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Wave-breaking phenomena, decay properties and limit behaviour of solutions of the Degasperis–Procesi equation

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We investigate some properties of solutions of the Degasperis–Procesi equation, which is an approximation to the incompressible Euler equation in shallow water theory. Sufficient conditions for wave breaking are found both on an infinite line and in a periodic domain by the method of characteristics. Moreover, we show that the solution enjoys the same decay property as the initial data. Finally, the weak and strong limits, respectively, of the solution as the dispersive parameter goes to zero are investigated.

1. Introduction

We are interested in the Cauchy problem of the Degasperis–Procesi (DP) equation

$$u_t - u_{txx} + 2\kappa u_x + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad t > 0, \quad x \in \mathbb{R}, \tag{1.1}$$

with initial data

$$u(0, x) = u_0(x), \tag{1.2}$$

where $u(t, x)$ represents the wave's height above the flat bottom, x denotes distance in the direction of propagation and t denotes the elapsed time. The parameter $\kappa \in \mathbb{R}$ in (1.1) is related to the critical shallow water speed.

The Degasperis–Procesi equation and the Camassa–Holm (CH) equation are two recently derived models for shallow water waves. The CH equation was first derived by Fokas and Fuchssteiner [28] as a bi-Hamiltonian equation, and then, in 1993, Camassa and Holm [5] derived it physically as a model for water waves. It was also found independently by Dai [22] as a model for nonlinear waves in cylindrical hyperelastic rods with $u(t, x)$ representing the radial stretch relative to a prestressed state (see also [47, 48]). Solitary waves in solids are very interesting from the point of view of applications, as they are easy to detect because they do not change their shape during propagation and can be used to determine material properties and to detect flaws [19]. The DP equation is also, in dimensionless space-time variables (x, t), an approximation to the incompressible Euler equations for shallow water [16, 25, 34, 35] and its asymptotic accuracy is the same as that of the CH equation. Both solutions $u(t, x)$ of the CH equation and the DP equation are considered as the horizontal component of the fluid velocity at time t in the spatial x-direction, but are evaluated at the different levels of the fluid domain [16]. It is worth pointing out that, in the context of the CH equation, continuation of a weak solution past the time of wave breaking was recently considered by Bressan and Constantin [2,3]. This continuation is not unique, but can be made unique either as a global conservative solution [2] or as a global dissipative solution [3].

The DP and CH equations have attracted much attention in recent years due to their integrable structure and intriguing properties caused by the balance between dispersion and nonlinearity [6, 7, 10, 27, 39, 40, 42, 45, 46]. These two equations show a variety of important properties. It is worth pointing out that, in the absence of linear dispersion (that is, in the limit case $\kappa \to 0$), the equations admit peaked solitary waves or 'peakons', which replicate a feature that is characteristic for the waves of great height (i.e. waves of largest amplitude that are exact solutions of the governing equations for water waves [14, 43]). Peaked solutions are true solitons that interact via elastic collisions under the CH dynamics, or the DP dynamics, respectively, and have proved to be orbitally stable in [18, 19, 38]. Another remarkable property of both equations is the presence of breaking waves (i.e. the solution remains bounded while its slope becomes unbounded in finite time). Therefore, as mentioned by Whitham [44], it is intriguing to know which mathematical models for shallow water waves exhibit both phenomena of soliton interaction and wave breaking.

It is easy to see that the DP equation with $\kappa = 0$ is reversible, i.e. invariant under transformation $u \mapsto -u$, $t \mapsto -t$. However, it is not Galilean invariant, i.e. not invariant under $u \mapsto u+\kappa$, $t \mapsto t$, $x \mapsto x+\kappa t$. It lies in a family of equations parametrized by the speed $\kappa \in \mathbb{R}$ of the Galilean frame, namely (1.1). Note that (1.1) with the linear dispersion is completely integrable for all $\kappa \in \mathbb{R}$, as it can be written as a compatibility condition of two linear systems (which is, sometimes, equivalent to the Lax pair; that is, a pair of matrices or operators $L(t)$ and $A(t)$ dependent on time and acting on a fixed Hilbert space, such that $\mathrm{d}L/\mathrm{d}t = [L, A]$, where $[L, A] = LA - AL$) [23, 24], that is,

$$\begin{gathered}(1 - \partial_x^2)\varPhi_x = \mu(y + \tfrac{2}{3}\kappa)\varPhi, \\ \varPhi_t + \frac{1}{\mu}\varPhi_{xx} + u\varPhi_x - u_x\varPhi = 0,\end{gathered}$$

where $y = u - \partial_x^2 u$ and the first equation above is the isospectral problem with a spectral parameter μ which is independent of time t. It is noted that unlike the CH equation, the isospectral problem of the DP equation is a third differential operator, which makes the isospectral analysis much more difficult than the case of the CH equation. It is worth pointing out that this problem was discussed recently in [21].

Constantin and Escher investigated the Cauchy problem for the CH equation on an infinite line $\mathbb{R}$ and on a unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ in [11] and [12], respectively. Local well-posedness and some global existence as well as blow-up criteria for solutions of this family of CH equations were given. Recently, we studied the Cauchy problem for (1.1) on $\mathbb{R}$ and on $\mathbb{S}$ in [29] and [30], respectively. The local and global well-posedness and blow-up scenarios were established. There are similarities to, but also structural differences from, the limiting case $\kappa = 0$, the rest Galilean frame. The analysis made in [41] showed that there are smooth solitary wave solutions to (1.1) for all $\kappa > 0$. When $\kappa \to 0$, these smooth travelling-wave solutions to (1.1) recover the existing peakon solitons to the DP equation. It is found that the periodic solution of (1.1) exists at any time if it is small initially in the Sobolev space $H^3(\mathbb{S})$ controlled by the parameter κ. This is very different from the limit case $\kappa = 0$, since the lifespan of the solution in the case $\kappa = 0$ is not affected by the smoothness or size of the initial profiles, but is affected by the shape of the initial profiles [30]. On the other hand, we showed in [30] that blow-up occurs in finite time if the initial profiles in L^2 are bounded away from zero depending on the parameter κ.

The goal of this paper is to study some properties of the solutions of (1.1) on the line $\mathbb{R}$ and in a periodic domain $\mathbb{S}$. In § 3, using the method of characteristics and a continuous family of diffeomorphisms of the line associated to (1.1) (see § 3), which was introduced by Constantin and Escher [8, 13] to study the CH equation, we present sufficient conditions guaranteeing the development of breaking waves in finite time on $\mathbb{R}$ and $\mathbb{S}$, respectively. Then, in § 4, it is shown that if the initial profile decays exponentially, the corresponding solution inherits this decay property. Finally, based on the global existence, the issue of passing to the limit as the dispersive parameter tends to zero for the solution of the DP-κ equation is investigated.

Notation. Throughout the paper, function spaces are assumed to be over $\mathbb{R}$ or $\mathbb{S}$, and both are dropped in function space notation if there is no ambiguity. We denote the convolution by $*$. For $1 \leqslant p < \infty$, the norm in the Lebesgue space L^p is

$$\|f\|_{L^p} = \left(\int |f(x)|^p \, \mathrm{d}x \right)^{1/p}.$$

The space L^∞ consists of all essentially bounded, Lebesgue measurable functions f equipped with the norm

$$\|f\|_{L^\infty} = \inf_{m(e)=0} \sup_{x \in \mathbb{R}(\text{or } \mathbb{S}) \setminus e} |f(x)|.$$

For a function f in the classical Sobolev spaces H^s, $s \geqslant 0$, the norm is denoted by $\|f\|_s$. We write $(\hat{f}_n)$ for the Fourier series of $f \in L^2$. The inner product in H^s is denoted by $\langle \cdot, \cdot \rangle_s$; in particular, the L^2 inner product is denoted by $\langle \cdot, \cdot \rangle$.

