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SMALL-TIME MODERATE DEVIATIONS FOR THE RANDOMISED HESTON MODEL

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Abstract

We extend previous large deviations results for the randomised Heston model to the case of moderate deviations. The proofs involve the Gärtner–Ellis theorem and sharp large deviations tools.

Keywords: Stochastic volatility; moderate deviations; Heston model; implied volatility; asymptotic expansion

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1. Introduction

Classical stochastic volatility models are known to provide an overall good fit of option price data (or of the so-called implied volatility surface), except for short maturities; in this particular region, adding jumps has historically provided a good patch, at the expense of complicated hedging, and more recently rough volatility models [2, 4, 11, 14, 16, 17, 23] have been shown to perform better while preserving the continuity of the sample paths. Building on the intuition that these refinements somehow capture a certain kind of uncertainty around the starting time of the process, a randomised version of the Heston model [20] was proposed in [21, 25], where the starting point of the variance process is considered random. The authors showed there that this extra source of randomness generates the desired behaviour of implied volatility for small times. Mathematically, this was proved showing that the underlying stock price process satisfies some large deviations principles with specific rates of convergence. Moderate deviations, however, although formally equivalent to large deviations, usually provide more efficient ways (from a numerical point of view) to compute limiting probabilities. Introduced in [27], they have become an increasingly useful tool in probability and in statistical physics, as discussed in [3, 8, 9, 24]. They have also recently appeared in mathematical finance in order to provide a different, yet somehow more useful, view on asymptotics, and important results in this direction can be studied in [5, 13, 22].

This paper builds upon the large deviations results from [21] and provides their moderate deviations counterparts, in the context of small-time behaviour of the randomised Heston model. Contrary to large deviations, the moderate deviations rate functions are available here in closed form, hence allowing for more efficient and quicker computations. In passing, we

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provide (Theorems 2 and 4) unusual examples of moderate deviations rate functions which do not have a quadratic form. We gather some technical results and background in the appendix.

1.1. Notation

Let $\mathbb{R}_+ := [0, \infty)$, $\mathbb{R}_+^* := (0, \infty)$, and $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$. For two functions f and g we write $f \sim g$ as x tends to x_0 if $\lim_{x\to x_0} f(x)/g(x) = 1$. We denote by $\sigma_t(x)$ the implied volatility for given maturity t and log-strike x. Finally, for a sequence $(Y_t)_{t\geq 0}$ satisfying a large deviations principle as t tends to zero with speed $g(t)$ and good rate function Λ we use the notation $Y \sim \mathrm{LDP}(g(t), \Lambda)$.

2. Model description

On a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P})$ supporting two independent Brownian motions $W^{(1)}$ and $W^{(2)}$, we consider the following dynamics for a log-stock price process $(X_t)_{t\geq 0}$:

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}V_t \mathrm{d}t + \sqrt{V_t}\Big(\rho\, \mathrm{d}W_t^{(1)} + \sqrt{1-\rho^2}\, \mathrm{d}W_t^{(2)}\Big), \quad && X_0 = 0,\\ \mathrm{d}V_t &= \kappa(\theta - V_t)\mathrm{d}t + \xi\sqrt{V_t}\mathrm{d}W_t^{(1)}, && V_0 = \mathcal{V}, \end{aligned} \tag{2.1}$$

with $\kappa, \theta, \xi > 0$ and $\rho \in [-1, 1]$. This corresponds to the randomised version of the classical Heston model [20], as recently proposed and analysed in [21, 25]. We assume that $\mathcal{V}$ is a continuous random variable independent of the filtration $(\mathcal{F}_t)_{t\geq 0}$, and that the interior of its support reads $(\mathfrak{v}_-, \mathfrak{v}_+) \subseteq \mathbb{R}_+^*$. We further assume that its moment-generating function $\mathrm{M}_{\mathcal{V}}(u) := \mathrm{E}[\mathrm{e}^{u\mathcal{V}}]$ is well defined on an open interval containing the origin, and denote $\mathfrak{m} := \sup\{u : \mathrm{E}[\mathrm{e}^{u\mathcal{V}}] < \infty\}$. We shall distinguish three separate behaviours for the randomisation $\mathcal{V}$: bounded support ($\mathfrak{v}_+ < \infty$), thin tail ($\mathfrak{m} = \infty$, $\mathfrak{v}_+ = \infty$), and fat tail ($\mathfrak{m} < \infty$, $\mathfrak{v}_+ = \infty$). Following [21], we introduce the following assumptions characterising the thin tail and fat tail cases:

Assumption 1. (Thin tails.) *$\mathfrak{v}_+ = \infty$ and $\mathcal{V}$ admits a smooth density f with $\log f(v) \sim -l_1 v^{l_2}$ as v tends to infinity, for some $(l_1, l_2) \in \mathbb{R}_+^* \times (1, \infty)$.*

