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PLURIPOLAR SETS, REAL SUBMANIFOLDS AND PSEUDOHOLOMORPHIC DISCS

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Abstract

We prove that a compact subset of full measure on a generic submanifold of an almost complex manifold is not a pluripolar set. Several related results on boundary behavior of plurisubharmonic functions are established. Our approach is based on gluing a family of complex discs to a generic manifold along a boundary arc and could admit further applications.

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1. Introduction

The foundations of the theory of almost complex structures go back to the classical work of Newlander and Nirenberg (see, for example, [1]), where a complete criterion of integrability of these structures was established. The modern period of development began after the famous paper by Gromov in [1], who discovered a deep connection between the almost complex and the symplectic geometry. Since then the analysis on almost complex manifolds has become a powerful tool of symplectic geometry.

From an analytic point of view (which is the focus of the present paper), the analysis on almost complex manifolds has several features not usual for the much better understood case of integrable almost complex structures. One of them is that for a ‘generic’ choice of an almost complex structure of complex dimension ≥ 2, the only holomorphic functions (even locally) are the constant ones. Contrarily, there are two objects (they can be viewed as dual ones) which always exist, at least, locally: pseudoholomorphic discs and plurisubharmonic functions. Both of them are important technical tools of the symplectic geometry. The theory of pseudoholomorphic discs is (relatively) well elaborated now (although there are many interesting open questions

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remaining). The theory of plurisubharmonic functions on almost complex manifolds is much younger and its development is rather recent; many quite natural questions remain open.

The goal of the present paper is to study some boundary properties of plurisubharmonic functions near real submanifolds of almost complex manifolds. Our main inspiration is the well-known paper by Sadullaev [16], where he established several useful results on boundary uniqueness for plurisubharmonic functions as well as the two-constants-type theorems in $\mathbb{C}^n$. His main technical tool is a construction (due to Pinchuk [12]) of a local family of holomorphic discs glued along the upper semicircle to a prescribed generic totally real manifold in $\mathbb{C}^n$. In the present paper we extend these results to the almost complex case. This first step was done recently in [17] but here we present considerably more advanced results. We hope that the almost complex analog of the Pinchuk–Sadullaev gluing disc construction elaborated in the present paper will have other applications.

The organization of the paper can be seen from the contents. Sections 2 and 3 are preliminary. Section 4 contains the main technical tool (the construction of pseudoholomorphic discs in the spirit of [12, 16]). The main results are contained in Section 5.

2. Almost complex manifolds and pseudoholomorphic discs

In this section we recall basic notions of the almost complex geometry making our presentation more convenient for specialists in analysis. Everywhere throughout this paper we assume that manifolds and almost complex structures are of class C^∞; notice that the main results remain true under considerably weaker regularity assumptions.

2.1. Almost complex manifolds. Let M be a smooth manifold of dimension $2n$. *An almost complex structure* J on M is a smooth map which associates to every point $p \in M$ a linear isomorphism $J(p) : T_pM \to T_pM$ of the tangent space T_pM such that $J(p)^2 = -I$; here I denotes the identity map of T_pM. Thus, every $J(p)$ is a linear complex structure on T_pM. A couple (M, J) is called *an almost complex manifold* of complex dimension n. Note that every almost complex manifold admits the canonical orientation represented by $(e_1, Je_1, \dots, e_n, Je_n)$, where $(e_1, \dots, e_n)$ is any complex basis of $(T_pM, J(p))$.

An important example is provided by the *standard complex structure* $J_{st} = J_{st}^{(2)}$ on $M = \mathbb{R}^2$, which is given in the canonical coordinates of $\mathbb{R}^2$ by the matrix

$$J_{st} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

More generally, the standard complex structure J_{st} on $\mathbb{R}^{2n}$ is represented by the block diagonal matrix $diag(J_{st}^{(2)}, \dots, J_{st}^{(2)})$ (usually we drop the notation of dimension because its value will be clear from the context). As usual, setting $iv := Jv$ for $v \in \mathbb{R}^{2n}$, we identify $(\mathbb{R}^{2n}, J_{st})$ with $\mathbb{C}^n$ using the notation $z = x + iy = x + Jy$ for the standard complex coordinates $z = (z_1, \dots, z_n) \in \mathbb{C}^n$.

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Let (M, J) and (M', J') be smooth almost complex manifolds. A C^1-map $f : M' \to M$ is called (J', J)-complex or (J', J)-holomorphic if it satisfies *the Cauchy–Riemann equations*

$$df \circ J' = J \circ df. \tag{2.1}$$

Note that a map $f : \mathbb{C}^n \to \mathbb{C}^m$ is (J_{st}, J_{st})-holomorphic if and only if each component of f is a usual holomorphic function.

Every almost complex manifold (M, J) can be viewed locally as the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ equipped with a small almost complex deformation of J_{st}. The following statement is often very useful.

Lemma 2.1. *Let (M, J) be an almost complex manifold. Then for every point $p \in M$, every $m \geq 0$ and $\lambda_0 > 0$ there exist a neighborhood U of p and a coordinate diffeomorphism $z : U \to \mathbb{B}$ such that $z(p) = 0$, $dz(p) \circ J(p) \circ dz^{-1}(0) = J_{st}$, and the direct image $z_*(J) := dz \circ J \circ dz^{-1}$ satisfies $\|z_*(J) - J_{st}\|_{C^m(\overline{\mathbb{B}})} \leq \lambda_0$.*

Proof. There exists a diffeomorphism z from a neighborhood U' of $p \in M$ onto $\mathbb{B}$ satisfying $z(p) = 0$; after an additional linear change of coordinates, one can achieve $dz(p) \circ J(p) \circ dz^{-1}(0) = J_{st}$ (this is a classical linear algebra). For $\lambda > 0$, consider the dilation $d_\lambda : t \mapsto \lambda^{-1} t$ in $\mathbb{R}^{2n}$ and the composition $z_\lambda = d_\lambda \circ z$. Then $\lim_{\lambda \to 0} \|(z_\lambda)_*(J) - J_{st}\|_{C^m(\overline{\mathbb{B}})} = 0$ for every $m \geq 0$. Setting $U = z_\lambda^{-1}(\mathbb{B})$ for $\lambda > 0$ small enough, we obtain the desired statement. □

In what follows we often denote the structure $z^*(J)$ again by J, viewing it as a local representation of J in the coordinate system (z).

Recall that an almost complex structure J is called *integrable* if (M, J) is locally biholomorphic in a neighborhood of each point to an open subset of $(\mathbb{C}^n, J_{st})$. In the case of a complex dimension > 1, integrable almost complex structures form a highly special subclass in the space of all almost complex structures on M.

