$$

and we are back to case 4′.

This completes the proof of the fundamental lemma. □

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