Assumption 2. (Fat tails.) *There exists $(\gamma_0, \gamma_1, \omega) \in \mathbb{R}^* \times \mathbb{R} \times \{1, 2\}$ such that the following asymptotics hold for the cumulant-generating function (cgf) of $\mathcal{V}$ as u tends to $\mathfrak{m}$ from below:*

$$\log \mathrm{M}_{\mathcal{V}}(u) = \begin{cases} \gamma_0 \log(\mathfrak{m} - u) + \gamma_1 + o(1) & \text{if } \omega = 1,\ \gamma_0 < 0,\\ \dfrac{\gamma_0[1 + \gamma_1(\mathfrak{m}-u)\log(\mathfrak{m}-u) + \mathcal{O}(\mathfrak{m}-u)]}{\mathfrak{m}-u} & \text{if } \omega = 2,\ \gamma_0 > 0, \end{cases}$$

and

$$\frac{\mathrm{M}'_{\mathcal{V}}(u)}{\mathrm{M}_{\mathcal{V}}(u)} = \begin{cases} \dfrac{|\gamma_0|}{\mathfrak{m}-u}(1 + o(1)) & \text{for } \omega = 1,\ \gamma_0 < 0,\\ \dfrac{\gamma_0}{(\mathfrak{m}-u)^2}(1 - \gamma_1(\mathfrak{m}-u) + o(\mathfrak{m}-u)) & \text{for } \omega = 2,\ \gamma_0 > 0. \end{cases}$$

Common continuous distributions fit into this framework, in particular the uniform distribution (bounded support), the folded Gaussian distribution, the gamma distribution (Assumption 2 with $\omega = 1$), and the noncentral chi-squared (Assumption 2 with $\omega = 2$).

Before stating the main results of the paper, let us recall some information on the cumulant-generating function of X_t, which will be essential for the rest of the analysis. As proved in [1], the moment-generating function of X_t in the standard Heston model (where $\mathcal{V}$ is a Dirac

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mass at $v_0 > 0$) admits the closed-form representation $\mathrm{M}(t, u) = \exp(\mathrm{C}(t, u) + \mathrm{D}(t, u)v_0)$, for any $u \in \mathcal{D}_{\mathrm{M}}^t \subset \mathbb{R}$, where

$$\begin{cases} \mathrm{C}(t, u) := \frac{\kappa\theta}{\xi^2}\left[(\kappa - \rho\xi u - d(u))t - 2\log\left(\frac{1 - g(u)\mathrm{e}^{-d(u)t}}{1 - g(u)}\right)\right], \\ \mathrm{D}(t, u) := \frac{\kappa - \rho\xi u - d(u)}{\xi^2}\frac{1 - \exp(-d(u)t)}{1 - g(u)\exp(-d(u)t)}, \\ \quad d(u) := ((\kappa - \rho\xi u)^2 + \xi^2 u(1 - u))^{1/2} \quad \text{and} \quad g(u) := \frac{\kappa - \rho\xi u - d(u)}{\kappa - \rho\xi u + d(u)}. \end{cases} \tag{2.2}$$

Introduce further the real numbers $u_- \leq 0$, $u_+ \geq 1$ and the function $\Lambda : (u_-, u_+) \to \mathbb{R}$:

$$\begin{aligned} u_- &:= \frac{2}{\xi\overline{\rho}}\arctan\left(\frac{\overline{\rho}}{\rho}\right)\mathbf{1}_{\{\rho<0\}} - \frac{\pi}{\xi}\mathbf{1}_{\{\rho=0\}} + \frac{2}{\xi\overline{\rho}}\left(\arctan\left(\frac{\overline{\rho}}{\rho}\right) - \pi\right)\mathbf{1}_{\{\rho>0\}}, \\ u_+ &:= \frac{2}{\xi\overline{\rho}}\left(\arctan\left(\frac{\overline{\rho}}{\rho}\right) + \pi\right)\mathbf{1}_{\{\rho<0\}} + \frac{\pi}{\xi}\mathbf{1}_{\{\rho=0\}} + \frac{2}{\xi\overline{\rho}}\arctan\left(\frac{\overline{\rho}}{\rho}\right)\mathbf{1}_{\{\rho>0\}}, \\ \Lambda(u) &:= \frac{u}{\xi(\overline{\rho}\cot(\xi\overline{\rho}u/2) - \rho)}. \end{aligned} \tag{2.3}$$

The pointwise limit of the (rescaled) cumulant-generating function of X_t then reads [12]

$$\lim_{t\downarrow 0} t\log \mathrm{M}\left(t, \frac{u}{t}\right) = \Lambda(u)v_0 \qquad \text{for any } u \in (u_-, u_+),$$

and Λ is well defined, smooth, strictly convex on (u_-, u_+), and infinite elsewhere.

3. From moderate to extra large deviations

Moderate deviations classically arise as rescaled large deviations; in our setting, they take the following form: for $\alpha \neq 0$, define the process $X^{(\alpha)}$ pathwise via $X_t^{(\alpha)} := t^{-\alpha}X_t$. Moderate deviations for the sequence $(X_t)_{t\geq 0}$ as t tends to zero are equivalent to large deviations for $(X_t^{(\alpha)})_{t\geq 0}$ and can, in our framework, be derived from finite-dimensional tools using the Gärtner–Ellis theorem. The assumptions on the behaviour of the randomisation $\mathcal{V}$ yield different rate functions and speed for the moderate deviations regime, which we analyse sequentially below.