In this paper we deal with standard classes of real submanifolds. A submanifold E of an almost complex n-dimensional (M, J) is called *totally real* if at every point $p \in E$ the tangent space T_pE does not contain nontrivial complex vectors, that is, $T_pE \cap JT_pE = \{0\}$. It is well known that the (real) dimension of a totally real submanifold of M is not bigger than n; we will consider in this paper only n-dimensional totally real submanifolds, that is, the case of maximal dimension. A real submanifold N of (M, J) is called *generic* if the complex span of T_pN is equal to the whole T_pM for each point $p \in N$. A real n-dimensional submanifold of (M, J) is generic if and only if it is totally real.

Lemma 2.2. *Let N be a generic $(n + d)$-dimensional $(d > 0)$ submanifold of an almost complex n-dimensional manifold (M, J). Suppose that K is a subset of N of nonzero Hausdorff $(n + d)$-measure. Then there exists a (local) foliation of N into a family (E_s), $s \in \mathbb{R}^d$ of totally real n-dimensional submanifolds such that the intersection $K \cap E_s$ has a nonzero Hausdorff n-measure for each s from some subset of nonzero Lebesgue measure in $\mathbb{R}^d$.*

Here the Hausdorff measure is defined with respect to any Riemannian metric on M; the assumption that K has a positive n-measure is independent of a choice of such metric.

Proof. Let p be a point of M such that K has a nonzero measure in each neighborhood of p. Choose local coordinates z near p such that $p = 0$ and $J(0) = J_{st}$. After a $\mathbb{C}$-linear change of coordinates, one has $N = \{x_j + o(|z|) = 0, j = n - d + 1, \ldots, n\}$. After a local diffeomorphism with the identical linear part at 0, we obtain that $N = \mathbb{R}^d(x_1, \ldots, x_d) \times i\mathbb{R}^n(y)$. In the new coordinates the condition $J(0) = J_s$ still holds and every slice $E_s = \{z \in N : x_1 = s_1, \ldots, x_d = s_d\}$ is totally real. Now we conclude by the Fubini theorem. □

A totally real manifold E can be defined as

$$E = \{p \in M : \rho_j(p) = 0, j = 1, \ldots, n\}, \tag{2.2}$$

where $\rho_j : M \to \mathbb{R}$ are smooth functions with nonvanishing gradients. The condition of total reality means that for every $p \in E$, the J-complex linear parts of the differentials $d\rho_j$ are (complex) linearly independent.

A subdomain

$$W = \{p \in M : \rho_j(p) < 0, j = 1, \ldots, n\} \tag{2.3}$$

is called *the wedge with the edge E*.

2.2. Pseudoholomorphic discs. Let (M, J) be an almost complex manifold of dimension $n > 1$. For a 'generic' choice of an almost complex structure, any holomorphic (even locally) function on M is constant. Similarly, M does not admit nontrivial J-complex submanifolds (that is, with tangent spaces invariant with respect to J) of complex dimension > 1. The only (but fundamentally important) exception arises in the case of pseudoholomorphic curves, that is, J-complex submanifolds of complex dimension 1: they always exist locally.

Usually pseudoholomorphic curves arise in connection with solutions f of (2.1) in the special case where M' has the complex dimension 1. These holomorphic maps are called J-complex (or J-holomorphic or *pseudoholomorphic*) curves. Note that we view here the curves as maps, that is, we consider parametrized curves. We use the notation $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ for the unit disc in $\mathbb{C}$ always assuming that it is equipped with the standard complex structure J_{st}. If in the equations (2.1) we have $M' = \mathbb{D}$, we call such a map f a *J-complex disc* or a *pseudoholomorphic disc* or just a holomorphic disc when the structure J is fixed.

A fundamental fact is that pseudoholomorphic discs always exist in a suitable neighborhood of any point of M; this is the classical Nijenhuis–Woolf theorem (see [1]). Here it is convenient to rewrite the equations (2.1) in local coordinates similarly to the complex version of the usual Cauchy–Riemann equations.

Everything will be local, so (as above) we are in a neighborhood Ω of 0 in $\mathbb{C}^n$ with the standard complex coordinates $z = (z_1, \ldots, z_n)$. We assume that J is an almost

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complex structure defined on Ω and $J(0) = J_{st}$. Let

$$z : \mathbb{D} \to \Omega,$$
$$z : \zeta \mapsto z(\zeta)$$

be a J-complex disc. Setting $\zeta = \xi + i\eta$, we write (2.1) in the form $z_\eta = J(z)z_\xi$. This equation can be in turn written as

$$z_{\overline{\zeta}} - A(z)\overline{z}_{\overline{\zeta}} = 0, \quad \zeta \in \mathbb{D}. \tag{2.4}$$

Here a smooth map $A : \Omega \to Mat(n, \mathbb{C})$ is defined by the equality $L(z)v = A\overline{v}$ for any vector $v \in \mathbb{C}^n$, and L is an $\mathbb{R}$-linear map defined by $L = (J_{st} + J)^{-1}(J_{st} - J)$. It is easy to check that the condition $J^2 = -Id$ is equivalent to the fact that L is $\overline{\mathbb{C}}$-linear. The matrix $A(z)$ is called *the complex matrix* of J in the local coordinates z. Locally the correspondence between A and J is one-to-one. Note that the condition $J(0) = J_{st}$ means that $A(0) = 0$.

If z' are other local coordinates and A' is the corresponding complex matrix of J', then, as it is easy to check, we have the following transformation rule:

$$A' = (z'_z A + z'_{\overline{z}})(\overline{z}'_{\overline{z}} + \overline{z}'_z A)^{-1}$$

(see [18]).

Recall that for any complex function $f \in C^r(\mathbb{D})$, $r > 0$, *the Cauchy–Green transform* is defined by

$$Tf(\zeta) = \frac{1}{2\pi i} \iint_{\mathbb{D}} \frac{f(\omega)\, d\omega \wedge d\overline{\omega}}{\omega - \zeta}. \tag{2.5}$$

It is classical that the operator T has the following properties:

(i) for every noninteger $r > 0$, the map $T : C^r(\mathbb{D}) \to C^{r+1}(\mathbb{D})$ is a bounded linear operator (a similar property holds in the Sobolev scales). Here we use the usual Hölder norm on the space $C^r(\mathbb{D})$;

(ii) $(Tf)_{\overline{\zeta}} = f$, that is, T solves the $\overline{\partial}$-equation in the unit disc;

(iii) the function Tf is holomorphic on $\mathbb{C} \setminus \overline{\mathbb{D}}$.

Fix a real noninteger $r > 1$. Let $z : \mathbb{D} \to \mathbb{C}^n$ be a J-complex disc. Since the operator

$$\Psi_J : z \longrightarrow w = z + TA(z)\overline{z}_{\overline{\zeta}}$$

takes the space $C^r(\overline{\mathbb{D}})$ into itself, we can write the equation (2.1) in the form $[\Psi_J(z)]_{\overline{\zeta}} = 0$. Thus, the disc z is J-holomorphic if and only if the map $\Psi_J(z) : \mathbb{D} \longrightarrow \mathbb{C}^n$ is J_{st}-holomorphic. When the norm of A is small enough (which is assured by Lemma 2.1), then by the implicit function theorem the operator Ψ_J is invertible and we obtain a bijective correspondence between J-holomorphic discs and usual holomorphic discs. This easily implies the existence of a J-holomorphic disc in a given tangent direction through a given point of M, as well as a smooth dependence of such a disc on a deformation of a point, a tangent vector and an almost complex structure; this also establishes the interior elliptic regularity of discs.

