A TOPOLOGICAL VARIATION OF THE RECONSTRUCTION CONJECTURE

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Abstract. This paper investigates topological reconstruction, related to the reconstruction conjecture in graph theory. We ask whether the homeomorphism types of subspaces of a space X which are obtained by deleting singletons determine X uniquely up to homeomorphism. If the question can be answered affirmatively, such a space is called reconstructible. We prove that in various cases topological properties can be reconstructed. As main result we find that familiar spaces such as the reals $\mathbb{R}$, the rationals $\mathbb{Q}$ and the irrationals $\mathbb{P}$ are reconstructible, as well as spaces occurring as Stone–Čech compactifications. Moreover, some non-reconstructible spaces are discovered, amongst them the Cantor set C.

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1. Introduction. In 1941, S.M. Ulam and P.J. Kelly proposed a conjecture which has been known since then as the *reconstruction conjecture*. Roughly speaking, the reconstruction conjecture asks whether every finite graph is uniquely determined by the structure of subgraphs which are obtained by deleting a single vertex and all incident edges. Information about the reconstruction conjecture can be found in the survey by J.A. Bondy, [**2**]. The conjecture remains unsolved – and is considered as one of the most challenging problems in graph theory.

In this paper we describe a topological version of the reconstruction problem: when do certain subspaces of topological spaces determine the space uniquely up to homeomorphism? Embracing terminology from graph theory, we say that a topological space Y is a *card* of another space X if Y is homeomorphic to $X \setminus \{x\}$ for some x in X. The *deck* of a space X is a transversal for the non-homeomorphic cards of X, i.e. an object recording the topologically distinct subspaces one can obtain by deleting singletons from X.

Formally, for a space X we denote by $[X]_\sim$ the homeomorphism type of X. The deck of X can then be defined as the set $\mathcal{D}(X) = \{[X \setminus \{x\}]_\sim : x \in X\}$, and cards of X correspond to elements of the deck of X. Note that we deliberately identify a card, which is a concrete topological space Y, with the class of spaces homeomorphic to Y. In practice this identification never causes confusion, and, as in the following examples, we will simply state which cards occur.

If Y is a card of the real line $\mathbb{R}$, then Y is homeomorphic to two copies of the real line. We write this as $\mathcal{D}(\mathbb{R}) = \{\mathbb{R} \oplus \mathbb{R}\}$. Similarly, $\mathcal{D}(\mathbb{R}^n) = \{\mathbb{R}^n \setminus \{0\}\}$. In the case of

the unit interval I we have $\mathcal{D}(I) = \{[0, 1), [0, 1) \oplus [0, 1)\}$. For the sphere, stereographic projection gives $\mathcal{D}(S^n) = \{\mathbb{R}^n\}$. The Cantor set has the deck $\mathcal{D}(C) = \{C \setminus \{0\}\}$. Lastly, Cantor's back-and-forth method gives $\mathcal{D}(\mathbb{Q}) = \{\mathbb{Q}\}$ and $\mathcal{D}(\mathbb{P}) = \{\mathbb{P}\}$ for the rationals $\mathbb{Q}$ and the irrationals $\mathbb{P}$.

We now introduce the central concept of this paper, the notion of reconstruction. Given topological spaces X and Z, we say that Z is a *reconstruction of* X if their decks agree. A topological space X is said to be *reconstructible* if the only reconstructions of it are the spaces homeomorphic to X. In the same spirit, we say that a property of topological spaces is reconstructible if it is preserved under reconstruction.

Formally, a space X is reconstructible if $\mathcal{D}(X) = \mathcal{D}(Z)$ implies $X \cong Z$ and a property $\mathcal{P}$ of topological spaces is reconstructible if $\mathcal{D}(X) = \mathcal{D}(Z)$ implies "X has $\mathcal{P}$ if and only if Z has $\mathcal{P}$".

We remark that one could just as well consider the deck of a space as multi-set instead of just a set – we define the *multi-deck* of a space X to be the multi-set $\mathcal{D}'(X) = \{[X \setminus \{x\}]_{\sim} : x \in X\}$. In other words, the multi-deck not only knows which cards occur, but also how often they occur. If a space is reconstructible from its multi-deck, we will say it is *weakly reconstructible*. The formal definitions are exactly the same as above, where $\mathcal{D}(X)$ is replaced by $\mathcal{D}'(X)$. Clearly, reconstructible spaces and properties are weakly reconstructible.

In Section 2, we give examples of some reconstructible spaces and present different techniques for reconstructing topological spaces. We also recall some classic topological characterisations of common topological spaces. Further, we present examples of non-reconstructible spaces and comment on which properties of them are not reconstructible, most importantly that neither compactness nor connectedness are reconstructible in general.

These examples inform and delimit the subsequent investigation into which properties are reconstructible. The recurring theme will be that for a space X having a topological property $\mathcal{P}$, if sufficiently many cards share $\mathcal{P}$ then every reconstruction will satisfy $\mathcal{P}$. Thus, in Section 3 we will show that the common separation axioms with the exception of normality are reconstructible. We will also discuss normality in some detail in that section. We will end this section with a short proof that in T_1-spaces local properties are reconstructible. In Section 4, we will show that cardinal invariants such as weight, character, density and spread are reconstructible. We will also investigate the number of isolated points of a space and show that spaces with a finite number of isolated points are reconstructible.

With these tools we show in Section 5 that in Hausdorff spaces, reconstructing compactness is equivalent to reconstructing the space. We prove that Stone–Čech spaces (Stone–Čech compactifications of non-compact Tychonoff spaces) and spaces that arise as maximal finite-point compactifications are reconstructible. We will also use these ideas to show that a variety of topological properties which can be expressed in terms of the Stone–Čech compactification are reconstructible.

Next, in Section 6 we take a brief look at compactness-like properties. By considering the weaker properties Lindelöfness and pseudocompactness we can show that if a compact Hausdorff space X contains a G_δ-point and a non-G_δ-point then it is reconstructible. We finish this section by showing that metrisability and complete metrisability is reconstructible.

In the final Section 7, we consider connectedness: although not reconstructible in general, we manage to show that connectedness is weakly reconstructible in separable Tychonoff spaces. The main interest here lies in the fact that cards of connected spaces

are usually no longer connected. It is also remarkable, that this is the only example where our reconstruction result depends on the deck being a multi-set. For all other results, it is not important to know how often a particular card occurs.