2. Preliminaries

In this section, we recall some results on the local/global well-posedness and blow-up results of (1.1) obtained in [29, 30].

Let $p(x) := \frac{1}{2}\mathrm{e}^{-|x|}$, $x \in \mathbb{R}$. Then $(1 - \partial_x^2)^{-1} f = p * f$ for all $f \in L^2(\mathbb{R})$ and $p * (u - u_{xx}) = u$. It is convenient to rewrite the initial-value problem (IVP) of (1.1) with the initial data (1.2) on the line $\mathbb{R}$ in its formally equivalent integral-differential form:

$$\left.\begin{aligned} u_t + uu_x + \partial_x p * (\tfrac{3}{2}u^2 + 2\kappa u) &= 0, \qquad t > 0, \quad x \in \mathbb{R}, \\ u(0, x) &= u_0(x), \qquad\qquad\quad x \in \mathbb{R}. \end{aligned}\right\} \tag{2.1}$$

On the other hand, set

$$G(x) := \frac{\cosh(x - [x] - \frac{1}{2})}{2 \sinh(\frac{1}{2})},$$

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where $[x]$ denotes the integer part of $x \in \mathbb{R}$. Then $(1-\partial_x^2)^{-1} f = G * f$ for all $f \in L^2(\mathbb{S})$ and $G * (u - u_{xx}) = u$. Using this identity, we can rewrite the periodic IVP (PIVP) of (1.1) with the initial data (1.2) on $\mathbb{S}$ as

$$\left.\begin{aligned} u_t + uu_x + \partial_x G * (\tfrac{3}{2}u^2 + 2\kappa u) = 0, &\qquad t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), &\qquad\qquad\quad x \in \mathbb{R}, \\ u(t,x) = u(t,x+1), &\qquad t \geqslant 0, \quad x \in \mathbb{R}. \end{aligned}\right\} \tag{2.2}$$

In [29, 30], we obtained the following result, which shows that the IVP (2.1) and the PIVP (2.2) are locally well posed (in Hadamard's sense: they show existence, uniqueness and continuous dependence).

THEOREM 2.1. *Given $u_0 \in H^s$, $s > \frac{3}{2}$, there exits a maximal $T = T(u_0, \kappa) > 0$ and a unique solution u to the IVP (2.1) or the PIVP (2.2), such that*

$$u = u(\cdot, u_0) \in C([0,T); H^s) \cap C^1([0,T); H^{s-1}).$$

Moreover, the flow map is continuous from H^s to the class defined above.

It is shown that the first blow-up can occur only in the form of wave breaking. More precisely, we have the following result.

THEOREM 2.2. *Assume $u_0 \in H^s (s > \frac{3}{2})$ and T is the existence time of the corresponding solution u to (2.1) (or (2.2)) with the initial data u_0. Then blow-up of the solution u to (2.1) (or (2.2)) in finite time $0 < T < \infty$ occurs if and only if*

$$\liminf_{t \uparrow T} \Big\{ \inf_{x \in \mathbb{R} (or\ \mathbb{S})} [u_x(t,x)] \Big\} = -\infty.$$

The above result was proved in [29, 30]. Here, for the IVP (2.1) with $s \geqslant 2$, we give another proof.

Indeed, set the momentum density $y = u - u_{xx}$. Then (1.1) can be rewritten as

$$y_t + uy_x + 3u_x y + 2\kappa u_x = 0, \quad t > 0, \quad x \in \mathbb{R}. \tag{2.3}$$

From the identity

$$\|y\|_{L^2}^2 = \int_{\mathbb{R}} (u - u_{xx})^2 \, dx = \int_{\mathbb{R}} (u^2 + 2u_x^2 + u_{xx}^2) \, dx,$$

we deduce that

$$\|u\|_2^2 \leqslant \|y\|_{L^2}^2 \leqslant 2\|u\|_2^2. \tag{2.4}$$

Due to (2.3) and integration by parts, we calculate

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} y^2(t,x) \, dx &= 2 \int_{\mathbb{R}} yy_t \, dx \\ &= 2 \int_{\mathbb{R}} y(-uy_x - 3u_x y - 2\kappa u_x) \, dx \\ &= -5 \int_{\mathbb{R}} u_x y^2 \, dx. \end{aligned}$$

If u_x is bounded below, one can deduce that the H^2-norm of u is also bounded by Gronwall's inequality and (2.4). Since

$$u(t,x) = p * y = \int_{\mathbb{R}} p(x-\eta) y(\eta)\,\mathrm{d}\eta,$$

taking into account (2.4), we obviously have

$$\|u_x\|_{L^\infty} \leqslant \left| \int_{\mathbb{R}} p(x-\eta) y(\eta)\,\mathrm{d}\eta \right| \leqslant \|p_x\|_{L^2} \|y\|_{L^2} \leqslant \|u\|_2.$$

This shows that if the H^2-norm of u is bounded, then so is $\|u_x\|_{L^\infty}$.

The following L^2- and L^∞-estimates of (2.1) and (2.2) play very important roles in studying breaking and permanent waves.

THEOREM 2.3. *Assume $u_0 \in H^s$, $s > \frac{3}{2}$. Let T be the maximal existence time of the solution u to (2.1) or (2.2) guaranteed by theorem 2.1. Then $E(u) = \int yv\,\mathrm{d}x$ is a conservation law, i.e.*

$$\int y(t,x) v(t,x)\,\mathrm{d}x = \int y_0(x) v_0(x)\,\mathrm{d}x,$$

where $y(t,x) = u(t,x) - u_{xx}(t,x)$ and $v(t,x) = (4 - \partial_x^2)^{-1} u$. Moreover, for any $t \in [0,T]$ we have two estimates,

$$\tfrac{1}{4}\|u_0\|_{L^2}^2 \leqslant \|u(t,\cdot)\|_{L^2}^2 \leqslant 4\|u_0\|_{L^2}^2 \tag{2.5}$$

and

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \leqslant 3(\|u_0\|_{L^2(\mathbb{R})}^2 + |\kappa|\|u_0\|_{L^2(\mathbb{R})})t + \|u_0\|_{L^\infty(\mathbb{R})} \tag{2.6}$$

or

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{S})} \leqslant At + \|u_0\|_{L^\infty(\mathbb{S})}, \tag{2.7}$$

where

$$A := 3\lambda\|u_0\|_{L^2(\mathbb{S})}^2 + 2|\kappa|\lambda\|u_0\|_{L^2(\mathbb{S})} \tag{2.8}$$

and

$$\lambda := \coth(\tfrac{1}{2}) = \frac{\cosh(\frac{1}{2})}{\sinh(\frac{1}{2})}.$$

Using a continuous family of diffeomorphisms of the line associated to (2.1) (see § 3), we can establish the global existence result.