Note that global pseudoholomorphic discs (that is, discs which are not contained in a small coordinate neighborhood) also have similar properties. The proof requires a considerably more subtle analysis of the integral operator Ψ_J; see [19].

Let (M, J) be an almost complex manifold and $E \subset M$ be a real submanifold of M. Suppose that a J-complex disc $f : \mathbb{D} \to M$ is continuous on $\overline{\mathbb{D}}$. With some abuse of terminology, we also call the image $f(\mathbb{D})$ simply a disc and we call the image $f(b\mathbb{D})$ the boundary of a disc. If $f(b\mathbb{D}) \subset E$, then we say that (the boundary of) the disc f is *glued* or *attached* to E or simply that f is attached to E. Sometimes such maps are called the *Bishop discs* for E and we employ this terminology. Of course, if p is a point of E, then the constant map $f \equiv p$ always satisfies this definition.

3. Plurisubharmonic functions on almost complex manifolds

This section discusses some basic properties of plurisubharmonic functions on almost complex manifolds.

3.1. Basic definitions. Let u be a real C^2 function on an open subset Ω of an almost complex manifold (M, J). Denote by $J^* \, du$ the differential form acting on a vector field X by $J^* \, du(X) := du(JX)$. Given a point $p \in M$ and a tangent vector $V \in T_p(M)$, consider a smooth vector field X in a neighborhood of p satisfying $X(p) = V$. The value of the *complex Hessian* (or the Levi form) of u with respect to J at p and V is defined by $H(u)(p, V) := -(dJ^* \, du)_p(X, JX)$. This definition is independent of the choice of a vector field X. For instance, if $J = J_{st}$ in $\mathbb{C}$, then $-dJ^* du = \Delta u \, d\xi \wedge d\eta$; here Δ denotes the Laplacian. In particular, $H_{J_{st}}(u)(0, \partial/\partial\xi) = \Delta u(0)$.

Recall some basic properties of the complex Hessian (see, for instance, [6]), as follows.

LEMMA 3.1. *Consider a real function u of class C^2 in a neighborhood of a point $p \in M$.*

(i) *Let $F : (M', J') \longrightarrow (M, J)$ be a (J', J)-holomorphic map, $F(p') = p$. For each vector $V' \in T_{p'}(M')$, we have $H_{J'}(u \circ F)(p', V') = H_J(u)(p, dF(p)(V'))$.*

(ii) *If $f : \mathbb{D} \longrightarrow M$ is a J-complex disc satisfying $f(0) = p$, and $df(0)(\partial/\partial\xi) = V \in T_p(M)$, then $H_J(u)(p, V) = \Delta(u \circ f)(0)$.*

Property (i) expresses the holomorphic invariance of the complex Hessian. Property (ii) is often useful in order to compute the complex Hessian on a given tangent vector V.

Let Ω be a domain in M. An upper semicontinuous function $u : \Omega \to [-\infty, +\infty[$ on (M, J) is *J-plurisubharmonic* (psh) if for every J-complex disc $f : \mathbb{D} \to \Omega$, the composition $u \circ f$ is a subharmonic function on $\mathbb{D}$. Of course, this definition makes sense because there are plenty of pseudoholomorphic discs in a neighborhood of each point of an almost complex manifold.

By Lemma 3.1, a C^2 function u is psh on Ω if and only if it has a positive semidefinite complex Hessian on Ω, that is, $H_J(u)(p, V) \geq 0$ for any $p \in \Omega$ and

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$V \in T_p(M)$. A real C^2 function $u : \Omega \to \mathbb{R}$ is called *strictly J-plurisubharmonic* on Ω if $H_J(u)(p, V) > 0$ for each $p \in M$ and $V \in T_p(M)\setminus\{0\}$. Obviously, these notions are local: an upper semicontinuous (respectively of class C^2) function on Ω is J-plurisubharmonic (respectively strictly) on Ω if and only if it is J-plurisubharmonic (respectively strictly) in some open neighborhood of each point of Ω. In what follows, we often write 'plurisubharmonic' instead of 'J-plurisubharmonic' when an almost complex structure J is prescribed.

A useful observation is that the Levi form of a function u at a point p in an almost complex manifold (M, J) coincides with the Levi form with respect to the standard structure J_{st} of $\mathbb{R}^{2n}$ if *suitable* local coordinates near p are chosen. Let us explain how to construct these adapted coordinate systems.

As above, by choosing local coordinates near p we may identify a neighborhood of p with a neighborhood of the origin and assume that J-holomorphic discs are solutions of (2.4).

Lemma 3.2. *There exists a degree-two polynomial local diffeomorphism Φ fixing the origin and with linear part equal to the identity such that in the new coordinates the complex matrix A of J (that is, A from the equation (2.4)) satisfies*

$$A(0) = 0, \quad A_z(0) = 0. \tag{3.1}$$

We conclude this section by several comments on Lemma 3.2.

(1) According to this lemma, by a suitable local change of coordinates one can remove the terms linear in z in the matrix A. We stress that in general it is impossible to get rid of first-order terms containing $\overline{z}$ since this would impose a restriction on the Nijenhuis tensor of J at the origin.

(2) I have learned this result from unpublished Chirka's notes; see [6] for the proof. In [18], it is shown that, in an almost complex manifold of (complex) dimension two, a similar normalization is possible along a given embedded J-holomorphic disc.

(3) As an example, consider a function $u(z) = \| z \|^2$ (we use the Euclidean norm) in the adapted coordinates (3.1). We conclude that this function is strictly J-plurisubharmonic near the origin. In particular, each almost complex manifold admits plenty of strictly J-psh functions locally.

(4) As another typical consequence, consider a totally real manifold E defined by (2.2). Then the function $u = \sum_{j=1}^n \rho_j^2$ is strictly J-plurisubharmonic in a neighborhood of E. Indeed, it suffices to choose local coordinates near $p \in M$ according to Lemma 3.2. This reduces the verification to the well-known case of J_{st}.

3.2. Envelopes of plurisubharmonic functions. It follows from the definitions that many of the elementary properties of plurisubharmonic functions can be directly transferred to the almost complex case. We mention here, for example, the maximum principle as well as the fundamental fact that the plurisubharmonicity is a local property: a function is plurisubharmonic on M if and only if it is plurisubharmonic in an open neighbourhood of every point of M.

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Let (M, J) be an almost complex manifold of complex dimension n. Fix a Riemannian metric on M; all norms and distances will be considered with respect to this metric. Of course, the results are independent of its choice.

As another typical example, we recall here a construction of an envelope of a family of plurisubharmonic functions following Bu and Schachermayer [2] (in the almost complex case, this construction was used in [4]).