Our terminology follows [4], except that we do no automatically include separation properties in the definition of compactness properties. Unless otherwise indicated, higher separation axioms always include lower separation axioms. The disjoint sum of topological spaces X_i, $i \in \kappa$ will be denoted by $\oplus_{i \in \kappa} X_i$. If all X_i are homeomorphic to X, we also write this as $\kappa \cdot X$. For a cardinal number κ, we denote by $\mathfrak{I}_\kappa$ the indiscrete space and by $\mathfrak{D}_\kappa$ the discrete space on κ elements.

2. Examples and counterexamples.

2.1. Examples of reconstructible spaces.
One important tool in showing that certain spaces are reconstructible is to characterise them and then show that each of the characterising properties is reconstructible. We will list here some of the spaces we show are reconstructible with forward pointers to the theorems that are needed to complete the proofs.

THEOREM 2.1. *For uncountable κ, the Cantor cube 2^κ and $[0, 1]^\kappa$ are reconstructible. None of their cards are compact, paracompact, Lindelöf or normal.*

Proof. For uncountable κ and $X = 2$ or $[0, 1]$, the space X^κ is a Stone–Čech compactification of a non-compact space [5, Theorem 2] and hence reconstructible by Theorem 5.4.

To see that none of its cards have the properties from above, note that by [4, 2.3.26 and 6.2.16], the space X^κ contains a (compact) copy of the Tychonoff plank $(\omega_1 + 1) \times (\omega + 1)$. Since X^κ is homogeneous, it follows that any card $X^\kappa \setminus \{x\}$ contains a closed copy of the deleted Tychonoff plank

$$(\omega_1 + 1) \times (\omega + 1) \setminus \{\langle \omega_1, \omega \rangle\},$$

well known for not being compact, paracompact, Lindelöf or normal [4, 3.12.20]. However, since all the properties mentioned in the theorem are hereditary with respect to closed subspaces, it follows that no card of X^κ can have any of named properties. □

THEOREM 2.2. *The space of real numbers $\mathbb{R}$, the space of rational numbers $\mathbb{Q}$ and the space of irrational numbers $\mathbb{P}$ are reconstructible spaces.*

Proof. The reals: Ward proved in [13] that every metric, locally connected, separable connected space, where every card has exactly two components, is homeomorphic to the space of real numbers. By Theorems 6.4, 3.3 and 4.2, any reconstruction of $\mathbb{R}$ is metrisable, locally connected and separable. Moreover, Corollary 7.5 gives that any reconstruction of $\mathbb{R}$ is connected. Thus, $\mathbb{R}$ is reconstructible.

The rationals: By Sierpinski's theorem (see [12] or [9, 1.9.6]), every dense-in-itself countable metrisable space is homeomorphic to the space of rational numbers. By Theorems 4.3 and 6.4, any reconstruction of $\mathbb{Q}$ shares these properties. Hence $\mathbb{Q}$ is reconstructible.

The irrationals: Alexandroff and Urysohn (see [1] or [9, 1.9.8]) showed that every topologically complete, zero-dimensional, separable metric space which does not contain a proper open compact subset is homeomorphic to the space $\mathbb{P}$ of irrational

numbers. If X is a reconstruction of $\mathbb{P}$ then by Theorems 6.5, 3.3 and 4.2, X is a completely metrisable, zero-dimensional, separable space.

Now suppose for a contradiction that X contains a non-empty compact open subset U. If $U \subsetneq X$, there exists an $x \in X$ such that $U \subset X \setminus \{x\} \cong \mathbb{P}$. This contradicts the fact that $\mathbb{P}$ has no proper open compact subsets. And if $U = X$, every card of X would inherit local compactness. This contradicts the fact that cards of X are homeomorphic to $\mathbb{P}$, and therefore not locally compact. Thus, X contains no proper open compact subset and therefore $X \cong \mathbb{P}$.   □

THEOREM 2.3. *The unit interval I and the spheres S^n are reconstructible.*

LEMMA 2.4 ([**8**] or [**3**, 6.10 and 6.11]). *(1) The only finite compactifications of $\mathbb{R}$ are S^1 and $[-\infty, \infty]$. (2) The only finite compactification of $\mathbb{R}^n$ ($n \geq 2$) is S^n.*   □

Proof of Theorem 2.3 **I**. I has a card $Y = [0, 1)$. Now any 2-point compactification of Y would be a 3-point compactification of $(0, 1)$, contradicting Lemma 2.4(1). Applying Theorem 5.3 gives that I is reconstructible.

S^1. Suppose X is a reconstruction of $\mathcal{D}(S^1) = \{\mathbb{R}\}$ which is not compact. As X must be locally compact Hausdorff by Theorems 3.3 and 3.1, there exists an Alexandroff 1-point compactification ωX which then is a 2-point compactification of $Y = \mathbb{R}$. Lemma 2.4(1) tells us that all such compactifications are unique, thus $\omega X \cong [0, 1]$. Since X must be connected, we have $X \cong [0, 1)$. But then $\mathcal{D}(X) \neq \{\mathbb{R}\}$. Hence X must be compact and therefore is the unique minimal compactification of $\mathbb{R}$, which is S^1.

S^n, $n \geq 2$. Lemma 2.4(2) gives that no card has a 2-point compactification, so we may apply Theorem 5.3.   □

Note that a similar method can be used in order to show that the Euclidean spaces $\mathbb{R}^n$ for $n \geq 2$ are reconstructible. In fact, one shows that $\mathbb{R}^n \setminus \{0\}$ has only a 1-point compactification or the 2-point compactification S^n. Then, it is not hard to see that if X is a reconstruction from $\mathcal{D}(\mathbb{R}^n)$ then X cannot be compact and $\omega X \cong S^n$.

2.2. Non-reconstructible spaces and properties. In the graph-theoretic case, the standard infinite counterexample to the reconstruction conjecture is the tree of countable infinite degree T, as its deck $\mathcal{D}(T) = \{\aleph_0 \cdot T\}$ is the same as of say $\aleph_0 \cdot T$ itself. In the topological world, this has a surprising analogy, namely the Cantor set C.