3.1. Distribution with bounded support

We first start with the case where the random initial distribution of $\mathcal{V}$ has bounded support, in which case the following holds:

Theorem 1. *If $\mathfrak{v}_+$ is finite then for any $\gamma \in (0, 1)$, $X^{(\alpha)} \sim \mathrm{LDP}(t^\gamma, \frac{x^2}{2\mathfrak{v}_+})$ holds with $\alpha := \frac{1}{2}(1 - \gamma)$.*

Since $\mathfrak{v}_+$ is finite, $\mathfrak{m}$ is infinite. One of the striking features of moderate deviations is that, contrary to classical large deviations, the rate function is usually available analytically, and is often of quadratic form [13, 18, 19]. Note here that, since $\gamma \in (0, 1)$, then $\alpha \in (0, \frac{1}{2})$, and we are in the realm of moderate deviations.

Remark 1. Following [12, 13], Theorem 1 implies that the European price satisfies

$$\lim_{t\downarrow 0} t^{1-2\alpha}\log \mathrm{E}\left[\left(\mathrm{e}^{X_t} - \mathrm{e}^{xt^\alpha}\right)_+\right] = -\frac{x^2}{2\mathfrak{v}_+}.$$

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Comparing it with the standard Heston case detailed in [13, Equation (5.11)] we see that in the bounded support case the upper bound $\mathfrak{v}_+$ replaces the role of fixed initial variance in the Heston model (at the leading order). It is also straightforward that $\lim_{t\downarrow 0}\sigma_t^2(xt^\alpha)=\mathfrak{v}_+$.

Proof. Let $\alpha, \gamma \in (0, 1)$. Notice that

$$\mathrm{M}(t, u) := \mathrm{E}\big[\mathrm{e}^{uX_t}\big] = \mathrm{E}\big[\mathrm{E}[\mathrm{e}^{uX_t} \mid \mathcal{V}]\big] = \mathrm{E}\big[\mathrm{e}^{\mathrm{C}(t,u)+\mathrm{D}(t,u)\mathcal{V}}\big] = \mathrm{e}^{\mathrm{C}(t,u)}\mathrm{M}_{\mathcal{V}}(\mathrm{D}(t, u)),$$

where the functions C and D are the components of the moment-generating function of the standard Heston model in (2.2). Then, for any $t > 0$, the rescaled cumulant-generating function of $X_t^{(\alpha)}$ reads

$$\Lambda_\gamma^{(\alpha)}\Big(t, \frac{u}{t^\gamma}\Big) := t^\gamma \log \mathrm{E}\Big[\exp\Big(\frac{uX_t^{(\alpha)}}{t^\gamma}\Big)\Big] = t^\gamma \log \mathrm{E}\Big[\exp\Big(\frac{uX_t}{t^{\gamma+\alpha}}\Big)\Big]$$
$$= t^\gamma \mathrm{C}\Big(t, \frac{u}{t^{\gamma+\alpha}}\Big) + t^\gamma \log \mathrm{M}_{\mathcal{V}}\Big(\mathrm{D}\Big(t, \frac{u}{t^{\gamma+\alpha}}\Big)\Big),$$

for all $u \in \mathbb{R}$ such that the left-hand side exists. Lemma 2 implies that $\gamma + \alpha$ has to be less than one in order to obtain a nontrivial behaviour. We have the following tail behaviour of the cumulant-generating function: for $\mathfrak{v}_+ < \infty$, $\lim_{u\uparrow\infty} u^{-1} \log \mathrm{M}_{\mathcal{V}}(u) = \mathfrak{v}_+$. Then, as t tends to zero, we deduce from Lemma 2 that

$$\Lambda_\gamma^{(\alpha)}\Big(t, \frac{u}{t^\gamma}\Big) = \begin{cases} \mathcal{O}(t^\gamma) + \mathfrak{v}_+\Lambda(u)t^{\gamma-1} & \text{if } \gamma+\alpha = 1, \text{for all } u \in (u_-, u_+), \\ o(t^\gamma) + \dfrac{\mathfrak{v}_+}{2}u^2 t^{1-\gamma-2\alpha} & \text{if } \gamma+\alpha < 1, \text{for all } u \in \mathbb{R}. \end{cases}$$

Since $\alpha \neq 0$, the nondegenerate result is obtained if and only if $1-\gamma-2\alpha = 0$, i.e. $\alpha = \frac{1-\gamma}{2}$, and the proof follows from application of the Gärtner–Ellis theorem [10, Theorem 2.3.6]. □

3.2. Thin-tail distribution

With l_1, l_2 given in Assumption 1, we introduce two special rates of convergence $\frac{1}{2} < \underline{\gamma} < 1 < \overline{\gamma}$, and two positive constants $\underline{\mathfrak{c}}, \overline{\mathfrak{c}}$:

$$\underline{\gamma} := \frac{l_2}{1+l_2}, \qquad \overline{\gamma} := \frac{l_2}{l_2-1}, \qquad \underline{\mathfrak{c}} := (2l_1l_2)^{\frac{1}{1+l_2}}, \qquad \overline{\mathfrak{c}} := (2l_1l_2)^{\frac{1}{1-l_2}},$$

and define the function $\underline{\Lambda}^*:\mathbb{R} \to \mathbb{R}_+$ by

$$\underline{\Lambda}^*(x) := \frac{\underline{\mathfrak{c}}}{2\underline{\gamma}}x^{2\underline{\gamma}} \quad \text{for any } x \text{ in } \mathbb{R}. \tag{3.1}$$