THEOREM 2.4. *Suppose that $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, and $\kappa \geqslant 0$. If $m_0 = u_0 - u_{0,xx} \geqslant -\frac{2}{3}\kappa$ on $\mathbb{R}$, then the IVP (2.1) admits a unique global solution*

$$u \in C([0,\infty)H^s(\mathbb{R})) \cap C^1([0,\infty); H^{s-1}(\mathbb{R})).$$

3. Wave-breaking phenomena

In this section, we shall use the method of characteristics to study the blow-up phenomena for solutions of the IVP (2.1) and the PIVP (2.2).

The associated Lagrangian scale of (1.1) is established by the Cauchy problem

$$\left.\begin{aligned} \frac{\mathrm{d}q}{\mathrm{d}t} &= u(t,q), \quad && t \in [0,T), \\ q(0,x) &= x, && x \in \mathbb{R}, \end{aligned}\right\} \tag{3.1}$$

where $u = u(\cdot, u_0) \in C([0,T); H^s) \cap C^1([0,T); H^{s-1})$ is the solution of the Cauchy problem (2.1) or (2.2) with initial data $u_0 \in H^s$ with $s > \frac{3}{2}$, and $T > 0$ is the maximal time of existence. Direct calculation yields $q_{tx}(t,x) = u_x(t, q(t,x)) q_x(t,x)$. Thus,

$$q_x(t,x) = \exp\left(\int_0^t u_x(s, q(s,x))\,\mathrm{d}s \right) > 0 \quad \text{for all } (t,x) \in [0,T) \times \mathbb{R}, \tag{3.2}$$

which implies that $q(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism of the line for every $t \in [0,T)$, and the L^∞-norm of any function $f(t,\cdot) \in L^\infty$, $t \in [0,T)$, is preserved under this family of diffeomorphisms, namely

$$\|f(t,\cdot)\|_{L^\infty} = \|f(t, q(t,\cdot))\|_{L^\infty}, \quad t \in [0,T).$$

Furthermore, setting $y = u - u_{xx}$, we have

$$(y(t, q(t,x)) + \tfrac{2}{3}\kappa) q_x^3(t,x) = y_0(x) + \tfrac{2}{3}\kappa \quad \text{for all } (t,x) \in [0,T) \times \mathbb{R}. \tag{3.3}$$

We note that, in the context of the CH dynamics, use of the q-family of functions was motivated by the fact that the CH equation can be recast as a geodesic equation [15, 20], using a similar method to that in which the Euler equation of hydrodynamics can be viewed as a geodesic equation [1, 17]. An analogous but slightly more involved geometric interpretation for the DP equation is given by Escher and Kolev [26]. More precisely, they show that the periodic Degasperis–Procesi equation can be regarded as the geodesic flow of a right-invariant symmetric linear connection on the diffeomorphism group of the circle.

3.1. An infinite line case

We obtain the following blow-up result for the IVP (2.1).

Theorem 3.1. *Let $\varepsilon > 0$ and let $u_0(x) \in H^s$, $s > \frac{3}{2}$. Suppose that T_1 is the smallest positive root of*

$$2tK(t) = \log\left(1 + \frac{2}{\varepsilon}\right), \tag{3.4}$$

where

$$\begin{aligned} K^2(t) = {} & \tfrac{27}{2}(\|u_0\|_{L^2}^2 + |\kappa|\|u_0\|_{L^2})^2 t^2 \\ & + (9\|u_0\|_{L^\infty} + 12|\kappa|)(\|u_0\|_{L^2}^2 + |\kappa|\|u_0\|_{L^2})t + (\tfrac{3}{2}\|u_0\|_{L^\infty}^2 + 4|\kappa|\|u_0\|_{L^\infty}). \end{aligned}$$

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Suppose, furthermore, that there exists an $x_0 \in \mathbb{R}$ *such that*

$$u_0'(x_0) \leqslant -(1+\varepsilon)K(T_1). \tag{3.5}$$

The the solution $u(t,x)$ *of the IVP (2.1) blows up in a finite time* $\tilde{T} \in (0, T_1)$ *in the sense of wave breaking.*

Proof. Using theorem 2.1 and a simple density argument, we need only to consider the case $s = 3$. Let $T > 0$ be the maximal time of existence of the solution $u(t,x)$ to (2.1) with the initial data $u_0 \in H^3$. By theorem 2.1 we know that $u \in C([0,T); H^3) \cap C^1([0,T); H^2)$.

Define $U(t,x) = u(t, q(t,x))$ and $V(t,x) = u_x(t, q(t,x))$, respectively, along the characteristics defined by (3.1). By theorem 2.1, $V(t,x)$ is absolutely continuous and almost everywhere (a.e.) differentiable on $(0,T) \times \mathbb{R}$.

In view of $\partial_x^2(p * f) = p * f - f$, for any $f \in L^2$, differentiating (2.1) with respect to x yields

$$u_{tx} + uu_{xx} = -u_x^2 + \tfrac{3}{2}u^2 + 2\kappa u - p * (\tfrac{3}{2}u^2 + 2\kappa u).$$

Since $p * (\frac{3}{2}u^2)(t, q(t,x)) \geqslant 0$ and $\|p\|_{L^1} = 1$, using the estimate (2.6) and Young's inequality, we obtain the *a priori* differential inequality along the characteristics

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= -V^2 + \tfrac{3}{2}U^2 + 2\kappa U - p * (\tfrac{3}{2}u^2 + 2ku) \\ &\leqslant -V^2 + \tfrac{3}{2}U^2 + 2|\kappa||U| + 2|\kappa||p * u| \\ &\leqslant -V^2 + \tfrac{3}{2}U^2 + 4|\kappa||U| \\ &\leqslant -V^2 + K^2(t)\end{aligned} \tag{3.6}$$

by involving the definition of $K(t)$.

Because $u_0'(x)$ is continuous and zero mean on $\mathbb{R}$, in view of (3.5), for fixed $\varepsilon > 0$ there exists an $\tilde{x}_0$ such that

$$V(0, \tilde{x}_0) = -(1+\varepsilon)K(T_1).$$

Noting (3.6), $V(t) := V(t, \tilde{x}_0)$ satisfies

$$\left.\begin{aligned}&\frac{\mathrm{d}V}{\mathrm{d}t} \leqslant -V^2(t) + K^2(T_1) \quad \text{a.e. } t \in [0,T_1] \cap [0,T), \\ &V(0) = -(1+\varepsilon)K(T_1).\end{aligned}\right\} \tag{3.7}$$

Consider another Cauchy problem

$$\left.\begin{aligned}&\frac{\mathrm{d}V_+}{\mathrm{d}t} = -V_+^2(t) + K^2(T_1), \quad t \in [0,T_1], \\ &V_+(0) = -(1+\varepsilon)K(T_1).\end{aligned}\right\} \tag{3.8}$$

This Cauchy problem has a solution that satisfies

$$\frac{V_+(t) + K(T_1)}{V_+(t) - K(T_1)} = \frac{V_+(0) + K(T_1)}{V_+(0) - K(T_1)} \mathrm{e}^{2K(T_1)t}, \quad t \in [0, T_1).$$

Let $t \uparrow T_1$ and recall T_1 is the smallest positive root of (3.4). We have

$$\frac{V_+(t) + K(T_1)}{V_+(t) - K(T_1)} = \frac{\varepsilon}{2+\varepsilon} \mathrm{e}^{2K(T_1)} t \to 1.$$

This implies that $\lim_{t\uparrow T_1} V_+(t) = -\infty$, since $V_+(t) < 0$ for $t \in [0, T_1] \cap [0, T)$, whereas, by the well-known comparison theorem for ordinary differential equations, from (3.7), (3.8) we get

$$V(t) \leqslant V_+(t) < 0 \quad \text{for all } t \in [0, T_1] \cap [0, T).$$

Hence, there exists a $\tilde{T} \in (0, T_1)$ such that $\lim_{t\uparrow\tilde{T}} V(t) = -\infty$. The proof is thus complete. □

3.2. A periodic domain case

We now turn our attention to PIVP (2.2). In a different form, we have the following blow-up result. As long as the solution u to the PIVP (2.2) is defined, we consider

$$m_1(t) = \min_{x\in\mathbb{S}}[u_x(t,x)], \qquad m_2(t) = \max_{x\in\mathbb{S}}[u_x(t,x)].$$

Furthermore, we let $x_1(t) \in \mathbb{S}$ and $x_2(t) \in \mathbb{S}$ be points where these extrema are attained, i.e. $m_i(t) = u_x(x_i(t), t)$, $i = 1, 2$. We shall make use of the following two lemmas.