EXAMPLE 2.5. We have $\mathcal{D}(C) = \mathcal{D}(C \setminus \{0\})$, hence the Cantor set is not reconstructible. In particular, the properties of compactness, countable compactness and pseudocompactness are non-reconstructible.

Proof. This follows easily from the observation that $C \setminus \{0\} \cong \aleph_0 \cdot C$.   □

Interestingly, the Cantor set is just one particular example of a whole family of non-reconstructible spaces. Spaces which only have λ different homeomorphism types amongst their open subspaces (for some cardinal λ) are said to be of *diversity* λ [**11**]. The Cantor set is a compact Hausdorff space of diversity 2, and it is easy to see that in fact every such space X is non-reconstructible, as $\mathcal{D}(X) = \mathcal{D}(X \setminus \{x\})$. Hence, for example, the Double Arrow space D and also the product $D \times C$ are non-reconstructible [**11**].

EXAMPLE 2.6. Connectedness is a non-reconstructible property. Also, there are spaces with an arbitrary large number of non-homeomorphic reconstructions.

We use the same idea as in the previous example: construct a space where every card is an infinite disjoint sum of the original space.

Construction. We present a sketch of the construction. Let κ be a cardinal number. The *open hedgehog of spininess* κ is the space $\kappa \times [0, 1)$ where we identify all left end-points (the quotient under the equivalent relation $\langle \alpha, x \rangle \sim \langle \beta, y \rangle$ iff $x = y = 0$).

Now start with the open unit interval and replace every point by the open hedgehog of spininess κ with the centre taking the place of the original point. We continue inductively, replacing every non-central point again by a hedgehog of spininess κ (with half-open spines). If we put the natural metric onto the resulting space, namely, the distance between two points is the " shortest walk" from one to the other along the spines, we obtain a connected metrisable space X with cards homeomorphic to $\kappa \cdot X$ (removing a point leaves the components: everything to the left along the spine containing the point; everything to the right along that spine; and everything on each of the spines starting at that point), showing that connectedness is not reconstructible.

For $\lambda \leq \kappa$, every card of $\lambda \cdot X$ is homeomorphic to $\kappa \cdot X$. Therefore, if $\kappa = \aleph_\mu$ then X has μ many distinct reconstructions. □

EXAMPLE 2.7. Lindelöfness is a non-reconstructible property.

Construction. We use the following, somewhat different topology, inspired by filtration spaces [**6**] and resolutions [**14**].

Fix a cardinal κ. For $n \in \omega$, let $L_n = (0, 1) \times (\kappa \times (0, 1))^n$ and set $X = \bigcup_n L_n$. Viewing elements of X as partial functions from ω, the notion of extension makes sense. For $x = (x_0, \lambda_1, x_1, \ldots, \lambda_n, x_n) \in X$ and $\epsilon > 0$ we write

$$(x - \epsilon, x + \epsilon) = \{y \in X : y \text{ equals or extends } (x_0, \ldots, \lambda_n, a),$$
$$\text{where } a \in (0, 1) \cap (x_n - \epsilon, x_n + \epsilon)\},$$

and similarly for $[x - \epsilon, x + \epsilon]$. For $x \in X$, $y_1, \ldots, y_n$ extending x and $\epsilon, \delta > 0$ define

$$\langle x, \epsilon, y_1, \delta, \ldots, y_n, \delta \rangle = (x - \epsilon, x + \epsilon) \setminus \bigcup [y_j - \delta, y_j + \delta].$$

Sets of this form are a neighbourhood basis at $x \in X$. It is Hausdorff, regular and since each L_n is Lindelöf, so is X. Every card is a disjoint sum of κ-many copies of X and hence every card is in fact also a reconstruction of X. But X is connected and Lindelöf whereas the cards are not (if κ is uncountable). □

Our last example, which was pointed out to the authors by Mika Göös, shows that Theorem 4.4, which states that T_1 spaces with finitely many isolated points are reconstructible, does not hold in the infinite case.

EXAMPLE 2.8. There exists a non-reconstructible space containing infinitely many isolated points.

Construction. Denote by O the open hedgehog of countable spininess. Consider the two non-homeomorphic spaces

$$X_1 = O - -O == O - -O == O - -O == \cdots$$

$$X_2 = O == O - -O == O - -O == O - - \cdots$$

where $O--O$ means connecting the centres with a simple copy of $[0, 1]$ and $O == O$ means connecting the centres with countably many copies of $[0, 1]$. Now for $n \in \mathbb{N}$ let G_i^n, $i = 1, 2$, be the initial sequences of the spaces X_i ending at the nth centre, e.g. $G_1^4 = O--O == O--O$. Finally, define G to be

$$G = (\aleph_0 \cdot (0, 1)) \oplus \bigoplus_{n \geq 1}(\aleph_0 \cdot G_1^n) \oplus \bigoplus_{n \geq 1}(\aleph_0 \cdot G_2^n).$$

Now consider a card of the space X_1. Deleting a point from one of the spines gives a card homeomorphic to $X_1 \oplus (0, 1)$. Deleting a point from a simple connection $O--O$ gives a copy of some $G_1^n \oplus X_2$ and deleting a point from a countable connection $O ==$ O leaves the space invariant. Finally, deleting a centre point of one of the hedgehogs gives a card homeomorphic to some $G_1^n \oplus X_1 \oplus (\aleph_0 \cdot (0, 1))$ or $G_1^n \oplus X_2 \oplus (\aleph_0 \cdot (0, 1))$, depending on whether the centre point was to the left or to the right of a countable connection $O == O$.

Similar considerations for points in G imply that $\mathcal{D}(X_1 \oplus G) = \{X_1 \oplus G, X_2 \oplus G\} = \mathcal{D}(X_2 \oplus G)$, hence both spaces are non-reconstructible. Similarly, the spaces $X_1 \oplus G \oplus \mathfrak{D}_{\aleph_0}$ and $X_2 \oplus G \oplus \mathfrak{D}_{\aleph_0}$ have the same deck and hence are non-reconstructible. $\square$

3. Separation properties.   In this section, we will see that all hereditary separation axioms are reconstructible. For information on separation axioms, see e.g. [**4**, Section 1.5]. Since the discrete space and the indiscrete space on two points are reconstructions of each other, we restrict our attention to spaces containing at least three points.