Introduce further

$$\overline{\Lambda}^*(x) := \sup_{u\in(u_-,u_+)}\Big\{ux - \frac{\overline{\mathfrak{c}}}{\overline{\gamma}}2^{\overline{\gamma}-1}\Lambda(u)^{\overline{\gamma}}\Big\} \quad \text{for all } x \in \mathbb{R},$$

with Λ, $u_\pm$ in (2.3). The rescaled deviations asymptotics then take the following form.

Theorem 2. *Under Assumption 1, the following statements hold as t tends to zero:*

(i) *For any* $\gamma \in (0, \overline{\gamma})$, $X^{(\alpha)} \sim \mathrm{LDP}(t^\gamma, \underline{\Lambda}^*)$ *with* $\alpha = \frac{1}{2}(1-\gamma/\underline{\gamma})$.

(ii) *If* $\gamma = \overline{\gamma}$, *then* $X^{(\alpha)} \sim \mathrm{LDP}(t^{\overline{\gamma}}, \overline{\Lambda}^*)$ *with* $\alpha = 1-\overline{\gamma}$.

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In case (i), it is easy to see that $\alpha < \frac{1}{2}$. In particular, for any fixed $l_2 > 1$, α is strictly positive if and only if $\gamma \in (0, \underline{\gamma})$, equal to zero if $\gamma = \underline{\gamma}$, and strictly negative otherwise; hence, only the first case really corresponds to moderate deviations, while the second case ($\gamma = \underline{\gamma}$) corresponds to classical large deviations. In (ii), α is strictly negative for any $l_2 > 1$. The behaviour when $\alpha < 0$ is not in the realm of moderate deviations: it is actually larger than large deviations (when $\alpha = 0$).

Let us first state and prove the following short technical lemma. Recall [6] that, for $\mathfrak{a} > 0$, a function $f:(\mathfrak{a}, \infty) \to \mathbb{R}_+^*$ is said to be regularly varying with index $l \in \mathbb{R}$ (and we write $f \in \mathcal{R}_l$) if $\lim_{x\uparrow\infty} f(\lambda x)/f(x) = \lambda^l$ for any $\lambda > 0$. When $l = 0$, the function is called slowly varying. We recall the following lemma, from [21, Lemma 4.7].

Lemma 1. *If* $|\log f| \in \mathcal{R}_l$ $(l > 1)$, *then* $\log \mathrm{M}_{\mathcal{V}}(z) \sim (l-1)(\frac{z}{l})^{\frac{l}{l-1}} \psi(z)$ *at infinity, with* $\psi \in \mathcal{R}_0$ *defined as*

$$\psi(z) := \left(\frac{z}{|\log f|^{\leftarrow}(z)}\right)^{\leftarrow} z^{\frac{l}{1-l}},$$

where $f^{\leftarrow}(x) := \inf\{y : f(y) > x\}$ *defines the generalised inverse.*

Proof of Theorem 2. Applying Lemma 2 and Lemma 1, if $\gamma + \alpha < 1$ then

$$\Lambda_\gamma^{(\alpha)}\left(t, \frac{u}{t^\gamma}\right) \sim \frac{\bar{\mathfrak{c}}}{2\overline{\gamma}} u^{2\overline{\gamma}} t^{\gamma + [1-2(\alpha+\gamma)]\overline{\gamma}} \quad \text{as } t \text{ tends to zero,for all } u \in \mathbb{R}.$$

The only nondegenerate result is obtained when $\alpha = \frac{1}{2}(1 - \gamma/\underline{\gamma})$, and the requirement that $\gamma + \alpha < 1$ implies that $\gamma < \overline{\gamma}$. The rest follows directly from the Gärtner–Ellis theorem.

If $\gamma + \alpha = 1$, then

$$\Lambda_\gamma^{(\alpha)}\left(t, \frac{u}{t^\gamma}\right) \sim \frac{\bar{\mathfrak{c}}}{\overline{\gamma}} 2^{\overline{\gamma}-1} \Lambda(u)^{\overline{\gamma}} t^{\gamma - \overline{\gamma}} \quad \text{as } t \text{ tends to zero, for all } u \in (u_-, u_+),$$

which imposes $\gamma = \overline{\gamma}$. Define now $f(u) := \overline{\gamma}^{-1}\bar{\mathfrak{c}} 2^{\overline{\gamma}-1}\Lambda(u)^{\overline{\gamma}}$ on (u_-, u_+); then

$$f'(u) = 2^{\overline{\gamma}-1}\bar{\mathfrak{c}} \frac{\Lambda'(u)}{\Lambda(u)^{1-\overline{\gamma}}} \qquad \text{and} \qquad f''(u) = 2^{\overline{\gamma}-1}\bar{\mathfrak{c}}\left[(\overline{\gamma} - 1)\frac{\Lambda'(u)^2}{\Lambda(u)^{2-\overline{\gamma}}} + \frac{\Lambda''(u)}{\Lambda(u)^{1-\overline{\gamma}}}\right].$$