Let

$$P_0\phi = \frac{1}{2\pi i}\int_{b\mathbb{D}} \phi(\omega)\frac{d\omega}{\omega} \tag{3.2}$$

denote the average of a real function ϕ over $b\mathbb{D}$,

Lemma 3.3. *Let v be an upper semicontinuous function on an almost complex manifold (M, J). Consider the sequence (v_m) defined as follows: $v_0 = v$ and, for $m \geq 1$, for $p \in M$,*

$$v_m(p) = \inf_f P_0(v_{m-1} \circ f),$$

where inf *is taken over all J-complex discs $f : \mathbb{D} \to M$ such that $f(0) = p$, f is of class $C^r(\overline{\mathbb{D}})$ with some (fixed) noninteger $r > 1$ and $f(\overline{\mathbb{D}}) \subset M$. Then the sequence (v_n) decreases pointwise to the largest J-plurisubharmonic function $DE[v]$ on M bounded from above by v.*

Proof. We proceed in several steps. Clearly, every v_m is correctly defined.

Step 1. The sequence (v_m) decreases. Indeed, for every p, the constant disc $f^0(\zeta) \equiv p$ is J-complex, so

$$v_m(p) = \inf_f P_0(v_{m-1} \circ f) \leq P_0(v_{m-1}(p)) = v_{m-1}(p).$$

Step 2. The function $DE[v]$ is upper semicontinuous. We proceed with the proof by induction on m. For $m = 0$, the statement is correct. Suppose that the function v_{m-1} is upper semicontinuous. Let $(p_k) \subset M$ be a sequence of points converging to $p_0 \in M$.

It follows from [19] that the following holds. Let $f : \mathbb{D} \to M$ be a J-complex disc of class $C^r(\mathbb{D})$ such that $f(0) = p_0$ and $f(\overline{\mathbb{D}}) \subset M$. Then there exists a sequence of J-complex discs $f_k : \mathbb{D} \to M$, of class $C^r(\mathbb{D})$, such that $f_k(\overline{\mathbb{D}}) \subset M$, $f_k(0) = p_k$ for every k and $f_k \longrightarrow f$ in $C^r(\mathbb{D})$.

Consider a compact set K containing the union $f(\overline{\mathbb{D}}) \cup (\cup_k f_k(\overline{\mathbb{D}}))$. Since v_{m-1} is an upper semicontinuous function, it is bounded from above on K by a constant C and

$$(v_{m-1} \circ f)(\zeta) \geq \limsup_{k\to\infty}(v_{m-1} \circ f_k)(\zeta), \quad \zeta \in \mathbb{D}.$$

So, by the Fatou lemma,

$$P_0(v_{m-1} \circ f) \geq \limsup_{k\to\infty} P_0(v_{m-1} \circ f_k) \geq \limsup_{k\longrightarrow\infty} v_m(p_k).$$

This implies that

$$v_m(p_0) = \inf_f P_0(v_{m-1} \circ f) \geq \limsup_{k\to\infty} v_m(p_k),$$

which proves the upper semicontinuity of v_m.

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Therefore, the function $DE[v]$ also is upper semicontinuous as a decreasing limit of upper semicontinuous functions.

Step 3. We prove by induction that for any J-plurisubharmonic function ϕ satisfying $\phi \leq v$, we have $\phi \leq v_n$ for any n. This is true for $m = 0$. Suppose that $\phi \leq v_{m-1}$. Fix an arbitrary point $p_0 \in M$. For every J-complex disc $f \in C^r(\mathbb{D})$ and satisfying $f(0) = p_0$, $f(\overline{\mathbb{D}}) \subset M$,

$$\phi(z_0) \leq P_0(\phi \circ f) \leq P_0(v_{m-1} \circ f).$$

Hence, $v_m(p_0) \geq \phi(p_0)$.

Step 4. We show that the restriction of $DE[v]$ on a J-complex disc is subharmonic. Given p_0 and f as above,

$$DE[v](p_0) = \lim_{m\to\infty} v_m(p_0) \leq \lim_{m\to\infty} P_0(v_{m-1} \circ f) = P_0(DE[v] \circ f)$$

by the Beppo Levi theorem. This concludes the proof of the lemma. □

We call the function $DE[v]$ *the disc envelope* of v. As a simple application, consider any family (u_α) of plurisubharmonic functions on (M, J) and the function $u = \sup_\alpha u_\alpha$. In general, u does not need to be upper semicontinuous, so we consider its *upper regularization*

$$u^*(p) = \lim_{\varepsilon\to 0+} \inf_{dist(q,p)\leq\varepsilon} u(q).$$

It is classical that u^* is the smallest upper semicontinuous function satisfying $u \leq u^*$. In order to prove that u^* is plurisubharmonic on M, consider the disc envelope $DE[u^*]$. It follows from Lemma 3.3 that $u_\alpha \leq DE[u^*]$ for all α, that is, $u \leq DE[u^*] \leq u^*$. Hence, $DE[u^*] = u^*$ and u^* is plurisubharmonic. Usually u^* is called the sup-*envelope* of the family (u_α).

Note that the usual (for $M = \mathbb{C}^n$) proofs of plurisubharmonicity of u^* do not go through directly in the almost complex case since they are based on regularization of plurisubharmonic functions by convolution (see, for example, [5]). This argument is not available in the general almost complex case because the Cauchy–Riemann equations (2.4) are only quasi-linear and not linear.

We point out here a difference of the above construction of the disc envelope and the argument of [2]. In [2], only linear complex discs are used. Of course, this does not make sense in the almost complex case and we need to consider all pseudoholomorphic discs. As a consequence, the set of discs under consideration is much larger and, from this point of view, we are closer to the construction of the disc envelope introduced by Poletsky [13]. He proved (in the case of $\mathbb{C}^n$) that the iteration process used in the proof of Lemma 3.3 stops already on the first step, that is, $v_1 = v_2 = \cdots = DE[v]$. His result was extended to the case of complex manifolds by Larusson and Sigurdsson [10] and Rosay [14]. To the best of my knowledge, it remains an open question if this is also true for almost complex manifolds in any dimension; the case of dimension two was settled by Kuzman [9].

3.3. Plurisuperharmonic measure. A function v is called *plurisuperharmonic* on M if the function $-v$ is plurisubharmonic on M.

Let Ω be a smoothly bounded domain in M with the boundary $b\Omega$. For $\alpha \geq 1$ and $q \in b\Omega$, set $A_\alpha(q) = \{p \in \Omega : dist(p,q) \leq \alpha d_q(p)\}$. Here *dist* denotes the distance on M and $d_q(p)$ denotes the distance from p to the tangent plane $T_q(b\Omega)$ to $b\Omega$ at q. In the case where $M = \mathbb{C}^n$ with the standard Euclidean distance, this is the intersection of Ω with a cone with vertex at w. In the general case $A_\alpha(q)$ is a region of Ω which approaches $b\Omega$ nontangentially at q.