Recall that a G_δ subset of topological space is a subset which can be expressed as a countable intersection of open sets, and an F_σ subset is a complement of a G_δ, a set which can be expressed as a countable union of closed subsets.

THEOREM 3.1. *If $|X| \geq 3$, then the property T_i is reconstructible for $i \neq 4$. For these i, X is T_i if and only if every card in the deck $\mathcal{D}(X)$ is T_i. In the T_4 case we have that if all cards $Y \in \mathcal{D}(X)$ are T_3 and one of them is T_4, then X is T_4 as well.*

Proof. For hereditary axioms T_i ($i \neq 4$) it suffices to prove the converse direction.

T₀ (Kolmogorov). Let $x \neq y \in X$ and find $z \in X \setminus \{x, y\}$. Then $x, y \in Y = X \setminus \{z\}$ and since Y is T_0 there exists an open set $U \subset X$ such that without loss of generality $x \in U \cap Y$ and $y \notin U \cap Y$. In X, we consequently have $x \in U$ and $y \notin U$, i.e. X is T_0.

T₁ (Frêchet). We need to show that all singletons $\{x\} \subset X$ are closed. Find $\{x, y, z\} \in X$. Since both $X \setminus \{y\}$ and $X \setminus \{z\}$ are T_1-spaces by assumption, $\{x\}$ is closed in these spaces. Now if $\{x\}$ was not closed in X, then by definition of the subspace topology the sets $\{x, y\}$ and $\{x, z\}$ must be closed in X. Thus, $\{x\} = \{x, y\} \cap \{x, z\}$ is closed, a contradiction.

T₂ (Hausdorff). Let $x \neq y \in X$ and find $z \in X \setminus \{x, y\}$. Then, $x, y \in Y = X \setminus \{z\}$ and since Y is T_2 there exist open sets $U, V \subset Y$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. Supposing that every card in our deck is T_2 we know by the previous part that X is T_1 and hence that Y must be an open subspace. Thus, U and V are open in X as well and therefore X is T_2.

T₃ (regular). Let $A \subset X$ be a closed set and let $x \in X \setminus A$. We have to separate x from A by open sets. Since the property T_2 is reconstructible, we may assume $|A| \geq 2$. Let $y \in A$ and $Y = X \setminus \{y\}$. Then $\emptyset \neq A \setminus \{y\} = A \cap Y$ is a closed subset of Y.

By assumption, Y is regular, so there exist disjoint open sets $U, V \subset Y$ such that $A \setminus \{y\} \subset U$ and $x \in V$. Because X is Hausdorff, the sets U and V are open in X. Further, we can separate x and y by disjoint open sets $M \ni x$ and $N \ni y$. Thus, we can separate x from A by $V \cap M \ni x$ and $U \cup N \supset A$.

$\mathbf{T_{3\frac{1}{2}}}$ (Tychonoff). Let $A \subset X$ be a closed set and let $x \in X \setminus A$. Since $T_{3\frac{1}{2}}$ implies T_3 we know by the previous part of this proof that X must be T_3. Hence we can find an open $U \subset X$ such that $x \in U \subset \overline{U} \subset (X \setminus A)$. Fix $y \in X \setminus \overline{U}$. By assumption, we can find a continuous mapping

$$f \colon X \setminus \{y\} \to [0, 1] \ \text{ such that } \ f(x) = 1 \ \text{ and } \ f((X \setminus U) \cap (X \setminus \{y\})) = 0.$$

The functions $g \colon X \setminus U \to \{0\}$ and $f|_{\overline{U}} \colon \overline{U} \to [0, 1]$ are both continuous, defined on closed subsets of X and coincide on their intersection. By the Pasting Lemma [**4**, 2.1.13], the continuous function $h \colon X \to [0, 1]$ coinciding with f and g where defined then separates the pair (A, x).

$\mathbf{T_4}$ (normal). Suppose all cards in $\mathcal{D}(X)$ are T_3 and $Y = X \setminus \{x\}$ is T_4. The previous parts show that X is T_3. If X was not T_4, there existed closed disjoint subsets $A, B \subset X$ which we cannot separate by open sets. If $x \notin A \cup B$, then we can separate A from B by open, disjoint $U, V \subset Y$. As Y is an open subspace, U and V are open in X, separating A from B. This is impossible, hence without loss of generality we may assume that $x \in A$. This time, we find disjoint open $U, V \subset Y$ separating $A \setminus \{x\}$ from B. Again, U and V are X-open as well. Further, since X is T_3, we can find an open $W \subset X$ such that $x \in W \subset \overline{W} \subset X \setminus B$. But now $U \cup W$ and $V \setminus \overline{W}$ separate A and B, a contradiction.

$\mathbf{T_5}$ (hereditarily normal). Recall that a space is T_5 if all its subspaces are T_4. Assume that every card in $\mathcal{D}(X)$ is T_5, but X is not. Then, there exists a subspace $Y \subset X$ which is not T_4. The previous part guarantees that X is T_4, thus $Y \subsetneq X$. With $x \in X \setminus Y$ the card corresponding to $X \setminus \{x\}$ contains the non-T_4 subspace Y, a contradiction.

$\mathbf{T_6}$ (perfectly normal). Recall that T_6 means that X is perfectly normal, i.e. that all closed subspaces are G_δ. Assume that every card in $\mathcal{D}(X)$ is T_6, but X is not. We may assume that X is T_1 but there exists some closed subset $A \subsetneq X$ which fails to be a G_δ-set. Choose $x \in X \setminus A$, then $Y = X \setminus \{x\}$ is T_6, so there exists a countable family of open sets $\mathcal{V}$ in Y such that $A = \bigcap \mathcal{V}$. But because X was T_1, all $V \in \mathcal{V}$ are X-open, a contradiction. $\qquad \square$

We remark that our reconstruction result about T_4-spaces is sufficient but not necessary. On the positive side, this last result applies to every normal space containing a G_δ-point, since normality is hereditary with respect to F_σ subspaces [**4**, 2.1.E]. In particular, normality is reconstructible in the realm of first-countable spaces.