Since $\overline{\gamma} > 1$, and since Λ is strictly convex and tends to infinity at $u_\pm$, then so does f. Consequently, for any $x \in \mathbb{R}$ the equation $x = f'(u)$ admits a unique solution in (u_-, u_+), and hence the function $\overline{\Lambda}^*$ is well defined on $\mathbb{R}$ and is a good rate function. The large deviations principle follows from the Gärtner–Ellis theorem. □

As mentioned above, in a mathematical finance context the case $\gamma < \underline{\gamma}$ belongs to the so-called regime of moderately out-of-the-money [13, 26], with time-dependent log-strike $x_t = xt^\alpha$ for $x \in \mathbb{R}_+^*$ and $\alpha \in (0, 1/2)$. In a thin-tail randomised environment, the rescaled limiting cgf does not satisfy [13, Assumption 6.1], in which the limit is assumed to have a quadratic form. Moreover, Theorem 2 implies that for the original process $(X_t)_{t\geq 0}$,

$$\mathrm{P}(X_t \geq x_t) = \mathrm{P}\big(X_t^{(\alpha)} \geq x\big) = \exp\left(-\frac{\underline{\Lambda}^*(x)}{t^\gamma}(1 + o(1))\right) \quad \text{as } t \text{ tends to zero.}$$

The moderate deviations result from Theorem 2(i) can in fact be strengthened in the following way.

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Theorem 3. *Let $s : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a slowly varying (at zero) function. Under Assumption 1, Theorem 2(i) still holds with $X^{(\alpha)}$ replaced by $X^{(\alpha)}/s(\cdot)$ but with speed $t^\gamma s(t)^{-2\underline{\gamma}}$. In particular, for $x_t := t^\alpha s(t)$, then*

$$\mathbb{P}(X_t \geq x_t) = \exp\left(-\frac{\underline{\Lambda}^*(s(t))}{t^\gamma}(1+o(1))\right) \quad \text{as } t \text{ tends to zero.}$$

This asymptotic behaviour can be translated into small-time behaviour of the implied volatility, following standard computations, and we obtain in particular that, with $\widehat{\gamma} := (1-2\alpha)(1-\underline{\gamma}) > 0$,

$$\lim_{t\downarrow 0} \frac{t^{\widehat{\gamma}} \sigma_t^2(x_t)}{s(t)^{2(1-\underline{\gamma})}} = \frac{\underline{\gamma}}{\underline{\mathfrak{c}}}.$$

Proof. The function $q : \mathbb{R}_+^* \to \mathbb{R}_+^*$ defined by $q(t) := s(t)^{-2\underline{\gamma}}$ is slowly varying at zero, and $\lim_{t\downarrow 0} t^\gamma q(t) = 0$. Since $\gamma + \alpha \in (\frac{1}{2}, 1)$, then $t = o(t^{\gamma+\alpha} q(t)s(t))$, and Lemma 2 implies that the rescaled cgf of $X_t/(s(t)t^\alpha)$ reads

$$\begin{aligned}
&t^\gamma q(t) \log \mathrm{M}\left(t, \frac{u}{t^{\gamma+\alpha} q(t)s(t)}\right) \\
&\quad = t^\gamma q(t) \mathrm{C}\left(t, \frac{u}{t^{\gamma+\alpha} q(t)s(t)}\right) + t^\gamma q(t) \log \mathrm{M}_{\mathcal{V}}\left(\mathrm{D}\left(t, \frac{u}{t^{\gamma+\alpha} q(t)s(t)}\right)\right) \\
&\quad = \mathcal{O}(t^{1+2\gamma+\alpha} q(t)^2 s(t)) + t^\gamma q(t) \log \mathrm{M}_{\mathcal{V}}\left(\frac{u^2 t(1+o(1))}{2t^{2(\gamma+\alpha)}(q(t)s(t))^2}\right).
\end{aligned}$$

Then, from Lemma 1, plugging in the expressions for α and the function q, the limit of the rescaled cgf reads

$$\lim_{t\downarrow 0} t^\gamma q(t) \log \mathrm{M}\left(t, \frac{u}{t^{\alpha+\gamma} q(t)s(t)}\right) = \frac{\overline{\mathfrak{c}}}{2\overline{\gamma}} u^{2\overline{\gamma}} \lim_{t\downarrow 0} t^\gamma q(t) \left(\frac{1+o(1)}{t^{\gamma/\overline{\gamma}}(q(t)s(t))^2}\right)^{\overline{\gamma}} = \frac{\overline{\mathfrak{c}}}{2\overline{\gamma}} u^{2\overline{\gamma}}.$$