Lemma 3.2 (Constantin and Escher [12, 13]). *Let $[0, T)$ be the maximal interval of existence of the solution $u(t)$ of (2.2) with the initial data $u_0 \in H^s$, $s > \frac{3}{2}$, as given by theorem 2.1. Then the function $m_i(t)$, $i = 1, 2$, is absolutely continuous on $(0, T)$ with*

$$\frac{\mathrm{d}m_i}{\mathrm{d}t} = u_{tx}(t, x_i(t)) \quad \text{a.e. on } (0, T).$$

Lemma 3.3 (Constantin and Escher [12]). *Let g be a monotone function on $[a, b]$. Furthermore, let f be a real continuous function on $[a, b]$. Then there exists a $\xi \in [a, b]$ such that*

$$\int_a^b f(s)g(s)\,\mathrm{d}s = g(a)\int_a^\xi f(s)\,\mathrm{d}s + g(b)\int_\xi^b f(s)\,\mathrm{d}s.$$

With these two lemmas in hand, we can prove the following blow-up result, which shows that for an initial profile with an asymmetric hump, the wave may break in finite time.

Theorem 3.4. *For $\kappa \in \mathbb{R}$ and $\delta > 0$. Assume $u_0 \in H^s (s > \frac{3}{2})$ such that*

$$m_1(0) + m_2(0) < \min\{-16|\kappa| - 2\sqrt{3}(AT_2 + \|u_0\|_{L^\infty}), -4|\kappa| - 2(1+\delta)Q(T_2)\}, \quad (3.9)$$

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then the solution of (2.2) with the initial data u_0 blows up in finite time, and the maximal time of existence is estimated above by T_2, where

$$T_2 = \left(\frac{-B + \sqrt{B^2 + 3A^2\ln^2(1 + 2/\delta)}}{6A^2}\right)^{1/2},$$
$$B = 4|\kappa|^2 + 3\|u_0\|^2_{L^\infty},$$
$$Q(t) = (3A^2t^2 + B)^{1/2}$$

and A is defined by (2.8).

Proof. As before, we only need to consider the case $s = 3$. Equation (2.2) can be rewritten as

$$u_t + uu_x = -2\kappa(G * u_x) - \partial_x G * (\tfrac{3}{2}u^2).$$

By differentiation we obtain that

$$u_{tx} + u_x^2 + uu_{xx} = -2\kappa(G * u_{xx}) + \tfrac{3}{2}u^2 - G * (\tfrac{3}{2}u^2).$$

In view of lemma 3.2 and the definitions of $m_i(t)$, $i = 1, 2$, setting $x = x_i(t)$, for $i = 1, 2$, we have that

$$\frac{\mathrm{d}m_i}{\mathrm{d}t} + m_i^2 + 2\kappa\int_0^1 G(y)u_{xx}(x_i - y, t)\,\mathrm{d}y = \tfrac{3}{2}u^2(x_i, t) - \frac{3}{2}\int_0^1 G(x_i - y)u^2(y, t)\,\mathrm{d}y \quad \text{a.e. on } (0, T), \tag{3.10}$$

as $u_{xx}(x_i(t), t) = 0$, $i = 1, 2$.

The function

$$g(y) = G(y) - \frac{1}{2\sinh(\frac{1}{2})}, \quad y \in \mathbb{R},$$

is continuous, decreasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$, with $g(\frac{1}{2}) = 0$, $g(0) = g(1) \leqslant \frac{1}{2}$. Note the spatial periodicity of u_{xx}. We find for $i = 1, 2$ that

$$\begin{aligned}\left|\int_0^1 G(y)u_{xx}(x_i - y, t)\,\mathrm{d}y\right| &= \left|\int_0^1 g(y)u_{xx}(x_i - y, t)\,\mathrm{d}y\right| \\ &\leqslant \left|\int_0^{1/2} g(y)u_{xx}(x_i - y, t)\,\mathrm{d}y\right| + \left|\int_{1/2}^1 g(y)u_{xx}(x_i - y, t)\,\mathrm{d}y\right|.\end{aligned}$$

To estimate each of the two integrals on the right-hand side, applying lemma 3.3, we find that

$$\left|\int_0^1 G(y)u_{xx}(x_i - y, t)\,\mathrm{d}y\right| \leqslant m_2(t) - m_1(t), \quad t \in [0, T]. \tag{3.11}$$

Note that $G(y) \geqslant \frac{1}{2}$ on $\mathbb{R}$. From (2.5) we may infer that

$$\int_0^1 G(x_i - y)u^2(y, t)\,\mathrm{d}y \geqslant \frac{1}{2}\int_{\mathbb{S}} u^2(y, t)\,\mathrm{d}y = \tfrac{1}{2}\|u\|^2_{L^2} \geqslant \tfrac{1}{8}\|u_0\|^2_{L^2}. \tag{3.12}$$

Due to the *a priori* estimate (2.7), we have

$$u^2(x_i,t) \leqslant [(3\lambda\|u_0\|_{L^2}^2 + 2\lambda|\kappa|\|u_0\|_{L^2})t + \|u\|_{L^\infty}]^2 = (At + \|u\|_{L^\infty})^2. \tag{3.13}$$

Combining (3.10)–(3.13), we deduce, for a.e. $t \in (0,T)$,

$$\left.\begin{aligned} \frac{\mathrm{d}m_1}{\mathrm{d}t} &\leqslant -m_1^2 + 2|\kappa|(m_2 - m_1) + \tfrac{3}{2}(At + \|u_0\|_{L^\infty})^2 - \tfrac{1}{8}\|u_0\|_{L^2}^2, \\ \frac{\mathrm{d}m_2}{\mathrm{d}t} &\leqslant -m_2^2 + 2|\kappa|(m_2 - m_1) + \tfrac{3}{2}(At + \|u_0\|_{L^\infty})^2 - \tfrac{1}{8}\|u_0\|_{L^2}^2. \end{aligned}\right\} \tag{3.14}$$