However, as noted in Theorem 2.1 there are compact Hausdorff spaces such that all cards are non-normal. Even more, it is consistent with the usual axioms of set theory ZFC that normality is not reconstructible: we have an example, requiring the Continuum Hypothesis, where normality is non-reconstructible. However, we do not yet know of a ZFC example of a normal space with a non-normal reconstruction.

QUESTION 3.2. Is there a ZFC example where normality is non-reconstructible? And under what additional assumptions is normality reconstructible (e.g. is every reconstruction of a separable normal space normal)?

When investigating other topological properties, it is often useful to assume some of the lower separation axioms. Some authors even include those in the definition of the relevant property. If this allows us to obtain certain results, we will freely assume separation axioms up to and including complete regularity. As an example, we can easily show that 'local' topological properties are reconstructible in T_1-spaces:

For a topological property $\mathcal{P}$, we will say that a space X is 'locally $\mathcal{P}$' and call 'locally $\mathcal{P}$' a local property if and only if every neighbourhood filter has a basis of sets which satisfy $\mathcal{P}$. In other words, X is locally $\mathcal{P}$ if and only if for every x and open $U \ni x$ there is $A \subseteq X$ such that $x \in \text{int}(A) \subseteq A \subseteq U$ and A is $\mathcal{P}$. Examples of local properties are locally compact, locally connected, locally metrisable (we note that every locally metrisable space is T_1) but also zero-dimensionality (in the presence of regularity locally zero-dimensional is equivalent to zero-dimensional and every T_1 zero-dimensional space is clearly regular).

THEOREM 3.3. *In the realm of T_1-spaces, local properties are reconstructible.*

Proof. We prove that for a topological property $\mathcal{P}$, X is locally $\mathcal{P}$ if and only if all cards are locally $\mathcal{P}$. For the direct implication, note that a local property is hereditary with respect to open subspaces. Conversely, take an arbitrary $x \in X$ and choose $y \neq x$. Now note that a neighbourhood basis of x in the card $X \setminus \{y\}$ is a neighbourhood basis of x in X.   □

4. Cardinal invariants.

It is clear that we can reconstruct the size of a space from its deck, as every card has exactly one element less than the original space. Other cardinal invariants are reconstructible as well. We focus on those we need for other results. Following [4], we assume all cardinal invariants other than the size of a space to have infinite values only.

THEOREM 4.1. *The weight $w(X)$ of a topological space X is reconstructible.*

Proof. We claim that $w(X) = \sup \{w(Y) \colon Y \in \mathcal{D}(X)\}$.
"$\geq$": This follows from the definition of the subspace topology.
"$\leq$": We show that if $w(Y) < \mathfrak{m}$ for all $Y \in \mathcal{D}(X)$ then $w(X) < \mathfrak{m}$. Choose $x_1 \neq x_2 \in X$ and let $Y_i = X \setminus \{x_i\}$. By assumption, there exist bases $\mathcal{B}_i$ of Y_i of cardinality less than $\mathfrak{m}$. For $U \in \mathcal{B}_i$ we define $\tilde{U}$ to be the open set it comes from, i.e. $\tilde{U} = U$ if U was X-open and $\tilde{U} = U \cup \{x_i\}$ otherwise. Let $\tilde{\mathcal{B}}_i = \{\tilde{U} \colon U \in \mathcal{B}_i\}$, then $\mathcal{B} = \tilde{\mathcal{B}}_1 \cup \tilde{\mathcal{B}}_2$ is a base of X with $|\mathcal{B}| < \mathfrak{m}$.   □

THEOREM 4.2. *The density $d(X)$ of a topological space X is reconstructible.*

Proof. We claim that $d(X) = \inf \{d(Y) \colon Y \in \mathcal{D}(X)\}$.
"$\leq$": Let $Y \in \mathcal{D}(X)$ i.e. $Y = X \setminus \{x\}$ for some $x \in X$ and let $A \subset Y$ be a dense subset, i.e. $\overline{A}^Y = Y$. We then have $\overline{A \cup \{x\}}^X = X$, yielding $d(X) \leq d(Y)$.
"$\geq$": Let $A \subset X$ be a dense subset. We may assume that $A \neq X$ as otherwise the claim is immediate. Now choose $x \notin A$ and let $Y = X \setminus \{x\}$. But then $\overline{A}^Y = \overline{A}^X \cap Y = Y$.   □

With similar methods one can, for example, show that the character, the cellularity and the spread of a space are reconstructible.

We now prove two theorems about the number of isolated points. We will use them frequently in the following.

THEOREM 4.3. *In the realm of T_1-spaces, the number $i(X)$ of isolated points is reconstructible. In particular, if no card of a space X has an isolated point then X has none either.*

Proof. Since X is a T_1-space, any isolated point of a card Y is isolated in X, too. It follows $i(Y) \leq i(X)$. Moreover, since we delete at most one point, we have that $i(X) - 1 \leq i(Y) \leq i(X)$ for all $Y \in \mathcal{D}(X)$. If we are dealing with infinitely many isolated points, we are clearly finished. Supposing that all cards have $i(Y) < \infty$, we encounter two cases. If there are $Y_1, Y_2 \in \mathcal{D}(X)$ with $i(Y_1) \neq i(Y_2)$ then $i(X) = \max\{i(Y)\}$. Otherwise, if for all $Y \in \mathcal{D}(X)$ we have $i(Y) = n$ then it is not hard to see that $i(X) = 0$ if $n = 0$ and $i(X) = n + 1$ if $n > 0$. □

THEOREM 4.4. *Suppose that X is a T_1-space with $|X| \geq 3$ containing a positive but finite number of isolated points. Then, X is reconstructible.*

Proof. Let X be a T_1-space with finitely many isolated points and let Z be a reconstruction of X. There exists a card $Y \in \mathcal{D}(X)$ with $i(X) - 1 = i(Y) = i(Z) - 1$. But this means that the card Y was obtained by deleting an isolated point. Hence $X \cong Y \oplus \mathfrak{D}_1 \cong Z$, showing that X is reconstructible. □

THEOREM 4.5. *Suppose that X is a T_1-space with $|X| \geq 3$ that contains isolated points and a finite number of non-isolated points. Then X is reconstructible.*

Proof. Using the same reasoning as in Theorem 4.3, we see that the number of non-isolated points of a space X is reconstructible. It follows that every card of X that has the same number of non-isolated points as X must have been obtained by deleting an isolated point. This implies as in the previous theorem that X is reconstructible. □

5. Classes of reconstructible spaces.
We start with an observation which proves that compact Hausdorff spaces containing an isolated point are reconstructible. As an immediate corollary we obtain that spaces such as the Tychonoff plank T or the converging sequence $\omega + 1$ are reconstructible.