The Gärtner–Ellis theorem implies that $(t^{-\alpha} X_t/s(t)) \sim \mathrm{LDP}(t^\gamma q(t), \underline{\Lambda}^*)$, with $\underline{\Lambda}^*$ in (3.1). Consequently,

$$-\inf_{x>1} \underline{\Lambda}^*(x) \leq \lim_{t\downarrow 0} t^\gamma q(t) \log \mathbb{P}(X_t \geq x_t) = \lim_{t\downarrow 0} t^\gamma q(t) \log \mathbb{P}\left(\frac{X_t}{t^\alpha s(t)} \geq 1\right) \leq -\inf_{x\geq 1} \underline{\Lambda}^*(x).$$

The proof then follows by noticing that $\dfrac{\underline{\Lambda}^*(1)}{q(t)} = \dfrac{\underline{\mathfrak{c}}}{2\underline{\gamma}} s(t)^{2\underline{\gamma}} = \underline{\Lambda}^*(s(t))$ for all $t > 0$. It is easy to check that the sequence $(t^\gamma x_t)_{t\geq 0}$ satisfies [7, Hypothesis 2.2], so that the small-time limit of the implied variance follows from [7, Theorem 2.3]:

$$\begin{aligned}
\sigma_t^2(x_t) &\sim \frac{2x_t}{t}\left[\sqrt{\frac{\underline{\mathfrak{c}}}{2\underline{\gamma}} \frac{s(t)^{2\underline{\gamma}-1}}{t^{\gamma+\alpha}}} - \sqrt{\frac{\underline{\mathfrak{c}}}{2\underline{\gamma}} \frac{s(t)^{2\underline{\gamma}-1}}{t^{\gamma+\alpha}} - 1}\right]^2 \\
&\sim \frac{2s(t)}{t^{1-\alpha}}\left[\sqrt{\frac{\underline{\mathfrak{c}}}{2\underline{\gamma}} \frac{s(t)^{2\underline{\gamma}-1}}{t^{\gamma+\alpha}}} + \sqrt{\frac{\underline{\mathfrak{c}}}{2\underline{\gamma}} \frac{s(t)^{2\underline{\gamma}-1}}{t^{\gamma+\alpha}} - 1}\right]^{-2} \sim \frac{\underline{\gamma} s(t)^{2(1-\underline{\gamma})}}{\underline{\mathfrak{c}} t^{\widehat{\gamma}}}.
\end{aligned}$$ □

For any fixed $x > 0$, the function $s(t) \equiv x$ is trivially regularly varying at the origin, and the following is thus an immediate consequence of Theorem 3.

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Corollary 1. *Consider the following two regimes:*

- *moderately out-of-the-money:* $(\alpha, x) \in (0, \frac{1}{2}) \times \mathbb{R}_+^*$;
- *small time and large strike:* $(\alpha, x) \in (1 - \overline{\gamma}, 0) \times \mathbb{R}_+^*$.

Under Assumption 1, for any $x > 0$, $\lim_{t\downarrow 0} t^{\widehat{\gamma}} \sigma_t^2(xt^\alpha) = \underline{\mathfrak{c}}^{-1} \underline{\gamma} x^{2(1-\underline{\gamma})}$.

3.3. Fat-tail distribution

The fat-tail distribution case yields some degeneracy, and forces us to analyse the asymptotic behaviour of the cumulant-generating function in more detail, in particular using sharp large deviations techniques for the rescaled process $(X_t^g)_{t\geq 0}$ defined by $X_t^g := g(t)^{-1} X_t$ for $t > 0$, where the function $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $g(t) = o(1)$ and $\sqrt{t} = o(g)$ as t tends to zero. For any rescaling function $h(t) := \sqrt{t}/g(t)$ $(= o(1))$, denote the rescaled cumulant-generating function as

$$\Lambda_t^g(u) := h(t) \log \mathrm{E}\Big[\exp\Big\{ \frac{u}{h(t)} X_t^g \Big\}\Big].$$

We provide higher-order asymptotic expansions for the European call option price with a time-dependent log-strike $x_t := xg(t)$, for any fixed $x \neq 0$, and translate this into small-time asymptotic behaviour of the implied volatility $\sigma_t(x_t)$. We discuss the case where the initial randomisation satisfies Assumption 2 with $\omega = 1$. The case where $\omega = 2$ can be processed in a similar fashion.

Theorem 4. *For any* $x \neq 0$, *as* t *tends to zero a European call with strike* e^{x_t} *satisfies*

$$\begin{aligned}\mathrm{E}\big[(\mathrm{e}^{X_t} - \mathrm{e}^{x_t})_+\big] =& (1 - \mathrm{e}^{x_t})_+ \\ &+ \exp\Big[-\sqrt{\frac{2\mathfrak{m}}{t}} |x_t| + \gamma_1 + x_t \Big] \frac{|x_t|^{|\gamma_0|-1} t^{1+\gamma_0/2} g(t)}{\Gamma(|\gamma_0|)(2\mathfrak{m})^{1-\gamma_0/2}} (1 + o(1)).\end{aligned}$$

Moreover, the implied volatility satisfies

$$\sigma_t^2(x_t) = \frac{|x_t|}{2\sqrt{2\mathfrak{m}t}} + \mathrm{h}_1(x) + \mathrm{h}_2 \log(t) + \frac{1}{4\mathfrak{m}} \log(g(t)) + o(1),$$

where

$$\begin{aligned}\mathrm{h}_1(x) &:= \frac{1}{8\mathfrak{m}} \Big\{ x_t - (2\gamma_0 + 1) \log|x_t| + \log\Big(\frac{16\pi \mathrm{e}^{2\gamma_1}}{\Gamma(|\gamma_0|)^2} \Big) - \Big(|\gamma_0| + \frac{1}{2} \Big) \log(2\mathfrak{m}) \Big\}, \\ \mathrm{h}_2 &:= \frac{1}{8\mathfrak{m}} \Big(\frac{1}{2} - |\gamma_0| \Big).\end{aligned}$$

Furthermore, under Assumption 2, $X^g \sim \mathrm{LDP}(h(t), \sqrt{2\mathfrak{m}}\,|x|)$.