Let u be a plurisuperharmonic function on Ω. Denote by u_* its *nontangential lower boundary extension*, which is defined as

$$u_*(q) = u(q), \quad q \in \Omega,$$
$$u_*(q) = \inf_{\alpha>1}(\lim \inf_{A_\alpha(q)\ni p\to q} u(p)), \quad q \in b\Omega.$$

Let K be a compact subset of $\overline{\Omega}$. Denote by $P(K)$ the class of all functions u plurisuperharmonic on Ω and such that $u(q) \geq 0$ for each $q \in \Omega$ and $u_*(q) \geq 1$ for each $q \in K$. *The plurisuperharmonic measure of K with respect to* Ω or simply *the P-measure* is the function

$$\omega_*(p, K, \Omega) = \lim \inf_{q\to p} \omega(q, K, \Omega),$$

where

$$\omega(q, K, \Omega) = \inf_{u\in P(K)} u(q).$$

Of course, in the one-variable case ω_* coincides with the usual harmonic measure.

Following [16], consider some basic properties of the P-measure. Recall that a subset $E \subset \Omega$ is called *pluripolar* if there exists a plurisuperharmonic on Ω function u nonidentically equal to $+\infty$ and such that $u|E = +\infty$. It follows from the results of Harvey and Lawson [7] that a pluripolar set is of measure zero.

Proposition 3.4. *We have the following results.*

(i) *$0 \leq \omega_*(p, K, \Omega) \leq 1$ for every $p \in \Omega$.*
(ii) *The function $p \mapsto \omega_*(p, K, \Omega)$ is plurisuperharmonic on Ω.*
(iii) *If $\Omega_1 \subset \Omega_2$ and $K_1 \subset K_2$, then $\omega_*(p, K_1, \Omega_1) \leq \omega_*(p, K_2, \Omega_2)$.*
(iv) *If $\omega_*(p^0, K, \Omega) = 0$ for some $p^0 \in \Omega$, then $\omega_*(p, K, \Omega) = 0$ for all $p \in \Omega$.*
(v) *Let $K \subset \overline{\Omega}$ be a pluripolar subset of some open neighborhood $\tilde{\Omega}$ of $\overline{\Omega}$. Then $\omega_*(p, K, \Omega) = 0$ for all $p \in \Omega$.*

The property (i) is obvious (the second inequality follows because the constant function $u = 1$ belongs to $P(K)$). The property (ii) follows from Section 3.2. The property (iii) is obvious; (iv) follows by the maximum principle. For the property (v), suppose that there exists a function u plurisuperharmonic on $\tilde{\Omega}$ and which is not

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equal to $+\infty$ identically such that $u|K = +\infty$. One can assume that $u \geq 0$ on Ω. For $m = 1, 2, \ldots$, the function

$$v_m(p) = \min(u(p)/m, 1)$$

is superharmonic on Ω and belongs to the class $P(K)$. Hence,

$$\omega(p, K, \Omega) \leq v(p) = \lim_{m\to\infty} v_m(p).$$

But $v(p) = 0$ when $u(p) \neq +\infty$ and $v(p) = 1$ when $u(p) = +\infty$. Therefore, the lower regularization $\omega_*(p, K, \Omega)$ vanishes identically.

As the first application, let us prove the following version of the two-constants theorem. Denote by u^* the *upper nontangent boundary extension* of a plurisubharmonic function u, that is,

$$u^*(q) = u(q), \quad q \in \Omega,$$
$$u^*(q) = \sup_{\alpha>1}(\lim \sup_{A_\alpha(q)\ni p\to q} u(p)), \quad q \in b\Omega.$$

Similarly to [16], we have the following result.

Proposition 3.5. *Let u be a plurisubharmonic function bounded above by C on a smoothly bounded domain Ω in (M, J). Suppose that for some compact subset $K \subset \overline{\Omega}$, we have $u^*(p) \leq c < C$ for every $p \in K$. Then*

$$u(p) \leq c\omega_*(p, K, \Omega) + C(1 - \omega_*(p, K, \Omega)).$$

For the proof, it suffices to note that $\omega_*(p, K, \Omega) \leq (C - u(p))/(C - c)$ because the function on the right-hand side is of class $P(K)$.

4. Construction of complex discs

This section presents our main technical tool. We fill a wedge W with a totally real edge E by a family of complex discs glued to the edge E along the upper semicircle. We apply the approach developed in [17], which requires some refinement suitable for our goals.

We will proceed in several steps.

(a) First consider the model case where $M = \mathbb{C}^n$ with $J = J_{st}$ and $E = i\mathbb{R}^n = \{x_j = 0, j = 1, \ldots, n\}$. Denote by W the standard wedge $W = \{z = x + iy : x_j < 0, j = 1, \ldots, n\}$.

Consider the family of linear complex maps

$$l : (c, t, \zeta) \mapsto (\zeta, \zeta t + ic). \tag{4.1}$$

Here $\zeta \in \mathbb{C}$; the variables $c = (c_2, \ldots, c_n) \in \mathbb{R}^{n-1}$ and $t \in \mathbb{R}^{n-1}_+ = \{t = (t_2, \ldots, t_n) \in \mathbb{R}^{n-1} : t_j > 0\}$) are viewed as parameters. Denote by V the wedge $V = \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}_+$. Also, let $\Pi = \{\mathrm{Re}\,\zeta < 0\}$ be the left half-plane; its boundary $b\Pi$ coincides with the imaginary axis $i\mathbb{R}$. The following properties of the above family are easy to check.

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(a1) The images $l(c,t)(b\Pi)$ form a family of real lines in $i\mathbb{R}^n = E$. For every fixed $t \in \mathbb{R}^{n-1}_+$, these lines are disjoint and

$$\cup_{c\in\mathbb{R}^{n-1}} l(c,t)(b\Pi) = E.$$

In other words, for every t this family (depending on the parameter c) forms a foliation of E by parallel lines.

(a2) One has

$$\cup_{(c,t)\in V} l(c,t)(\Pi) = W.$$

(a3) For every fixed $t \in \mathbb{R}^{n-1}_+$,

$$\cup_{c\in\mathbb{R}^{n-1}} l(c,t)(\Pi) = E_t = \{z \in \mathbb{C}^n : \operatorname{Re}(z_j - t_j z_1) = 0, j = 2, \dots, n\} \cap W$$

and the union is disjoint. Every E_t is a real linear $(n+1)$-dimensional half-space contained in W and $bE_t = E$.

(a4) The family (E_t), $t \in \mathbb{R}^{n-1}_+$, is disjoint in W and its union coincides with W.