THEOREM 5.1. *Suppose X is a Hausdorff space with one compact card Y. Then X is reconstructible.*

Proof. Let $Y = X \setminus \{x\}$ be a compact subspace of X. Because in Hausdorff spaces compact subspaces are closed, we know that Y was obtained by deleting an isolated point. Thus $X \cong Y \oplus \mathfrak{D}_1$ is reconstructible. □

THEOREM 5.2. *If X is a compact Hausdorff space such that every reconstruction of X is compact then X is reconstructible.*

Proof. If X contains an isolated point, the claim follows from Theorem 5.1. In the other case, the claim follows from the fact that every reconstruction must be the 1-point compactification of some (and in fact every) card, so the result follows from the uniqueness of the 1-point compactification. □

This theorem explains why reconstructing compactness is so hard: for Hausdorff spaces, reconstructing compactness is in fact equivalent to reconstructing the space itself.

We now prove a pair of theorems, showing that certain "maximal" compactifications are reconstructible:

THEOREM 5.3. *Suppose X is a compact Hausdorff space. If there is a space Y such that X is the maximal finite-point compactification of Y (i.e. X is a compactification of Y with finite remainder of size n and there is no $n+1$-point compactification of Y), then X is reconstructible. In particular, if one card of X does not have a 2-point compactification then X is reconstructible.*

Proof. We may assume that X has no isolated points, as otherwise it is reconstructible by Theorem 5.1. Now assume that $Y = X \setminus \{x_0, \ldots, x_n\}$ is as above. Observe that $Y' = X \setminus \{x_0\}$ is locally compact and that any compactification of Y' is a compactification of Y (as $x_1, \ldots, x_n$ are not isolated). Hence Y' does not have a 2-point compactification.

Now assume that Z is a non-compact reconstruction from $\mathcal{D}(X)$. Then Z must be locally compact Hausdorff and contain no isolated points. Hence Z has a one-point compactification ωZ. But then ωZ would be a two-point compactification of Y', a contradiction.

Hence every reconstruction of X is compact Hausdorff and by Theorem 5.2 the result follows.   □

We call a space Z a *Stone–Čech space* if it occurs as a Stone–Čech compactification, meaning there exists a non-compact Tychonoff space X such that $Z = \beta X$. A similar idea as above allows us to prove that Stone–Čech spaces are reconstructible:

THEOREM 5.4. *Every Stone-Čech space $Z = \beta X$ is reconstructible.*

Proof. If Z contains an isolated point, then it is reconstructible by Theorem 5.1. So we may assume that Z contains no isolated points. Once we have shown that one card has no 2-point compactification, our claim follows from Theorem 5.3. Choose $z \in Z \setminus X$. Then, $Y = Z \setminus \{z\}$ lies in between X and its Stone–Čech compactification, yielding $\beta Y = Z$ [4, 3.6.9]. Now since the Stone–Čech compactification of Y coincides with its 1-point compactification, it follows from the maximality of the Stone–Čech compactification that Y cannot have 2-point compactifications.   □

This theorem raises the question whether there is an elegant way that allows us to decide when a compact Hausdorff space is a Stone–Čech space. To our knowledge, this question has only been answered in special cases. One of these cases has be investigated in [5, Theorem 2]. In this paper, Glicksberg showed that when κ is uncountable, then both the Cantor cube $\{0, 1\}^\kappa$ and the Hilbert cube I^κ are Stone–Čech spaces. Thus, they are reconstructible.

We now describe a method which deals with the reconstruction of topological properties that can be characterised in terms of the relation between a Tychonoff space X and its Stone–Čech compactification. Suppose $\mathcal{P}$ is a topological property which is hereditary with respect to cards. We call $\mathcal{P}$ *Stone–Čech characterisable* if there exists a property $\mathcal{Q}$ of subspaces such that

$$X \vDash \mathcal{P} \ (X \text{ has } \mathcal{P}) \quad \text{if and only if} \quad (X, \alpha X) \vDash \mathcal{Q}$$

for every compactification αX of X. The property $\mathcal{Q}$ is *finitely additive* if whenever $(Y_1, Z) \vDash \mathcal{Q}$ and $(Y_2, Z) \vDash \mathcal{Q}$ then as well $(Y_1 \cup Y_2, Z) \vDash \mathcal{Q}$.

LEMMA 5.5. *Let X be a $T_{3\frac{1}{2}}$-space. Suppose that $\mathcal{P}$ is a Stone–Čech characterisable property such that the corresponding $\mathcal{Q}$ is finitely additive. Then, $\mathcal{P}$ is reconstructible.*

Proof. We show that a space X has property $\mathcal{P}$ if and only if all cards have property $\mathcal{P}$. First, by assumption, property $\mathcal{P}$ is hereditary with respect to cards.

For the converse, assume that all cards of X have $\mathcal{P}$. By Theorem 4.5 we may assume that X contains at least two non-isolated points x_1 and x_2. Consider the cards $Y_i = X \setminus \{x_i\}$. Since both cards are dense in X, it follows that βX is a compactification of both Y_i. By assumption, we have $(Y_i, \beta X) \vDash \mathcal{Q}$. As $\mathcal{Q}$ is finitely additive, we have $(Y_1 \cup Y_2, \beta X) \vDash \mathcal{Q}$. However, $Y_1 \cup Y_2 = X$, so finally $X \vDash \mathcal{P}$, completing the proof. $\square$

THEOREM 5.6.   *Čech-completeness is reconstructible, i.e. a $T_{3\frac{1}{2}}$-space is Čech-complete if and only if all cards $Y \in \mathcal{D}(X)$ are Čech-complete.*

Proof. Let $\mathcal{P}$ denote Čech-completeness. Recall that a space X has $\mathcal{P}$ if X is a G_δ-set in every ($\Leftrightarrow$ one) compactification [**4**, 3.9.1]. We show that $\mathcal{P}$ satisfies all assumptions in Lemma 5.5. If $Y \subset X$ is an open subspace and $X \subset \beta X$ is a G_δ-set then it is clear that Y is a G_δ-set in βX as well. So Y is a G_δ-set in $\overline{Y} \subset \beta X$ which is a compactification of Y. This shows that Y is Čech-complete, and hence cards inherit $\mathcal{P}$. Moreover, the union of two G_δ-sets is again a G_δ-set. This shows that $\mathcal{Q}$ is finitely additive, as required. $\square$

6. Compactness-like properties and metrisability.   Compactness is surely one of the strangest properties with respect to reconstruction. We have already seen that compactness itself is not reconstructible. At the same time, compactness-like properties play an essential role in our affirmative results about reconstruction.