Proof of Theorem 4. The proof is close to that of [21, Theorem 4.11], so we only sketch the highlights. Notice that $\sqrt{t} = o(h(t))$. Following similar steps to [21, Lemma D.1], it is easy to show that for $x \neq 0$ and $t > 0$ small, the equation $\partial_u \Lambda_t^g(u) = x$ admits a unique solution $u_t^*(x)$ satisfying $u_t^*(x) = \mathrm{sgn}(x)\sqrt{2\mathfrak{m}} - \frac{|\gamma_0|}{x} h(t) + \mathcal{O}(h(t)^2 + \sqrt{t})$. Then, as t tends to zero, direct computations yield

$$\exp\Big\{ \frac{-xu_t^*(x) + \Lambda_t^g(u_t^*(x))}{h(t)} \Big\} = \exp\Big\{ -\sqrt{\frac{2\mathfrak{m}}{t}} |x_t| - \gamma_0 + \gamma_1 \Big\} \Big(\frac{|\gamma_0|\sqrt{2\mathfrak{m}t}}{|x_t|} \Big)^{\gamma_0} (1 + o(1)).$$

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For fixed $x \neq 0$ and small $t > 0$, define the time-dependent measure $\mathbb{Q}_t$ by

$$\frac{\mathrm{d}\mathbb{Q}_t}{\mathrm{d}\mathbb{Q}} := \exp\left\{\frac{u_t^*(x)X_t^g - \Lambda_t^g(u_t^*(x))}{h(t)}\right\},$$

so that, for $x > 0$,

$$\begin{aligned}
&\mathrm{E}\big[(\mathrm{e}^{X_t} - \mathrm{e}^{x_t})_+\big] \\
&\quad = \mathrm{E}^{\mathbb{Q}_t}\bigg[\mathrm{e}^{x_t}\big(\mathrm{e}^{g(t)(X_t^g - x)} - 1\big)_+ \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}_t}\bigg] \\
&\quad = \exp\left\{\frac{-xu_t^*(x) + \Lambda_t^g(u_t^*(x))}{h(t)}\right\}\mathrm{e}^{x_t}\mathrm{E}^{\mathbb{Q}_t}\bigg[\exp\left\{\frac{-u_t^*(x)Z_t}{h(t)}\right\}\big(\mathrm{e}^{g(t)Z_t} - 1\big)_+\bigg],
\end{aligned} \tag{3.2}$$

with $Z_t := X_t^g - x$. From Lemma 3, under $\mathbb{Q}_t$, the characteristic function of Z_t satisfies

$$\Psi_t(u) := \mathrm{E}^{\mathbb{Q}_t}\big[\mathrm{e}^{\mathrm{i}uZ_t}\big] = \mathrm{e}^{-\mathrm{i}ux}\bigg(1 - \frac{\mathrm{i}ux}{|\gamma_0|}\bigg)^{\gamma_0}(1 + o(1)) \quad \text{as } t \text{ tends to zero.}$$

By Fourier inversion, we can therefore write, for small $t > 0$,

$$\begin{aligned}
&\mathrm{E}^{\mathbb{Q}_t}\bigg[\exp\bigg(\frac{-u_t^*(x)Z_t}{h(t)}\bigg)(\mathrm{e}^{g(t)Z_t} - 1)_+\bigg] \\
&\qquad = \frac{t}{2\pi}\int_{-\infty}^{\infty}\frac{\Psi_t(u)\,\mathrm{d}u}{[u_t^*(x) + (\mathrm{i}u - g(t))h(t)][u_t^*(x) + \mathrm{i}uh(t)]} \\
&\qquad = \frac{tf_\Gamma(x)}{2\mathfrak{m}}(1 + o(1)),
\end{aligned}$$

with

$$f_\Gamma(y) := \frac{y^{|\gamma_0|-1}}{\Gamma(|\gamma_0|)}\exp\bigg(-\bigg|\frac{\gamma_0}{x}\bigg|y\bigg)\bigg(\bigg|\frac{\gamma_0}{x}\bigg|\bigg)^{|\gamma_0|} \quad \text{for } y > 0,$$

and the result follows by plugging this back into (3.2). The case $x < 0$ follows by put–call parity. Finally, a direct application of [15, Corollary 7.2] yields the asymptotics for the implied volatility. □