Summing these expressions yields

$$\begin{aligned} \frac{\mathrm{d}(m_1 + m_2)}{\mathrm{d}t} &\leqslant -m_1^2 - m_2^2 + 4|\kappa|(m_1 + m_2) \\ &\qquad - 8|\kappa|m_1 + 3(At + \|u_0\|_{L^\infty})^2 - \tfrac{1}{4}\|u_0\|_{L^2}^2 \end{aligned} \tag{3.15}$$

for a.e. $t \in (0,T)$. Hence, if $(m_1 + m_2) \leqslant -16|\kappa| - 2\sqrt{3}(AT_2 + \|u\|_{L^\infty})$ initially, it remains so for all $t \in (0,T) \cap (0,T_2]$. Indeed, since $(m_1 + m_2) \leqslant 0$ and $m_1 \leqslant m_2$, we have $m_1 \leqslant -8|\kappa| - \sqrt{3}(AT_2 + \|u\|_{L^\infty})$, and the right-hand side of (3.15) is less than $[-m_1^2 - 8|\kappa|m_1 + 3(At + \|u_0\|_{L^\infty})^2 - \frac{1}{4}\|u_0\|_{L^2}^2]$, which is negative by the estimate obtained on m_1. Substituting the information obtained into the first equation of (3.14), we infer that

$$\begin{aligned} \frac{\mathrm{d}m_1(t)}{\mathrm{d}t} &\leqslant -m_1^2 + 2|\kappa|(m_1 + m_2) - 4|\kappa|m_1 + \tfrac{3}{2}(At + \|u_0\|_{L^\infty})^2 - \tfrac{1}{8}\|u_0\|_{L^2}^2 \\ &\leqslant -m_1^2 - 4|\kappa|m_1 + \tfrac{3}{2}(At + \|u_0\|_{L^\infty})^2 - \tfrac{1}{8}\|u_0\|_{L^2}^2 \\ &= -(m_1 + 2|\kappa|)^2 + 4|\kappa|^2 + \tfrac{3}{2}(At + \|u_0\|_{L^\infty})^2 - \tfrac{1}{8}\|u_0\|_{L^2}^2 \end{aligned}$$

for a.e. $t \in (0,T) \cap (0,T_2]$. Let $M(t) := m_1(t) + 2|\kappa|$ for a.e. $t \in (0,T) \cap (0,T_2]$. Then

$$\begin{aligned} \frac{\mathrm{d}M(t)}{\mathrm{d}t} &\leqslant -M^2(t) + 4|\kappa|^2 + \tfrac{3}{2}(At + \|u_0\|_{L^\infty})^2 - \tfrac{1}{8}\|u_0\|_{L^2}^2 \\ &\leqslant -M^2(t) + 3A^2t^2 + 4|\kappa|^2 + 3\|u_0\|_{L^\infty}^2 \\ &\leqslant -M^2(t) + 3A^2T_2^2 + B \\ &= -M^2(t) + Q^2(T_2), \end{aligned} \tag{3.16}$$

by recalling the definition of $Q(t)$. Using this assumption, it is easy to check that

$$2Q(T_2)T_2 - \ln\left(1 + \frac{2}{\delta}\right) \geqslant 0 \tag{3.17}$$

and

$$M(0) < -(1+\delta)Q(T_2).$$

Clearly,

$$0 < \frac{M(0) - Q(T_2)}{M(0) + Q(T_2)} = 1 - \frac{2Q(T_2)}{M(0) + Q(T_2)} \leqslant 1 + \frac{2}{\delta}.$$

This inequality and (3.16) lead to

$$\frac{1}{2Q(T_2)}\ln\frac{M(0)-Q(T_2)}{M(0)+Q(T_2)} \leqslant T_2. \tag{3.18}$$

From (3.9), we obtain $M(0) < -(1+\delta)Q(T_2) < -Q(T_2)$. Therefore, using (3.18) we can prove that $M(t) < -Q(T_2)$ by the argument of continuous deduction. Otherwise, since $M(t)$ is continuous on $[0, T_2)$, there exists some $t_0 \in (0, T_2)$ such that $Q^2(T_2) < M^2(T_2)$ on $[0, t_0)$, while $Q^2(T_2) = M^2(t_0)$. Thus,

$$\frac{\mathrm{d}M}{\mathrm{d}t} < 0 \quad \text{a.e. } t \in (0, t_0).$$

Being locally Lipschitz, the function M is absolutely continuous on $[0, t_0]$, and therefore an integration of the previous inequality would lead to

$$M(t_0) \leqslant M(0) < -M(T_1),$$

which contradicts our assumption $Q^2(T_2) = M^2(t_0)$.

Solving the inequality (3.16) gives

$$\frac{M(0)+Q(T_2)}{M(0)-Q(T_2)}\mathrm{e}^{2Q(T_2)t} - 1 \leqslant \frac{2Q(T_2)}{M(t)-Q(T_2)} \leqslant 0.$$

As

$$0 < \frac{M(0)+Q(T_2)}{M(0)-Q(T_2)} < 1,$$

we deduce that there exists a T^* satisfying

$$0 < T^* < \frac{1}{2Q(T_2)}\ln\frac{M(0)-Q(T_2)}{M(0)+Q(T_2)} \leqslant T_2$$

such that $\lim_{t\uparrow T^*} M(t) = -\infty$ and this leads to blow-up. □

Remark 3.5. Using the mollifier, it is not difficult to construct initial data satisfying the above blow-up condition. For $n \geqslant 4$, let ρ_n be the usual mollifier on $\mathbb{R}$, i.e. $\rho_n(x) = n\rho(nx)$, $x \in \mathbb{R}$, where

$$\rho(x) = \begin{cases} \alpha \exp\left(-\dfrac{1}{1-x^2}\right), & |x| < 1, \\ 0, & |x| \geqslant 1, \end{cases}$$

and $\alpha > 0$ satisfies

$$\int_{\mathbb{R}} \rho(x)\,\mathrm{d}x = 1.$$

Now define the function $f_n \in C^\infty(\mathbb{S})$ by

$$f_n(x) = \begin{cases} -\rho_n\left(x + \dfrac{1}{n}\right), & x \in \left[0, \dfrac{2}{n}\right], \\ 0, & x \in \left[\dfrac{2}{n}, \dfrac{1}{2}\right], \\ \rho_4(x + \frac{3}{4}), & x \in [\frac{1}{2}, 1]. \end{cases}$$

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Then

$$\min_{x\in\mathbb{S}} f_n(x) = -\frac{n\alpha}{e}, \qquad \max_{x\in\mathbb{S}} f_n(x) = \frac{4\alpha}{e},$$

and therefore

$$m_1 + m_2 = \frac{(4-n)\alpha}{e}$$

such that for sufficiently large n the blow-up condition is satisfied.

4. A decay property

Himonas *et al*. studied the persistence properties of the CH equation in [33]. They proved that a strong solution of the CH equation, initially decaying exponentially together with its spacial derivative, must be identically equal to zero if it also decays exponentially at a later time. We shall follow their idea to study the decay property of solutions of the IVP (2.1). However, we have a different proof by the method of characteristics here. The main result in this section is as follows.

THEOREM 4.1. *Let* $u_0(x) \in H^s$ *with* $s > \frac{3}{2}$*, where* $u \in C([0,T]; H^s)$ *is a strong solution of the IVP (2.1) for some* $T > 0$*. If there exists a* $\theta \in (0,1)$ *such that*

$$|u_0(x)|, |\partial_x u_0(x)| \sim O(\mathrm{e}^{-\theta x}) \quad \text{as } x \uparrow \infty, \tag{4.1}$$

then

$$|u(t,x)|, |u_x(t,x)| \sim O(\mathrm{e}^{-\theta x}) \quad \text{as } x \uparrow \infty \tag{4.2}$$

uniformly in $[0,T]$.