It is clear that if one card of a space X is compact (Lindelöf, countably compact, pseudocompact) then so is X. In the case of Lindelöfness and pseudocompactness, we can use this observation in certain circumstances. Since Lindelöfness is inherited by F_σ-sets, it is reconstructible in spaces having a G_δ-point, e.g. in T_6 and in first-countable spaces. Further, recall that a space X is pseudocompact if in some compactification αX of X every closed G_δ-set intersects X. This yields that pseudocompactness is reconstructible in spaces containing a non-G_δ-point. Together, these two observations shed light on the question under which conditions compactness is reconstructible.

THEOREM 6.1. *Suppose X is a compact T_2-space containing points x_1 and x_2 such that x_1 is a G_δ-point and x_2 is not. Then X is reconstructible.*

Proof. Again we may assume that X does not contain isolated points. The card $Y_1 = X \setminus \{x_1\}$ is Lindelöf and normal. Theorem 3.1 and the previous remarks give that every reconstruction of X is T_4, Lindelöf and pseudocompact. Since in T_4-spaces, pseudocompactness and countable compactness are equivalent [**4**, 3.10.21], it follows that every reconstruction of X is Lindelöf and countably compact, hence compact. Now use Theorem 5.2. $\square$

Another reason that makes compactness-like properties so interesting in our context is that they can be used to obtain results about metrisability. Our first result in this direction is about paracompactness. It is worth noting this result's symmetry to the partial reconstruction of the T_4 axiom in Theorem 3.1.

LEMMA 6.2. *Suppose that every card $Y \in \mathcal{D}(X)$ is T_3 and that one card is paracompact. Then X is paracompact.*

Proof. We use [**4**, 5.1.11]: For a T_3 space X we have the following equivalence: X is paracompact if and only if every open cover has a locally finite refinement consisting of arbitrary sets. For the proof, let $\mathcal{U}$ be an arbitrary open cover. Identify $x \in X$ such that $X \setminus \{x\}$ is paracompact. Choose $A \in \mathcal{U}$ such that $x \in A$. Now $X \setminus A$ is a closed subset of a paracompact space $X \setminus \{x\}$, hence paracompact. Choose a locally finite open refinement $\mathcal{U}'$. Now let $\mathcal{U}'' = \{A\} \cup \{V \cap (X \setminus A) : V \in \mathcal{U}'\}$. This is a locally finite refinement (consisting of arbitrary sets) of $\mathcal{U}$, as for all $x \in A$, this A is a neighbourhood intersecting $\mathcal{U}''$ only once. For all other points, local finiteness is inherited from $\mathcal{U}'$. Since X being T_3 is reconstructible by Theorem 3.1, the characterisation given in the beginning of this proof validates our claim. $\qquad\square$

THEOREM 6.3. *In the realm of T_3-spaces containing one singleton $\{x\} \subset X$ which is a G_δ-set, paracompactness is reconstructible.*

Proof. Lemma 6.2 together with the fact that paracompactness is hereditary with respect to F_σ sets ([**4**, 5.1.28]). $\qquad\square$

THEOREM 6.4. *Metrisability is reconstructible. That is, X is metrisable if and only if every card in $\mathcal{D}(X)$ is metrisable.*

Proof. Every reconstruction of a metrisable space X is locally metrisable by Theorem 3.3. Further, in a metric space every point is a G_δ. Thus, by Theorem 6.3 every reconstruction of X is paracompact. Now the Smirnov Metrisation Theorem (X is metrisable if and only if X is paracompact and locally metrisable) [**4**, 5.4.A] gives that X is metrisable. $\qquad\square$

THEOREM 6.5. *Complete metrisability is reconstructible, i.e. X is completely metrisable if and only if all cards $Y \in \mathcal{D}(X)$ are completely metrisable.*

Proof. A space is completely metrisable if and only if it is metrisable and Čech-complete [**4**, 4.3.26]. The claim now follows from Theorems 6.4 and 5.6. $\qquad\square$

7. Connectedness properties. In this section, we are again facing the difficulty that cards of connected spaces do not have to be connected. As in the case of compactness it is not hard to see that a space with at least three elements is connected if all cards are connected. A bit stronger: if X has no isolated points then one connected card guarantees that X is connected. Still, we already saw that connectedness is non-reconstructible. On the other side, various kinds of disconnectedness are reconstructible. These results base strongly on the following lemma by Kline.

LEMMA 7.1. ([**7**]) *A connected space X with $|X| \geq 3$ cannot have more than one dispersion point. Moreover, if one card of X is totally disconnected, then all other cards are connected.* $\qquad\square$

THEOREM 7.2. *Let $|X| \geq 3$. Then, total disconnectedness is reconstructible, i.e. X is totally disconnected if and only if every card is totally disconnected.*

Proof. The direct implication follows from the fact that total disconnectedness is hereditary. For the converse we prove the contraposition. Suppose that X is not totally disconnected. If one component $A \subset X$ has $|A| \geq 3$, we can apply Lemma 7.1 and see that for some $a \in A$, the card corresponding to $X \setminus \{a\}$ is not totally disconnected. As a result, we may assume that all components of X contain at most two points, and further that there is at least one component $A \subset X$ with $|A| = 2$. But now, for a

point $x \in X \setminus A$, the card $X \setminus \{x\}$ contains the connected set A, and hence is not totally disconnected. $\qquad \square$

Although we have seen that in general connectedness is not reconstructible and in fact may not be inherited by any of the cards, traces of connectedness remain in the cards: removing a point will, with few exceptions, not leave behind a very disconnected space. We can use a precise version of this together with the fact that connected Tychonoff spaces have cardinality at least continuum to reconstruct connectedness in certain circumstances.