Let $K \subset E$ be a compact subset of nonzero Hausdorff n-measure (one can consider it with respect to the standard metric). Consider the set Σ_t of $c \in \mathbb{R}^{n-1}$ such that the real line $l(c,t)(b\Pi)$ intersects K in a subset of nonzero 1-measure. It follows by (a1) and the Fubini theorem that for every $t \in \mathbb{R}^{n-1}_+$, the set Σ_t has a nonzero $(n-1)$-measure. Again by the Fubini theorem and (a3), the set $\cup_{c\in\Sigma_t} l(c,t)(\Pi)$ is a subset of E_t of nonzero $(n+1)$-measure. Finally, by (a4):

(a5) $\cup_{t\in\mathbb{R}^{n-1}_+} \cup_{c\in\Sigma_t} l(c,t)(\Pi)$ is a subset of W of nonzero $2n$-measure.

In what follows, we will use these properties locally, that is, in a neighborhood of the origin. It is convenient to reparametrize the family of complex half-lines $l(c,t)$ by complex discs.

We represent the family of discs (4.1) (after suitable reparametrization) as a general solution of an integral equation.

Let

$$S\phi(\zeta) = \frac{1}{2\pi i}\int_{b\mathbb{D}} \frac{\omega+\zeta}{\omega-\zeta}\phi(\omega)\frac{d\omega}{\omega} \tag{4.2}$$

denote the Schwarz integral. In terms of the Cauchy transform

$$Kf(\zeta) = \frac{1}{2\pi i}\int_{b\mathbb{D}} \frac{f(\omega)\,d\omega}{\omega-\zeta},$$

we have the following relation: $S = 2K - P_0$. As a consequence, the boundary properties of the Schwarz integral are the same as the classical properties of the Cauchy integral.

For a *noninteger* $r > 1$, consider the Banach spaces $C^r(b\mathbb{D})$ and $C^r(\mathbb{D})$ (with the usual Hölder norm). It is classical that K and S are bounded linear mappings in these classes of functions. For a real function $\phi \in C^r(b\mathbb{D})$, the Schwarz integral $S\phi$ is a function of class $C^r(\mathbb{D})$ holomorphic in $\mathbb{D}$; the trace of its real part on the boundary coincides with ϕ and its imaginary part vanishes at the origin. In particular, every holomorphic function $f \in C^r(\mathbb{D})$ satisfies the Schwarz formula $f = S\,\mathrm{Re}\, f + iP_0 f$ (recall that P_0 is defined by (3.2)).

We are going to fill W by complex discs glued to $i\mathbb{R}^n$ along the (closed) upper semicircle $b\mathbb{D}^+ = \{e^{i\theta} : \theta \in [0, \pi]\}$; let also $b\mathbb{D}^- := b\mathbb{D} \setminus b\mathbb{D}^+$.

Fix a smooth real function $\phi : b\mathbb{D} \to \mathbb{R}$ such that $\phi|b\mathbb{D}^+ = 0$ and $\phi|b\mathbb{D}^- < 0$.

Consider now a real $2n$-parameter family of holomorphic discs $z^0 = (z^0_1, \dots, z^0_n) : \mathbb{D} \to \mathbb{C}^n$ with components

$$z^0_j(c,t)(\zeta) = x_j(\zeta) + iy_j(\zeta) = t_j S\phi(\zeta) + ic_j, \quad j = 1, \dots, n. \tag{4.3}$$

Here $t_j > 0$ and $c_j \in \mathbb{R}$ are parameters, $t = (t_2, \dots, t_n)$, $c = (c_2, \dots, c_n)$; in (4.3) we formally set $t_1 = 1$ and $c_1 = 0$.

Obviously, every $z^0(c,t)(\mathbb{D})$ is a subset of $l(c,t)(\Pi)$ and $z^0(b\mathbb{D}^+) = l(c,t)(b\Pi)$. Thus, the family $z^0(c,t)$ is a (local) biholomorphic reparametrization of the family $l(c,t)$. As a consequence, the properties (a1)–(a5) also hold for the family $z^0(c,t)$. Notice also the following obvious properties of this family:

(a6) for every j, one has $x_j|b\mathbb{D}^+ = 0$ and $x_j(\zeta) < 0$ when $\zeta \in \mathbb{D}$ (by the maximum principle for harmonic functions);

(a7) the evaluation map $Ev_0 : (c,t,\zeta) \mapsto z^0(c,t)(\zeta)$ is one-to-one from $V \times \mathbb{D}$ to W.

Now we construct an analog of this family in the general case.

(b) In order to write an integral equation defining a required family of discs, we need to employ an analog of the Schwarz formula and to choose suitable local coordinates.

We have the following Green–Schwarz formula (see the proof, for example, in [18], although, of course, it can be found in the vast list of classical works). Let $f = \phi + i\psi : \overline{\mathbb{D}} \to \mathbb{C}$ be a function of class $C^r(\mathbb{D})$. Then, for each $\zeta \in \overline{\mathbb{D}}$,

$$f(\zeta) = S\phi(\zeta) + iP_0\psi + Tf_{\overline{\zeta}}(\zeta) - \overline{Tf_{\overline{\zeta}}(1/\overline{\zeta})}.$$

Here we use the integral operators (2.5), (3.2) and (4.2). Note that because of the 'symmetrization' the real part of the sum of two terms containing the Cauchy–Green operator T vanishes on the unit circle. Notice also that the last term is holomorphic on $\mathbb{D}$.

Now let (M, J) be an almost complex manifold of complex dimension n and E be a totally real n-dimensional submanifold of M. We assume that E and W are given by (2.2) and (2.3), respectively.

First, according to Section 2, we choose local coordinates z such that $p = 0$ and the complex matrix A of J satisfies (3.1). For every $\tau > 0$ small enough and $C > 0$ big enough, the functions

$$\tilde{\rho}_j := \rho_j - \tau \sum_{k \neq j} \rho_k + C \sum_{k=1}^n \rho_k^2$$

are strictly J-plurisubharmonic in a neighborhood of the origin and the 'truncated' wedge $W_\tau = \{\tilde{\rho}_j < 0, j = 1, \dots, n\}$ is contained in W. After a $\mathbb{C}$-linear (with respect to J_{st}) change of coordinates, one can assume that $\tilde{\rho}_j = x_j + o(|z|)$. Consider now a local diffeomorphism

$$\Phi : z = x_j + iy_j \mapsto z' = x'_j + iy'_j = \tilde{\rho}_j + iy_j.$$

Then $\Phi(0) = 0$, $d\Phi(0) = 0$ and, in the new coordinates $\tilde{\rho}_j = x_j$ (we drop the primes), $E = i\mathbb{R}^n$ and $W_\tau = \{x_j < 0, j = 1, \ldots, n\}$. We keep the notation J for the direct image $\Phi_*(J)$. Then in the chosen coordinates the complex matrix of J still satisfies $A(0) = 0$. Note also that the coordinate functions x_j are strictly plurisubharmonic for J.

Finally, similarly to the proof of Lemma 2.1, for $\lambda > 0$ consider the isotropic dilations $d_\lambda : z \mapsto \lambda^{-1}z$ and the direct images $J_\lambda := (d_\lambda)_*(J)$. Denote by $A(z, \lambda)$ the complex matrix of J_λ.