Before proceeding with the proof, a few comments are in order. Firstly, by theorem 2.1, the IVP (2.1) admits a unique strong solution with the initial data $u_0 \in H^s$, $s > \frac{3}{2}$, which inherits at any later time the spatial smoothness of the initial data, measured on a Sobolev space scale with integer index s. Secondly, for the CH and DP equations, with $\kappa = 0$, the finite propagation speed was established by Constantin [9] and Henry [31], respectively. They claimed that any classical solution of both equations will have compact support if the initial datum has this property. Their proof relied on the invariance of the scale $y(t,x)$ associated to (2.1). However, due to (3.3), (1.1) does not have this property; it seems that any classical solution of this equation cannot propagate at finite speed. Henry [32] showed that certain exponential decay properties of the initial data persist as long as the solution of the IVP (2.1) exists. Note that, for the CH equation with $\kappa \neq 0$, an (inverse scattering transform) approach which makes great use of the integrability was taken in [37] to investigate the infinite speed property. Finally, the persistence of decay properties to the b-family of equations was recently studied in [4].

Proof. We introduce the notation

$$F(u) = \tfrac{3}{2}u^2 + 2\kappa u \tag{4.3}$$

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and the weight function

$$\phi_N(x) = \begin{cases} 1, & x \leqslant 0, \\ e^{\theta x}, & 0 < x < N, \\ e^{\theta N}, & x \geqslant N, \end{cases} \tag{4.4}$$

where N is a positive integer. Obviously,

$$0 \leqslant \phi_N'(x) \leqslant \theta\phi_N(x) \quad \text{for a.e. } x \in \mathbb{R}. \tag{4.5}$$

Along the characteristic defined by (3.1), set

$$\tilde{U}(t) = u(t, q(t,x))\phi_N(q(t,x)) \tag{4.6}$$

and

$$\tilde{V}(t) = u_x(t, q(t,x))\phi_N(q(t,x)). \tag{4.7}$$

A direct calculation shows that there exists a $c_0 > 0$, which depends only on θ, such that, for any positive integer N,

$$\phi_N(x)\int_{-\infty}^{\infty} e^{-|x-y|}\frac{1}{\phi_N(y)}\,dy \leqslant 2c_0 = \frac{4}{1-\theta}. \tag{4.8}$$

Thus, for any appropriate function f we have

$$\begin{aligned} |\phi_N\partial_x p * f^2(x)| &= \left|\tfrac{1}{2}\phi_N(x)\int_{-\infty}^{\infty}\operatorname{sgn}(x-y)e^{-|x-y|}f^2(y)\,dy\right| \\ &\leqslant \tfrac{1}{2}\phi_N(x)\int_{-\infty}^{\infty} e^{-|x-y|}\frac{1}{\phi_N(y)}\phi_N(y)f(y)f(y)\,dy \\ &\leqslant \frac{1}{2}\left(\phi_N(x)\int_{-\infty}^{\infty} e^{-|x-y|}\frac{1}{\phi_N(y)}\,dy\right)\|\phi_N f\|_{L^\infty}\|f\|_{L^\infty} \\ &\leqslant c_0\|\phi_N f\|_{L^\infty}\|f\|_{L^\infty} \end{aligned} \tag{4.9}$$

and

$$|\phi_N\partial_x p * f(x)| \leqslant c_0\|\phi_N f\|_{L^\infty}. \tag{4.10}$$

Noting (4.3), multiplying ϕ_N on both side of (2.1) gives

$$u_t\phi_N + uu_x\phi_N = -\phi_N\partial_x p * F(u).$$

By (3.1) and (4.6), we calculate

$$\begin{aligned} \frac{d\tilde{U}}{dt} &= u_t\phi_N + u_x q_t\phi_N + u\phi_N' q_t \\ &= u_t\phi_N + uu_x\phi_N + u^2\phi_N' \\ &= -\phi_N\partial_x p * F(u) + u^2\phi_N'. \end{aligned}$$

In view of (4.3) and (4.5), by (4.10) we have

$$\left|\frac{d\tilde{U}}{dt}\right| \leqslant (\tfrac{3}{2}c_0 M + 2|\kappa|c_0 + \theta M)\|\phi_N u\|_{L^\infty} \triangleq c_1\|\phi_N u\|_{L^\infty},$$

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where $M = \sup_{t\in[0,T]} \|u(t,\cdot)\|_s$. In the remainder of the proof, let c_i, $i = 2, 3, \dots$, be the positive constants depending on M, κ, c_0, θ and T. Therefore,

$$\|\tilde{U}(t)\|_{L^\infty} \leqslant c_1 \int_0^t \|u\phi_N(s)\|_{L^\infty}\, \mathrm{d}s + \|\tilde{U}(0)\|_{L^\infty}. \tag{4.11}$$

Differentiating (2.1) with respect to x and multiplying by ϕ_N, and noting that $\partial_x^2(p * f) = p * f - f$ for any $f \in L^2(\mathbb{R})$, yields

$$u_{tx}\phi_N + uu_{xx}\phi_N = -u_x^2\phi_N + F(u)\phi_N - p * F(u)\phi_N.$$

With (3.1), from (4.7) and the above identity we deduce that

$$\begin{aligned}\frac{\mathrm{d}\tilde{V}}{\mathrm{d}t} &= u_{xt}\phi_N + u_{xx}q_t\phi_N + u_x\phi_N' q_t \\ &= u_{xt}\phi_N + u_{xx}u\phi_N + uu_x\phi_N' \\ &= uu_x\phi_N' - u_x^2\phi_N + F(u)\phi_N - p * F(u)\phi_N.\end{aligned}$$

Then

$$\left|\frac{\mathrm{d}\tilde{V}}{\mathrm{d}t}\right| \leqslant c_2(\|u\phi_N\|_{L^\infty} + \|u_x\phi_N\|_{L^\infty})$$

and

$$\|\tilde{V}(t)\|_{L^\infty} \leqslant c_2 \int_0^t (\|u\phi_N\|_{L^\infty} + \|u_x\phi_N\|_{L^\infty})(s)\, \mathrm{d}s + \|\tilde{V}(0)\|_{L^\infty}. \tag{4.12}$$

Combining (4.11) and (4.12) leads to

$$\begin{aligned}&\|\tilde{U}(t)\|_{L^\infty} + \|\tilde{V}(t)\|_{L^\infty} \\ &\qquad \leqslant c_3 \int_0^t (\|u\phi_N\|_{L^\infty} + \|u_x\phi_N\|_{L^\infty})(s)\, \mathrm{d}s + \|\tilde{U}(0)\|_{L^\infty} + \|\tilde{V}(0)\|_{L^\infty}.\end{aligned}$$

Gronwall's inequality shows that

$$\|\tilde{U}(t)\|_{L^\infty} + \|\tilde{V}(t)\|_{L^\infty} \leqslant (\|\tilde{U}(0)\|_{L^\infty} + \|\tilde{V}(0)\|_{L^\infty})\mathrm{e}^{c_3 t} \quad \text{for all } t \in [0,T],$$

or, equivalently,

$$\|u\phi_N\|_{L^\infty} + \|u_x\phi_N\|_{L^\infty} \leqslant \mathrm{e}^{c_3 T}(\|u_0 \max\{1, \mathrm{e}^{\theta x}\}\|_{L^\infty} + \|u_{0,x} \max\{1, \mathrm{e}^{\theta x}\}\|_{L^\infty}).$$

Since $q(t,\cdot)\colon \mathbb{R} \to \mathbb{R}$ is a diffeomorphism of the line for every $t \in [0,T]$, letting $N \to \infty$ yields

$$\|u\mathrm{e}^{\theta x}\|_{L^\infty} + \|u_x\mathrm{e}^{\theta x}\|_{L^\infty} \leqslant \mathrm{e}^{c_3 T}(\|u_0 \max\{1, \mathrm{e}^{\theta x}\}\|_{L^\infty} + \|u_{0,x} \max\{1, \mathrm{e}^{\theta x}\}\|_{L^\infty}),$$

which is the desired result. □

5. Limit behaviour as $\kappa \to 0$

In this final section, based on the global existence result (theorem 2.4), we shall study the weak- and strong-limit behaviour of the solutions to the IVP (2.1) as $\kappa \to$

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0, respectively. In the following, let u^κ be the solution of the IVP (2.1) corresponding to κ in $H^s(\mathbb{R})$ with $s > \frac{3}{2}$. We get the following result concerning the weak limit of the solutions to the IVP (2.1).