We note that this is the only proof of this paper which needs the deck to be a multiset instead of a set. It is also the only property which we have shown to be reconstructible without any card having it.

The following is a slight generalisation of a theorem of Whyburn about compact connected metrisable spaces in [**15**]. The proof is a modern, slightly simplified and adapted version of Whyburn's original proof.

THEOREM 7.3. *If X is a T_1 connected separable space then for all but countably many points x of X, the subspace $X \setminus \{x\}$ has at most two components.*

Proof. Let E be the subset of X consisting of all points x such that the corresponding card $Y_x = X \setminus \{x\}$ has at least three components. Assume for a contradiction that E is uncountable.

Let D be a countable dense subset of X. For each point x in E, there is a partition of Y_x into disjoint non-empty Y_x-clopen sets $C_{x,1}$, $C_{x,2}$ and $C_{x,3}$. Since X is T_1, every $C_{x,i}$ is X-open. In particular, we may pick points $d_{x,i} \in C_{x,i} \cap D$. Further, since X is connected, the closure of $C_{x,i}$ in X equals $C_{x,i} \cup \{x\}$ and is connected: this last observation holds, as any $\overline{C_{x,i}}$-clopen subset not containing x would be X-clopen, contradicting connectedness of X.

By the pigeon-hole principle, there must be distinct points x and y in E such that $\{d_{x,1}, d_{x,2}, d_{x,3}\} = \{d_{y,1}, d_{y,2}, d_{y,3}\}$. Without loss of generality, we may assume $d_{x,i} = d_{y,i}$, and hence that $C_{x,i}$ intersects $C_{y,i}$ for all i. It follows that the connected set $C_{x,1} \cup C_{x,2} \cup \{x\}$ must contain y, since a subset of Y_y intersecting both $C_{y,1}$ and $C_{y,2}$ must be disconnected.

But by a symmetric argument, also $C_{x,1} \cup C_{x,3} \cup \{x\}$ and $C_{x,2} \cup C_{x,3} \cup \{x\}$ have to contain y, from which we conclude that $x = y$, a contradiction. $\qquad \square$

THEOREM 7.4. *Being a Tychonoff, separable and connected space is weakly reconstructible.*

Proof. First note that being a Tychonoff separable space is reconstructible, and hence also weakly reconstructible. Next, if X has a connected card, then every reconstruction is connected, since $i(X) = 0$ is reconstructible in T_1-spaces by Theorem 4.3. Hence we may assume that X is a cut-space, i.e. that every card is disconnected. We now claim:

Claim: If X is a connected, separable, Tychonoff space without isolated points and without a connected card, then there is a card $X \setminus \{x_0\}$ having precisely two components C_1 and C_2 such that both have uncountably many cut-points.

Proof of claim: Let E again denote the set of points x such that the corresponding card $Y_x = X \setminus \{x\}$ has at least three components. Then E is countable by Theorem 7.3. As X is connected Tychonoff, so uncountable, we can find some $x_0 \notin E$ such that Y_{x_0}

has indeed two components C_1 and C_2. Both components are non-trivial connected Tychonoff spaces, so uncountable.

To establish the claim, it suffices to show that all points in $C_1 \setminus E$ are cut-points of C_1. Assume for a contradiction that $y \in C_1 \setminus E$ is not a cut-point of C_1. As before, the card Y_y has two components D_1 and D_2. Since C_2 is a connected subset of Y_y, we may assume without loss of generality that $C_2 \subset D_2$. It follows that closure of C_2 in X, equal to $C_2 \cup \{x_0\}$, is also a connected subset of Y_y, and hence $C_2 \cup \{x_0\} \subset D_2$.

However, the closure of $C_1 \setminus \{y\}$ in X, a connected set by assumption on y, also contains x_0. Since it intersects D_2 in x_0, it must be contained in D_2. Thus D_1 is empty, a contradiction. This establishes the claim.

To conclude the proof, let $X \setminus \{x_0\}$ be the card mentioned in the claim and assume Z is a disconnected reconstruction of X. Find $z_0 \in Z$ such that $Z \setminus \{z_0\} \cong X \setminus \{x_0\}$. It follows that $Z \setminus \{z_0\}$ has two components F_1 and F_2 that correspond to C_1 and C_2 of $X \setminus \{x_0\}$.

Note that without loss of generality, $z_0 \notin \overline{F_2}$, as otherwise Z would be connected. Hence, F_2 and $F_1 \cup \{z\}$ are clopen subsets of Z. But F_2 has uncountably many cut-points y such that $F_2 \setminus \{y\}$ has components $C_{y,1}$ and $C_{y,2}$. Thus, for all those y, the card $Z \setminus \{y\}$ has a partition into three non-empty open sets, namely $F_1 \cup \{z\}$, $C_{y,1}$ and $C_{y,2}$. We conclude that the multi-deck of Z contains uncountably many cards having at least three components. Therefore, by Theorem 7.3, Z cannot be a reconstruction of X. $\square$

It is easy to see that for spaces having only one card, the notions of reconstruction and weak reconstruction coincide. Hence, we obtain the following corollary.

COROLLARY 7.5. *Every reconstruction of the reals $\mathbb{R}$ is connected.* $\square$

All of our counterexamples of non-reconstructible spaces in Section 2 rely on the failure of connectedness in that a possible different reconstruction has many components. Given the previous theorem and also Theorem 5.2 it is therefore interesting to ask:

QUESTION 7.6. *If X is connected and every reconstruction of X is connected, is X reconstructible?*

If not, are separable connected Tychonoff spaces reconstructible?

If not, are compact connected metrisable spaces reconstructible?

For the last question, note that by the Non-cutpoint Existence Theorem [**10**, 6.6], every compact connected metrisable space contains a connected card. Since by Theorem 4.3, no reconstruction of such a space can contain isolated points, it follows that every reconstruction of a compact connected metrisable space must be connected.

In a forthcoming paper, we prove that indeed all compact connected metrisable spaces are reconstructible.

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