4. A numerical example

We illustrate numerically the impact of the fat-tail random initialisation of the variance process on the implied volatility. In the Heston model (2.1), we consider the parameters $(\kappa, \theta, \rho, \xi) = (2.1, 0.05, -0.6, 0.1)$, which correspond to sensible values on equity markets, and choose the parameters of $\mathcal{V}$ such that $\mathrm{E}[\sqrt{\mathcal{V}}] = \sqrt{0.06} \approx 24.5\%$. In Figure 1 we plot the map $t \mapsto \sigma_t(x_t)$ with $x_t := xt^{0.1}$, $x = 0.2$ (left), and $x = 0.4$ (right), where $\mathcal{V} \sim \exp(13.09)$, i.e. $\mathfrak{m} = 13.09$. The blue and cyan dashed lines are first- and second-order asymptotics, and the yellow dashed lines represent the true implied volatilities calculated by fast Fourier transform. The small-time explosion is observed and is very different from the standard Heston case illustrated in [13, Section 5], where the implied volatility has a finite small-time limit. Furthermore, it shows similar exploding patterns as observed for rough volatility models, as detailed in [5], for example.

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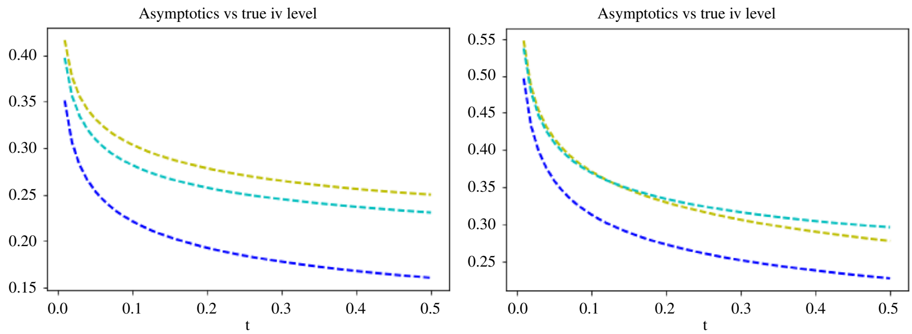

FIGURE 1. Numerical illustration – see Section 4.

Appendix A. Useful results

We recall the following small-time expansion of the (rescaled) functions C and D from [21, Appendix C].

Lemma 2. *The following asymptotic behaviours hold as t tends to zero:*

$$\mathrm{C}\left(t, \frac{u}{h(t)}\right) = \begin{cases} \textit{undefined} & u \neq 0, \quad \textit{if } h(t) = o(t), \\ \mathcal{O}(1) & u \in \mathcal{U}, \quad \textit{if } h(t) = t + \mathcal{O}(t^2), \\ h^2(t)\mathcal{E}_{\mathrm{C}}(t) + \dfrac{\kappa\theta u^2}{4}\left(\dfrac{t}{h(t)}\right)^2 [1 + \mathcal{E}_{\mathrm{C}}(t)] & u \in \mathbb{R}, \quad \textit{if } t = o(h(t)); \end{cases}$$

$$\mathrm{D}\left(t, \frac{u}{h(t)}\right) = \begin{cases} 0 & u = 0, \quad \textit{for any function } h, \\ \textit{undefined} & u \neq 0, \quad \textit{if } h(t) = o(t), \\ t^{-1}\Lambda(u) + \mathcal{O}(1) & u \in \mathcal{U}, \quad \textit{if } h(t) = t + \mathcal{O}(t^2), \\ \dfrac{u^2 t}{2h^2(t)}\left[1 - \dfrac{h(t)}{u} + \dfrac{\rho\xi u t}{2h(t)} + \mathcal{E}_{\mathrm{D}}(t)\right] & u \in \mathbb{R}, \quad \textit{if } t = o(h(t)), \end{cases}$$

where we denote $\mathcal{U} := (u_-, u_+)$,

$$\mathcal{E}_{\mathrm{C}}(t) := \mathcal{O}\left(h(t) + \frac{t}{h(t)}\right), \quad \textit{and} \quad \mathcal{E}_{\mathrm{D}}(t) := \mathcal{O}\left(t + h^2(t) + \frac{t^2}{h^2(t)}\right).$$

We also recall the following lemma.

Lemma 3. *[[21, Lemma D.3]] For any* $x \neq 0$, *let* $Z_t := (X_t - x)/\vartheta(t)$, *where* $\vartheta(t) := \mathbf{1}_{\{\omega=1\}} + \mathbf{1}_{\{\omega=2\}} t^{1/8}$. *Under Assumption 2, as t tends to zero,*

$$\Psi_t(u) := \mathrm{E}^{\mathbb{Q}_t}[\mathrm{e}^{\mathrm{i}uZ_t}] = \begin{cases} \mathrm{e}^{-\mathrm{i}ux}\left(1 - \dfrac{\mathrm{i}ux}{|\gamma_0|}\right)^{\gamma_0}(1 + o(1)) & \textit{for } \omega = 1, \\ \exp\left(\dfrac{-u^2\zeta^2(x)}{2}\right)(1 + o(1)) & \textit{for } \omega = 2, \end{cases}$$

where $\zeta(x) := \sqrt{2}\left(\dfrac{2\mathrm{m}}{\gamma_0^2}\right)^{1/8}|x|^{3/4}$.

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