THEOREM 5.1. *Suppose that* $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, *and* $\kappa \geqslant 0$. *If* $y_0 = u_0 - u_{0,xx} \geqslant -\frac{2}{3}\kappa$ *on* $\mathbb{R}$, *then, for any fixed* $T > 0$,

$$\sup_{t\in[0,T]} \|(u^{\kappa_1} - u^{\kappa_2})(t,\cdot)\|_{L^2} \to 0 \quad \text{as } \kappa_1,\kappa_2 \to 0. \tag{5.1}$$

To prove this theorem, we recall Kato and Ponce's result [36] on estimating commutators.

LEMMA 5.2. *Let* $\Lambda^s = (1-\partial_x^2)^{s/2}$. *If* $s \geqslant 0$, $1 < p < \infty$, $f, g \in S(\mathbb{R}^n)$, *then there exists a constant* $C = C(s,n,p)$ *such that*

$$\|[\Lambda^s, f]g\|_{L^p} \leqslant C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}) \tag{5.2}$$

and

$$\|\Lambda(fg)\|_{L^p} \leqslant C(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}), \tag{5.3}$$

where $1 < p_2, p_3 < \infty$ *and* $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$.

Proof of theorem 5.1. Since $y_0 + \frac{2}{3}\kappa$ remains non-negative on $\mathbb{R}$, according to theorem 2.4, the IVP (2.1) admits a unique global strong solution. Throughout this section, fix $T > 0$, and let C denote generic positive constants depending only on T and $\|u_0\|_s$ (but not on κ), whose meaning may change from line to line. Since $\kappa \to 0$, we restrict our proof to the case $0 < \kappa \leqslant 1$.

The proof will completed in three steps.

STEP 1. We first prove that, for any $t \in [0,T]$,

$$\|u_x(t,\cdot)\|_{L^\infty} \leqslant C. \tag{5.4}$$

Indeed, from the relation $u = p * y$, we deduce that

$$u(t,x) + \tfrac{2}{3}\kappa = \tfrac{1}{2}\mathrm{e}^{-x}\int_{-\infty}^{x} \mathrm{e}^{\eta}(y(t,\eta) + \tfrac{2}{3}\kappa)\,\mathrm{d}\eta + \tfrac{1}{2}\mathrm{e}^{x}\int_{x}^{\infty} \mathrm{e}^{-\eta}(y(t,\eta) + \tfrac{2}{3}\kappa)\,\mathrm{d}\eta$$

and

$$u_x(t,x) = -\tfrac{1}{2}\mathrm{e}^{-x}\int_{-\infty}^{x} \mathrm{e}^{\eta}(y(t,\eta) + \tfrac{2}{3}\kappa)\,\mathrm{d}\eta + \tfrac{1}{2}\mathrm{e}^{x}\int_{x}^{\infty} \mathrm{e}^{-\eta}(y(t,\eta) + \tfrac{2}{3}\kappa)\,\mathrm{d}\eta.$$

Then

$$u(t,x) + u_x(t,x) + \tfrac{2}{3}\kappa = \mathrm{e}^{x}\int_{x}^{\infty} \mathrm{e}^{-\eta}(y(t,\eta) + \tfrac{2}{3}\kappa)\,\mathrm{d}\eta$$

and

$$u(t,x) - u_x(t,x) + \tfrac{2}{3}\kappa = \mathrm{e}^{-x}\int_{-\infty}^{x} \mathrm{e}^{\eta}(y(t,\eta) + \tfrac{2}{3}\kappa)\,\mathrm{d}\eta.$$

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Since $y_0 + \frac{2}{3}\kappa \geqslant 0$, it follows from (3.3) that $y + \frac{2}{3}\kappa \geqslant 0$ on $\mathbb{R}$ for all $t \in [0, T]$. It follows from the combination of the two identities above that

$$-u(t,x) - \tfrac{2}{3}\kappa \leqslant u_x(t,x) \leqslant u(t,x) + \tfrac{2}{3}\kappa \quad \text{for all } (t,x) \in [0,T) \times \mathbb{R}.$$

In view of the L^∞-estimate (2.6), we then get

$$\begin{aligned} |u_x(t,x)| &\leqslant |u(t,x)| + \tfrac{2}{3}\kappa \\ &\leqslant 3(\|u_0\|_{L^2}^2 + \kappa\|u_0\|_{L^2})t + \|u_0\|_{L^\infty} + \tfrac{2}{3}\kappa \\ &\leqslant 3(\|u_0\|_{L^2}^2 + \|u_0\|_{L^2})T + \|u_0\|_{L^\infty} + \tfrac{2}{3} \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}. \end{aligned}$$

Step 2. We next prove that, for any $t \in [0, T]$, the solution $u(t, x)$ of (2.1) remains bounded in the space $H^s(\mathbb{R})$, $s > \frac{3}{2}$.

Indeed, taking the H^s inner product with u on both sides of (2.1) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_s^2 = -\langle u, (u^2)_x\rangle_s - (3 + 4\kappa)\langle u, \partial_x p * (u^2)\rangle_s.$$

Note that, by (5.4), applying lemma 5.2 and the Schwarz inequality, we obtain the estimates

$$\begin{aligned} |\langle u, (u^2)_x\rangle_s| &= |2\langle u, uu_x\rangle_s| \\ &= |2\langle \Lambda^s(uu_x), \Lambda^s u\rangle| \\ &\leqslant 2|\langle u\Lambda^s u_x, \Lambda^s u\rangle| + |\langle [\Lambda^s, u]u_x, \Lambda^s u\rangle| \\ &\leqslant C(\|u_x\|_{L^\infty}\|u\|_s^2 + \|[\Lambda^s, u]u_x\|_{L^2}\|\Lambda^s u\|_{L^2}) \\ &\leqslant C\|u_x\|_{L^\infty}(\|u\|_s^2 + \|\Lambda^s u\|_{L^2}^2) \leqslant C\|u\|_s^2 \end{aligned} \tag{5.5}$$

and

$$|\langle u, \partial_x p * (u^2)\rangle_s| \leqslant \|u\|_s\|\partial_x p * (u^2)\|_s \leqslant \|u\|_s\|u^2\|_{s-1} \leqslant C\|u\|_s^2. \tag{5.6}$$

The combination of (5.5) and (5.6) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,\cdot)\|_s^2 \leqslant C\|u(t,\cdot)\|_s^2.$$