For $\lambda > 0$ small enough, we are looking for the solutions $z : \mathbb{D} \to \mathbb{C}^n$ of the Bishop-type integral equation

$$z(\zeta) = h(z(\zeta), c, t, \lambda) \tag{4.4}$$

with

$$h(z(\zeta), c, t, \lambda) = tS\phi(\zeta) + ic + TA(z, \lambda)\bar{z}_{\bar{\zeta}}(\zeta) - \overline{TA(z, \lambda)\bar{z}_{\bar{\zeta}}(1/\bar{\zeta})},$$

where $t = (t_2, \ldots, t_n)$, $t_j > 0$ and $c \in \mathbb{R}^{n-1}$ as well as λ are viewed as real parameters. Of course, we assume implicitly that all parameters are close to the origin.

Note that the first and the last terms in the right-hand side are holomorphic on $\mathbb{D}$. Therefore, any solution of (4.4) satisfies the Cauchy–Riemann equations (2.4), that is, is a J-complex disc. Furthermore, $x_j(\zeta)$ vanishes on $b\mathbb{D}^+$ (that is, $z(b\mathbb{D}^+) \subset E$) and is negative on $b\mathbb{D}^-$. Since the function $z \mapsto x_j$ is strictly J-plurisubharmonic, by the maximum principle the image $z(\overline{\mathbb{D}})$ is contained in $\overline{W}_\tau$.

The existence of solutions follows by the implicit function theorem. Note that for $\lambda = 0$, the equation (4.4) admits the solution (4.3). Consider the smooth map of Banach spaces

$$H : C^r(\mathbb{D}) \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R} \longrightarrow C^r(\mathbb{D})$$
$$H : (z, c, t, \lambda) \mapsto h(z(\zeta), c, t, \lambda).$$

Obviously, the partial derivative of H in z vanishes: $(D_zH)(z^0, c, t, 0) = 0$, where z^0 is a disc given by (4.3). By the implicit function theorem, for every (c, t, λ) close enough to the origin, the equation (4.4) admits a unique solution

$$(c, t, \zeta) \mapsto z(c, t)(\zeta) \tag{4.5}$$

of class $C^r(\mathbb{D})$, smoothly depending on the parameters (c, t) (as well as λ, of course).

Fixing $\lambda > 0$, consider the smooth evaluation map

$$Ev_\lambda : (c, t, \zeta) \mapsto z(c, t)(\zeta)$$

which associates to each parameter a point of the corresponding disc. Note that for $\lambda = 0$, we obtain the linear mapping $Ev_0(c, t)(\zeta)$ that appeared already in (a7)

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for the model case of the standard structure. Indeed, in this case the family (4.5) coincides with the family (4.3) (by the uniqueness of solutions assured by the implicit function theorem). Notice also that when $t = 0$ for every $\lambda > 0$ and every $c \in \mathbb{R}^{n-1}$, the equation (4.4) has the unique solution $z^0(c, 0)(\zeta) = (\zeta, ic)$ (compare with (4.3)). Hence, $Ev_\lambda(c, 0)(\zeta) = Ev_0(c, 0)(\zeta)$.

By (a7), $Ev_0(V \times \mathbb{D})$ coincides with W_τ in a neighborhood of the origin. Furthermore, $Ev_\lambda(\{(c, t) : c \in \mathbb{R}^{n-1}, t = 0\} \times b\mathbb{D}^+) = E$ for $\lambda > 0$. Also, for $\alpha > 0$ the 'truncated' wedge $W_\alpha = \{x_j - \alpha \sum_{k \neq j} x_k < 0\}$ with the edge E is contained in W_τ. The faces of the boundary of W_α are transversal to the face of W_τ. Since this property is stable under small perturbations, we conclude that $W_\alpha \subset Ev_\lambda(V)$ for all λ small enough. In terms of the initial defining functions ρ_j, we have $\{z : \rho_j - \delta \sum_{k \neq j} \rho_k < 0\} \subset W_\varepsilon$ when $\tau + \varepsilon < \delta$.

Concerning the regularity of manifolds and almost complex structures, it suffices to require the class C^r with real $r > 2$ and the argument goes through. We skip the details.

Fix $\delta > 0$. Since the properties of linear discs (a1)–(a5) are stable under small perturbations, the obtained family of discs admits similar properties. For the reader's convenience, we list them.

(b1) The images $z(c, t)(bD^+)$ form a family of real curves in E. For every fixed $t \in \mathbb{R}^{n-1}_+$, these curves are disjoint and

$$\cup_{c \in \mathbb{R}^{n-1}} z(c, t)(b\mathbb{D}^+) = E.$$

In other words, for every t this family (depending on the parameter t) forms a foliation of E. Furthermore, every disc is contained in W.

(b2) One has

$$W_\delta = \Big\{z : \rho_j - \delta \sum_{k \neq j} \rho_k < 0\Big\} \subset \cup_{(c,t) \in V} z(c, t)(\mathbb{D}).$$

(b3) For every fixed $t \in \mathbb{R}^{n-1}_+$, the union

$$E_t := \cup_{c \in \mathbb{R}^{n-1}} z(c, t)(\mathbb{D}) \subset W$$

is a real $(n + 1)$-dimensional manifold with boundary $bE_t = E$.

(b4) The family (E_t), $t \in \mathbb{R}^{n-1}_+$, is disjoint and its union contains W_δ.

Similar to (a5):

(b5) let $K \subset E$ be a compact subset of nonzero Hausdorff n-measure. The discs $z(c, t)$ whose boundaries intersect K in a set of positive 1-measure fill a subset of W_δ of nonzero Hausdorff $2n$-measure.

Consider now the important special case where a totally real n-dimensional manifold E of the form (2.2) is contained in the boundary $b\Omega$ of a domain Ω. We assume that $b\Omega$ is a smooth real hypersurface defined in a neighborhood of a point $p \in E \subset b\Omega$ by $b\Omega = \{\rho = 0\}$, where ρ is a smooth real function with nonvanishing gradient; one can assume also that $\Omega = \{\rho < 0\}$. Then we can always choose a wedge

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W of the form (2.3) such that $W \subset \Omega$ and its faces $\{\rho_j = 0\}$ are transverse to $b\Omega$. Then each complex disc from the family constructed above also is transverse to $b\Omega$. More precisely, we have the following property.

(b6) Every disc from the family (4.5) is nontangent to $b\Omega$ at every boundary point.

5. Boundary behavior of plurisubharmonic functions

Now we are able to prove our main results.

THEOREM 5.1. *Let Ω be a smoothly bounded domain in an almost complex n-dimensional manifold (M, J). Suppose that $E \subset b\Omega$ is a generic submanifold and that $K \subset E$ is a subset with nonempty interior (with respect to E). Then $\omega_*(p, K, \Omega)$ does not vanish identically.*

Here the Hausdorff measure is considered with respect to any Riemannian metric on M; as we already have pointed out, the condition of nonvanishing of the Hausdorff measure of K is independent of a choice of the Riemannian metric.