Gronwall's inequality guarantees that there exists a constant $K(\|u_0\|_s, T) > 0$ such that

$$\|u(t,\cdot)\|_s \leqslant K \quad \text{for all } t \in [0,T]. \tag{5.7}$$

Step 3. Finally, we prove that $\|(u^{\kappa_1} - u^{\kappa_2})(t,\cdot)\|_{L^2} \to 0$ uniformly for all $t \in [0, T]$, as $\kappa_1, \kappa_2 \to 0$. For simplicity, set $u = u^{\kappa_1}(t, x)$ and $v = u^{\kappa_2}(t, x)$ and $w = u - v$. Then w satisfies

$$w_t = -uw_x - v_x w - 2\kappa_1\partial_x p * w - 2(\kappa_1 - \kappa_2)\partial_x p * v - \tfrac{3}{2}\partial_x p * ((u + v)w).$$

We multiply the above equation by w and integrate over $\mathbb{R}$ with respect to x. Integration by parts, Hölder's and Young's inequalities, yield, by the estimate (2.5) and (5.4), that

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 &= -\int_{\mathbb{R}} uww_x\,\mathrm{d}x - \int_{\mathbb{R}} w^2 v_x\,\mathrm{d}x - 2\kappa_1\int_{\mathbb{R}} w\partial_x p * w\,\mathrm{d}x \\
&\quad - 2(\kappa_1-\kappa_2)\int_{\mathbb{R}} w\partial_x p * v\,\mathrm{d}x - \frac{3}{2}\int_{\mathbb{R}} w\partial_x p * (w(u+v))\,\mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}} u_x w^2\,\mathrm{d}x - \int_{\mathbb{R}} w^2 v_x\,\mathrm{d}x - 2\kappa_1\int_{\mathbb{R}} w\partial_x p * w\,\mathrm{d}x \\
&\quad - 2(\kappa_1-\kappa_2)\int_{\mathbb{R}} w\partial_x p * v\,\mathrm{d}x - \frac{3}{2}\int_{\mathbb{R}} w\partial_x p * (w(u+v))\,\mathrm{d}x \\
&\leqslant C(\|u_x\|_{L^\infty}\|w\|_{L^2}^2 + \|v_x\|_{L^\infty}\|w\|_{L^2}^2 + \|w\|_{L^2}\|\partial_x p * (w(u+v))\|_{L^2} \\
&\qquad + \|w\|_{L^2}\|\partial_x p * w\|_{L^2} + |\kappa_1-\kappa_2|\|w\|_{L^2}\|\partial_x p * v\|_{L^2}) \\
&\leqslant C(\|u_x\|_{L^\infty}\|w\|_{L^2}^2 + \|v_x\|_{L^\infty}\|w\|_{L^2}^2 + \|w\|_{L^2}\|w(u+v)\|_{L^1}\|\partial_x p\|_{L^2} \\
&\qquad + \|w\|_{L^2}^2\|\partial_x p\|_{L^1} + |\kappa_1-\kappa_2|\|w\|_{L^2}\|v\|_{L^2}\|\partial_x p\|_{L^1}) \\
&\leqslant C(\|u_x\|_{L^\infty}\|w\|_{L^2}^2 + \|v_x\|_{L^\infty}\|w\|_{L^2}^2 + \|w\|_{L^2}^2\|u+v\|_{L^2}\|\partial_x p\|_{L^2} \\
&\qquad + \|w\|_{L^2}^2\|\partial_x p\|_{L^1} + |\kappa_1-\kappa_2|\|w\|_{L^2}\|v\|_{L^2}\|\partial_x p\|_{L^1}) \\
&\leqslant C(\|w\|_{L^2}^2 + |\kappa_1-\kappa_2|^2).
\end{aligned}
$$

Therefore, by Gronwall's inequality, we obtain

$$\|u-v\|_{L^2}^2 = \|w\|_{L^2}^2 \leqslant C|\gamma_1-\gamma_2|^2,$$

which implies that $\{u^\kappa\}$ is a Cauchy sequence in L^2 as $\kappa \to 0$, uniformly with respect to $t \in [0,T]$. □

We now consider the strong-limit behaviour as $\kappa \to 0$. Using theorem 5.1, as $\kappa \to 0$, it is claimed that the Cauchy sequence of solutions of (2.1) locally strongly converges to the solution of the Cauchy problem of the DP equation.

We thus consider the corresponding Cauchy problem for the DP equation, namely,

$$
\left.
\begin{aligned}
u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad t>0, \quad x\in\mathbb{R}, \\
u(0,x) = u_0(x), \qquad\qquad\qquad x\in\mathbb{R}.
\end{aligned}
\right\} \tag{5.8}
$$

THEOREM 5.3. *Under the assumptions of theorem 5.1, u^κ converges to the solution of the Cauchy problem (5.8) in $H^s(\mathbb{R})$ as $\kappa \to 0$.*

Proof. According to theorem 5.1, $\{u^\kappa\}$ is a Cauchy sequence in L^2 as $\kappa \to 0$. Let $u^\star = \lim_{\kappa\to 0} u^\kappa$ in L^2. We now prove that $u^\star$ is the strong solution of (5.8).

Since u^κ is the solution of (2.1), we may infer that

$$u^\kappa(t) = S^\kappa(t)u_0 + \int_0^t S^\kappa(t-\tau)G(u^\kappa)\,\mathrm{d}\tau, \tag{5.9}$$

where

$$
\left.
\begin{aligned}
S^\kappa(t)v &= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{i(\xi x - \kappa\xi t)}\hat{v}(\xi)\,\mathrm{d}\xi, \\
G(v) &= -vv_x - \partial_x p * (\tfrac{3}{2}v^2 + 2\kappa v).
\end{aligned}
\right\} \tag{5.10}
$$

Moreover, $S^\kappa(t)$ satisfies $\|S^\kappa(t)v\|_s = \|v\|_s$ for all $v \in H^s$, $s \geqslant 0$.

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For $t \in [0,T]$ and $\tau \in [0,t]$, by (5.4) and (5.7) we may infer that

$$\begin{aligned} |S^\kappa(t-\tau)G(u^\kappa)| &\leqslant \|S^\kappa(t-\tau)G(u^\kappa)\|_{s-1} \\ &= \|G(u^\kappa)\|_{s-1} \\ &= \|-u^\kappa u^\kappa_x - \partial_x p * (\tfrac{3}{2}(u^\kappa)^2 + 2\kappa u^\kappa)\|_{s-1} \\ &\leqslant \|u^\kappa\|_\infty \|u^\kappa_x\|_{s-1} + \|\tfrac{3}{2}(u^\kappa)^2 + 2\kappa u^\kappa\|_{s-2} \\ &\leqslant (\tfrac{5}{2}\|u^\kappa\|_\infty + 2\kappa)\|u^\kappa\|_s \\ &\leqslant CK. \end{aligned}$$

Let $\kappa \to 0$ in (5.9). In view of the above estimate, it follows from the Lebesgue dominated convergence theorem that

$$u^\star(t) = S(t)u_0 + \int_0^t S(t-\tau)G(u^\star)\,\mathrm{d}\tau,$$

where $S(t)v = S_0(t)v$. Hence, $u^\star \in L^\infty([0,T];L^2)$ is the solution of (5.8). However, by theorem 2.1, we know that (5.8) admits a unique solution in $C([0,\infty);H^s)$, $s > \frac{3}{2}$. This implies that $u^\star$ is the strong solution of (5.8) and completes the proof. □

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