In view of Lemma 2.2, without loss of generality one can assume that E is totally real. Consider the family (4.5) of discs for E constructed in Section 4; recall that these discs are not tangent to the boundary $b\Omega$ in view of (b6). Since the construction is local and K has a nonempty interior in E, one can assume that $K = E$. Using the properties (b4) and (b6), we see that the discs fill an open subset X of Ω. Each disc f intersects K along $b\mathbb{D}^+$; hence, for every $\zeta \in \mathbb{D}$ with $\operatorname{Im} \zeta > 0$,

$$\omega(f(\zeta), K, \Omega) \geq \omega(\zeta, b\mathbb{D}^+, \mathbb{D}) \geq c > 0,$$

where c is a universal constant. We obtain that $\omega(p, K, \Omega) \geq c > 0$ for each $p \in X$. We conclude that $\omega_*(p, K, \Omega) > 0$ on X; that is, it does not vanish identically.

The next result is the following uniqueness principle.

THEOREM 5.2. *Let Ω be a smoothly bounded domain in an almost complex n-dimensional manifold (M, J). Suppose that $E \subset b\Omega$ is a generic submanifold and that $K \subset E$ is a compact subset of nonzero n-Hausdorff measure. Assume that u is a bounded from above plurisubharmonic function in Ω such that $u^*(p) = -\infty$ for each $p \in K$. Then $u \equiv -\infty$.*

Indeed, for every disc f from the family (4.5) the composition $u \circ f$ is an upper bounded subharmonic function in $\mathbb{D}$. Notice again that the boundary of every disc is transverse to $b\Omega$ (see (b6)). Consider now the discs f from these families intersecting K along a subset $l_f \subset b\mathbb{D}$ of positive measure. For each disc f and $\zeta \in \mathbb{D}$,

$$\omega(f(\zeta), K, \Omega) \geq \omega(\zeta, l_f, \mathbb{D}) > 0.$$

The above discs fill a subset X of Ω of positive $2n$-measure and with nonempty interior in Ω according to (b5). By applying to every $u \circ f$ the two-constants theorem for subharmonic functions, we conclude that $u \circ f \equiv -\infty$. Since these discs fill a set of positive measure in Ω, we conclude that $u \equiv= -\infty$.

In particular, we obtain the following far-reaching generalization of one of the results of Rosay [15].

Corollary 5.3. *Let E be a totally real n-dimensional submanifold of an almost complex n-dimensional manifold (M, J). Suppose that K is a closed subset of E of nonzero Hausdorff n-measure. Then E is not contained in a pluripolar set.*

We need some additional properties of discs (4.5) constructed in Section 4. We use the notation from that section.

(a) Once again we begin with the standard case where $M = \mathbb{C}^n$ with $J = J_{st}$ and $E = i\mathbb{R}^n = \{x_j = 0, j = 1, \ldots, n\}$. Consider the family $l(c, t)$ of the form (4.3) (that is, (4.1) after a reparametrization); these complex maps attach the left half-plane Π to E along the imaginary axis. Our goal is to study more carefully those maps from this family whose boundary does not touch the origin in $i\mathbb{R}^n$. This means that $c_j \neq 0$ for at least one $j \in \{2, \ldots, n\}$. Each point $z = x + iy \in W = \{x_j < 0, j = 1, \ldots, n\}$ belongs to the disc $l(c, t)$ with $t_j = x_1/x_j > 0$ and $c_j = y_j - y_1 x_1/x_j$. For each $j = 2, \ldots, n$, consider a real smooth hypersurface $\Gamma_j = \{z \in W : y_j x_j - y_1 x_1 = 0\}$ in W. We obtain the following result.

(a8) The discs (4.1) whose boundaries do not touch the origin fill an open dense subset $W \setminus \cup_{j=2}^{n} \Gamma_j$ of W.

Now we pass to the general case.

(b) Assume that E is contained in the boundary of a smoothly bounded domain Ω and consider discs (4.5) such that (b6) holds. Since the property (a8) is stable under perturbation, its analog holds for the family of pseudoholomorphic discs (4.5). We consider only nonconstant discs with $t \in \mathbb{R}^n_+$.

(b7) The discs $z(c, t)$ of the family (4.5) whose boundary does not touch a point $p \in E$ fill an open dense subset X of W_δ.

(b8) In particular, under the assumptions of (b6), the closure $\overline{X}$ of X contains any nontangential region $A_\alpha(p)$ in a domain Ω.

The following property that we need is obvious as well.

(b9) Under the assumptions of (b7), the arcs $z(c, t)(b\mathbb{D}^-)$ fill a compact subset $Y \Subset W_\delta$.

Our next result is inspired by the work of Khurumov [8]. Another motivation arises from the work by Levenberg *et al.* [11]. They proved that the direct analog of the Lindelöf–Chirka principle [3] fails for bounded from above plurisubharmonic functions in complex dimension > 1. The following results can be viewed as a partial analog of the two-constants theorem and the Lindelöf–Chirka principle for plurisubharmonic functions.

Theorem 5.4. *Let Ω be a smoothly bounded domain in an almost complex n-dimensional manifold (M, J) and let $E \subset \overline{\Omega}$ be a smooth totally real n-dimensional manifold. Suppose that a function u is plurisubharmonic on Ω and $u^*|_{E \setminus \{p\}} \leq C$ for*

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some $p \in E \cap b\Omega$*. Then for every* $\varepsilon > 0$ *there exists a neighborhood* U *of* p *such that* $u \leq C + \varepsilon$ *on* $A_\alpha(p) \cap U$ *for every* $\alpha > 1$.

Proof. For the proof, it suffices to use properties (b7)–(b9) of the family (4.5) and to apply the two-constants theorem to the restriction of u on every disc. □

Corollary 5.5. *Let* Ω *be a smoothly bounded domain in an almost complex* n*-dimensional manifold* (M, J) *and let* $E \subset \overline{\Omega}$ *be a smooth totally real* n*-dimensional manifold. Suppose that a function* u *is plurisubharmonic and bounded from above on* Ω *and* $\limsup_{E \ni q \to p} u^*(q) = -\infty$ *for some* $p \in E \cap b\Omega$*. Then, for every* $\alpha > 1$*, one has* $\lim_{A_\alpha(p) \ni q \to p} u(q) = -\infty$.

Finally, we have the following result.

Corollary 5.6. *Let* Ω *be a smoothly bounded domain in an almost complex* n*-dimensional manifold* (M, J) *and let* $L \Subset \Omega$ *be a generic* $(n + 1)$*-dimensional manifold with the totally real boundary* $N \subset b\Omega$*. Suppose that a function* u *is plurisubharmonic and bounded above on* Ω *and* $\limsup_{L \ni q \to N} u(q) = -\infty$*. Then* $u \equiv -\infty$.

Indeed, for any point $p \in N$, consider a totally real n-dimensional manifold $E \subset L \cup \{p\}$ containing p and apply Corollary 5.5.

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